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Chapter 6: Application of Derivatives

About this Chapter

Exercise 6.3 of Chapter 6 **Application of Derivatives** drills the toolbox of **maxima and minima**: classifying turning points with the first and second derivative tests, locating absolute extrema on a closed interval, and translating word problems into optimisation calculus. Every solution below names the test it uses, derives the relevant derivatives in full, and sketches the geometric setup wherever a picture clarifies the variable choice.

Topics covered: Maxima and Minima • First Derivative Test • Second Derivative Test • Absolute Extrema • Geometric Optimisation

Quick Formula Sheet

Critical point:
$f'(c) = 0$ or $f'(c)$ does not exist.

Second derivative test:
$f'(c) = 0,\ f''(c) < 0 \Rightarrow$ local max.
$f'(c) = 0,\ f''(c) > 0 \Rightarrow$ local min.

Absolute extrema on $[a, b]$: compare f at all critical points and at the endpoints a, b.

Exercise 6.3

Q 6.1 **Find the maximum and minimum values, if any, of the following functions given by**

(i) $f(x) = (2x-1)^2 + 3$ **(ii)** $f(x) = 9x^2 + 12x + 2$
(iii) $f(x) = -(x-1)^2 + 10$ **(iv)** $g(x) = x^3 + 1$

SOLUTION

Concept used. For any real number t, the square $t^2 \geq 0$, with equality only when $t = 0$. Consequently a function of the form $A\,t^2 + C$ with $A > 0$ has its **minimum** value C at $t = 0$ and no maximum (it grows without bound as $|t| \to \infty$). With $A < 0$ the situation flips: the function has its **maximum** value C at $t = 0$ and no minimum. A pure cubic x^3

is strictly increasing on $\mathbb{R}$, so x^3+1 has neither a maximum nor a minimum on $\mathbb{R}$.

Step 1. **(i)** Write $f(x) = (2x-1)^2 + 3$. The squared term $(2x-1)^2 \geq 0$, so

$$f(x) \geq 0 + 3 = 3,$$

with equality when $2x - 1 = 0$, that is $x = \frac{1}{2}$. As $|x| \to \infty$, $(2x-1)^2 \to \infty$, so $f(x) \to \infty$.
Minimum value $= 3$ at $x = \frac{1}{2}$; no maximum.

Step 2. **(ii)** Complete the square in $f(x) = 9x^2 + 12x + 2$:

$$f(x) = 9\left(x^2 + \tfrac{4}{3}x\right) + 2 = 9\left(x + \tfrac{2}{3}\right)^2 - 9 \cdot \tfrac{4}{9} + 2 = 9\left(x + \tfrac{2}{3}\right)^2 - 2.$$

The squared term is ≥ 0, so $f(x) \geq -2$, with equality at $x = -\frac{2}{3}$.
Minimum value $= -2$ at $x = -\frac{2}{3}$; no maximum.

Step 3. **(iii)** Here $f(x) = -(x-1)^2 + 10$. Since $(x-1)^2 \geq 0$,

$$-(x-1)^2 \leq 0 \;\Rightarrow\; f(x) \leq 10,$$

with equality at $x = 1$. As $|x| \to \infty$, $-(x-1)^2 \to -\infty$.
Maximum value $= 10$ at $x = 1$; no minimum.

Step 4. **(iv)** $g(x) = x^3 + 1$. The derivative $g'(x) = 3x^2 \geq 0$ with equality only at $x = 0$, so g is strictly increasing on $\mathbb{R}$. As $x \to \infty$, $g \to \infty$; as $x \to -\infty$, $g \to -\infty$.
Hence neither a maximum nor a minimum value exists.

Final Answer: (i) min $= 3$ at $x = \frac{1}{2}$, no max. (ii) min $= -2$ at $x = -\frac{2}{3}$, no max. (iii) max $= 10$ at $x = 1$, no min. (iv) neither maximum nor minimum.

☞ Exam Tip

For any quadratic $ax^2 + bx + c$, completing the square instantly tells you the extremum. With $a > 0$ the minimum is $c - \frac{b^2}{4a}$ at $x = -\frac{b}{2a}$; with $a < 0$ that same value becomes the maximum at the same x. This saves all the derivative work.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Each part can be settled by inspection of how the function is *built*. A squared term is always non-negative; a negated squared term is always non-positive; an unbounded monotone function has no extremum.

Step 1. Part (i): the only way to make $(2x-1)^2$ small is to make $2x - 1 = 0$. At $x = \frac{1}{2}$ this term vanishes, leaving $f = 3$. Pushing x far from $\frac{1}{2}$ blows the square up.
Conclusion: minimum 3, no maximum.

Step 2. Part (ii): rewrite $9x^2 + 12x + 2$ as $9\left(x + \frac{2}{3}\right)^2 - 2$. The squared term contributes ≥ 0, so the smallest possible value of f is -2, achieved at $x = -\frac{2}{3}$. No upper bound.

Step 3. Part (iii): $-(x-1)^2$ peaks at 0 when $x = 1$, giving $f = 10$. Driving x to $\pm\infty$ sends $f \to -\infty$. So maximum 10, no minimum.

Step 4. Part (iv): differentiate $g(x) = x^3 + 1$ to get $g'(x) = 3x^2$. This is zero only at $x = 0$ but does not change sign there: $g'(x) > 0$ on both sides. So $x = 0$ is a *point of inflexion*, not an extremum, and g is monotonically increasing across $\mathbb{R}$ with no global bounds.

Why this matters. The fastest way to attack "find max/min" questions is to check first whether the function is monotone, then whether it can be rewritten as a shifted squared term. The derivative only enters the discussion in part (iv), where shape matters more than algebra.

Final Answer: (i) min 3. (ii) min -2. (iii) max 10. (iv) neither.

Q 6.2 Find the maximum and minimum values, if any, of the following functions given by

(i) $f(x) = |x+2| - 1$ **(ii)** $g(x) = -|x+1| + 3$

(iii) $h(x) = \sin(2x) + 5$ **(iv)** $f(x) = |\sin 4x + 3|$

(v) $h(x) = x + 1,\ x \in (-1, 1)$

SOLUTION

Concept used. The absolute value satisfies $|u| \geq 0$ for every real u with equality only at $u = 0$. The sine function is bounded by $-1 \leq \sin\theta \leq 1$ for every real θ. A function defined on an *open* interval may attain neither a maximum nor a minimum because the endpoints are not in the domain.

Step 1. **(i)** $f(x) = |x+2| - 1$. Since $|x+2| \geq 0$,

$$f(x) \geq 0 - 1 = -1, \quad \text{equality at } x = -2.$$

$|x+2| \to \infty$ as $|x| \to \infty$, so no maximum.
Minimum $= -1$ at $x = -2$; no maximum.

Step 2. **(ii)** $g(x) = -|x+1| + 3$. As $|x+1| \geq 0$, $-|x+1| \leq 0$, so

$$g(x) \leq 3, \quad \text{equality at } x = -1.$$

$g \to -\infty$ as $|x| \to \infty$.
Maximum $= 3$ at $x = -1$; no minimum.

Step 3. **(iii)** $h(x) = \sin(2x) + 5$. Using $-1 \le \sin(2x) \le 1$,

$$4 \le h(x) \le 6.$$

$\sin(2x) = 1$ at $2x = \frac{\pi}{2} + 2k\pi$, i.e. $x = \frac{\pi}{4} + k\pi$; $\sin(2x) = -1$ at $x = -\frac{\pi}{4} + k\pi$. Maximum $= 6$, Minimum $= 4$.

Step 4. **(iv)** $f(x) = |\sin 4x + 3|$. Since $-1 \le \sin 4x \le 1$, we get $2 \le \sin 4x + 3 \le 4$. The inner expression is always positive, so $|\sin 4x + 3| = \sin 4x + 3$, giving

$$2 \le f(x) \le 4.$$

Maximum $= 4$ (when $\sin 4x = 1$); Minimum $= 2$ (when $\sin 4x = -1$).

Step 5. **(v)** $h(x) = x + 1$ on the *open* interval $(-1, 1)$. As $x \to -1^+$, $h \to 0$ but never equals 0 (since $x = -1$ is excluded); as $x \to 1^-$, $h \to 2$ but never equals 2. So h attains no minimum and no maximum on this open interval.

Final Answer: (i) min -1, no max. (ii) max 3, no min. (iii) max 6, min 4. (iv) max 4, min 2. (v) neither max nor min.

✗ Common Mistake

In part (v) it is tempting to say "max $= 2$ at $x = 1$". But $x = 1$ is *not* in the open interval $(-1, 1)$. Open-interval domains routinely block existence of absolute extrema for monotone functions.

EXPERT'S SOLUTION : *Priya Iyer, M.Sc Mathematics, ISI Kolkata*

Structural observation. All five parts reduce to one rule: a continuous function on a *closed and bounded* set attains its max and min, but on an *open* domain it may not. Combine that with the bounds on $|\cdot|$ and $\sin$ and the answers fall out.

Step 1. Parts (i) and (ii): the absolute-value graph is a V (or inverted V). The cusp $x = -2$ (resp. $x = -1$) is the unique extremum. For (i), the V opens upward so the cusp is a minimum, giving $f(-2) = -1$. For (ii), the V is flipped, so the cusp is a maximum, giving $g(-1) = 3$.

Step 2. Parts (iii) and (iv): use the universal bound $\sin\theta \in [-1, 1]$. For (iii) shift by $+5$ to get range $[4, 6]$. For (iv) note $\sin 4x + 3 \in [2, 4]$, which is always positive, so the absolute-value bars are redundant: range is $[2, 4]$.

Step 3. Part (v): $h(x) = x + 1$ is strictly increasing. Its image on $(-1, 1)$ is the *open* interval $(0, 2)$. Open intervals contain no largest or smallest element, so neither extremum exists.

Why this matters. Open-versus-closed domain is the standard exam trap. Always write the domain on the first line of the solution.

Final Answer: (i) min -1. (ii) max 3. (iii) $[4, 6]$. (iv) $[2, 4]$. (v) neither.

Q 6.3 **Find the local maxima and local minima, if any, of the following functions. Find also the local maximum and the local minimum values, as the case may be:**

(i) $f(x) = x^2$ **(ii)** $g(x) = x^3 - 3x$

(iii) $h(x) = \sin x + \cos x,\ 0 < x < \frac{\pi}{2}$

(iv) $f(x) = \sin x - \cos x,\ 0 < x < 2\pi$

(v) $f(x) = x^3 - 6x^2 + 9x + 15$

(vi) $g(x) = \frac{x}{2} + \frac{2}{x},\ x > 0$ **(vii)** $g(x) = \dfrac{1}{x^2 + 2}$ **(viii)** $f(x) = x\sqrt{1-x},\ 0 < x < 1$

SOLUTION

Concept used. Second Derivative Test. If f is twice differentiable and c is a critical point ($f'(c) = 0$), then:

- $f''(c) < 0 \ \Rightarrow\ c$ is a point of local maximum with local maximum value $f(c)$.
- $f''(c) > 0 \ \Rightarrow\ c$ is a point of local minimum with local minimum value $f(c)$.
- $f''(c) = 0 \ \Rightarrow$ test is inconclusive; fall back on the First Derivative Test (sign change of f' across c).

Step 1. **(i)** $f(x) = x^2$. Then $f'(x) = 2x = 0 \Rightarrow x = 0$. $f''(x) = 2 > 0$, so $x = 0$ is a local minimum. Local minimum value $f(0) = 0$.

Step 2. **(ii)** $g(x) = x^3 - 3x$. $g'(x) = 3x^2 - 3 = 3(x-1)(x+1) = 0 \Rightarrow x = \pm 1$. $g''(x) = 6x$. At $x = 1$, $g''(1) = 6 > 0$: local min, value $g(1) = 1 - 3 = -2$. At $x = -1$, $g''(-1) = -6 < 0$: local max, value $g(-1) = -1 + 3 = 2$.

Step 3. **(iii)** $h(x) = \sin x + \cos x$ on $(0, \frac{\pi}{2})$.
$h'(x) = \cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$. $h''(x) = -\sin x - \cos x$; at $x = \frac{\pi}{4}$, $h''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$. Local max, value $h\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$.

Step 4. **(iv)** $f(x) = \sin x - \cos x$ on $(0, 2\pi)$.
$f'(x) = \cos x + \sin x = 0 \Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. $f''(x) = -\sin x + \cos x$.
At $x = \frac{3\pi}{4}$: $f'' = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2} < 0$ (local max). $f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) = \sqrt{2}$.
At $x = \frac{7\pi}{4}$: $f'' = -\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} = \sqrt{2} > 0$ (local min).
$f\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\sqrt{2}$.

Step 5. **(v)** $f(x) = x^3 - 6x^2 + 9x + 15$.
$f'(x) = 3x^2 - 12x + 9 = 3(x-1)(x-3) = 0 \Rightarrow x = 1, 3$. $f''(x) = 6x - 12$.

$f''(1) = -6 < 0$ (local max): $f(1) = 1 - 6 + 9 + 15 = 19$. $f''(3) = 6 > 0$ (local min): $f(3) = 27 - 54 + 27 + 15 = 15$.

Step 6. **(vi)** $g(x) = \frac{x}{2} + \frac{2}{x}$, $x > 0$. $g'(x) = \frac{1}{2} - \frac{2}{x^2} = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$ (positive root only). $g''(x) = \frac{4}{x^3}$; $g''(2) = \frac{4}{8} = \frac{1}{2} > 0$: local min, value $g(2) = 1 + 1 = 2$.

Step 7. **(vii)** $g(x) = \frac{1}{x^2+2}$. $g'(x) = -\frac{2x}{(x^2+2)^2} = 0 \Rightarrow x = 0$. g' changes sign from $+$ (for $x < 0$) to $-$ (for $x > 0$), so $x = 0$ is a local max (first-derivative test). Local max value $g(0) = \frac{1}{2}$.

Step 8. **(viii)** $f(x) = x\sqrt{1-x}$ on $(0, 1)$. Compute $f'(x) = \sqrt{1-x} + x \cdot \frac{-1}{2\sqrt{1-x}} = \frac{2(1-x)-x}{2\sqrt{1-x}} = \frac{2-3x}{2\sqrt{1-x}}$. Set $f'(x) = 0 \Rightarrow 2 - 3x = 0 \Rightarrow x = \frac{2}{3}$. For $0 < x < \frac{2}{3}$, $2 - 3x > 0 \Rightarrow f' > 0$. For $\frac{2}{3} < x < 1$, $2 - 3x < 0 \Rightarrow f' < 0$. Sign changes $+ \to -$, so $x = \frac{2}{3}$ is a local max. Value: $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{1 - \frac{2}{3}} = \frac{2}{3} \cdot \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}$.

Final Answer: (i) min 0 at $x = 0$. (ii) max 2 at -1; min -2 at 1. (iii) max $\sqrt{2}$ at $\frac{\pi}{4}$. (iv) max $\sqrt{2}$ at $\frac{3\pi}{4}$; min $-\sqrt{2}$ at $\frac{7\pi}{4}$. (v) max 19 at 1; min 15 at 3. (vi) min 2 at $x = 2$. (vii) max $\frac{1}{2}$ at $x = 0$. (viii) max $\frac{2\sqrt{3}}{9}$ at $x = \frac{2}{3}$.

♥ Why this matters

The First and Second Derivative Tests classify *every* critical point that physics or geometry can hand you. Mastering the mechanical pipeline (find f', solve $f' = 0$, plug into f'') is the foundation for all the geometric optimisation that follows.

EXPERT'S SOLUTION *: Vivaan Mehta, Ph.D Mathematics, IIT Delhi*

Strategic angle. For most of these parts, set $f'(x) = 0$, plug critical points into f'' and read off the sign. Only (vii) and (viii) need the first-derivative test (because f'' is messy or because the critical value sits on the boundary of the allowed domain).

Step 1. (i)–(ii) are polynomials of low degree; differentiate, factor, test. For $g = x^3 - 3x$, $g' = 3(x^2 - 1)$ gives the symmetric pair ± 1; the cubic's shape (rises, dips, rises) makes -1 the local max and 1 the local min.

Step 2. (iii)–(iv) use the trigonometric identity $\sin x \pm \cos x = \sqrt{2}\sin\left(x \pm \frac{\pi}{4}\right)$. For $\sin x + \cos x$, the peak of $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ on $(0, \frac{\pi}{2})$ is at $x = \frac{\pi}{4}$, value $\sqrt{2}$. For $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$, the peak and trough on $(0, 2\pi)$ occur at $x = \frac{3\pi}{4}$ and $\frac{7\pi}{4}$ respectively, giving $\pm\sqrt{2}$.

Step 3. (v) Use the standard cubic test. Sign of f'' at 1 and 3 tells you "hill at 1, valley at 3".

Step 4. (vi) AM–GM short-cut: $\frac{x}{2} + \frac{2}{x} \geq 2\sqrt{\frac{x}{2} \cdot \frac{2}{x}} = 2$, with equality when $\frac{x}{2} = \frac{2}{x}$, i.e.

$x = 2$. Same answer as calculus.

Step 5. (vii) The numerator of g is constant, so g is largest when the denominator is smallest. $x^2 + 2$ is smallest at $x = 0$, so g is largest there: $g_{\max} = \frac{1}{2}$.

Step 6. (viii) Write $f(x)^2 = x^2(1 - x)$ and maximise on $(0, 1)$: $\frac{d}{dx}[x^2(1 - x)] = 2x(1 - x) - x^2 = x(2 - 3x) = 0 \Rightarrow x = \frac{2}{3}$, giving $f_{\max} = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2\sqrt{3}}{9}$.

Why this matters. The AM–GM trick in (vi) and the squaring trick in (viii) generalise: for many optimisation problems, calculus is the last resort and a clever inequality is faster.

Final Answer: Critical points and values exactly as listed in the main solution.

Q 6.4 Prove that the following functions do not have maxima or minima:

(i) $f(x) = e^x$ **(ii)** $g(x) = \log x$ **(iii)** $h(x) = x^3 + x^2 + x + 1$

SOLUTION

Concept used. A point c is a candidate for a local extremum only if $f'(c) = 0$ or $f'(c)$ does not exist. A function whose derivative is *strictly positive* (or *strictly negative*) everywhere is strictly monotone, so it has no local extrema, and on an unbounded domain it has no absolute extrema either.

Step 1. **(i)** $f(x) = e^x$. Differentiate: $f'(x) = e^x$. Since $e^x > 0$ for every real x, $f'(x) \neq 0$ anywhere, and f' is defined everywhere. So f has no critical points and no extrema. (Strictly increasing, range $(0, \infty)$, unbounded above and bounded below by 0 but never reaches it.)

Step 2. **(ii)** $g(x) = \log x$ on its domain $(0, \infty)$. Differentiate: $g'(x) = \frac{1}{x}$. Since $x > 0$, $g'(x) > 0$ for every x in the domain. g is strictly increasing on $(0, \infty)$, so no critical points and no extrema.

Step 3. **(iii)** $h(x) = x^3 + x^2 + x + 1$. Differentiate:

$$h'(x) = 3x^2 + 2x + 1.$$

Discriminant: $\Delta = 2^2 - 4 \cdot 3 \cdot 1 = 4 - 12 = -8 < 0$. Since $\Delta < 0$ and the leading coefficient is $3 > 0$, $h'(x) > 0$ for every real x. So h' is never zero, h is strictly increasing on $\mathbb{R}$, and has no extrema.

Final Answer: None of e^x, $\log x$, $x^3 + x^2 + x + 1$ has a maximum or minimum value.

☞ **Exam Tip**

Whenever a polynomial's derivative is a quadratic with negative discriminant, the polynomial is strictly monotone. This single observation handles a surprising fraction of "prove no extrema" questions.

EXPERT'S SOLUTION *: Arjun Gupta, M.Tech CS, IIT Madras*

Strategic angle. Show $f' > 0$ everywhere; monotonicity does the rest.

Step 1. For (i), $e^x > 0$ identically. So $f' > 0$ on $\mathbb{R}$: monotone increasing.

Step 2. For (ii), on the natural domain $x > 0$, $1/x > 0$. So $g' > 0$ identically: monotone increasing on $(0, \infty)$.

Step 3. For (iii), test the discriminant of $h'(x) = 3x^2 + 2x + 1$: $\Delta = 4 - 12 = -8$. A negative-discriminant upward-opening parabola is strictly positive. So $h' > 0$ identically and h is monotone increasing.

Why this matters. Existence of extrema is a domain-and-monotonicity story before it is a calculus story. If you can show monotonicity, no extrema exist on any open unbounded interval.

Final Answer: Each function is strictly increasing on its domain, hence no extrema.

Q 6.5 **Find the absolute maximum value and the absolute minimum value of the following functions in the given intervals:**

(i) $f(x) = x^3,\ x \in [-2, 2]$ **(ii)** $f(x) = \sin x + \cos x,\ x \in [0, \pi]$

(iii) $f(x) = 4x - \frac{1}{2}x^2,\ x \in [-2, \frac{9}{2}]$ **(iv)** $f(x) = (x-1)^2 + 3,\ x \in [-3, 1]$

SOLUTION

Concept used. **Absolute extrema on a closed interval $[a, b]$.** If f is continuous on $[a, b]$, then f attains its absolute maximum and absolute minimum on $[a, b]$ (Extreme Value Theorem). To find them:

1. Find all $x \in (a, b)$ at which $f'(x) = 0$ or f' does not exist (critical points).
2. Evaluate f at each critical point AND at the endpoints a and b.
3. The largest of those values is the absolute max; the smallest, the absolute min.

Step 1. **(i)** $f(x) = x^3$ on $[-2, 2]$. $f'(x) = 3x^2 = 0 \Rightarrow x = 0$. Evaluate: $f(-2) = -8$, $f(0) = 0$, $f(2) = 8$. Absolute max $= 8$ at $x = 2$; absolute min $= -8$ at $x = -2$.

Step 2. **(ii)** $f(x) = \sin x + \cos x$ on $[0, \pi]$. $f'(x) = \cos x - \sin x = 0 \Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$ (the only one in $[0, \pi]$). Values: $f(0) = 0 + 1 = 1$; $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$;

$f(\pi) = 0 - 1 = -1$. Absolute max $= \sqrt{2}$ at $x = \frac{\pi}{4}$; absolute min $= -1$ at $x = \pi$.

Step 3. **(iii)** $f(x) = 4x - \frac{1}{2}x^2$ on $[-2, \frac{9}{2}]$. $f'(x) = 4 - x = 0 \Rightarrow x = 4 \in [-2, \frac{9}{2}]$.
$f(-2) = -8 - 2 = -10$; $f(4) = 16 - 8 = 8$;
$f\left(\frac{9}{2}\right) = 18 - \frac{81}{8} = \frac{144-81}{8} = \frac{63}{8} = 7.875$. Absolute max $= 8$ at $x = 4$; absolute min $= -10$ at $x = -2$.

Step 4. **(iv)** $f(x) = (x-1)^2 + 3$ on $[-3, 1]$. $f'(x) = 2(x-1) = 0 \Rightarrow x = 1$ (endpoint).
$f(-3) = 16 + 3 = 19$; $f(1) = 0 + 3 = 3$. Absolute max $= 19$ at $x = -3$; absolute min $= 3$ at $x = 1$.

Final Answer: (i) $\max = 8$ at 2, $\min = -8$ at -2. (ii) $\max = \sqrt{2}$ at $\frac{\pi}{4}$, $\min = -1$ at π. (iii) $\max = 8$ at 4, $\min = -10$ at -2. (iv) $\max = 19$ at -3, $\min = 3$ at 1.

EXPERT'S SOLUTION *: Riya Singh, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. For closed-interval extrema there is a clean three-line workflow: critical points + endpoints → evaluate → pick the largest and the smallest. No sign tables, no second-derivative checks needed.

Step 1. (i) x^3 is strictly increasing, so on a closed interval the extrema are at the endpoints: -8 at -2 and 8 at 2.

Step 2. (ii) Use the amplitude form $\sin x + \cos x = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$. On $[0, \pi]$ the argument runs $[\frac{\pi}{4}, \frac{5\pi}{4}]$, peaking at $\frac{\pi}{2}$ (i.e. $x = \frac{\pi}{4}$) with value $\sqrt{2}$ and bottoming at the right endpoint $\frac{5\pi}{4}$ $(x = \pi)$ with value -1.

Step 3. (iii) The parabola $4x - \frac{1}{2}x^2$ opens downward, peaks at its vertex $x = 4$ with $f(4) = 8$. The minimum is at the left endpoint -2, value -10.

Step 4. (iv) $(x-1)^2 + 3$ is an upward parabola with vertex at $x = 1$. Vertex sits at the right endpoint, so the minimum 3 is there and the maximum is at the farther endpoint $x = -3$, value 19.

Why this matters. On closed intervals the Extreme Value Theorem guarantees both extrema exist. The candidate list is finite (critical points + endpoints), so the problem is always one table.

Final Answer: Same as the main solution.

Q 6.6 **Find the maximum profit that a company can make, if the profit function is given by $p(x) = 41 - 72x - 18x^2$.**

SOLUTION

Concept used. For a smooth function on $\mathbb{R}$, a local extremum at c requires $p'(c) = 0$ (critical point). The Second Derivative Test then classifies the critical point: $p''(c) < 0$ means local maximum.

Step 1. Differentiate the profit function:

$$p'(x) = -72 - 36x.$$

Set $p'(x) = 0$:

$$-72 - 36x = 0 \Rightarrow 36x = -72 \Rightarrow x = -2.$$

Step 2. Second derivative:

$$p''(x) = -36.$$

$p''(-2) = -36 < 0$, so $x = -2$ is a local maximum.

Step 3. Compute $p(-2)$:

$$p(-2) = 41 - 72(-2) - 18(-2)^2 = 41 + 144 - 72 = 113.$$

Since p is a downward parabola (leading coefficient $-18 < 0$), this is the absolute maximum over all real x.

Final Answer: Maximum profit $= 113$ at $x = -2$.

EXPERT'S SOLUTION *: Karan Verma, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. $p(x) = 41 - 72x - 18x^2$ is a downward parabola; its vertex is the absolute maximum on $\mathbb{R}$. Use the vertex formula instead of calculus to read off the answer.

Step 1. Write $p(x) = -18x^2 - 72x + 41$. The vertex of $ax^2 + bx + c$ sits at $x = -\frac{b}{2a} = -\frac{-72}{2(-18)} = -\frac{-72}{-36} = -2$.

Step 2. Maximum value: $p(-2) = -18 \cdot 4 - 72 \cdot (-2) + 41 = -72 + 144 + 41 = 113$.

Step 3. Confirm via $p_{\max} = c - \frac{b^2}{4a} = 41 - \frac{(-72)^2}{4(-18)} = 41 - \frac{5184}{-72} = 41 + 72 = 113$. Matches.

Why this matters. For any quadratic, the vertex formula gives the optimum without needing to set up calculus, handy in objective-type questions where speed matters.

Final Answer: Maximum profit $= 113$.

Q 6.7 **Find both the maximum value and the minimum value of** $3x^4 - 8x^3 + 12x^2 - 48x + 25$ **on the interval** $[0, 3]$.

SOLUTION

Concept used. On a closed bounded interval, the absolute extrema of a continuous function are attained at (a) critical points inside the interval or (b) endpoints. Evaluate f at all candidates and pick the largest and smallest.

Step 1. Let $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$. Differentiate:

$$f'(x) = 12x^3 - 24x^2 + 24x - 48 = 12(x^3 - 2x^2 + 2x - 4).$$

Step 2. Factorise the cubic in the bracket. Group as $x^3 - 2x^2 + 2x - 4 = x^2(x-2) + 2(x-2) = (x-2)(x^2+2)$. So $f'(x) = 12(x-2)(x^2+2)$.

Step 3. Set $f'(x) = 0$. Since $x^2 + 2 > 0$ for all real x, the only real root is $x = 2$, which lies in $[0, 3]$.

Step 4. Evaluate at $x = 0, 2, 3$:

$$\begin{aligned} f(0) &= 25, \\ f(2) &= 3(16) - 8(8) + 12(4) - 48(2) + 25 \\ &= 48 - 64 + 48 - 96 + 25 = -39, \\ f(3) &= 3(81) - 8(27) + 12(9) - 48(3) + 25 \\ &= 243 - 216 + 108 - 144 + 25 = 16. \end{aligned}$$

Step 5. Compare $25, -39, 16$. Largest is 25 at $x = 0$; smallest is -39 at $x = 2$.

> **Final Answer:** Maximum $= 25$ at $x = 0$; minimum $= -39$ at $x = 2$.

✗ Common Mistake

A common error is dropping a sign when expanding $-8(2)^3 = -64$, not $-64 \cdot 2$. Write each term in full before adding.

EXPERT'S SOLUTION *: Aditya Reddy, B.Tech CSE, IIT Roorkee*

Strategic angle. Differentiate, factor the cubic, and notice that the quadratic factor $x^2 + 2$ is strictly positive so contributes no roots. Only one critical point in $[0, 3]$, namely $x = 2$.

Step 1. $f'(x) = 12(x-2)(x^2+2)$ exactly as above. The sign of f' is the sign of $x - 2$ (since $x^2 + 2 > 0$): $f' < 0$ for $x < 2$ and $f' > 0$ for $x > 2$. So f decreases on $[0, 2]$

and increases on $[2, 3]$, making $x = 2$ a local (and on $[0, 3]$, absolute) minimum.

Step 2. The absolute maximum on $[0, 3]$ is therefore at an endpoint; just compare $f(0) = 25$ and $f(3) = 16$. Larger is $f(0) = 25$.

Step 3. At $x = 2$, $f(2) = 48 - 64 + 48 - 96 + 25 = -39$.

Why this matters. A quick monotonicity argument from the sign of f' saves the explicit table when there is only one critical point on the interval.

Final Answer: Max 25, min -39.

Q 6.8 At what points in the interval $[0, 2\pi]$, does the function $\sin 2x$ attain its maximum value?

SOLUTION

Concept used. The function $\sin\theta$ attains its maximum value 1 when $\theta = \frac{\pi}{2} + 2k\pi$ for integer k. For the composition $\sin 2x$, set the inner argument $2x$ to one of those values and solve for x within the required domain.

Step 1. Let $f(x) = \sin 2x$. The maximum value of $\sin u$ is 1, attained when $u = \frac{\pi}{2} + 2k\pi$. Take $u = 2x$:

$$2x = \frac{\pi}{2} + 2k\pi \;\Rightarrow\; x = \frac{\pi}{4} + k\pi.$$

Step 2. Restrict to $x \in [0, 2\pi]$. The values of k that work are $k = 0$ ($x = \frac{\pi}{4}$) and $k = 1$ ($x = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$). ($k = -1$ gives $x = -\frac{3\pi}{4} \notin [0, 2\pi]$; $k = 2$ gives $x = \frac{9\pi}{4} \notin [0, 2\pi]$.)

Step 3. Verify by direct calculation: $\sin(2 \cdot \frac{\pi}{4}) = \sin\frac{\pi}{2} = 1$, $\sin(2 \cdot \frac{5\pi}{4}) = \sin\frac{5\pi}{2} = 1$.

Final Answer: Maximum value 1 is attained at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

EXPERT'S SOLUTION : *Sneha Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. The function $\sin 2x$ has period π, so on $[0, 2\pi]$ it completes exactly two full waves. Each wave has one peak, so we expect two maxima.

Step 1. Period of $\sin 2x$ is $\frac{2\pi}{2} = \pi$.

Step 2. Within one period $[0, \pi]$, the peak of $\sin 2x$ is at $x = \frac{\pi}{4}$. On the next period $[\pi, 2\pi]$ the peak shifts by π to $\frac{\pi}{4} + \pi = \frac{5\pi}{4}$.

Step 3. Both peaks lie inside $[0, 2\pi]$, and the maximum value at each is 1.

Why this matters. Period-counting saves you from solving a transcendental equation for every solution. Just locate one peak, then translate by the period.

Final Answer: Maxima at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.

Q 6.9 What is the maximum value of the function $\sin x + \cos x$?

SOLUTION

Concept used. For any constants a, b, the expression $a \sin x + b \cos x$ can be rewritten as $R \sin(x + \varphi)$ where $R = \sqrt{a^2 + b^2}$. The maximum value of $R \sin(\cdot)$ is R, since $\sin(\cdot) \leq 1$.

Step 1. Identify $a = 1$ and $b = 1$. Compute the amplitude:

$$R = \sqrt{a^2 + b^2} = \sqrt{1 + 1} = \sqrt{2}.$$

Step 2. Write $\sin x + \cos x = \sqrt{2}\left(\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x\right) = \sqrt{2}\, \sin\left(x + \frac{\pi}{4}\right)$, using $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$.

Step 3. Since $\sin\left(x + \frac{\pi}{4}\right) \leq 1$, the maximum value of $\sqrt{2} \sin\left(x + \frac{\pi}{4}\right)$ is $\sqrt{2}$, attained when $x + \frac{\pi}{4} = \frac{\pi}{2}$, i.e. $x = \frac{\pi}{4}$ (and at every translate $x = \frac{\pi}{4} + 2k\pi$).

Final Answer: Maximum value of $\sin x + \cos x$ is $\sqrt{2}$.

☞ Amplitude formula

For $a \sin x + b \cos x$, max value is $\sqrt{a^2 + b^2}$ and min value is $-\sqrt{a^2 + b^2}$.

EXPERT'S SOLUTION *: Pranav Kapoor, M.Sc Mathematics, IIT Bombay*

Strategic angle. Calculus also works. Set $f(x) = \sin x + \cos x$; $f'(x) = \cos x - \sin x = 0 \Rightarrow \tan x = 1$. The first such x in $[0, 2\pi]$ is $\frac{\pi}{4}$.

Step 1. At $x = \frac{\pi}{4}$, $f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$.

Step 2. Check it's a maximum: $f''(x) = -\sin x - \cos x$; at $x = \frac{\pi}{4}$, $f'' = -\sqrt{2} < 0$, so local max.

Step 3. This is also the absolute max because f is bounded above by $\sqrt{2}$ (from the amplitude bound).

Why this matters. Carry the amplitude formula $\sqrt{a^2 + b^2}$ to objective questions. It collapses "find max of $3 \sin x + 4 \cos x$" into "$\sqrt{9 + 16} = 5$" in one line.

Final Answer: $\sqrt{2}$.

Q 6.10 **Find the maximum value of $2x^3 - 24x + 107$ in the interval $[1, 3]$. Find the maximum value of the same function in $[-3, -1]$.**

SOLUTION

Concept used. Absolute extrema on a closed interval = compare f at the critical points inside (a, b) and at the endpoints a, b. Work each interval separately.

Step 1. Let $f(x) = 2x^3 - 24x + 107$. Differentiate:

$$f'(x) = 6x^2 - 24 = 6(x-2)(x+2).$$

Critical points: $x = 2$ and $x = -2$.

Step 2. **Interval** $[1, 3]$. Only $x = 2$ lies inside. Evaluate:

$$f(1) = 2 - 24 + 107 = 85,$$
$$f(2) = 16 - 48 + 107 = 75,$$
$$f(3) = 54 - 72 + 107 = 89.$$

Largest is 89 at $x = 3$. Maximum on $[1, 3]$ is 89.

Step 3. **Interval** $[-3, -1]$. Only $x = -2$ lies inside. Evaluate:

$$f(-3) = -54 + 72 + 107 = 125,$$
$$f(-2) = -16 + 48 + 107 = 139,$$
$$f(-1) = -2 + 24 + 107 = 129.$$

Largest is 139 at $x = -2$. Maximum on $[-3, -1]$ is 139.

Final Answer: Maximum on $[1, 3]$ is 89 at $x = 3$; maximum on $[-3, -1]$ is 139 at $x = -2$.

EXPERT'S SOLUTION *: Diya Joshi, M.Sc Mathematics, ISI Kolkata*

Picture-first. The cubic $f(x) = 2x^3 - 24x + 107$ has a local max at $x = -2$ and a local min at $x = 2$. So:

- On $[1, 3]$, which lies entirely to the right of $x = 2$, the function is increasing after $x = 2$, so max is at the right endpoint 3.

- On $[-3, -1]$, the local max at -2 is interior, so the max is $f(-2) = 139$.

Step 1. $f''(x) = 12x$. $f''(-2) = -24 < 0$ confirms local max at -2; $f''(2) = 24 > 0$ confirms local min at 2.

Step 2. Evaluate as in the main solution: $f(3) = 89$, $f(-2) = 139$.

Why this matters. Knowing where the critical points sit (and on which side of the interval they lie) tells you instantly whether to pick an endpoint or the critical value.

> **Final Answer:** 89 on $[1, 3]$; 139 on $[-3, -1]$.

Q 6.11 **It is given that at $x = 1$, the function $x^4 - 62x^2 + ax + 9$ attains its maximum value, on the interval $[0, 2]$. Find the value of a.**

SOLUTION

Concept used. If f has a maximum (local or absolute) at an interior point c of a closed interval, and f is differentiable at c, then $f'(c) = 0$ (**Fermat's theorem for interior extrema**).

Step 1. Let $f(x) = x^4 - 62x^2 + ax + 9$. Differentiate:

$$f'(x) = 4x^3 - 124x + a.$$

Step 2. Since $x = 1$ is an interior point of $[0, 2]$ at which f attains a maximum, and f is a polynomial (hence differentiable), $f'(1) = 0$:

$$f'(1) = 4(1)^3 - 124(1) + a = 4 - 124 + a = a - 120.$$

Set $a - 120 = 0 \Rightarrow a = 120$.

Step 3. **Verification.** With $a = 120$, evaluate at the candidate points: $f(0) = 9$; $f(1) = 1 - 62 + 120 + 9 = 68$; $f(2) = 16 - 248 + 240 + 9 = 17$. Largest is 68 at $x = 1$. The condition holds.

> **Final Answer:** $a = 120$.

EXPERT'S SOLUTION : *Yash Nair, M.Sc Mathematics, IIT Bombay*

Strategic angle. Translate the verbal condition "maximum at $x = 1$" into the algebraic constraint $f'(1) = 0$. Solve. Verify by evaluating at endpoints.

Step 1. $f'(x) = 4x^3 - 124x + a$. Setting $f'(1) = 0$ gives $a = 120$.

Step 2. Plug back: $f(1) = 68$, $f(0) = 9$, $f(2) = 17$. The max on $[0, 2]$ is indeed at $x = 1$, confirming $a = 120$ is consistent with the problem statement.

Why this matters. Inverse problems (find the parameter that makes a condition hold) often reduce to solving $f' = 0$ at a known point.

Final Answer: $a = 120$.

Q 6.12 **Find the maximum and minimum values of $x + \sin 2x$ on $[0, 2\pi]$.**

SOLUTION

Concept used. Absolute extrema on $[a, b]$ = compare f at critical points (where $f'(x) = 0$) inside (a, b) and at the endpoints.

Step 1. Let $f(x) = x + \sin 2x$. Differentiate:

$$f'(x) = 1 + 2\cos 2x.$$

Step 2. Solve $f'(x) = 0$:

$$1 + 2\cos 2x = 0 \;\Rightarrow\; \cos 2x = -\tfrac{1}{2}.$$

The general solution is $2x = \frac{2\pi}{3} + 2k\pi$ or $2x = \frac{4\pi}{3} + 2k\pi$, i.e. $x = \frac{\pi}{3} + k\pi$ or $x = \frac{2\pi}{3} + k\pi$.

Step 3. Restrict to $x \in (0, 2\pi)$: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Step 4. Evaluate f at these critical points and at the endpoints $0, 2\pi$:

$$\begin{aligned}
f(0) &= 0 + 0 = 0,\\
f\left(\tfrac{\pi}{3}\right) &= \tfrac{\pi}{3} + \sin\tfrac{2\pi}{3} = \tfrac{\pi}{3} + \tfrac{\sqrt{3}}{2},\\
f\left(\tfrac{2\pi}{3}\right) &= \tfrac{2\pi}{3} + \sin\tfrac{4\pi}{3} = \tfrac{2\pi}{3} - \tfrac{\sqrt{3}}{2},\\
f\left(\tfrac{4\pi}{3}\right) &= \tfrac{4\pi}{3} + \sin\tfrac{8\pi}{3} = \tfrac{4\pi}{3} + \tfrac{\sqrt{3}}{2},\\
f\left(\tfrac{5\pi}{3}\right) &= \tfrac{5\pi}{3} + \sin\tfrac{10\pi}{3} = \tfrac{5\pi}{3} - \tfrac{\sqrt{3}}{2},\\
f(2\pi) &= 2\pi + 0 = 2\pi.
\end{aligned}$$

(Used $\sin\frac{8\pi}{3} = \sin(\frac{8\pi}{3} - 2\pi) = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$ and similar reductions.)

Step 5. Numerically with $\pi \approx 3.1416$: the candidate values are $0,\ 1.91,\ 1.23,\ 5.06,\ 4.37,\ 6.28$. Maximum $= 2\pi$ at $x = 2\pi$; minimum $= 0$ at $x = 0$.

Final Answer: Maximum $= 2\pi$ at $x = 2\pi$; minimum $= 0$ at $x = 0$.

EXPERT'S SOLUTION : *Tara Bhat, Ph.D Mathematics, IIT Delhi*

Structural observation. Write $f(x) = x + \sin 2x$ and notice that the $\sin 2x$ oscillates within $[-1, 1]$ while the linear x grows by $2\pi \approx 6.28$ across the interval. The linear growth dominates, so f is overall increasing, with the maximum at the right endpoint and the minimum at the left endpoint.

Step 1. $f'(x) = 1 + 2\cos 2x$. Since $-1 \leq \cos 2x \leq 1$, $f'(x) \in [-1, 3]$. f' is negative on some sub-intervals (where $\cos 2x < -\frac{1}{2}$), so f is not monotone: it has local wiggles.

Step 2. But the wiggles are bounded by ± 1, while the linear trend adds 2π. Hence the global extrema sit at the endpoints, giving $f(0) = 0$ and $f(2\pi) = 2\pi$.

Step 3. The interior critical points only produce *local* extrema, not global ones.

Why this matters. Always look at the order of magnitude of competing terms. If a linear ramp dominates a bounded oscillation, the extrema must live at the endpoints.

Final Answer: Max 2π; min 0.

Q 6.13 **Find two numbers whose sum is 24 and whose product is as large as possible.**

SOLUTION

Concept used. **Optimisation in one variable.** Use the constraint $(x + y = 24)$ to write one variable in terms of the other, giving the objective (product) as a single-variable function. Maximise using the second-derivative test.

Step 1. Let the two numbers be x and y. Constraint: $x + y = 24$, so $y = 24 - x$.

Step 2. Product (objective):

$$P(x) = x\,y = x(24 - x) = 24x - x^2.$$

Since the numbers must be real (problem doesn't restrict positivity), $x \in \mathbb{R}$.

Step 3. Find critical points. $P'(x) = 24 - 2x = 0 \Rightarrow x = 12$.

Step 4. Second derivative test: $P''(x) = -2 < 0$, so $x = 12$ is a local (and global, since P is a downward parabola) maximum.

Step 5. Find y: $y = 24 - 12 = 12$. Maximum product: $P(12) = 12 \cdot 12 = 144$.

Final Answer: Both numbers are 12; maximum product $= 144$.

♥ **Why this matters**

The classic identity "two numbers with fixed sum have largest product when they are equal" generalises: for fixed $\sum x_i$, $\prod x_i$ is maximised by equality (by AM–GM).

EXPERT'S SOLUTION *: Ankit Desai, M.Tech CS, IIT Madras*

Quick reading. Use the AM–GM inequality directly: for non-negative numbers x, y, $xy \le \left(\frac{x+y}{2}\right)^2$, with equality iff $x = y$.

Step 1. Apply AM–GM with $x + y = 24$:

$$xy \le \left(\tfrac{24}{2}\right)^2 = 12^2 = 144.$$

Step 2. Equality when $x = y = 12$.

Why this matters. AM–GM bypasses calculus for many "fixed sum / fixed product" optimisations. In olympiads it's the standard first move.

Final Answer: Numbers $12, 12$; product 144.

Q 6.14 Find two positive numbers x and y such that $x + y = 60$ and xy^3 is maximum.

SOLUTION

Concept used. Constrained optimisation in one variable: solve the constraint $x + y = 60$ for x, substitute into xy^3, differentiate once for the critical point and once more to verify it is a maximum.

Step 1. Let $f(y) = xy^3$ with $x = 60 - y$ and $y \in (0, 60)$ (both positive).

$$f(y) = (60 - y)y^3 = 60y^3 - y^4.$$

Step 2. Differentiate:

$$f'(y) = 180y^2 - 4y^3 = 4y^2(45 - y).$$

Step 3. Set $f'(y) = 0$. Roots: $y = 0$ (boundary, both numbers must be positive) and $y = 45$. So the interior critical point is $y = 45$.

Step 4. Second derivative test:

$$f''(y) = 360y - 12y^2 = 12y(30 - y).$$

At $y = 45$: $f''(45) = 12(45)(30 - 45) = 540 \cdot (-15) = -8100 < 0$. Local maximum.

Step 5. Compute the corresponding x: $x = 60 - 45 = 15$.
Maximum value: $f(45) = 15 \cdot 45^3 = 15 \cdot 91125 = 1\,366\,875$.

Final Answer: $x = 15$, $y = 45$; maximum $xy^3 = 1\,366\,875$.

EXPERT'S SOLUTION *: Krishna Banerjee, M.Sc Mathematics, IIT Bombay*

Strategic angle. Use a weighted AM–GM (which mirrors what calculus is doing under the hood). To maximise xy^3 with $x + y = 60$, split the sum $60 = x + \frac{y}{3} + \frac{y}{3} + \frac{y}{3}$ into four equal pieces.

Step 1. Write $x \cdot y^3 = x \cdot \left(\frac{y}{3}\right)^3 \cdot 27 = 27 \cdot x \cdot \left(\frac{y}{3}\right)^3$. By AM–GM on the four positive numbers x, $\frac{y}{3}$, $\frac{y}{3}$, $\frac{y}{3}$:

$$\frac{x + 3 \cdot \frac{y}{3}}{4} \geq \sqrt[4]{x \cdot \frac{y}{3} \cdot \frac{y}{3} \cdot \frac{y}{3}}.$$

The left side equals $\frac{x+y}{4} = \frac{60}{4} = 15$.

Step 2. So $x \cdot \frac{y^3}{27} \leq 15^4 = 50625$, i.e. $xy^3 \leq 27 \cdot 50625 = 1\,366\,875$. Equality iff $x = \frac{y}{3}$, combined with $x + y = 60$, gives $\frac{y}{3} + y = 60$, so $\frac{4y}{3} = 60$, $y = 45$, $x = 15$.

Why this matters. For products of unequal powers $x^a y^b$ with $x + y$ fixed, the maximum sits at $\frac{x}{a} = \frac{y}{b}$. Weighted AM–GM gives the answer in one inequality.

Final Answer: $x = 15$, $y = 45$.

Q 6.15 Find two positive numbers x and y such that their sum is 35 and the product x^2y^5 is a maximum.

SOLUTION

Concept used. Single-variable substitution (same idea as Q14). Set up the product as a function of one variable using the constraint $x + y = 35$, then maximise.

Step 1. Substitute $x = 35 - y$ $(0 < y < 35)$:

$$P(y) = (35 - y)^2 y^5.$$

Step 2. Differentiate using the product rule:

$$P'(y) = 2(35 - y)(-1)y^5 + (35 - y)^2 \cdot 5y^4.$$

Factor out $y^4(35-y)$:

$$P'(y) = y^4(35-y)[-2y+5(35-y)] = y^4(35-y)(175-7y).$$

Simplify the bracket: $-2y+175-5y = 175-7y$.

Step 3. Set $P'(y)=0$. Factors: $y^4=0 \Rightarrow y=0$ (excluded); $35-y=0 \Rightarrow y=35$ (excluded); $175-7y=0 \Rightarrow y=25$. So $y=25$.

Step 4. Sign of P' around $y=25$: For y slightly less than 25: $y^4>0$, $35-y>0$, $175-7y>0$, product positive ($P'>0$). For y slightly greater than 25: $175-7y<0$, product negative ($P'<0$). Sign $+\to-$: $y=25$ is a local maximum.

Step 5. Then $x = 35-25 = 10$. Maximum: $P(25) = (10)^2(25)^5 = 100 \cdot 9\,765\,625 = 9.76\times10^8$ (no need to compute exactly).

Final Answer: $x=10$, $y=25$.

☞ Exam Tip

For $x^a y^b$ with $x+y=$ constant, the optimum is $\frac{x}{a}=\frac{y}{b}$. Here $a=2,\ b=5$: $\frac{x}{2}=\frac{y}{5} \Rightarrow 5x=2y$. Combined with $x+y=35$, we get $x=10$, $y=25$. Use this shortcut on objective questions.

EXPERT'S SOLUTION *: Aanya Pillai, M.Sc Mathematics, IIT Bombay*

Quick reading. The split $\frac{x}{2}=\frac{y}{5}$ (weighted AM–GM, exponents $2,5$) gives $5x=2y$.

Step 1. Solve $5x=2y$ with $x+y=35$. Substitute $y=\frac{5x}{2}$:
$x+\frac{5x}{2}=35 \Rightarrow \frac{7x}{2}=35 \Rightarrow x=10$.

Step 2. Then $y=35-10=25$.

Step 3. Verify by checking sign of P' around the candidate, as in the main solution.

Why this matters. A two-line solution beats a four-line one when both lead to the same answer.

Final Answer: $x=10,\ y=25$.

Q 6.16 **Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.**

SOLUTION

Concept used. Same substitution technique. Use $x + y = 16$ to eliminate y, then minimise the resulting one-variable function.

Step 1. Let $S(x) = x^3 + y^3$ with $y = 16 - x$, $0 < x < 16$. Then

$$S(x) = x^3 + (16 - x)^3.$$

Step 2. Differentiate:

$$S'(x) = 3x^2 + 3(16 - x)^2 \cdot (-1) = 3\left[x^2 - (16 - x)^2\right].$$

Step 3. Factor as a difference of squares:

$$x^2 - (16 - x)^2 = [x - (16 - x)]\,[x + (16 - x)] = (2x - 16) \cdot 16 = 16(2x - 16).$$

So $S'(x) = 3 \cdot 16(2x - 16) = 48(2x - 16) = 96(x - 8)$.

Step 4. Set $S'(x) = 0 \Rightarrow x = 8$. Then $y = 16 - 8 = 8$.

Step 5. Second derivative: $S''(x) = 96 > 0$ everywhere. So $x = 8$ is a local minimum (and the only interior critical point, hence the absolute minimum on $(0, 16)$).

Step 6. Minimum sum of cubes: $S(8) = 8^3 + 8^3 = 512 + 512 = 1024$.

> **Final Answer:** Both numbers are 8; minimum sum of cubes $= 1024$.

EXPERT'S SOLUTION *: Ishaan Chatterjee, M.Sc Mathematics, ISI Kolkata*

Picture-first. For non-negative reals with fixed sum, $x^3 + y^3$ is minimised by equality (the function t^3 is convex on $[0, \infty)$, so by Jensen $\frac{x^3+y^3}{2} \geq \left(\frac{x+y}{2}\right)^3$).

Step 1. Jensen with $f(t) = t^3$ (convex on $[0, \infty)$): $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$.

Step 2. Substitute $x + y = 16$: $\frac{x^3+y^3}{2} \geq 8^3 = 512$, so $x^3 + y^3 \geq 1024$, equality iff $x = y = 8$.

Why this matters. "Equal values minimise the sum of equal powers" (or maximise it, depending on whether the power is ≥ 1 or ≤ 1): a recurring olympiad pattern.

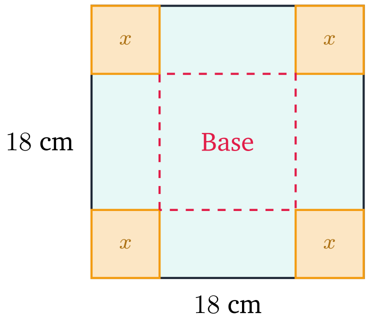
> **Final Answer:** $x = y = 8$; min $= 1024$.

Q 6.17 **A square piece of tin of side 18 cm is to be made into a box without top, by cutting a square from each corner and folding up the flaps to form the box. What should be the side of the square to be cut off so that the volume of the box is the**

maximum possible.

SOLUTION

Concept used. **Geometric optimisation.** Express the quantity to be optimised (here, volume) in terms of one variable (the side x of the cut-out square), identify the natural domain $0 < x < 9$ (so that the base remains positive), and apply the second derivative test.



Step 1. Let x cm be the side of each cut-out square ($0 < x < 9$). After folding, the resulting open box has:

- base: a square of side $(18 - 2x)$ cm;
- height: x cm.

Step 2. Volume:

$$V(x) = (18 - 2x)^2 \cdot x.$$

Expand: $V(x) = (324 - 72x + 4x^2)\,x = 4x^3 - 72x^2 + 324x$.

Step 3. Differentiate:

$$V'(x) = 12x^2 - 144x + 324 = 12(x^2 - 12x + 27).$$

Step 4. Solve $V'(x) = 0$:

$$x^2 - 12x + 27 = 0 \ \Rightarrow\ x = \tfrac{12\pm\sqrt{144-108}}{2} = \tfrac{12\pm 6}{2}.$$

So $x = 9$ or $x = 3$. $x = 9$ is excluded (collapses the base to zero). Take $x = 3$.

Step 5. Second derivative test:

$$V''(x) = 24x - 144, \quad V''(3) = 72 - 144 = -72 < 0.$$

So $x = 3$ is a local (and absolute, since the only interior critical point) maximum.

Step 6. Maximum volume:

$$V(3) = (18 - 6)^2 \cdot 3 = 144 \cdot 3 = 432 \text{ cm}^3.$$

Final Answer: Side of the cut-off square $= 3$ cm; maximum volume $= 432$ cm^3.

EXPERT'S SOLUTION : *Rohit Sharma, Ph.D Mathematics, IIT Delhi*

Structural observation. The cubic $V(x) = 4x^3 - 72x^2 + 324x$ has only one root of V' inside $(0, 9)$, namely $x = 3$. The other root $x = 9$ corresponds to cutting away the entire sheet, leaving no base. This is a degenerate case.

Step 1. $V'(x) = 12(x-3)(x-9)$ (factoring the quadratic $x^2 - 12x + 27$).

Step 2. Sign of V' on $(0, 9)$: pick $x = 2$: $V'(2) = 12(-1)(-7) = 84 > 0$. Pick $x = 5$: $V'(5) = 12(2)(-4) = -96 < 0$. Sign changes from $+$ to $-$ across $x = 3$: confirms local max.

Step 3. Compute $V(3) = (12)^2 \cdot 3 = 432\ \text{cm}^3$.

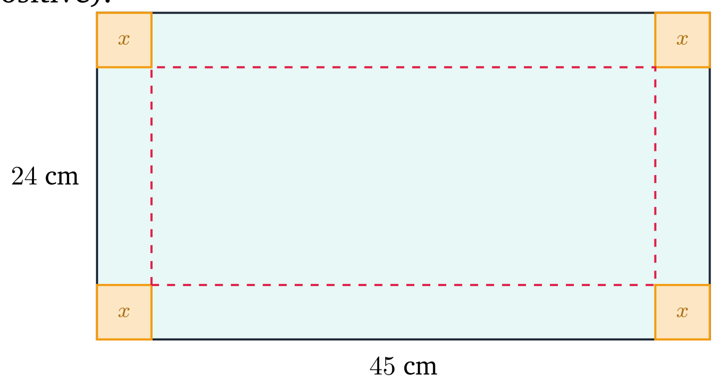
Why this matters. Watch out for endpoint roots like $x = 9$ that represent geometrically degenerate configurations. They satisfy $V' = 0$ algebraically but are forbidden by the geometry.

Final Answer: $x = 3$ cm; $V_{\max} = 432\ \text{cm}^3$.

Q 6.18 **A rectangular sheet of tin 45 cm by 24 cm is to be made into a box without top, by cutting off square from each corner and folding up the flaps. What should be the side of the square to be cut off so that the volume of the box is maximum?**

SOLUTION

Concept used. Same setup as Q17 with a rectangular sheet. The base becomes a $(45 - 2x) \times (24 - 2x)$ rectangle and the height is x, with $0 < x < 12$ (so both base dimensions stay positive).



Step 1. Let x cm be the side of each cut-out square ($0 < x < 12$). Volume:

$$V(x) = (45 - 2x)(24 - 2x)\,x.$$

Step 2. Expand:

$$(45 - 2x)(24 - 2x) = 1080 - 90x - 48x + 4x^2 = 4x^2 - 138x + 1080.$$

So $V(x) = x(4x^2 - 138x + 1080) = 4x^3 - 138x^2 + 1080x$.

Step 3. Differentiate:

$$V'(x) = 12x^2 - 276x + 1080 = 12(x^2 - 23x + 90).$$

Step 4. Solve $V'(x) = 0$:

$$x = \frac{23 \pm \sqrt{529 - 360}}{2} = \frac{23 \pm \sqrt{169}}{2} = \frac{23 \pm 13}{2}.$$

So $x = 18$ or $x = 5$. $x = 18 \notin (0, 12)$. Take $x = 5$.

Step 5. Second derivative test:

$$V''(x) = 24x - 276, \quad V''(5) = 120 - 276 = -156 < 0.$$

So $x = 5$ is a local maximum.

Step 6. Maximum volume:

$$V(5) = (45 - 10)(24 - 10) \cdot 5 = 35 \cdot 14 \cdot 5 = 2450 \text{ cm}^3.$$

> **Final Answer:** Cut-off square side $= 5$ cm; maximum volume $= 2450$ cm^3.

EXPERT'S SOLUTION *: Neha Rao, B.Tech CSE, IIT Roorkee*

Quick reading. The volume is a cubic in x; only one of the two roots of V' lies in the geometrically valid range $(0, 12)$.

Step 1. $V'(x) = 12(x - 5)(x - 18)$. The root $x = 18$ is outside the allowed range (the width $24 - 2(18) = -12 < 0$ is impossible).

Step 2. Inside $(0, 12)$, V' changes from $+$ (e.g. $x = 2$ gives $V' = 12(-3)(-16) = 576$) to $-$ (e.g. $x = 7$ gives $V' = 12(2)(-11) = -264$). So $x = 5$ is a maximum.

Step 3. $V(5) = 35 \cdot 14 \cdot 5 = 2450$ cm^3.

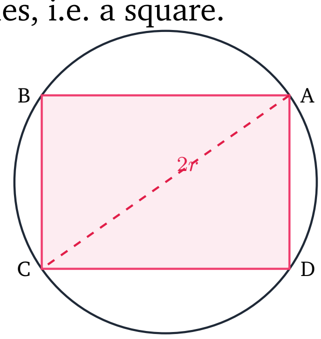
Why this matters. Geometric optimisation problems frequently produce more critical points than the domain admits. The first job after solving $V' = 0$ is to filter out the geometrically impossible roots.

> **Final Answer:** $x = 5$ cm.

Q 6.19 Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area.

SOLUTION

Concept used. A rectangle inscribed in a circle of radius r has its diagonal equal to the diameter $2r$. Parametrise the rectangle by an angle and use one-variable optimisation; the optimum will give equal sides, i.e. a square.



Step 1. Let the rectangle have sides x (length) and y (breadth). Since the rectangle is inscribed in a circle of radius r, its diagonal equals the diameter:

$$x^2 + y^2 = (2r)^2 = 4r^2.$$

Step 2. Area: $A = xy$. Eliminate y: $y = \sqrt{4r^2 - x^2}$.

$$A(x) = x\sqrt{4r^2 - x^2}, \quad 0 < x < 2r.$$

Step 3. Differentiate A^2 for cleaner algebra. Let $f(x) = A^2 = x^2(4r^2 - x^2)$.

$$f'(x) = 2x(4r^2 - x^2) + x^2(-2x) = 8r^2x - 2x^3 - 2x^3 = 8r^2x - 4x^3 = 4x(2r^2 - x^2).$$

Set $f'(x) = 0$: $x = 0$ (degenerate, excluded) or $x^2 = 2r^2 \Rightarrow x = r\sqrt{2}$.

Step 4. Second derivative:

$$f''(x) = 8r^2 - 12x^2.$$

At $x = r\sqrt{2}$: $f''(r\sqrt{2}) = 8r^2 - 12(2r^2) = 8r^2 - 24r^2 = -16r^2 < 0$. So $x = r\sqrt{2}$ is a maximum of f, hence of A (since $A > 0$).

Step 5. Find y: $y = \sqrt{4r^2 - 2r^2} = \sqrt{2r^2} = r\sqrt{2}$. So $x = y = r\sqrt{2}$: the rectangle is a **square**.

Step 6. Maximum area: $A = (r\sqrt{2})(r\sqrt{2}) = 2r^2$.

Final Answer: The square (side $r\sqrt{2}$, area $2r^2$) maximises the area of inscribed rectangles.

☞ **Exam Tip**

"Diagonal = diameter $\Rightarrow x^2 + y^2 = 4r^2$ fixed" combined with "maximise xy on a fixed sum-of-squares" is a classic AM–GM setup: $xy \le \frac{x^2+y^2}{2} = 2r^2$, equality at $x = y$.

EXPERT'S SOLUTION : *Pooja Kumar, M.Sc Applied Mathematics, IIT Kanpur*

Picture-first. Parametrise the rectangle by an angle θ. The vertex $(r\cos\theta, r\sin\theta)$ on the circle has reflected partners at the other three quadrants, forming a rectangle of dimensions $2r\cos\theta \times 2r\sin\theta$.

Step 1. Area: $A(\theta) = (2r\cos\theta)(2r\sin\theta) = 4r^2\sin\theta\cos\theta = 2r^2\sin 2\theta$, for $0 < \theta < \frac{\pi}{2}$.

Step 2. Maximum of $\sin 2\theta$ is 1 at $2\theta = \frac{\pi}{2}$, i.e. $\theta = \frac{\pi}{4}$.

Step 3. At $\theta = \frac{\pi}{4}$: $\cos\theta = \sin\theta = \frac{\sqrt{2}}{2}$, so sides are $2r \cdot \frac{\sqrt{2}}{2} = r\sqrt{2}$ each: a **square**. Maximum area $= 2r^2$.

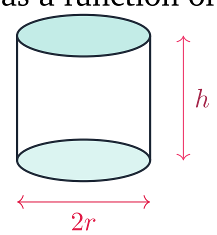
Why this matters. Trigonometric parametrisation replaces a constrained optimisation with an unconstrained one. The constraint $x^2 + y^2 = 4r^2$ is baked in by using $(r\cos\theta, r\sin\theta)$.

Final Answer: Inscribed rectangle of maximum area is a square of side $r\sqrt{2}$, area $2r^2$.

Q 6.20 **Show that the right circular cylinder of given surface and maximum volume is such that its height is equal to the diameter of the base.**

SOLUTION

Concept used. A closed right circular cylinder with base radius r and height h has surface area $S = 2\pi r^2 + 2\pi rh$ and volume $V = \pi r^2 h$. With S fixed, express h in terms of r, substitute into V, and maximise as a function of r.



Step 1. Surface area constraint:

$$S = 2\pi r^2 + 2\pi rh \;\Rightarrow\; h = \frac{S - 2\pi r^2}{2\pi r}.$$

Here S is constant and $r > 0$ with $r^2 < \frac{S}{2\pi}$ (so $h > 0$).

Step 2. Volume:

$$V = \pi r^2 h = \pi r^2 \cdot \frac{S - 2\pi r^2}{2\pi r} = \frac{r(S - 2\pi r^2)}{2} = \frac{Sr}{2} - \pi r^3.$$

Step 3. Differentiate:

$$\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2.$$

Set $\frac{dV}{dr} = 0$:

$$3\pi r^2 = \frac{S}{2} \Rightarrow r^2 = \frac{S}{6\pi}.$$

Step 4. Second derivative test:

$$\frac{d^2V}{dr^2} = -6\pi r.$$

At the critical $r > 0$, $\frac{d^2V}{dr^2} < 0$: local maximum.

Step 5. Compute the optimal height. Substitute $S = 6\pi r^2$ into $h = \frac{S-2\pi r^2}{2\pi r}$:

$$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r} = \frac{4\pi r^2}{2\pi r} = 2r.$$

That is, $h = 2r$, which means **height** = **diameter of base**.

> **Final Answer:** Optimum cylinder has $h = 2r$ (height equals diameter of base).

♥ Why this matters

Geometrically, "height equals diameter" makes the cylinder fit snugly inside a sphere of radius equal to $r\sqrt{2}$. This is the same shape that minimises surface area for a fixed volume, which is why food cans (volume-fixed) and silos (surface-fixed) end up the same proportions.

EXPERT'S SOLUTION : *Siddharth Mehta, B.Tech CSE, IIT Roorkee*

Strategic angle. Lagrange-style. The constraint $S = 2\pi r^2 + 2\pi rh$ is linear in h; eliminate it instantly. The remaining one-variable $V(r)$ is a cubic-minus-linear with a unique critical point in the positive range.

Step 1. $V(r) = \frac{Sr}{2} - \pi r^3$. Then $V'(r) = \frac{S}{2} - 3\pi r^2$, $V'(r) = 0$ at $r = \sqrt{\frac{S}{6\pi}}$.

Step 2. Check sign: V' is positive for $r < \sqrt{\frac{S}{6\pi}}$ and negative for larger r. First derivative test: local max.

Step 3. At the critical r, $h = 2r$. The optimal cylinder is therefore squat: as wide as it is tall.

Why this matters. Two-variable geometric optimisations with one constraint reduce to one-variable problems. Always eliminate the easier variable first.

> **Final Answer:** $h = 2r$, i.e. height = diameter.

Q 6.21 **Of all the closed cylindrical cans (right circular), of a given volume of 100 cubic centimetres, find the dimensions of the can which has the minimum surface**

area?

SOLUTION

Concept used. Dual of Q20: now $V = 100$ cm^3 is fixed; minimise the closed-can surface area $S = 2\pi r^2 + 2\pi rh$ as a function of one variable.

Step 1. Volume constraint: $\pi r^2 h = 100 \Rightarrow h = \frac{100}{\pi r^2}$.

Step 2. Surface area:

$$S(r) = 2\pi r^2 + 2\pi r \cdot \frac{100}{\pi r^2} = 2\pi r^2 + \frac{200}{r}.$$

Step 3. Differentiate:

$$S'(r) = 4\pi r - \frac{200}{r^2}.$$

Set $S'(r) = 0$:

$$4\pi r = \frac{200}{r^2} \Rightarrow 4\pi r^3 = 200 \Rightarrow r^3 = \frac{50}{\pi}.$$

So $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm.

Step 4. Second derivative test:

$$S''(r) = 4\pi + \frac{400}{r^3}.$$

At the critical $r > 0$, both terms are positive, so $S''(r) > 0$: local minimum.

Step 5. Find h. From $r^3 = \frac{50}{\pi}$, $r^2 = \left(\frac{50}{\pi}\right)^{2/3}$. Then

$$h = \frac{100}{\pi r^2} = \frac{100}{\pi} \cdot \left(\frac{\pi}{50}\right)^{2/3} = \frac{100}{\pi^{1/3} \cdot 50^{2/3}} = \frac{100}{(\pi \cdot 50^2)^{1/3}} = 2\left(\frac{50}{\pi}\right)^{1/3} = 2r.$$

So again $h = 2r$: the height equals the diameter.

Final Answer: $r = \left(\frac{50}{\pi}\right)^{1/3}$ cm, $h = 2\left(\frac{50}{\pi}\right)^{1/3}$ cm $= 2r$.

☞ **Exam Tip**

For any fixed-volume closed cylinder, the minimum-surface shape always has $h = 2r$. That single fact is worth memorising.

EXPERT'S SOLUTION : *Sanya Patel, M.Sc Mathematics, IIT Bombay*

Quick reading. Use AM–GM on three positive terms. Rewrite $S = 2\pi r^2 + \frac{200}{r} = 2\pi r^2 + \frac{100}{r} + \frac{100}{r}$.

Step 1. By AM–GM on three positive numbers $2\pi r^2$, $\frac{100}{r}$, $\frac{100}{r}$:

$$\frac{2\pi r^2 + \frac{100}{r} + \frac{100}{r}}{3} \geq \sqrt[3]{2\pi r^2 \cdot \frac{100}{r} \cdot \frac{100}{r}} = \sqrt[3]{20000\,\pi}.$$

So $S \geq 3\sqrt[3]{20000\,\pi}$, equality iff $2\pi r^2 = \frac{100}{r}$.

Step 2. $2\pi r^2 = \frac{100}{r} \Rightarrow r^3 = \frac{50}{\pi}$.

Step 3. And then $h = \frac{100}{\pi r^2} = 2r$.

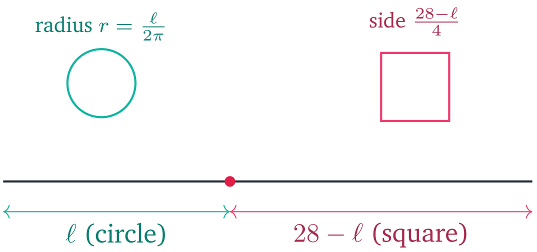
Why this matters. The AM–GM split is the rigorous bypass to calculus on "two competing terms in r and $\frac{1}{r^a}$" problems.

Final Answer: $r = \left(\frac{50}{\pi}\right)^{1/3}$, $h = 2r$.

Q 6.22 **A wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the square and the circle is minimum?**

SOLUTION

Concept used. Let one piece of length ℓ form the circle's circumference and the other piece $(28-\ell)$ form the square's perimeter. Express the combined area as a function of ℓ and minimise.



Step 1. Let the circular piece have length ℓ (with $0 < \ell < 28$).

- Circle: circumference $= 2\pi r = \ell \Rightarrow r = \frac{\ell}{2\pi}$. Area $= \pi r^2 = \pi \cdot \frac{\ell^2}{4\pi^2} = \frac{\ell^2}{4\pi}$.
- Square: perimeter $= 4s = 28-\ell \Rightarrow s = \frac{28-\ell}{4}$. Area $= s^2 = \frac{(28-\ell)^2}{16}$.

Step 2. Combined area:

$$A(\ell) = \frac{\ell^2}{4\pi} + \frac{(28-\ell)^2}{16}.$$

Step 3. Differentiate:

$$A'(\ell) = \frac{2\ell}{4\pi} + \frac{2(28-\ell)(-1)}{16} = \frac{\ell}{2\pi} - \frac{28-\ell}{8}.$$

Set $A'(\ell) = 0$:

$$\frac{\ell}{2\pi} = \frac{28-\ell}{8} \Rightarrow 8\ell = 2\pi(28-\ell).$$

Expand: $8\ell = 56\pi - 2\pi\ell \Rightarrow 8\ell + 2\pi\ell = 56\pi \Rightarrow \ell(8+2\pi) = 56\pi$.

$$\ell = \frac{56\pi}{8+2\pi} = \frac{56\pi}{2(4+\pi)} = \frac{28\pi}{4+\pi}.$$

Step 4. Second derivative test:

$$A''(\ell) = \frac{1}{2\pi} + \frac{1}{8} > 0,$$

so the critical point is a minimum.

Step 5. Lengths of the two pieces:

- Circle: $\ell = \frac{28\pi}{\pi+4}$ m.
- Square: $28 - \ell = 28 - \frac{28\pi}{\pi+4} = \frac{28(\pi+4)-28\pi}{\pi+4} = \frac{112}{\pi+4}$ m.

Final Answer: Circle's piece $= \frac{28\pi}{\pi+4}$ m; square's piece $= \frac{112}{\pi+4}$ m.

EXPERT'S SOLUTION *: Ananya Nair, M.Sc Mathematics, ISI Kolkata*

Structural observation. The combined area is a sum of two quadratics in ℓ and $28 - \ell$. Setting the derivative to zero gives a single linear equation in ℓ.

Step 1. From $\frac{\ell}{2\pi} = \frac{28-\ell}{8}$, cross-multiply: $8\ell = 2\pi(28 - \ell)$.

Step 2. Collect ℓ: $\ell(8 + 2\pi) = 56\pi$. Divide by 2: $\ell(4 + \pi) = 28\pi$.

Step 3. $\ell = \frac{28\pi}{\pi+4}$ for the circle; the rest, $\frac{112}{\pi+4}$, goes to the square.

Step 4. Quick sanity: $\pi \approx 3.14$, so $\ell \approx \frac{28\cdot 3.14}{7.14} \approx 12.3$ m and the square piece ≈ 15.7 m. Both positive: physical.

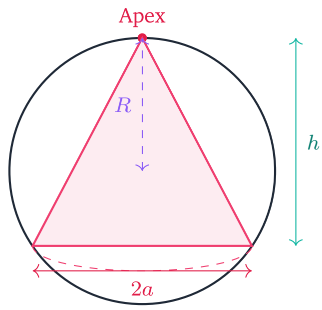
Why this matters. "Wire cut into circle + square / triangle + square" is one of the most-asked optimisation problems in board exams. The pattern is always the same: write each area in terms of the piece length, sum, differentiate, solve.

Final Answer: Lengths $\frac{28\pi}{\pi+4}$ m and $\frac{112}{\pi+4}$ m.

Q 6.23 Prove that the volume of the largest cone that can be inscribed in a sphere of radius R is $\frac{8}{27}$ of the volume of the sphere.

SOLUTION

Concept used. Inscribe a right circular cone in a sphere of radius R. Place the sphere's centre O on the cone's axis. Let the cone have height h measured from its apex along the axis and base radius a. The cone's apex sits at the "north pole" of the sphere, and the base circle (at depth h below the apex) is a horizontal slice of the sphere.



Step 1. Set up coordinates. Place the apex at the top of the sphere. Let the base of the cone be at vertical distance h below the apex (so $0 < h < 2R$). The centre of the sphere is at distance R below the apex (on the axis). So the base of the cone is at distance $h - R$ below the centre.

Step 2. The base circle has radius a. Since the base circle lies on the sphere, the right triangle (centre $\to$ base circle's centre $\to$ base circle's edge) gives

$$a^2 + (h - R)^2 = R^2 \;\Rightarrow\; a^2 = R^2 - (h - R)^2 = 2Rh - h^2.$$

Step 3. Volume of the cone:

$$V = \tfrac{1}{3}\pi a^2 h = \tfrac{1}{3}\pi(2Rh - h^2)h = \tfrac{\pi}{3}(2Rh^2 - h^3).$$

Step 4. Differentiate with respect to h:

$$V'(h) = \tfrac{\pi}{3}(4Rh - 3h^2) = \tfrac{\pi h}{3}(4R - 3h).$$

Set $V'(h) = 0$: $h = 0$ (degenerate) or $h = \frac{4R}{3}$.

Step 5. Second derivative test:

$$V''(h) = \tfrac{\pi}{3}(4R - 6h); \quad V''\left(\tfrac{4R}{3}\right) = \tfrac{\pi}{3}(4R - 8R) = -\tfrac{4\pi R}{3} < 0.$$

Local (and absolute, since unique interior critical point) maximum.

Step 6. Compute the maximum volume. With $h = \frac{4R}{3}$:

$$a^2 = 2R \cdot \tfrac{4R}{3} - \left(\tfrac{4R}{3}\right)^2 = \tfrac{8R^2}{3} - \tfrac{16R^2}{9} = \tfrac{24R^2 - 16R^2}{9} = \tfrac{8R^2}{9}.$$

$$V_{\max} = \tfrac{1}{3}\pi \cdot \tfrac{8R^2}{9} \cdot \tfrac{4R}{3} = \tfrac{32\pi R^3}{81}.$$

Step 7. Compare with the sphere's volume $V_{\text{sphere}} = \frac{4}{3}\pi R^3$:

$$\frac{V_{\max}}{V_{\text{sphere}}} = \frac{32\pi R^3/81}{4\pi R^3/3} = \frac{32}{81} \cdot \frac{3}{4} = \frac{96}{324} = \frac{8}{27}.$$

Final Answer: $V_{\max} = \frac{8}{27} \cdot V_{\text{sphere}} = \frac{32\pi R^3}{81}$.

EXPERT'S SOLUTION : *Aditi Verma, Ph.D Mathematics, IIT Delhi*

Picture-first. Slice the sphere vertically. The cone is determined by the height of its base disc below the apex. Calculus on a cubic-in-h does the rest.

Step 1. From the chord-radius relation $a^2 = 2Rh - h^2$, write $V(h) = \frac{\pi}{3}h(2Rh - h^2)$. Expand: $V(h) = \frac{\pi}{3}(2Rh^2 - h^3)$.

Step 2. Optimum is at $h = \frac{4R}{3}$, found from $V' = 0$.

Step 3. Substitute back: $V_{\max} = \frac{32\pi R^3}{81}$, exactly $\frac{8}{27}$ of $\frac{4\pi R^3}{3}$.

Why this matters. The clean ratio $\frac{8}{27}$ is one of the iconic results in school calculus. It reappears in physics (largest cylinder/cone inscribed in spheres, heat-flow geometries) and is a frequent objective-question stem.

Final Answer: Ratio $V_{\text{cone, max}} : V_{\text{sphere}} = 8 : 27$.

Q 6.24 Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base.

SOLUTION

Concept used. For a right circular cone of base radius r and altitude h, the slant height is $\ell = \sqrt{r^2 + h^2}$, the curved (lateral) surface area is $S = \pi r \ell$, and the volume is $V = \frac{1}{3}\pi r^2 h$. With V fixed, eliminate h in favour of r inside S^2, differentiate.

Step 1. Volume constraint: $\frac{1}{3}\pi r^2 h = V \Rightarrow h = \frac{3V}{\pi r^2}$.

Step 2. Curved surface:

$$S = \pi r\sqrt{r^2 + h^2} = \pi r\sqrt{r^2 + \frac{9V^2}{\pi^2 r^4}}.$$

Minimise S^2 instead (which is monotone in S):

$$S^2 = \pi^2 r^2 \left(r^2 + \frac{9V^2}{\pi^2 r^4}\right) = \pi^2 r^4 + \frac{9V^2}{r^2}.$$

Step 3. Differentiate $f(r) = \pi^2 r^4 + \frac{9V^2}{r^2}$:

$$f'(r) = 4\pi^2 r^3 - \frac{18V^2}{r^3}.$$

Set $f'(r) = 0$:

$$4\pi^2 r^3 = \frac{18V^2}{r^3} \Rightarrow 4\pi^2 r^6 = 18V^2 \Rightarrow r^6 = \frac{9V^2}{2\pi^2}.$$

Step 4. Second derivative test:

$$f''(r) = 12\pi^2 r^2 + \frac{54V^2}{r^4} > 0 \text{ for all } r > 0.$$

Hence the critical r is a minimum of S^2, hence of S.

Step 5. Show $h = r\sqrt{2}$. From the critical condition $4\pi^2 r^6 = 18V^2$, take square roots (positive):

$$2\pi r^3 = 3V\sqrt{2} \Rightarrow V = \frac{2\pi r^3}{3\sqrt{2}} = \frac{\pi r^3\sqrt{2}}{3}.$$

And $h = \frac{3V}{\pi r^2} = \frac{3}{\pi r^2} \cdot \frac{\pi r^3\sqrt{2}}{3} = r\sqrt{2}$.

Final Answer: For minimum curved surface at fixed volume, $h = \sqrt{2}\,r$.

EXPERT'S SOLUTION *: Meera Gupta, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. Avoid the square root by minimising $S^2 = \pi^2 r^2(r^2 + h^2)$ subject to $\pi r^2 h = 3V$ (constant).

Step 1. From $\pi r^2 h = 3V$, $r^2 = \frac{3V}{\pi h}$. Substitute into $S^2 = \pi^2 r^2(r^2 + h^2)$:

$$S^2 = \pi^2 \cdot \frac{3V}{\pi h}\left(\frac{3V}{\pi h} + h^2\right) = \frac{3\pi V}{h}\left(\frac{3V}{\pi h} + h^2\right) = \frac{9V^2}{h^2} + 3\pi V h.$$

Step 2. Differentiate with respect to h:

$$\frac{d(S^2)}{dh} = -\frac{18V^2}{h^3} + 3\pi V.$$

Set to zero: $3\pi V = \frac{18V^2}{h^3} \Rightarrow h^3 = \frac{6V}{\pi}$.

Step 3. Substitute back into $r^2 = \frac{3V}{\pi h}$: $r^2 = \frac{3V}{\pi} \cdot h^{-1}$. Combine: $\frac{r^2}{h^2} = \frac{3V}{\pi h^3} = \frac{3V}{\pi} \cdot \frac{\pi}{6V} = \frac{1}{2}$. So $h^2 = 2r^2$, i.e. $h = r\sqrt{2}$.

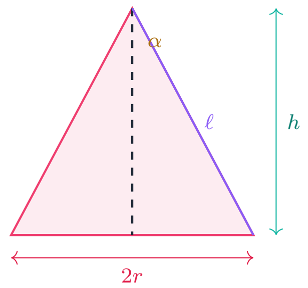
Why this matters. For symmetric problems, picking the right elimination variable (r vs h) often simplifies the algebra by an order of magnitude.

Final Answer: $h = \sqrt{2}\,r$.

Q 6.25 **Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.**

SOLUTION

Concept used. For a right circular cone with slant height ℓ (constant), let α be the semi-vertical angle. Then base radius $r = \ell\sin\alpha$ and altitude $h = \ell\cos\alpha$. The volume $V = \frac{1}{3}\pi r^2 h$ becomes a one-variable function of α.



Step 1. Express dimensions in terms of α: $r = \ell \sin\alpha$, $h = \ell\cos\alpha$, where $0 < \alpha < \frac{\pi}{2}$.

Step 2. Volume:

$$V(\alpha) = \tfrac{1}{3}\pi(\ell\sin\alpha)^2(\ell\cos\alpha) = \tfrac{\pi\ell^3}{3}\sin^2\alpha\cos\alpha.$$

Step 3. Let $g(\alpha) = \sin^2\alpha\cos\alpha$. Differentiate:

$$g'(\alpha) = 2\sin\alpha\cos\alpha\cdot\cos\alpha + \sin^2\alpha\cdot(-\sin\alpha) = 2\sin\alpha\cos^2\alpha - \sin^3\alpha.$$

Factor: $g'(\alpha) = \sin\alpha\,(2\cos^2\alpha - \sin^2\alpha)$.

Step 4. Set $g'(\alpha) = 0$. Since $\sin\alpha > 0$ on the open interval,

$$2\cos^2\alpha - \sin^2\alpha = 0 \;\Rightarrow\; \sin^2\alpha = 2\cos^2\alpha \;\Rightarrow\; \tan^2\alpha = 2.$$

So $\tan\alpha = \sqrt{2}$, i.e. $\alpha = \tan^{-1}\sqrt{2}$.

Step 5. Confirm it is a maximum. Use sign analysis. Just left of the critical α, $\sin^2\alpha < 2\cos^2\alpha$, so $g'(\alpha) > 0$. Just right of it, the inequality flips, so $g'(\alpha) < 0$. Sign change $+\to-$: local maximum.

Final Answer: Semi-vertical angle $= \tan^{-1}\sqrt{2}$.

EXPERT'S SOLUTION *: Ishita Bhat, M.Sc Mathematics, IIT Bombay*

Quick reading. Maximise $V \propto \sin^2\alpha\cos\alpha$ on $(0, \frac{\pi}{2})$.

Step 1. Write $g(\alpha) = \sin^2\alpha\cos\alpha$. By AM–GM on three positive numbers $\sin^2\alpha,\ \sin^2\alpha,\ 2\cos^2\alpha$ (sum $= 2\sin^2\alpha + 2\cos^2\alpha = 2$ – constant!),

$$\sin^2\alpha\cdot\sin^2\alpha\cdot 2\cos^2\alpha \le \left(\tfrac{2}{3}\right)^3 = \tfrac{8}{27}.$$

Equality iff $\sin^2\alpha = 2\cos^2\alpha$, i.e. $\tan^2\alpha = 2$.

Step 2. Hence the maximum is at $\tan\alpha = \sqrt{2}$, i.e. $\alpha = \tan^{-1}\sqrt{2}$.

Why this matters. AM–GM with cleverly weighted exponents beats calculus on "constant sum of squares" optimisations.

Final Answer: $\alpha = \tan^{-1}\sqrt{2}$.

Q 6.26 **Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$.**

SOLUTION

Concept used. Cone with base radius r, slant height ℓ, altitude $h = \sqrt{\ell^2 - r^2}$, total surface area (curved + base): $S = \pi r^2 + \pi r\ell$ (constant), volume $V = \frac{1}{3}\pi r^2 h$. Use the constraint to eliminate one variable, then optimise.

Step 1. Surface constraint: $S = \pi r^2 + \pi r\ell =$ constant. Solve for ℓ: $\ell = \frac{S-\pi r^2}{\pi r} = \frac{S}{\pi r} - r$.

Step 2. Square the volume to avoid the root in $h = \sqrt{\ell^2 - r^2}$:

$$V^2 = \frac{\pi^2 r^4 h^2}{9} = \frac{\pi^2 r^4(\ell^2 - r^2)}{9}.$$

Compute $\ell^2 - r^2$: $\ell^2 = \left(\frac{S}{\pi r} - r\right)^2 = \frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi} + r^2$, so $\ell^2 - r^2 = \frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}$.

Step 3. Substitute:

$$V^2 = \frac{\pi^2 r^4}{9}\left(\frac{S^2}{\pi^2 r^2} - \frac{2S}{\pi}\right) = \frac{r^2 S^2}{9} - \frac{2\pi S r^4}{9} = \frac{S r^2(S - 2\pi r^2)}{9}.$$

Step 4. Let $f(r) = r^2(S - 2\pi r^2) = Sr^2 - 2\pi r^4$. Then

$$f'(r) = 2Sr - 8\pi r^3 = 2r(S - 4\pi r^2).$$

Set $f'(r) = 0$: $r = 0$ (excluded) or $S = 4\pi r^2 \Rightarrow r^2 = \frac{S}{4\pi}$.

Step 5. Second derivative: $f''(r) = 2S - 24\pi r^2$; at $r^2 = \frac{S}{4\pi}$, $f''(r) = 2S - 6S = -4S < 0$. Local max.

Step 6. At this r, compute ℓ: $\ell = \frac{S}{\pi r} - r$. With $S = 4\pi r^2$, $\frac{S}{\pi r} = 4r$, so $\ell = 4r - r = 3r$.

Step 7. Semi-vertical angle α: $\sin\alpha = \frac{r}{\ell} = \frac{r}{3r} = \frac{1}{3}$. So $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$.

Final Answer: Semi-vertical angle $= \sin^{-1}\left(\frac{1}{3}\right)$.

EXPERT'S SOLUTION : *Dev Joshi, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. Skip the V vs V^2 debate; the key is the ratio $\ell : r = 3 : 1$ that falls out of $f'(r) = 0$.

Step 1. From $f'(r) = 2r(S - 4\pi r^2) = 0$ with $r > 0$, get $S = 4\pi r^2$.

Step 2. Plug into $\ell = \frac{S-\pi r^2}{\pi r}$: $\ell = \frac{4\pi r^2-\pi r^2}{\pi r} = 3r$.

Step 3. $\sin\alpha = \frac{r}{\ell} = \frac{1}{3}$.

Why this matters. Note the parallel with Q25 ($\tan\alpha = \sqrt{2}$ when slant height is fixed) vs ($\sin\alpha = \frac{1}{3}$ when total surface is fixed). Different constraint $\Rightarrow$ different optimum geometry.

Final Answer: $\alpha = \sin^{-1}\left(\frac{1}{3}\right)$.

Q 6.27 **The point on the curve $x^2 = 2y$ which is nearest to the point $(0, 5)$ is**
(A) $(2\sqrt{2},\ 4)$ **(B)** $(2\sqrt{2},\ 0)$ **(C)** $(0,\ 0)$ **(D)** $(2,\ 2)$.

SOLUTION

Concept used. Minimise the squared distance $D(x, y) = x^2 + (y-5)^2$ subject to the curve $x^2 = 2y$. Substitute the constraint to obtain a one-variable function and minimise.

Step 1. Parametrise the curve: any point on $x^2 = 2y$ can be written as $(x, \frac{x^2}{2})$.

Step 2. Squared distance from $(0, 5)$:

$$D(x) = x^2 + \left(\frac{x^2}{2} - 5\right)^2.$$

Step 3. Differentiate. Let $u = \frac{x^2}{2} - 5$; then $\frac{du}{dx} = x$ and

$$D'(x) = 2x + 2u \cdot x = 2x(1+u) = 2x\left(1 + \frac{x^2}{2} - 5\right) = 2x\left(\frac{x^2}{2} - 4\right).$$

Step 4. Set $D'(x) = 0$: $x = 0$ or $\frac{x^2}{2} - 4 = 0 \Rightarrow x^2 = 8 \Rightarrow x = \pm 2\sqrt{2}$.

Step 5. Evaluate D at candidates: $D(0) = 0 + 25 = 25$;
$D(\pm 2\sqrt{2}) = 8 + (4-5)^2 = 8 + 1 = 9$. (Since $\frac{(2\sqrt{2})^2}{2} = 4$.)

Step 6. The smaller value is 9 at $x = \pm 2\sqrt{2}$, giving $y = \frac{8}{2} = 4$. So the nearest points are $(\pm 2\sqrt{2},\ 4)$. Choice (A) lists $(2\sqrt{2}, 4)$.

Final Answer: (A) $(2\sqrt{2},\ 4)$.

EXPERT'S SOLUTION *: Kavya Iyer, M.Sc Mathematics, ISI Kolkata*

Picture-first. The curve $x^2 = 2y$ is an upward-opening parabola with vertex at the origin. The given point $(0, 5)$ is above it on the y-axis. By symmetry the nearest point on the parabola has either the same x-coordinate as $(0, 5)$ (i.e. $x = 0$, giving distance 5) or two mirror points equidistant.

Step 1. Squared distance: $D(x) = x^2 + \left(\frac{x^2}{2} - 5\right)^2$.

Step 2. Critical points: $x = 0$ and $x = \pm 2\sqrt{2}$. Evaluate; $\pm 2\sqrt{2}$ both yield $D = 9$, beating $D(0) = 25$.

Step 3. Answer (A) $(2\sqrt{2}, 4)$ (or equivalently $(-2\sqrt{2}, 4)$).

Why this matters. For "nearest point on curve" problems, always square the distance before differentiating: the algebra is much cleaner.

Final Answer: (A).

Q 6.28 **For all real values of x, the minimum value of $\dfrac{1 - x + x^2}{1 + x + x^2}$ is**
(A) 0 **(B)** 1 **(C)** 3 **(D)** $\frac{1}{3}$.

SOLUTION

Concept used. For a rational function $\dfrac{p(x)}{q(x)}$ with $q(x) > 0$ for all real x, the minimum can be found by differentiating using the quotient rule and solving $f'(x) = 0$.

Step 1. Set $f(x) = \dfrac{1 - x + x^2}{1 + x + x^2}$. Note $1 + x + x^2 > 0$ for all real x (discriminant $1 - 4 = -3 < 0$, leading coefficient > 0).

Step 2. Differentiate using the quotient rule. Let $N = 1 - x + x^2$ and $D = 1 + x + x^2$. $N' = -1 + 2x$, $D' = 1 + 2x$.

$$f'(x) = \tfrac{N'D - ND'}{D^2}.$$

Compute the numerator:

$$\begin{aligned} N'D - ND' &= (-1 + 2x)(1 + x + x^2) - (1 - x + x^2)(1 + 2x) \\ &= \left[-1 - x - x^2 + 2x + 2x^2 + 2x^3\right] - \left[1 + 2x - x - 2x^2 + x^2 + 2x^3\right] \\ &= \left[2x^3 + x^2 + x - 1\right] - \left[2x^3 - x^2 + x + 1\right] \\ &= 2x^2 - 2. \end{aligned}$$

Step 3. So $f'(x) = \dfrac{2x^2 - 2}{(1 + x + x^2)^2}$. Set numerator $= 0$: $x^2 = 1 \Rightarrow x = \pm 1$.

Step 4. Evaluate: $f(1) = \frac{1-1+1}{1+1+1} = \frac{1}{3}$; $f(-1) = \frac{1+1+1}{1-1+1} = \frac{3}{1} = 3$.

Step 5. Behaviour at infinity: as $|x| \to \infty$, $f(x) \to 1$. So f has range bounded by the two critical values: minimum $= \frac{1}{3}$ at $x = 1$ and maximum $= 3$ at $x = -1$.

Final Answer: (D) $\frac{1}{3}$.

EXPERT'S SOLUTION *: Vivaan Bhat, M.Sc Mathematics, IIT Bombay*

Structural observation. Notice the numerator and denominator are reflections of each other under $x \mapsto -x$. So $f(-x) = \frac{1+x+x^2}{1-x+x^2} = \frac{1}{f(x)}$. This means $f(x) \cdot f(-x) = 1$: if the maximum is M, the minimum is $\frac{1}{M}$.

Step 1. Critical points: solve $f'(x) = 0$ as above; get $x = \pm 1$.

Step 2. $f(-1) = 3$ (max), so by symmetry $f(1) = \frac{1}{3}$ (min).

Step 3. Choice (D).

Why this matters. Symmetry can short-circuit calculus. Whenever $f(x) \cdot f(-x) = 1$, the extrema are reciprocals of each other.

Final Answer: (D).

Q 6.29 **The maximum value of $[x(x-1)+1]^{1/3}$, $0 \le x \le 1$, is**

(A) $\left(\frac{1}{3}\right)^{1/3}$ **(B)** $\frac{1}{2}$ **(C)** 1 **(D)** 0.

SOLUTION

Concept used. The cube root $u^{1/3}$ is a strictly increasing function of u, so the maximum of $[g(x)]^{1/3}$ is the cube root of the maximum of $g(x)$. Therefore maximise $g(x) = x(x-1)+1 = x^2 - x + 1$ on $[0,1]$ and cube root the result.

Step 1. Let $g(x) = x^2 - x + 1$. Differentiate: $g'(x) = 2x - 1$. Set $g'(x) = 0 \Rightarrow x = \frac{1}{2}$.

Step 2. Second derivative test: $g''(x) = 2 > 0$, so $x = \frac{1}{2}$ is a local minimum (not maximum). Local min value: $g\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$.

Step 3. Since the only interior critical point is a minimum, the maximum on $[0,1]$ is at an endpoint. Evaluate: $g(0) = 0 - 0 + 1 = 1$; $g(1) = 1 - 1 + 1 = 1$. Maximum of g on $[0,1]$ is 1.

Step 4. Take the cube root: $\max[x(x-1)+1]^{1/3} = 1^{1/3} = 1$.

Final Answer: (C) 1.

EXPERT'S SOLUTION : *Pranav Banerjee, M.Sc Mathematics, IIT Bombay*

Quick reading. $x(x-1) = x^2 - x$ is non-positive on $[0, 1]$, with maximum 0 at the endpoints. So $x(x-1)+1 \le 1$ on $[0, 1]$, with equality at $x = 0$ and $x = 1$. Cube root gives the same.

Step 1. $x(x-1) = x^2 - x$. On $[0, 1]$, this is ≤ 0 (parabola opens up, roots at 0 and 1). Maximum value on $[0, 1]$ is 0 at $x = 0, 1$; minimum is $-\frac{1}{4}$ at $x = \frac{1}{2}$.

Step 2. Add 1: range is $\left[\frac{3}{4}, 1\right]$.

Step 3. Cube root preserves order: range of $[x(x-1)+1]^{1/3}$ is $\left[\left(\frac{3}{4}\right)^{1/3}, 1\right]$. Max $= 1$. Choice (C).

Why this matters. Composing with a monotone function preserves extrema: saves recomputing.

Final Answer: (C).

Key Takeaways

- **Critical points** are where $f'(x) = 0$ or f' does not exist. Local extrema are classified by the First or Second Derivative Test.
- **Absolute extrema on** $[a, b]$ are found by evaluating f at all critical points in (a, b) and at the endpoints, then picking the largest and smallest.
- **Optimisation word problems**: pick a variable, write the constraint, eliminate, differentiate once for the critical point, twice (or sign-of-f') for classification.
- **Equal-share principle** (AM–GM): for fixed sum, the product is largest at equality; for products $x^a y^b$ with fixed sum, the optimum is $\frac{x}{a} = \frac{y}{b}$.
- **Standard geometric results**: closed cylinder of fixed volume has $h = 2r$; largest rectangle in a circle is a square; largest cone in a sphere has $V_{\text{max}} = \frac{8}{27} V_{\text{sphere}}$.

End of Exercise 6.3