Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 5: Continuity and Differentiability

About this Chapter

The Miscellaneous Exercise stitches together every technique from the chapter: chain rule, product/quotient/log differentiation, inverse-trig substitutions, parametric and implicit differentiation, and second-order derivatives. The questions are longer and require choosing the right tool first.

Topics covered: Composite functions • Logarithmic differentiation • Inverse trig substitutions • Parametric & implicit second derivatives

Quick Formula Sheet

Curvature-style formula (Q15): $\dfrac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$ is the (signed) radius of curvature of the curve $y = f(x)$.

Key identities: $1 \pm \sin x = (\cos x/2 \pm \sin x/2)^2$, $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$.

Generic trick: Wherever an inverse-trig expression hides a multiple-angle, the cleanest path is a $\tan\theta, \sin\theta$ or $\cos\theta$ substitution.

Miscellaneous Exercise on Chapter 5

Q 5.1 **Differentiate** $(3x^2 - 9x + 5)^9$ **with respect to** x.

SOLUTION

Concept used. Chain rule on a power: $\dfrac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$.

Step 1. Let $u = 3x^2 - 9x + 5$. Then $u' = 6x - 9 = 3(2x - 3)$.

Step 2. Apply chain rule:

$$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 \cdot (6x - 9) = 27(2x - 3)(3x^2 - 9x + 5)^8.$$

Final Answer: $\frac{dy}{dx} = 27(2x - 3)(3x^2 - 9x + 5)^8.$

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Outer power and inner quadratic; multiply $9 \cdot (6x - 9) = 9(6x - 9)$ and factor.

Step 1. Factor: $9(6x - 9) = 27(2x - 3)$.

Final Answer: $27(2x - 3)(3x^2 - 9x + 5)^8$.

Q 5.2 **Differentiate** $\sin^3 x + \cos^6 x$ **with respect to** x.

SOLUTION

Concept used. Chain rule on each term: $(f(x))^n \mapsto n(f(x))^{n-1} f'(x)$.

Step 1. $\frac{d}{dx}(\sin^3 x) = 3\sin^2 x \cos x$.

Step 2. $\frac{d}{dx}(\cos^6 x) = 6\cos^5 x(-\sin x) = -6\cos^5 x \sin x$.

Step 3. Sum:

$$\frac{dy}{dx} = 3\sin^2 x \cos x - 6\cos^5 x \sin x = 3\sin x \cos x(\sin x - 2\cos^4 x).$$

Final Answer: $\frac{dy}{dx} = 3\sin x \cos x(\sin x - 2\cos^4 x)$.

EXPERT'S SOLUTION : *Sneha Iyer, Ph.D Mathematics, IIT Delhi*

Quick reading. Two power-chain rules summed.

Step 1. $3\sin^2 x \cos x$ from the cube of sine.

Step 2. $-6\cos^5 x \sin x$ from the sixth power of cosine.

Final Answer: $3\sin^2 x\cos x - 6\cos^5 x\sin x$.

Q 5.3 **Differentiate $(5x)^{3\cos 2x}$ with respect to x.**

SOLUTION

Concept used. Variable base AND variable exponent: take $\log$ of both sides.

Step 1. Let $y = (5x)^{3\cos 2x}$. Take $\log$:

$$\log y = 3\cos 2x \cdot \log(5x).$$

Step 2. Differentiate (product rule on right):

$$\frac{1}{y}\frac{dy}{dx} = -6\sin 2x \cdot \log(5x) + 3\cos 2x \cdot \frac{1}{5x} \cdot 5 = -6\sin 2x\log(5x) + \frac{3\cos 2x}{x}.$$

Step 3. Multiply by $y = (5x)^{3\cos 2x}$:

$$\frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x\log(5x)\right].$$

Final Answer: $\frac{dy}{dx} = (5x)^{3\cos 2x}\left[\frac{3\cos 2x}{x} - 6\sin 2x\log(5x)\right]$.

EXPERT'S SOLUTION *: Vivaan Gupta, M.Tech CS, IIT Madras*

Quick reading. Log-diff with product rule on $3\cos 2x \cdot \log(5x)$.

Step 1. $\log y = 3\cos 2x\log(5x)$.

Step 2. Differentiate; multiply by y.

Final Answer: $(5x)^{3\cos 2x}\left[3\cos 2x/x - 6\sin 2x\log(5x)\right]$.

Q 5.4 **Differentiate $\sin^{-1}(x\sqrt{x})$ with respect to x, $0 \le x \le 1$.**

SOLUTION

Concept used. Chain rule. $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\cdot u'$. Here $u = x\sqrt{x} = x^{3/2}$, $u' = \frac{3}{2}x^{1/2}$.

Step 1. Note $x\sqrt{x} = x^{3/2}$, so $u^2 = x^3$.

Step 2. $\frac{du}{dx} = \frac{3}{2}\sqrt{x}$.

Step 3. Chain rule:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^3}}\cdot\frac{3\sqrt{x}}{2} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}}.$$

Final Answer: $\frac{dy}{dx} = \frac{3\sqrt{x}}{2\sqrt{1-x^3}}\ (0 \le x < 1)$.

EXPERT'S SOLUTION *: Aanya Mehta, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. Treat $x\sqrt{x}$ as $x^{3/2}$ before differentiating.

Step 1. $u = x^{3/2}$, $u' = (3/2)x^{1/2}$.

Step 2. Chain rule with $\sin^{-1}$.

Final Answer: $3\sqrt{x}/(2\sqrt{1-x^3})$.

Q 5.5 **Differentiate $\frac{\cos^{-1}(x/2)}{\sqrt{2x+7}}$ with respect to x, $-2 < x < 2$.**

SOLUTION

Concept used. Quotient rule with chain rule on each piece.

Step 1. Numerator derivative:

$$\frac{d}{dx}\cos^{-1}\left(\frac{x}{2}\right) = -\frac{1}{\sqrt{1-x^2/4}}\cdot\frac{1}{2} = -\frac{1}{\sqrt{4-x^2}}.$$

(Multiplied numerator and denominator by 2.)

Step 2. Denominator derivative:

$$\frac{d}{dx}\sqrt{2x+7} = \frac{1}{2\sqrt{2x+7}}\cdot 2 = \frac{1}{\sqrt{2x+7}}.$$

Step 3. Quotient rule with $u = \cos^{-1}(x/2)$, $v = \sqrt{2x+7}$:

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{-\dfrac{\sqrt{2x+7}}{\sqrt{4-x^2}} - \cos^{-1}(x/2)\cdot\dfrac{1}{\sqrt{2x+7}}}{2x+7}.$$

Step 4. Compute the two pieces of the quotient rule separately:

$$\frac{u'v}{v^2} = \frac{-\sqrt{2x+7}/\sqrt{4-x^2}}{2x+7} = -\frac{1}{(2x+7)^{1/2}\sqrt{4-x^2}}.$$

$$-\frac{uv'}{v^2} = -\frac{\cos^{-1}(x/2)/\sqrt{2x+7}}{2x+7} = -\frac{\cos^{-1}(x/2)}{(2x+7)^{3/2}}.$$

Add:

$$\frac{dy}{dx} = -\frac{1}{\sqrt{(2x+7)(4-x^2)}} - \frac{\cos^{-1}(x/2)}{(2x+7)^{3/2}}.$$

Final Answer: $\dfrac{dy}{dx} = -\dfrac{1}{\sqrt{(2x+7)(4-x^2)}} - \dfrac{\cos^{-1}(x/2)}{(2x+7)^{3/2}}$.

EXPERT'S SOLUTION : *Priya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Quotient rule with separate derivative for each piece.

Step 1. Compute u' and v' via chain rule.

Step 2. Plug into quotient formula and simplify.

Final Answer: See main.

Q 5.6 **Differentiate the function below with respect to x for $0 < x < \pi/2$:**

$$\cot^{-1}\left[\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right].$$

SOLUTION

Concept used. Simplify the bracket using $1 \pm \sin x = \left(\cos\frac{x}{2} \pm \sin\frac{x}{2}\right)^2$, then use the cotangent of a difference.

Step 1. For $0 < x < \pi/2$, both $\cos(x/2) > \sin(x/2) > 0$. Then

$$\sqrt{1+\sin x} = \cos(x/2) + \sin(x/2), \qquad \sqrt{1-\sin x} = \cos(x/2) - \sin(x/2).$$

Step 2. Substitute into the bracket:

$$\text{Numerator} = [\cos(x/2) + \sin(x/2)] + [\cos(x/2) - \sin(x/2)] = 2\cos(x/2),$$
$$\text{Denominator} = [\cos(x/2) + \sin(x/2)] - [\cos(x/2) - \sin(x/2)] = 2\sin(x/2).$$

Hence the bracket simplifies to

$$\frac{2\cos(x/2)}{2\sin(x/2)} = \cot(x/2).$$

Step 3. Therefore

$$y = \cot^{-1}(\cot(x/2)) = \frac{x}{2},$$

since $0 < x/2 < \pi/4$ lies in the principal range $(0, \pi)$ of $\cot^{-1}$.

Step 4. Differentiate:

$$\frac{dy}{dx} = \frac{1}{2}.$$

Final Answer: $\frac{dy}{dx} = \frac{1}{2}$.

☞ Exam Tip

Whenever you see $\sqrt{1 \pm \sin x}$, rewrite using half-angle squares. The simplification often turns a fearsome expression into a trivial constant or linear function.

EXPERT'S SOLUTION *: Karan Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. Half-angle squares collapse the bracket; the answer is then immediate.

Step 1. Bracket $\to \cot(x/2)$.

Step 2. $y = x/2$, $dy/dx = 1/2$.

Final Answer: $1/2$.

Q 5.7 **Differentiate** $(\log x)^{\log x}$ **with respect to** x, $x > 1$.

SOLUTION

Concept used. Variable base, variable exponent both equal to $\log x$: take $\log$.

Step 1. Let $y = (\log x)^{\log x}$. Take $\log$:

$$\log y = \log x \cdot \log(\log x).$$

Step 2. Differentiate (product rule):

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\log(\log x) + \log x \cdot \frac{1}{\log x} \cdot \frac{1}{x} = \frac{\log(\log x)}{x} + \frac{1}{x}.$$

Step 3. Multiply by y:

$$\frac{dy}{dx} = \frac{(\log x)^{\log x}}{x}\big[1 + \log(\log x)\big].$$

Final Answer: $\frac{dy}{dx} = \frac{(\log x)^{\log x}\big[1 + \log(\log x)\big]}{x}$ $(x > 1)$.

EXPERT'S SOLUTION *: Tara Pillai, Ph.D Mathematics, IIT Delhi*

Quick reading. The expression is a^a with $a = \log x$.

Step 1. Generic formula: $\frac{d}{dx}a^a = a^a(1 + \log a) \cdot a'$ where $a = a(x)$.

Step 2. Here $a = \log x$, $a' = 1/x$.

Final Answer: $(\log x)^{\log x}(1 + \log\log x)/x$.

Q 5.8 **Differentiate $\cos(a\cos x + b\sin x)$ with respect to x, where a, b are constants.**

SOLUTION

Concept used. Chain rule with inner $a\cos x + b\sin x$.

Step 1. Let $u = a\cos x + b\sin x$. Then $u' = -a\sin x + b\cos x$.

Step 2. Chain rule on $\cos u$:

$$\frac{dy}{dx} = -\sin u \cdot u' = -\sin(a\cos x + b\sin x)(b\cos x - a\sin x).$$

Final Answer: $\frac{dy}{dx} = -(b\cos x - a\sin x)\sin(a\cos x + b\sin x) = (a\sin x - b\cos x)\sin(a\cos x + b\sin x)$.

EXPERT'S SOLUTION : *Krishna Rao, M.Sc Mathematics, IIT Bombay*

Quick reading. Outer $\cos$, inner linear combination of $\sin$ and $\cos$.

Step 1. $u' = -a\sin x + b\cos x$.

Step 2. Chain: $-\sin u \cdot u'$.

Final Answer: $(a\sin x - b\cos x)\sin(a\cos x + b\sin x)$.

Q 5.9 **Differentiate** $(\sin x - \cos x)^{(\sin x - \cos x)}$ **with respect to** x, $\frac{\pi}{4} < x < \frac{3\pi}{4}$.

SOLUTION

Concept used. Variable base, variable exponent both $= \sin x - \cos x$: take $\log$.

Step 1. In $(\pi/4, 3\pi/4)$, $\sin x - \cos x > 0$, so $\log$ is well-defined.

Step 2. Let $y = (\sin x - \cos x)^{\sin x - \cos x}$, $a = \sin x - \cos x$. Then $y = a^a$ and $\log y = a\log a$. Differentiate:

$$\frac{1}{y}\frac{dy}{dx} = a'\log a + a\cdot\frac{a'}{a} = a'(1+\log a) = (\cos x + \sin x)(1 + \log(\sin x - \cos x)).$$

Step 3. Multiply by y:

$$\frac{dy}{dx} = (\sin x - \cos x)^{\sin x - \cos x}(\cos x + \sin x)(1 + \log(\sin x - \cos x)).$$

Final Answer: $\frac{dy}{dx} = (\sin x - \cos x)^{\sin x - \cos x}(\sin x + \cos x)\left[1 + \log(\sin x - \cos x)\right]$.

EXPERT'S SOLUTION : *Meera Chatterjee, M.Sc Mathematics, ISI Kolkata*

Structural observation. a^a form with $a = \sin x - \cos x$; standard log-diff applies.

Step 1. $a' = \cos x + \sin x$.

Step 2. $\dfrac{d}{dx}(a^a) = a^a(1+\log a)a'$.

Final Answer: See main.

Q 5.10 **Differentiate $x^x + x^a + a^x + a^a$ w.r.t. x, for fixed $a > 0$ and $x > 0$.**

SOLUTION

Concept used. Differentiate term-by-term.

Step 1. $\dfrac{d}{dx}x^x = x^x(1+\log x)$ (log-diff, Ex 5.5 Q4).

Step 2. $\dfrac{d}{dx}x^a = ax^{a-1}$ (power rule, a constant).

Step 3. $\dfrac{d}{dx}a^x = a^x \log a$ (exponential rule).

Step 4. $\dfrac{d}{dx}a^a = 0$ (constant).

Step 5. Sum:

$$\frac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a.$$

Final Answer: $\dfrac{dy}{dx} = x^x(1+\log x) + ax^{a-1} + a^x \log a.$

EXPERT'S SOLUTION *: Rohit Verma, M.Sc Mathematics, ISI Kolkata*

Quick reading. Four distinct shapes; recognise each.

Step 1. x^x: log-diff.

Step 2. x^a: power rule.

Step 3. a^x: exponential.

Step 4. a^a: constant.

Final Answer: Sum as in main.

Q 5.11 **Differentiate $x^{x^2-3} + (x-3)^{x^2}$ w.r.t. x, for $x > 3$.**

SOLUTION

Concept used. Two variable-base/variable-exponent terms; log-diff each separately.

Step 1. Let $u = x^{x^2-3}$**.** $\log u = (x^2-3)\log x$. Differentiate:

$$\frac{1}{u}\frac{du}{dx} = 2x\log x + (x^2-3)\cdot\frac{1}{x} = 2x\log x + x - \frac{3}{x}.$$

Hence

$$\frac{du}{dx} = x^{x^2-3}\left[2x\log x + x - \frac{3}{x}\right].$$

Step 2. Let $v = (x-3)^{x^2}$**.** $\log v = x^2\log(x-3)$. Differentiate:

$$\frac{1}{v}\frac{dv}{dx} = 2x\log(x-3) + x^2\cdot\frac{1}{x-3} = 2x\log(x-3) + \frac{x^2}{x-3}.$$

Hence

$$\frac{dv}{dx} = (x-3)^{x^2}\left[2x\log(x-3) + \frac{x^2}{x-3}\right].$$

Step 3. Sum:

$$\frac{dy}{dx} = x^{x^2-3}\left[2x\log x + x - \frac{3}{x}\right] + (x-3)^{x^2}\left[2x\log(x-3) + \frac{x^2}{x-3}\right].$$

Final Answer: See expression above.

EXPERT'S SOLUTION *: Pranav Kapoor, M.Sc Mathematics, IIT Bombay*

Strategic angle. Two log-diffs, then add.

Step 1. $u' = u[2x\log x + x - 3/x]$.

Step 2. $v' = v[2x\log(x-3) + x^2/(x-3)]$.

Final Answer: Sum as in main.

Q 5.12 **Find** $\frac{dy}{dx}$ **if** $y = 12(1-\cos t)$, $x = 10(t-\sin t)$, $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

SOLUTION

Concept used. Parametric differentiation with half-angle identities.

Step 1. Differentiate:

$$\frac{dx}{dt} = 10(1 - \cos t), \qquad \frac{dy}{dt} = 12 \sin t.$$

Step 2. Use half-angle: $1 - \cos t = 2\sin^2(t/2)$ and $\sin t = 2\sin(t/2)\cos(t/2)$. Then

$$\frac{dy}{dx} = \frac{12 \cdot 2\sin(t/2)\cos(t/2)}{10 \cdot 2\sin^2(t/2)} = \frac{12\cos(t/2)}{10\sin(t/2)} = \frac{6}{5}\cot(t/2).$$

Final Answer: $\frac{dy}{dx} = \frac{6}{5}\cot(t/2)$.

EXPERT'S SOLUTION *: Aditi Banerjee, M.Sc Mathematics, ISI Kolkata*

Quick reading. Half-angle cancellation.

Step 1. Numerator $\sin t = 2\sin(t/2)\cos(t/2)$.

Step 2. Denominator $1 - \cos t = 2\sin^2(t/2)$.

Step 3. Ratio $= (6/5)\cot(t/2)$.

Final Answer: $(6/5)\cot(t/2)$.

Q 5.13 **Find** $\frac{dy}{dx}$ **if** $y = \sin^{-1} x + \sin^{-1}\sqrt{1 - x^2}$, $0 < x < 1$.

SOLUTION

Concept used. Recognise that $\sin^{-1}\sqrt{1 - x^2} = \cos^{-1} x$ for $0 < x < 1$, and $\sin^{-1} x + \cos^{-1} x = \pi/2$.

Step 1. Let $x = \sin\theta$ with $\theta \in (0, \pi/2)$. Then $\sqrt{1 - x^2} = \cos\theta$, and $\sin^{-1}(\cos\theta) = \pi/2 - \theta = \pi/2 - \sin^{-1} x = \cos^{-1} x$.

Step 2. Substitute:

$$y = \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

Step 3. Differentiate the constant:

$$\frac{dy}{dx} = 0.$$

Final Answer: $\frac{dy}{dx} = 0$.

♥ **Complement identity**

For $-1 \le x \le 1$, $\sin^{-1} x + \cos^{-1} x = \pi/2$ identically. So any expression that recombines them is just a constant in disguise.

EXPERT'S SOLUTION *: Aditya Patel, M.Sc Mathematics, IIT Bombay*

Strategic angle. Spot the identity; the sum is constant.

Step 1. $\sin^{-1} \sqrt{1-x^2} = \cos^{-1} x$ on $(0, 1)$.

Step 2. Sum: $\sin^{-1} x + \cos^{-1} x = \pi/2$.

Step 3. Derivative: 0.

Final Answer: 0.

Q 5.14 **If** $x\sqrt{1+y} + y\sqrt{1+x} = 0$**, for** $-1 < x < 1$**, prove that** $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$**.**

SOLUTION

Concept used. Solve the implicit equation for y as an explicit function of x, then differentiate.

Step 1. Rewrite:

$$x\sqrt{1+y} = -y\sqrt{1+x}.$$

Square both sides:

$$x^2(1+y) = y^2(1+x).$$

Step 2. Expand:

$$x^2 + x^2y = y^2 + xy^2 \implies x^2 - y^2 + x^2y - xy^2 = 0.$$

Factor:

$$(x-y)(x+y) + xy(x-y) = (x-y)(x+y+xy) = 0.$$

Step 3. Either $x = y$ or $x + y + xy = 0$. The first contradicts the original (substitute $y = x$: $2x\sqrt{1+x} = 0 \Rightarrow x = 0$ trivially). The non-trivial branch:

$$x + y + xy = 0 \implies y(1+x) = -x \implies y = -\frac{x}{1+x}.$$

Step 4. Differentiate this explicit formula (quotient rule):

$$\frac{dy}{dx} = -\frac{(1)(1+x) - x(1)}{(1+x)^2} = -\frac{1}{(1+x)^2}.$$

Final Answer: $\dfrac{dy}{dx} = -\dfrac{1}{(1+x)^2}$.

EXPERT'S SOLUTION *: Ishita Reddy, Ph.D Mathematics, IIT Delhi*

Structural observation. The square trick converts the radical equation to a polynomial that factors.

Step 1. Square to $x^2(1+y) = y^2(1+x)$.

Step 2. Factor as $(x-y)(x+y+xy) = 0$.

Step 3. Use $y = -x/(1+x)$.

Final Answer: $-1/(1+x)^2$.

Q 5.15 **If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that**

$$\frac{[1+(dy/dx)^2]^{3/2}}{d^2y/dx^2}$$

is a constant independent of a and b.

SOLUTION

Concept used. Implicit differentiation twice. The result is the (negative) radius of curvature of the circle, which is the constant c (or $-c$).

Step 1. Differentiate implicitly:

$$2(x-a) + 2(y-b)\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x-a}{y-b}.$$

Step 2. Differentiate again. Quotient rule on $-(x-a)/(y-b)$:

$$\frac{d^2y}{dx^2} = -\frac{(1)(y-b) - (x-a)(dy/dx)}{(y-b)^2} = -\frac{y-b-(x-a)\big(-(x-a)/(y-b)\big)}{(y-b)^2}.$$

Simplify the numerator:

$$y - b + \frac{(x-a)^2}{y-b} = \frac{(y-b)^2 + (x-a)^2}{y-b} = \frac{c^2}{y-b}.$$

Therefore

$$\frac{d^2y}{dx^2} = -\frac{c^2}{(y-b)^3}.$$

Step 3. Compute $1 + (dy/dx)^2$:

$$1 + \frac{(x-a)^2}{(y-b)^2} = \frac{(y-b)^2 + (x-a)^2}{(y-b)^2} = \frac{c^2}{(y-b)^2}.$$

So $[1 + (dy/dx)^2]^{3/2} = \dfrac{c^3}{|y-b|^3}$.

Step 4. Take the ratio:

$$\frac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} = \frac{c^3/|y-b|^3}{-c^2/(y-b)^3} = -\frac{c^3(y-b)^3}{c^2|y-b|^3}.$$

Since $(y-b)^3/|y-b|^3 = \text{sgn}(y-b)$, the ratio is $\pm c$. Taking $y - b > 0$ (the upper semicircle), the ratio equals $-c$, a constant independent of a, b.

Final Answer: The expression equals $-c$ (constant), independent of a, b.

EXPERT'S SOLUTION : *Kavya Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Use the identity $(x-a)^2 + (y-b)^2 = c^2$ at every step to collapse expressions.

Step 1. $dy/dx = -(x-a)/(y-b)$.

Step 2. $d^2y/dx^2 = -c^2/(y-b)^3$.

Step 3. Ratio: $\pm c$, constant.

Final Answer: Constant $= \pm c$.

Q 5.16 **If** $\cos y = x\cos(a+y)$ **with** $\cos a \neq \pm 1$**, prove that** $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$**.**

SOLUTION

Concept used. Solve for x in terms of y first, then differentiate $\dfrac{dx}{dy}$ and use $\dfrac{dy}{dx} = 1/\dfrac{dx}{dy}$.

Step 1. From $\cos y = x\cos(a+y)$:

$$x = \frac{\cos y}{\cos(a+y)}.$$

Step 2. Differentiate w.r.t. y using quotient rule:

$$\frac{dx}{dy} = \frac{-\sin y\cos(a+y) - \cos y \cdot (-\sin(a+y))}{\cos^2(a+y)}$$
$$= \frac{-\sin y\cos(a+y) + \cos y\sin(a+y)}{\cos^2(a+y)}.$$

Step 3. Recognise the numerator as $\sin((a+y)-y) = \sin a$:

$$\frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}.$$

Step 4. Invert:

$$\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}.$$

Final Answer: $\dfrac{dy}{dx} = \dfrac{\cos^2(a+y)}{\sin a}$.

EXPERT'S SOLUTION *: Yash Nair, M.Tech CS, IIT Madras*

Strategic angle. Differentiate w.r.t. y (not x); much cleaner here.

Step 1. $x = \cos y/\cos(a+y)$.

Step 2. $dx/dy = \sin a/\cos^2(a+y)$.

Step 3. Invert.

Final Answer: $\cos^2(a+y)/\sin a$.

Q 5.17 **If** $x = a(\cos t + t\sin t)$ **and** $y = a(\sin t - t\cos t)$**, find** $\dfrac{d^2y}{dx^2}$**.**

SOLUTION

Concept used. Parametric second derivative: $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right) \cdot \dfrac{dt}{dx}$.

Step 1. From Ex 5.6 Q10: $dx/dt = at\cos t$, $dy/dt = at\sin t$, so $dy/dx = \tan t$.

Step 2. Differentiate $\tan t$ w.r.t. x:

$$\frac{d^2y}{dx^2} = \frac{d(\tan t)}{dt} \cdot \frac{dt}{dx} = \sec^2 t \cdot \frac{1}{at\cos t} = \frac{\sec^3 t}{at}.$$

Final Answer: $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3 t}{at}$.

EXPERT'S SOLUTION : *Krishna Banerjee, Ph.D Mathematics, IIT Delhi*

Strategic angle. Build on Ex 5.6 Q10's result $dy/dx = \tan t$.

Step 1. $d(\tan t)/dt = \sec^2 t$.

Step 2. Divide by $dx/dt = at\cos t$ to get $\sec^2 t/(at\cos t) = \sec^3 t/(at)$.

Final Answer: $\sec^3 t/(at)$.

Q 5.18 **If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.**

SOLUTION

Concept used. Write piecewise: $f(x) = x^3$ for $x \geq 0$, $f(x) = -x^3$ for $x < 0$. Compute f' and f'' on each piece; check at 0.

Step 1. For $x > 0$: $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$.

Step 2. For $x < 0$: $f(x) = -x^3$, $f'(x) = -3x^2$, $f''(x) = -6x = 6|x|$.

Step 3. At $x = 0$: LHD of f uses $-x^3$ and gives $\lim_{h\to 0^-}(-h^3)/h = 0$; RHD uses x^3 giving $\lim_{h\to 0^+} h^3/h = 0$. So $f'(0) = 0$. Similarly for f'': from left $f''(x) = -6x \to 0$, from right $f''(x) = 6x \to 0$. Combined with $f'(0) = 0$, the second-derivative limit is 0, so $f''(0) = 0$.

Step 4. Unified formula: $f''(x) = 6|x|$ for all real x.

Final Answer: $f''(x) = 6|x|$ for every real x.

EXPERT'S SOLUTION : *Vivaan Mehta, Ph.D Mathematics, IIT Delhi*

Picture-first. $f(x) = |x|^3$ is even, smooth except possibly at 0. The first derivative $3x|x|$ exists everywhere (slope 0 at origin); the second derivative is $6|x|$ which is continuous at 0 with value 0.

Step 1. Piecewise formulas join smoothly at 0 through second derivative.

Final Answer: $f''(x) = 6|x|$.

Q 5.19 **Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and differentiation, obtain the sum formula for cosines.**

SOLUTION

Concept used. Differentiate the given identity with respect to a chosen variable; the derivative is the partner cosine identity.

Step 1. Start with the identity $\sin(A+B) = \sin A \cos B + \cos A \sin B$, treating B as a constant and A as the variable.

Step 2. Differentiate w.r.t. A. Use $\frac{d}{dA}\sin(A+B) = \cos(A+B)$:

$$\cos(A+B) = \cos A \cos B + (-\sin A)\sin B = \cos A \cos B - \sin A \sin B.$$

Step 3. This is the sum formula for cosines.

Final Answer: $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

EXPERT'S SOLUTION *: Karan Joshi, M.Sc Mathematics, IIT Bombay*

Why this matters. Each trig identity is paired with another by differentiation. Knowing one lets you recover the other without re-deriving from scratch.

Step 1. Differentiate $\sin(A+B)$ formula w.r.t. A.

Step 2. Result is $\cos(A+B)$ formula.

Final Answer: Verified.

Q 5.20 **Does there exist a function which is continuous everywhere but not differentiable at exactly two points? Justify your answer.**

SOLUTION

Concept used. Continuity is preserved by sums and compositions; modulus functions $|x-a|$ are continuous but not differentiable at $x = a$.

Step 1. Yes. Consider $f(x) = |x| + |x-1|$.

Step 2. **Continuity.** Each modulus is continuous on $\mathbb{R}$; sum of continuous functions is continuous. Hence f is continuous everywhere.

Step 3. **Differentiability.** $|x|$ is not differentiable at $x=0$; $|x-1|$ is not differentiable at $x=1$. At every other point both modulus functions are differentiable, so the sum is differentiable there. Hence f fails to be differentiable at exactly the two points $x=0$ and $x=1$.

Step 4. This is an explicit example proving the existence.

Final Answer: Yes; e.g. $f(x) = |x|+|x-1|$ is continuous on $\mathbb{R}$ but not differentiable at $x=0$ and $x=1$.

EXPERT'S SOLUTION *: Sneha Bhat, Ph.D Mathematics, IIT Delhi*

Picture-first. Two V-vertices at $x=0$ and $x=1$, with linear segments connecting them. Two corners, two non-differentiable points.

Step 1. Show example $f = |x| + |x-1|$.

Step 2. Continuous (sum of continuous).

Step 3. Non-differentiable only at the two corner abscissae.

Final Answer: Yes, $|x| + |x-1|$ works.

Q 5.21 **If** $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$**, prove that** $\dfrac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$.

SOLUTION

Concept used. Expand the determinant along the first row. The constants l, m, n, a, b, c in the lower two rows do not depend on x; only the first row varies.

Step 1. Cofactor expansion of y along row 1:

$$y = f(x)(mc - nb) - g(x)(lc - na) + h(x)(lb - ma).$$

Note: $mc - nb$, $lc - na$, $lb - ma$ are constants.

Step 2. Differentiate term-by-term:

$$\frac{dy}{dx} = f'(x)(mc - nb) - g'(x)(lc - na) + h'(x)(lb - ma).$$

Step 3. Re-assemble into a 3×3 determinant. The constants $mc - nb$, etc., are exactly the cofactors of the first row of a determinant with the same lower two rows. Hence

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

Final Answer: Verified.

EXPERT'S SOLUTION : *Aditi Verma, M.Sc Mathematics, IIT Bombay*

Why this matters. Determinants are linear in each row; differentiating a row $\Rightarrow$ differentiating each row separately. With only one variable row, the rule is the special case shown.

Step 1. Linearity of determinant in row 1.

Step 2. Differentiate row 1 entries.

Final Answer: See main.

Q 5.22 **If $y = e^{a\cos^{-1}x}$, $-1 \le x \le 1$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0$.**

SOLUTION

Concept used. Differentiate, multiply by $\sqrt{1-x^2}$, differentiate again; use $(\cos^{-1}x)' = -1/\sqrt{1-x^2}$.

Step 1. First derivative (chain rule):

$$\frac{dy}{dx} = e^{a\cos^{-1}x} \cdot a \cdot \left(-\frac{1}{\sqrt{1-x^2}}\right) = -\frac{ay}{\sqrt{1-x^2}}.$$

Square both sides:

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = a^2y^2. \qquad \text{...}(*)$$

Step 2. Differentiate $(*)$ w.r.t. x:

$$-2x\left(\frac{dy}{dx}\right)^2 + (1-x^2) \cdot 2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2a^2y\frac{dy}{dx}.$$

Step 3. Divide through by $2\,dy/dx$ (assuming non-zero, the case $dy/dx = 0$ holds trivially):

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = a^2y.$$

Step 4. Rearrange:

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0.$$

Final Answer: $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0.$

EXPERT'S SOLUTION : *Ishaan Patel, B.Tech Engineering Physics, IIT Bombay*

Strategic angle. Square the first derivative equation to eliminate $\sqrt{1-x^2}$, then differentiate.

Step 1. $\sqrt{1-x^2}\,y' = -ay$, square to get rid of root.

Step 2. Differentiate, simplify, divide by $2y'$.

Final Answer: ODE verified.

Key Takeaways

- The Miscellaneous problems mix every technique; identify the right tool (chain, log-diff, parametric, implicit) before computing.
- Half-angle identities collapse $\sqrt{1 \pm \sin x}$, $1 \pm \cos\theta$ expressions; complement identities like $\sin^{-1}x + \cos^{-1}x = \pi/2$ collapse pairs of inverse-trig terms.
- For ODE proofs, multiply by the right factor (often $(1-x^2)$, $1+x^2$ or x) at the right moment to make subsequent differentiation telescope.

End of Miscellaneous Exercise on Chapter 5