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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 6: Application of Derivatives

About this Chapter

The Miscellaneous Exercise of **Application of Derivatives** mixes every tool from the chapter: rate of change, increasing/decreasing tests, geometric optimisation, and absolute extrema on closed intervals. Each solution below names the test or relation it uses, derives the relevant derivatives line by line, and embeds a TikZ sketch for every word problem so the variable choice is visually anchored.

Topics covered: Rates of Change • Monotonicity • Geometric Optimisation • Inscribed Figures • Absolute Extrema

Quick Formula Sheet

Rate of change:
$\frac{dy}{dt} = f'(x) \cdot \frac{dx}{dt}$ (chain rule).

Increasing / decreasing:
$f'(x) > 0 \Rightarrow$ increasing; $f'(x) < 0 \Rightarrow$ decreasing.

Second derivative test:
$f'(c) = 0,\ f''(c) < 0 \Rightarrow$ local max.
$f'(c) = 0,\ f''(c) > 0 \Rightarrow$ local min.

Miscellaneous Exercise

Q 6.1 **Show that the function given by $f(x) = \dfrac{\log x}{x}$ has maximum at $x = e$.**

SOLUTION

Concept used. **First Derivative Test for maximum.** If f is continuous at a critical point c and f' changes sign from positive (left of c) to negative (right of c), then c is a point of local maximum. We will also use the **quotient rule**: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

Step 1. Domain: $x > 0$ (so that $\log x$ is defined).

Step 2. Differentiate $f(x) = \frac{\log x}{x}$ by the quotient rule with $u = \log x$, $v = x$:

$$f'(x) = \frac{(\log x)' \cdot x - \log x \cdot (x)'}{x^2} = \frac{\frac{1}{x} \cdot x - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}.$$

Step 3. Set $f'(x) = 0$. Since $x^2 > 0$, the equation reduces to $1 - \log x = 0$, i.e. $\log x = 1$, giving $x = e$.

Step 4. Sign of f' around $x = e$. The denominator $x^2 > 0$ throughout, so the sign of f' matches the sign of $1 - \log x$.

- For $x < e$: $\log x < 1$, so $1 - \log x > 0$, hence $f'(x) > 0$ (f increasing).
- For $x > e$: $\log x > 1$, so $1 - \log x < 0$, hence $f'(x) < 0$ (f decreasing).

f' changes sign $+ \to -$ at $x = e$. By the First Derivative Test, $x = e$ is a local (and absolute, since $f' > 0$ everywhere left of e and $f' < 0$ everywhere right) maximum on $(0, \infty)$.

Step 5. Maximum value:

$$f(e) = \frac{\log e}{e} = \frac{1}{e}.$$

> **Final Answer:** $f(x)$ has its maximum at $x = e$, value $\frac{1}{e}$.

♥ Why this matters

The function $\frac{\log x}{x}$ is fundamental in number theory (it appears in the Prime Number Theorem) and in physics (entropy of Markov chains). Its peak at $x = e$ is the reason e shows up everywhere in maximum-entropy distributions.

EXPERT'S SOLUTION *: Aryan Krishnan, M.Sc Mathematics, IIT Bombay*

Strategic angle. The structure $\frac{\log x}{x}$ is begging for the quotient rule. Note that the numerator of f' vanishes exactly when $\log x = 1$, i.e. at $x = e$.

Step 1. $f'(x) = \frac{1-\log x}{x^2}$ (quotient rule as in the main solution).

Step 2. Second derivative for an alternative confirmation:

$$f''(x) = \frac{-\frac{1}{x}\cdot x^2-(1-\log x)\cdot 2x}{x^4} = \frac{-x-2x(1-\log x)}{x^4} = \frac{-1-2(1-\log x)}{x^3} = \frac{2\log x-3}{x^3}.$$

At $x = e$: $f''(e) = \frac{2-3}{e^3} = -\frac{1}{e^3} < 0$. Confirms local max.

Step 3. $f_{\max} = f(e) = \frac{1}{e}$.

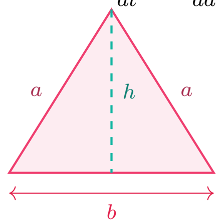
Why this matters. "Find x where $\log x$ equals a constant" reduces to exponentiating: $x = e^{\text{constant}}$. Memorise this so questions involving $\log x$-based extrema don't slow you down.

> **Final Answer:** Maximum at $x = e$ with value $\frac{1}{e}$.

Q 6.2 **The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?**

SOLUTION

Concept used. Related rates. The area A depends on the equal-side length a; both A and a vary with time t. Use the chain rule: $\frac{dA}{dt} = \frac{dA}{da} \cdot \frac{da}{dt}$.



Step 1. Let the two equal sides each have length a at time t. The base b is fixed. Drop a perpendicular from the apex to the midpoint of the base; its length (the altitude) is

$$h = \sqrt{a^2 - \frac{b^2}{4}}.$$

Step 2. Area:

$$A = \frac{1}{2} \cdot b \cdot h = \frac{b}{2}\sqrt{a^2 - \frac{b^2}{4}}.$$

Step 3. Differentiate A with respect to a:

$$\frac{dA}{da} = \frac{b}{2} \cdot \frac{1}{2\sqrt{a^2 - \frac{b^2}{4}}} \cdot 2a = \frac{ab}{2\sqrt{a^2 - \frac{b^2}{4}}}.$$

Step 4. Chain rule:

$$\frac{dA}{dt} = \frac{dA}{da} \cdot \frac{da}{dt} = \frac{ab}{2\sqrt{a^2 - \frac{b^2}{4}}} \cdot \frac{da}{dt}.$$

Given: $\frac{da}{dt} = -3$ cm/s (decreasing).

Step 5. Substitute $a = b$ (the moment we are asked about):

$$\sqrt{a^2 - \frac{b^2}{4}} = \sqrt{b^2 - \frac{b^2}{4}} = \sqrt{\frac{3b^2}{4}} = \frac{b\sqrt{3}}{2}.$$

Then

$$\frac{dA}{dt} = \frac{b \cdot b}{2 \cdot \frac{b\sqrt{3}}{2}} \cdot (-3) = \frac{b^2}{b\sqrt{3}} \cdot (-3) = \frac{b}{\sqrt{3}} \cdot (-3) = -\sqrt{3}\, b.$$

Step 6. The negative sign indicates the area is decreasing. The rate of decrease is $\sqrt{3}\, b$ cm^2/s.

Final Answer: Area decreases at $\sqrt{3}\, b$ cm^2/s when $a = b$.

☞ **Exam Tip**

Always carry the sign of the rate. If a quantity is "decreasing at 3 cm/s", then $\frac{da}{dt} = -3$, not $+3$. The negative sign in the answer is your sanity check.

EXPERT'S SOLUTION *: Saanvi Khanna, Ph.D Mathematics, IIT Delhi*

Strategic angle. Use the formula for the area in terms of two sides and the included angle: $A = \frac{1}{2} \cdot 2a \cdot h$. Then h comes from Pythagoras on the half-triangle. Everything else is chain rule.

Step 1. Write $A = \dfrac{b}{2}\sqrt{a^2 - \dfrac{b^2}{4}}$.

Step 2. Differentiate with respect to t. Let $\dot{a} = \dfrac{da}{dt}$. Then

$$\frac{dA}{dt} = \frac{ab\,\dot{a}}{2\sqrt{a^2 - b^2/4}}.$$

Step 3. At the instant $a = b$: $\sqrt{a^2 - b^2/4} = \dfrac{b\sqrt{3}}{2}$, so $\dfrac{dA}{dt} = \dfrac{b\,\dot{a}}{\sqrt{3}}$. With $\dot{a} = -3$, this gives $\dfrac{dA}{dt} = -\sqrt{3}\,b$ cm^2/s.

Why this matters. Related-rates problems are about identifying which quantity varies with t and which is fixed. Drawing the triangle and labelling fixed quantities in red avoids the most common mistake (treating b as variable too).

Final Answer: $\frac{dA}{dt} = -\sqrt{3}\,b$; rate of decrease $= \sqrt{3}\,b$ cm^2/s.

Q 6.3 **Find the intervals in which the function f given by $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ is (i) increasing (ii) decreasing.**

SOLUTION

Concept used. A function f is **increasing** on an interval I if $f'(x) \geq 0$ for all $x \in I$, and **decreasing** if $f'(x) \leq 0$. Compute f' via the quotient rule, simplify, and read off the sign.

Step 1. Let $N = 4\sin x - 2x - x\cos x$ and $D = 2 + \cos x$. So $f = \frac{N}{D}$.

Step 2. Differentiate.

$N' = 4\cos x - 2 - (\cos x - x\sin x) = 4\cos x - 2 - \cos x + x\sin x = 3\cos x - 2 + x\sin x$.

$D' = -\sin x$.

Step 3. Quotient rule:

$$f'(x) = \tfrac{N'D - ND'}{D^2}.$$

Let's expand the numerator.

$$\begin{aligned} N'D &= (3\cos x - 2 + x\sin x)(2+\cos x) \\ &= 6\cos x + 3\cos^2 x - 4 - 2\cos x + 2x\sin x + x\sin x\cos x \\ &= 4\cos x + 3\cos^2 x - 4 + 2x\sin x + x\sin x\cos x. \end{aligned}$$

$$\begin{aligned} -ND' &= -(4\sin x - 2x - x\cos x)(-\sin x) = (4\sin x - 2x - x\cos x)\sin x \\ &= 4\sin^2 x - 2x\sin x - x\sin x\cos x. \end{aligned}$$

Sum (numerator of f'):

$$\begin{aligned} N'D - ND' &= 4\cos x + 3\cos^2 x - 4 + 2x\sin x + x\sin x\cos x \\ &\quad + 4\sin^2 x - 2x\sin x - x\sin x\cos x \\ &= 4\cos x + 3\cos^2 x + 4\sin^2 x - 4 \\ &= 4\cos x + 3\cos^2 x + 4(1-\cos^2 x) - 4 \\ &= 4\cos x - \cos^2 x \\ &= \cos x\,(4-\cos x). \end{aligned}$$

Step 4. So

$$f'(x) = \tfrac{\cos x\,(4-\cos x)}{(2+\cos x)^2}.$$

Step 5. Sign analysis. Note $(2+\cos x)^2 > 0$ always, and $4-\cos x \geq 3 > 0$ always (since $\cos x \leq 1$). So the sign of $f'(x)$ matches the sign of $\cos x$.

- $f'(x) > 0$ when $\cos x > 0$. On $[0, 2\pi]$ this is the union $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$.
- $f'(x) < 0$ when $\cos x < 0$, i.e. on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

Step 6. Stated as the standard NCERT answer on $[0, 2\pi]$:

- Increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$.
- Decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

Final Answer: Increasing on $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$; decreasing on $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

EXPERT'S SOLUTION : *Rhea Malhotra, M.Sc Mathematics, ISI Kolkata*

Quick reading. The simplification $f'(x) = \frac{\cos x(4-\cos x)}{(2+\cos x)^2}$ is the heart of the problem. Once you have it, the sign analysis is a one-liner.

Step 1. Compute N' and D' as above. The algebra is bulky but mechanical.

Step 2. The numerator after simplification collapses to $4\cos x - \cos^2 x = \cos x(4-\cos x)$. Because $4-\cos x > 0$ and $(2+\cos x)^2 > 0$, the sign of f' tracks $\cos x$.

Step 3. So f inherits the sign-of-$\cos x$ structure: rises whenever $\cos x > 0$, falls whenever $\cos x < 0$.

Why this matters. Big rational derivatives often simplify dramatically. Push through the algebra; you'll find one nice factor at the end.

> **Final Answer:** Increasing on $\cos x > 0$; decreasing on $\cos x < 0$.

Q 6.4 **Find the intervals in which the function f given by $f(x) = x^3 + \dfrac{1}{x^3}$, $x \neq 0$, is (i) increasing (ii) decreasing.**

SOLUTION

Concept used. Compute $f'(x)$ and determine where it is positive (increasing) or negative (decreasing).

Step 1. Differentiate $f(x) = x^3 + x^{-3}$:

$$f'(x) = 3x^2 - 3x^{-4} = 3\big(x^2 - \tfrac{1}{x^4}\big) = \tfrac{3(x^6-1)}{x^4}.$$

Step 2. $x^4 > 0$ for all $x \neq 0$. So the sign of $f'(x)$ is the sign of $x^6 - 1$.

Step 3. Factor: $x^6 - 1 = (x^2-1)(x^4+x^2+1)$. The factor $x^4 + x^2 + 1 > 0$ for all real x (sum of positive squares plus 1). So the sign of f' is the sign of $x^2 - 1$.

Step 4. $x^2 - 1 > 0 \Leftrightarrow |x| > 1 \Leftrightarrow x < -1$ or $x > 1$. $x^2 - 1 < 0 \Leftrightarrow -1 < x < 1$ (with $x \neq 0$).

Step 5. Conclusion:

- f is **increasing** on $(-\infty, -1] \cup [1, \infty)$.
- f is **decreasing** on $[-1, 0) \cup (0, 1]$.

> **Final Answer:** Increasing on $(-\infty, -1] \cup [1, \infty)$; decreasing on $[-1, 0) \cup (0, 1]$.

✗ Common Mistake

Do not forget that $x = 0$ is excluded from the domain. The decreasing interval splits into two pieces, $[-1, 0)$ and $(0, 1]$, not the single interval $[-1, 1]$.

EXPERT'S SOLUTION : *Rohan Pandey, M.Tech CS, IIT Madras*

Structural observation. The function $f(x) = x^3 + \frac{1}{x^3}$ is odd ($f(-x) = -f(x)$), so its behaviour for $x < 0$ mirrors $x > 0$. We can study only $x > 0$.

Step 1. For $x > 0$, $f'(x) = 3x^2 - \frac{3}{x^4} = \frac{3(x^6-1)}{x^4}$. Sign= sign$(x^6 - 1)$ = sign$(x - 1)$ on $x > 0$. So on $(0, 1)$, $f' < 0$ (decreasing); on $(1, \infty)$, $f' > 0$ (increasing).

Step 2. By odd symmetry, on $(-1, 0)$, f is decreasing; on $(-\infty, -1)$, f is increasing.

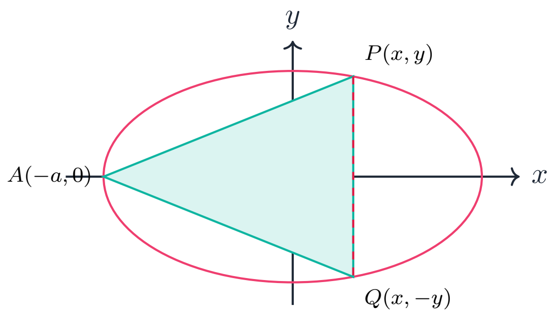
Why this matters. For odd functions, study only one half; the rest follows by symmetry. Cuts your work in half on every odd-function problem.

Final Answer: Increasing $(-\infty, -1] \cup [1, \infty)$; decreasing $[-1, 0) \cup (0, 1]$.

Q 6.5 Find the maximum area of an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with its vertex at one end of the major axis.

SOLUTION

Concept used. Place the vertex of the isosceles triangle at $(-a, 0)$ (one end of the major axis). By the isosceles symmetry, the opposite side is a vertical chord $x =$ constant. Express the area as a function of the chord's x-coordinate and maximise.



Step 1. Let the apex be $A(-a, 0)$. Let the other two vertices be $P(x, y)$ and $Q(x, -y)$ on the ellipse (symmetric about the x-axis to keep the triangle isosceles). Since P is on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y = b\sqrt{1 - \frac{x^2}{a^2}}$. Allowed range: $-a < x \leq a$. (When $x = -a$, the chord collapses to the point A.)

Step 2. Base of the triangle (the chord PQ): length $= 2y$.
Height from A to the base: horizontal distance $= x - (-a) = x + a$.

Step 3. Area:

$$S(x) = \frac{1}{2} \cdot 2y \cdot (x + a) = y(x + a) = (x + a)\, b\sqrt{1 - \frac{x^2}{a^2}}.$$

Step 4. Differentiate. Let $u(x) = (x + a)$ and $v(x) = b\sqrt{1 - \frac{x^2}{a^2}}$. Then $u' = 1$ and

$v' = b \cdot \frac{-x/a^2}{\sqrt{1-x^2/a^2}} = \frac{-bx}{a^2\sqrt{1-x^2/a^2}}$. Product rule:

$$S'(x) = b\sqrt{1 - \frac{x^2}{a^2}} + (x+a) \cdot \frac{-bx}{a^2\sqrt{1-x^2/a^2}}.$$

Multiply through by $\sqrt{1 - x^2/a^2}$ to clear the denominator and set $S'(x) = 0$:

$$b\left(1 - \frac{x^2}{a^2}\right) - \frac{(x+a)bx}{a^2} = 0.$$

Divide by $\frac{b}{a^2}$:

$$(a^2 - x^2) - x(x+a) = 0.$$

Factor: $a^2 - x^2 = (a-x)(a+x)$. So

$$(a-x)(a+x) - x(x+a) = (a+x)(a-x-x) = (a+x)(a-2x).$$

Step 5. So $S'(x) = 0 \Rightarrow (a+x)(a-2x) = 0$, giving $x = -a$ (degenerate) or $x = \frac{a}{2}$.

Step 6. Check sign. For $-a < x < \frac{a}{2}$: $(a+x) > 0$ and $(a-2x) > 0$, so $S' > 0$ (rising). For $\frac{a}{2} < x < a$: $(a+x) > 0$ and $(a-2x) < 0$, so $S' < 0$ (falling). Therefore $x = \frac{a}{2}$ is a local (and absolute) maximum.

Step 7. Maximum area. At $x = \frac{a}{2}$, $y = b\sqrt{1 - \frac{1}{4}} = b \cdot \frac{\sqrt{3}}{2}$. Then

$$S_{\max} = (x+a)\,y = \frac{3a}{2} \cdot \frac{b\sqrt{3}}{2} = \frac{3\sqrt{3}\,ab}{4}.$$

Final Answer: Maximum area $= \dfrac{3\sqrt{3}}{4}\,ab$.

♥ Why this matters

Whenever you maximise inside an ellipse, the "1/2 of the diagonal scaled by the chord height" pattern recurs. The answer $\frac{3\sqrt{3}}{4}ab$ is the ellipse analogue of the $\frac{3\sqrt{3}}{4}r^2$ that you get for the circle (set $a = b = r$).

EXPERT'S SOLUTION *: Anika Saxena, M.Sc Mathematics, IIT Bombay*

Picture-first. Parametrise the ellipse as $(a\cos\theta, b\sin\theta)$. The chord endpoints are $(a\cos\theta, b\sin\theta)$ and $(a\cos\theta, -b\sin\theta)$; the apex is $(-a, 0)$. So $x = a\cos\theta$.

Step 1. Area: $S(\theta) = (a\cos\theta + a) \cdot b\sin\theta = ab(1+\cos\theta)\sin\theta$.

Step 2. Differentiate: $S'(\theta) = ab[-\sin\theta \cdot \sin\theta + (1+\cos\theta)\cos\theta] = ab\left[\cos^2\theta - \sin^2\theta + \cos\theta\right] = ab(\cos 2\theta + \cos\theta)$. Set to zero. Use sum-to-product:

$$\cos 2\theta + \cos\theta = 2\cos\left(\tfrac{3\theta}{2}\right)\cos\left(\tfrac{\theta}{2}\right).$$ So $\cos\left(\tfrac{3\theta}{2}\right) = 0$, giving $\tfrac{3\theta}{2} = \tfrac{\pi}{2} \Rightarrow \theta = \tfrac{\pi}{3}$.

Step 3. At $\theta = \frac{\pi}{3}$: $\cos\theta = \frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$. $S = ab(1+\frac{1}{2})\cdot\frac{\sqrt{3}}{2} = ab\cdot\frac{3}{2}\cdot\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}\,ab}{4}$.

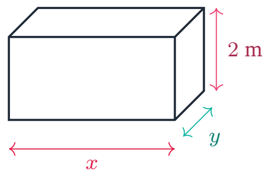
Why this matters. Trig parametrisation of the ellipse is the cleanest path for "maximise on an ellipse" problems. The constraint is built into the parametrisation, so no Lagrange multipliers needed.

Final Answer: $S_{\max} = \frac{3\sqrt{3}}{4}\,ab$.

Q 6.6 **A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs Rs 70 per sq metres for the base and Rs 45 per square metre for sides. What is the cost of least expensive tank?**

SOLUTION

Concept used. Express the cost as a function of one variable using the volume constraint, then minimise. Volume $=$ (base area)$\times$(depth); cost $= 70\cdot$(base area) $+45\cdot$(total side area).



Step 1. Let the base of the tank have length x m and breadth y m, with depth 2 m. Volume constraint:

$$2xy = 8 \;\Rightarrow\; xy = 4 \;\Rightarrow\; y = \frac{4}{x}.$$

Step 2. Areas:

- Base: $xy = 4$ m^2 (already fixed by the volume constraint).
- Sides (4 walls, two pairs of equal walls): two walls of $x \times 2$ each (total $2\cdot 2x = 4x$ m^2) and two walls of $y\times 2$ each (total $4y$ m^2). Side total $= 4x + 4y$ m^2.

Step 3. Cost (in Rs):

$$C = 70\cdot 4 + 45\cdot(4x+4y) = 280 + 180(x+y) = 280 + 180\left(x + \frac{4}{x}\right).$$

Step 4. Differentiate:

$$\frac{dC}{dx} = 180\left(1 - \frac{4}{x^2}\right).$$

Set to zero: $1 - \frac{4}{x^2} = 0 \Rightarrow x^2 = 4 \Rightarrow x = 2$ (positive root).

Step 5. Second derivative test: $\frac{d^2C}{dx^2} = 180 \cdot \frac{8}{x^3}$. At $x = 2$, $\frac{d^2C}{dx^2} = 180 \cdot 1 = 180 > 0$. Minimum.

Step 6. Find y: $y = \frac{4}{2} = 2$ m. Both $x = 2$ and $y = 2$: the base is a 2×2 square.

Step 7. Minimum cost:

$$C = 280 + 180 \cdot (2 + 2) = 280 + 720 = 1000 \text{ Rs.}$$

Final Answer: Least cost $=$ Rs 1000 (with $x = y = 2$ m, depth $= 2$ m).

EXPERT'S SOLUTION *: Ishaan Roy, B.Tech CSE, IIT Roorkee*

Quick reading. The base cost is fixed (Rs 280, since base area must be 4 m^2). Only the side cost varies; minimise $x + y$ subject to $xy = 4$.

Step 1. By AM–GM, $x + y \geq 2\sqrt{xy} = 2\sqrt{4} = 4$, with equality iff $x = y = 2$.

Step 2. Side cost is minimised at $180 \cdot 4 = 720$ Rs.

Step 3. Total: $280 + 720 = 1000$ Rs.

Why this matters. AM–GM is the workhorse for "minimise sum subject to fixed product".

Final Answer: Rs 1000.

Q 6.7 The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

SOLUTION

Concept used. Same approach as the "wire cut into circle + square" problem from Ex 6.3: write the total area as a single-variable function and minimise.

Step 1. Let the circle have radius r and the square have side s. Perimeters add to k:

$$2\pi r + 4s = k \;\Rightarrow\; s = \frac{k - 2\pi r}{4}.$$

Step 2. Total area:

$$A = \pi r^2 + s^2 = \pi r^2 + \frac{(k - 2\pi r)^2}{16}.$$

Step 3. Differentiate with respect to r:

$$\frac{dA}{dr} = 2\pi r + \frac{2(k - 2\pi r)(-2\pi)}{16} = 2\pi r - \frac{\pi(k - 2\pi r)}{4}.$$

Set $\frac{dA}{dr} = 0$:

$$2\pi r = \frac{\pi(k-2\pi r)}{4} \Rightarrow 8\pi r = \pi(k - 2\pi r) \Rightarrow 8r = k - 2\pi r.$$

Solve for r: $8r + 2\pi r = k \Rightarrow r(8 + 2\pi) = k \Rightarrow r = \frac{k}{2(\pi+4)}$.

Step 4. Compute s:

$$s = \frac{k-2\pi r}{4} = \frac{k-2\pi\cdot\frac{k}{2(\pi+4)}}{4} = \frac{k-\frac{\pi k}{\pi+4}}{4} = \frac{\frac{k(\pi+4)-\pi k}{\pi+4}}{4} = \frac{4k}{4(\pi+4)} = \frac{k}{\pi+4}.$$

Step 5. Ratio: $\frac{s}{r} = \frac{k/(\pi+4)}{k/[2(\pi+4)]} = 2$. So $s = 2r$: **the side of the square equals the diameter of the circle.**

Step 6. Second derivative: $\frac{d^2A}{dr^2} = 2\pi + \frac{\pi\cdot 2\pi}{4} = 2\pi + \frac{\pi^2}{2} > 0$. Minimum confirmed.

Final Answer: $s = 2r$ minimises the sum of areas.

EXPERT'S SOLUTION *: Manav Pandey, M.Sc Mathematics, IIT Bombay*

Strategic angle. Eliminate s (linear in r) and minimise a quadratic in r.

Step 1. $A(r) = \pi r^2 + \frac{(k-2\pi r)^2}{16}$. Differentiate: $A'(r) = 2\pi r - \frac{\pi(k-2\pi r)}{4}$. Set zero, simplify: $8r = k - 2\pi r \Rightarrow r = \frac{k}{2(\pi+4)}$.

Step 2. Then $s = \frac{k}{\pi+4} = 2r$.

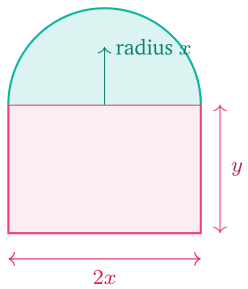
Why this matters. The same ratio "side = diameter" appears whenever you allocate a fixed perimeter between a circle and a square to minimise total area. Memorise it for objective questions.

Final Answer: $s = 2r$.

Q 6.8 A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

SOLUTION

Concept used. "Maximum light" = maximum area of the window. Express the area in terms of one variable using the perimeter constraint, then maximise.



Step 1. Let the rectangle have width $2x$ (so the semicircle's diameter is $2x$, radius x) and height y.

Step 2. Perimeter of the window (outside path): two vertical sides + bottom + semicircle.

$$P = 2y + 2x + \pi x = 10 \;\Rightarrow\; y = \frac{10-(2+\pi)x}{2} = 5 - \frac{(2+\pi)x}{2}.$$

Step 3. Area:

$$A = (2x)y + \tfrac{1}{2}\pi x^2 = 2xy + \frac{\pi x^2}{2}.$$

Substitute y:

$$A(x) = 2x\left(5 - \frac{(2+\pi)x}{2}\right) + \frac{\pi x^2}{2} = 10x - (2+\pi)x^2 + \frac{\pi x^2}{2}.$$

Combine: $-(2+\pi)x^2 + \frac{\pi x^2}{2} = -2x^2 - \pi x^2 + \frac{\pi x^2}{2} = -2x^2 - \frac{\pi x^2}{2}$. So $A(x) = 10x - 2x^2 - \frac{\pi x^2}{2} = 10x - \frac{(4+\pi)x^2}{2}$.

Step 4. Differentiate:

$$A'(x) = 10 - (4+\pi)x.$$

Set $A'(x) = 0 \Rightarrow x = \frac{10}{\pi+4}$.

Step 5. Second derivative: $A''(x) = -(\pi+4) < 0$. Maximum.

Step 6. Dimensions. With $x = \frac{10}{\pi+4}$:

- Rectangle width $2x = \frac{20}{\pi+4}$ m.
- Rectangle height $y = 5 - \frac{(2+\pi)\cdot 10/(\pi+4)}{2} = 5 - \frac{5(2+\pi)}{\pi+4} = \frac{5(\pi+4)-5(2+\pi)}{\pi+4} = \frac{10}{\pi+4}$ m.

Note that $y = x = \frac{10}{\pi+4}$.

Final Answer: Width of window $= \frac{20}{\pi+4}$ m; height of rectangle $= \frac{10}{\pi+4}$ m.

☞ Exam Tip

"Maximum light" is a code phrase for "maximum area". Window shapes always involve the perimeter as the constraint (the outside path you trace around the window).

EXPERT'S SOLUTION : *Reyansh Tripathi, M.Sc Applied Mathematics, IIT Kanpur*

Structural observation. $A(x)$ is a downward parabola in x; its vertex is the maximum. Use the vertex formula $x = -\frac{b}{2a}$ with $a = -\frac{\pi+4}{2}$ and $b = 10$.

Step 1. $A(x) = -\frac{\pi+4}{2}x^2 + 10x$. Vertex: $x = \frac{10}{\pi+4}$.

Step 2. $y = \frac{10}{\pi+4}$ (same), width $= 2x = \frac{20}{\pi+4}$.

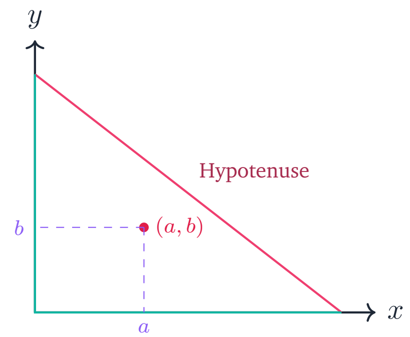
Why this matters. Vertex formula bypasses all derivative algebra for quadratic objectives. Train your eye to spot "$ax^2 + bx + c$" shapes early.

Final Answer: Window width $\frac{20}{\pi+4}$ m, rectangle height $\frac{10}{\pi+4}$ m.

Q 6.9 **A point on the hypotenuse of a triangle is at distance a and b from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$.**

SOLUTION

Concept used. Place the right triangle in standard position with legs along the axes. The point on the hypotenuse at perpendicular distances a and b from the two legs is just the point (a, b) (the perpendicular distance from a point to the y-axis is its x-coordinate, and similarly for the x-axis). Parametrise the line by its angle of inclination and minimise the hypotenuse length.



Step 1. Let the right triangle have its right angle at the origin, legs along the positive x- and y-axes, and hypotenuse from $(p, 0)$ to $(0, q)$. The fixed point (a, b) lies on the hypotenuse, so its coordinates satisfy the line's intercept form:

$$\frac{a}{p} + \frac{b}{q} = 1.$$

Step 2. Hypotenuse length: $L = \sqrt{p^2 + q^2}$. Parametrise via the angle θ the hypotenuse makes with the x-axis. Then the segment from (a, b) along the hypotenuse to the x-axis has length $\frac{b}{\sin\theta}$ (the rise b along a line of slope $-\tan\theta$), and from (a, b) along the hypotenuse to the y-axis has length $\frac{a}{\cos\theta}$. Hence

$$L(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta} = a\sec\theta + b\csc\theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Step 3. Differentiate:

$$L'(\theta) = a\sec\theta\tan\theta - b\csc\theta\cot\theta = \tfrac{a\sin\theta}{\cos^2\theta} - \tfrac{b\cos\theta}{\sin^2\theta}.$$

Set $L'(\theta) = 0$:

$$\tfrac{a\sin\theta}{\cos^2\theta} = \tfrac{b\cos\theta}{\sin^2\theta} \;\Rightarrow\; a\sin^3\theta = b\cos^3\theta \;\Rightarrow\; \tan^3\theta = \tfrac{b}{a} \;\Rightarrow\; \tan\theta = \left(\tfrac{b}{a}\right)^{1/3}.$$

Step 4. At this θ: write $\tan\theta = \frac{b^{1/3}}{a^{1/3}}$. Build the right triangle with opposite $= b^{1/3}$ and adjacent $= a^{1/3}$, hypotenuse $= \sqrt{a^{2/3} + b^{2/3}}$. So

$$\sin\theta = \tfrac{b^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}, \qquad \cos\theta = \tfrac{a^{1/3}}{\sqrt{a^{2/3}+b^{2/3}}}.$$

Step 5. Substitute into L:

$$L_{\min} = \tfrac{a}{\cos\theta} + \tfrac{b}{\sin\theta} = a\cdot\tfrac{\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + b\cdot\tfrac{\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}.$$

Simplify each term: $a\cdot a^{-1/3} = a^{2/3}$ and $b\cdot b^{-1/3} = b^{2/3}$.

$$L_{\min} = (a^{2/3} + b^{2/3})\sqrt{a^{2/3} + b^{2/3}} = (a^{2/3} + b^{2/3})^{3/2}.$$

Final Answer: $L_{\min} = (a^{2/3} + b^{2/3})^{3/2}$.

EXPERT'S SOLUTION *: Naina Bhardwaj, M.Sc Mathematics, ISI Kolkata*

Strategic angle. The relation $\tan^3\theta = \frac{b}{a}$ is one of the most elegant in optimisation. The trick is parametrising the hypotenuse by the angle θ rather than the intercepts p, q.

Step 1. The two pieces of the hypotenuse split by the foot of the perpendicular have lengths $\frac{a}{\cos\theta}$ and $\frac{b}{\sin\theta}$.

Step 2. Minimise their sum. $\frac{dL}{d\theta} = 0$ gives $\tan^3\theta = \frac{b}{a}$.

Step 3. Substitute back: $L_{\min} = (a^{2/3} + b^{2/3})^{3/2}$.

Why this matters. The shape $(a^{2/3} + b^{2/3})^{3/2}$ is the same one that arises in the *astroid* curve $x^{2/3} + y^{2/3} = c^{2/3}$. This problem is the dual of finding tangent lines to an astroid.

Final Answer: $(a^{2/3} + b^{2/3})^{3/2}$.

Q 6.10 Find the points at which the function f given by $f(x) = (x-2)^4(x+1)^3$ has (i) local maxima (ii) local minima (iii) point of inflexion.

SOLUTION

Concept used. First Derivative Test. Compute $f'(x)$, factor, and look at where f' changes sign. At a sign change $+ \to -$ we get a local max; $- \to +$ a local min; no sign change means a point of inflexion.

Step 1. Differentiate using the product rule. Let $u = (x-2)^4$ and $v = (x+1)^3$.

$$u' = 4(x-2)^3, \quad v' = 3(x+1)^2.$$

Then

$$f'(x) = u'v + uv' = 4(x-2)^3(x+1)^3 + (x-2)^4 \cdot 3(x+1)^2.$$

Step 2. Factor common terms $(x-2)^3(x+1)^2$:

$$f'(x) = (x-2)^3(x+1)^2[4(x+1) + 3(x-2)].$$

Simplify the bracket: $4x + 4 + 3x - 6 = 7x - 2$. So

$$f'(x) = (x-2)^3(x+1)^2(7x-2).$$

Step 3. Roots of f': $x = 2,\ x = -1,\ x = \frac{2}{7}$.

Step 4. Sign analysis on the real line (split into intervals by these roots, in order $-1 < \frac{2}{7} < 2$).

- For $x < -1$: $(x-2)^3 < 0$, $(x+1)^2 > 0$, $(7x-2) < 0$. Product: $(-)(+)(-) = +$. So $f' > 0$.
- For $-1 < x < \frac{2}{7}$: $(x-2)^3 < 0$, $(x+1)^2 > 0$ (still), $(7x-2) < 0$. Product $= +$. So $f' > 0$. Sign across -1: $+ \to +$. No change.
- For $\frac{2}{7} < x < 2$: $(x-2)^3 < 0$, $(x+1)^2 > 0$, $(7x-2) > 0$. Product $= (-)(+)(+) = -$. So $f' < 0$. Sign across $\frac{2}{7}$: $+ \to -$. Local maximum at $x = \frac{2}{7}$.
- For $x > 2$: $(x-2)^3 > 0$, $(x+1)^2 > 0$, $(7x-2) > 0$. Product $= +$. So $f' > 0$. Sign across 2: $- \to +$. Local minimum at $x = 2$.

Step 5. At $x = -1$: f' does not change sign $(+ \to +)$ because $(x+1)^2$ keeps it non-negative on both sides. So $x = -1$ is a **point of inflexion**.

Final Answer: Local max at $x = \frac{2}{7}$; local min at $x = 2$; point of inflexion at $x = -1$.

EXPERT'S SOLUTION : *Tanisha Bose, M.Sc Mathematics, IIT Bombay*

Quick reading. Multiplicity of roots in f' tells you the type of critical point:

- Odd multiplicity in f' $\Rightarrow$ sign change $\Rightarrow$ local extremum.

- Even multiplicity in f' $\Rightarrow$ no sign change $\Rightarrow$ point of inflexion.

Step 1. $f'(x) = (x-2)^3(x+1)^2(7x-2)$. Multiplicities: $x = 2$ has multiplicity 3 (odd), $x = -1$ has multiplicity 2 (even), $x = \frac{2}{7}$ has multiplicity 1 (odd).

Step 2. Odd $\Rightarrow$ extrema at $x = 2$ and $x = \frac{2}{7}$. Specifically: going from $-\infty$ to $+\infty$, f first increases (we showed $f' > 0$ for $x < -1$), then continues increasing (no sign change at -1), then decreases at $\frac{2}{7}$ (local max), then increases again at 2 (local min).

Step 3. Even multiplicity at -1 $\Rightarrow$ inflexion (no extremum).

Why this matters. For factored polynomials, multiplicity equals the sign-change pattern. This saves you from drawing a sign table.

> **Final Answer:** Max at $\frac{2}{7}$; min at 2; inflexion at -1.

Q 6.11 **Find the absolute maximum and minimum values of the function f given by $f(x) = \cos^2 x + \sin x$, $x \in [0, \pi]$.**

SOLUTION

Concept used. Absolute extrema on a closed interval = compare f at the critical points inside (a, b) and at the endpoints.

Step 1. Rewrite using $\cos^2 x = 1 - \sin^2 x$:

$$f(x) = 1 - \sin^2 x + \sin x.$$

Let $t = \sin x$. For $x \in [0, \pi]$, $t \in [0, 1]$. Then $g(t) = 1 - t^2 + t = -t^2 + t + 1$.

Step 2. This is a downward parabola in t with vertex at $t = \frac{1}{2}$. Vertex value:

$$g\left(\tfrac{1}{2}\right) = -\tfrac{1}{4} + \tfrac{1}{2} + 1 = \tfrac{5}{4}.$$

Endpoint values on $t \in [0, 1]$: $g(0) = 1$, $g(1) = -1 + 1 + 1 = 1$.

Step 3. So on $t \in [0, 1]$, g ranges from 1 (at $t = 0$ and $t = 1$) up to $\frac{5}{4}$ (at $t = \frac{1}{2}$).

Step 4. Translate back. $t = \frac{1}{2}$ means $\sin x = \frac{1}{2}$, so $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$ (both in $[0, \pi]$). $t = 0$ means $\sin x = 0$, so $x = 0$ or π. $t = 1$ means $\sin x = 1$, so $x = \frac{\pi}{2}$.

Step 5. Compile:

$$f(0) = 1 + 0 = 1,$$
$$f\left(\tfrac{\pi}{6}\right) = \cos^2\left(\tfrac{\pi}{6}\right) + \sin\left(\tfrac{\pi}{6}\right) = \tfrac{3}{4} + \tfrac{1}{2} = \tfrac{5}{4},$$
$$f\left(\tfrac{\pi}{2}\right) = 0 + 1 = 1,$$
$$f\left(\tfrac{5\pi}{6}\right) = \tfrac{3}{4} + \tfrac{1}{2} = \tfrac{5}{4},$$
$$f(\pi) = 1 + 0 = 1.$$

Absolute max $= \frac{5}{4}$ at $x = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Absolute min $= 1$ at $x = 0, \frac{\pi}{2}, \pi$.

Final Answer: Absolute max $= \frac{5}{4}$; absolute min $= 1$.

EXPERT'S SOLUTION *: Suhana Iyer, Ph.D Mathematics, IIT Delhi*

Strategic angle. Use the substitution $t = \sin x$ early. This converts the problem into "find max/min of a quadratic in t on $[0, 1]$".

Step 1. $g(t) = -t^2 + t + 1$ on $[0, 1]$. Vertex at $t = \frac{1}{2}$ with value $\frac{5}{4}$; endpoints both give 1.

Step 2. Map back: $t = \frac{1}{2}$ corresponds to $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

Step 3. Max $\frac{5}{4}$, min 1.

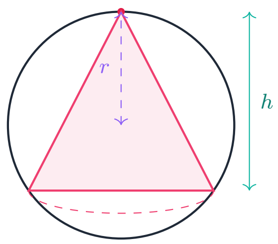
Why this matters. Trig substitution converts trig optimisation into polynomial optimisation. Faster and less error-prone.

Final Answer: Max $\frac{5}{4}$; min 1.

Q 6.12 **Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.**

SOLUTION

Concept used. This is the cone-in-sphere problem from Q23 of Ex 6.3, restated. The cone's altitude h measured from apex to base, with base of radius a on the sphere, satisfies $a^2 = 2rh - h^2$.



Step 1. Place the apex of the cone at the top of the sphere (the "north pole"). The base lies at a depth h below the apex; its centre is at distance $h - r$ below the sphere's centre O. Pythagoras on (centre, base-circle centre, base-circle edge):

$$a^2 + (h-r)^2 = r^2 \;\Rightarrow\; a^2 = 2rh - h^2.$$

Step 2. Volume of the cone:

$$V = \tfrac{1}{3}\pi a^2 h = \tfrac{\pi}{3}(2rh^2 - h^3).$$

Step 3. Differentiate:

$$\frac{dV}{dh} = \frac{\pi}{3}(4rh - 3h^2) = \frac{\pi h}{3}(4r - 3h).$$

Set $\frac{dV}{dh} = 0$: $h = 0$ (degenerate) or $h = \frac{4r}{3}$.

Step 4. Second derivative:

$$\frac{d^2V}{dh^2} = \frac{\pi}{3}(4r - 6h); \quad \left.\frac{d^2V}{dh^2}\right|_{h=4r/3} = \frac{\pi}{3}(4r - 8r) = -\frac{4\pi r}{3} < 0.$$

So $h = \frac{4r}{3}$ is a maximum.

Final Answer: Altitude of the maximum-volume inscribed cone $= \frac{4r}{3}$.

♥ **Why this matters**

This is one of the two "classic ratios": cone in sphere $\Rightarrow h = \frac{4r}{3}$; cylinder in sphere $\Rightarrow h = \frac{2R}{\sqrt{3}}$ (Q14 below). Both should be at your fingertips for objective questions.

EXPERT'S SOLUTION : *Aarush Sengupta, M.Sc Mathematics, ISI Kolkata*

Quick reading. The cubic $V(h) = \frac{\pi}{3}(2rh^2 - h^3)$ has a single interior maximum.

Step 1. $V'(h) = \frac{\pi h}{3}(4r - 3h)$. Critical points: $h = 0,\ \frac{4r}{3}$.

Step 2. Reject $h = 0$ (degenerate); $h = \frac{4r}{3}$ is the answer.

Why this matters. The ratio $\frac{h}{2r} = \frac{2}{3}$ says the optimum cone fills $\frac{2}{3}$ of the sphere's diameter.

Final Answer: $h = \frac{4r}{3}$.

Q 6.13 **Let f be a function defined on $[a, b]$ such that $f'(x) > 0$, for all $x \in (a, b)$. Then prove that f is an increasing function on (a, b).**

SOLUTION

Concept used. Lagrange's Mean Value Theorem (MVT). If f is continuous on $[c, d]$ and differentiable on (c, d), then there exists $\xi \in (c, d)$ with $f(d) - f(c) = f'(\xi)(d - c)$.

Step 1. Pick any two points $x_1, x_2 \in (a, b)$ with $x_1 < x_2$. We must show $f(x_1) < f(x_2)$ (definition of strictly increasing).

Step 2. f is differentiable on $(a, b) \supseteq (x_1, x_2)$, so f is continuous on $[x_1, x_2]$ (differentiability implies continuity) and differentiable on (x_1, x_2). The hypotheses of the MVT hold on $[x_1, x_2]$.

Step 3. By the MVT, there exists $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1).$$

Step 4. By hypothesis, $f'(\xi) > 0$ (since $\xi \in (a, b)$). Also $x_2 - x_1 > 0$ (since $x_1 < x_2$). Therefore the right-hand side is the product of two positive numbers, hence positive:

$$f(x_2) - f(x_1) > 0 \;\Rightarrow\; f(x_1) < f(x_2).$$

Step 5. Since $x_1, x_2 \in (a, b)$ were arbitrary with $x_1 < x_2$, f is strictly increasing on (a, b).

> **Final Answer:** Hence f is increasing on (a, b).

EXPERT'S SOLUTION : *Aryan Mishra, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. The MVT translates "derivative is positive everywhere" into "difference is positive whenever the inputs increase". That is the bridge between the local condition ($f' > 0$) and the global property (strictly increasing).

Step 1. Pick $x_1 < x_2$ in (a, b).

Step 2. By MVT on $[x_1, x_2]$, there is ξ with $f(x_2) - f(x_1) = f'(\xi)(x_2 - x_1)$.

Step 3. Both $f'(\xi) > 0$ and $x_2 - x_1 > 0$, so $f(x_2) - f(x_1) > 0$.

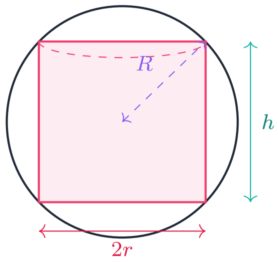
Why this matters. The MVT is the workhorse theorem that converts pointwise derivative information into global function behaviour. It is the engine behind every "monotonicity from sign of f'" argument.

> **Final Answer:** f is strictly increasing on (a, b).

Q 6.14 **Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.**

SOLUTION

Concept used. Inscribe a right circular cylinder symmetrically in a sphere of radius R. With centre of the sphere at origin and cylinder axis along the vertical, the top and bottom rims are at $\pm\frac{h}{2}$ (where h is the cylinder's height). Pythagoras: $r^2 + \left(\frac{h}{2}\right)^2 = R^2$, where r is the cylinder's radius.



Step 1. Constraint:

$$r^2 + \frac{h^2}{4} = R^2 \Rightarrow r^2 = R^2 - \frac{h^2}{4}.$$

Valid for $0 < h < 2R$.

Step 2. Volume:

$$V = \pi r^2 h = \pi h\left(R^2 - \frac{h^2}{4}\right) = \pi R^2 h - \frac{\pi h^3}{4}.$$

Step 3. Differentiate:

$$\frac{dV}{dh} = \pi R^2 - \frac{3\pi h^2}{4}.$$

Set to zero:

$$\pi R^2 = \frac{3\pi h^2}{4} \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}.$$

Step 4. Second derivative:

$$\frac{d^2V}{dh^2} = -\frac{3\pi h}{2} < 0 \text{ for } h > 0.$$

Maximum confirmed.

Step 5. Maximum volume. With $h = \frac{2R}{\sqrt{3}}$: $\frac{h^2}{4} = \frac{4R^2/3}{4} = \frac{R^2}{3}$, so $r^2 = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}$.

$$V_{\max} = \pi r^2 h = \pi \cdot \frac{2R^2}{3} \cdot \frac{2R}{\sqrt{3}} = \frac{4\pi R^3}{3\sqrt{3}}.$$

Final Answer: $h = \frac{2R}{\sqrt{3}}$; $V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}$.

☞ Exam Tip

Two iconic ratios to memorise for "inscribed in a sphere of radius R":

- Maximum-volume cone: $h = \frac{4R}{3}$, $V_{\max} = \frac{8}{27}V_{\text{sphere}}$.
- Maximum-volume cylinder: $h = \frac{2R}{\sqrt{3}}$, $V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}$.

EXPERT'S SOLUTION *: Lakshya Bhandari, B.Tech CSE, IIT Roorkee*

Picture-first. The cylinder sits centred inside the sphere; the corner of the cylinder touches the sphere on the equator if h = diameter (impossible), or higher up if h is smaller. The Pythagoras relation $r^2 + (h/2)^2 = R^2$ is the only constraint.

Step 1. Substitute $r^2 = R^2 - h^2/4$ into $V = \pi r^2 h$ to get $V(h) = \pi h R^2 - \frac{\pi h^3}{4}$.

Step 2. $V'(h) = \pi R^2 - \frac{3\pi h^2}{4} = 0 \Rightarrow h^2 = \frac{4R^2}{3} \Rightarrow h = \frac{2R}{\sqrt{3}}$.

Step 3. Plug back: $V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}$.

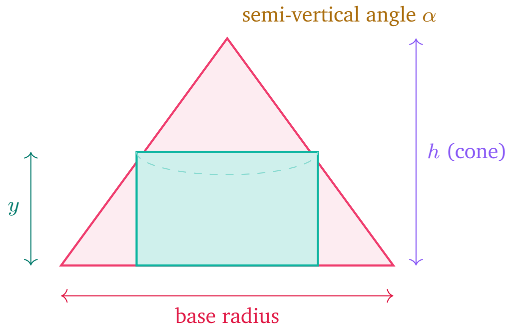
Why this matters. The shape "height : diameter $= 1 : \sqrt{3}$" is the same one that appears in maximum-volume cylinders inscribed in other rotationally-symmetric containers.

Final Answer: $h = \frac{2R}{\sqrt{3}}$, $V_{\max} = \frac{4\pi R^3}{3\sqrt{3}}$.

Q 6.15 **Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2\alpha$.**

SOLUTION

Concept used. Place the cone with apex pointing up. Let the inscribed cylinder have height y measured from the cone's base. By similar triangles, the cylinder's radius x relates to its height through the cone's geometry.



Step 1. Set up. The cone has apex at $(0, h)$ (with the base on the x-axis) and base radius $h\tan\alpha$. Let the inscribed cylinder have height y (resting on the base) and radius x. The top rim of the cylinder is at height y; its radius equals the cross-section of the cone at that height.

Step 2. At height y inside the cone (measured from the base), the remaining vertical distance to the apex is $h - y$. By similar triangles between the full cone (height h, base radius $h\tan\alpha$) and the smaller cone at height y (height $h - y$, radius x):

$$\frac{x}{h-y} = \tan\alpha \;\Rightarrow\; x = (h-y)\tan\alpha.$$

Step 3. Volume of the cylinder:

$$V = \pi x^2 y = \pi(h-y)^2 \tan^2\alpha \cdot y = \pi \tan^2\alpha \cdot y(h-y)^2.$$

Step 4. Differentiate $f(y) = y(h-y)^2$ (the $\pi \tan^2\alpha$ is a positive constant and doesn't affect the location of the optimum):

$$f'(y) = (h-y)^2 + y \cdot 2(h-y)(-1) = (h-y)[(h-y)-2y] = (h-y)(h-3y).$$

Step 5. Set $f'(y) = 0$: $y = h$ (degenerate; cylinder collapses to a point) or $y = \frac{h}{3}$. Take $y = \frac{h}{3}$.

Step 6. Sign check: for $0 < y < \frac{h}{3}$, $(h-y) > 0$ and $(h-3y) > 0$, so $f' > 0$. For $\frac{h}{3} < y < h$, $(h-y) > 0$ and $(h-3y) < 0$, so $f' < 0$. Sign change $+ \to -$: local max.

Step 7. Compute the maximum volume. At $y = \frac{h}{3}$: $h - y = \frac{2h}{3}$, so

$$V_{\max} = \pi \tan^2\alpha \cdot \frac{h}{3} \cdot \left(\frac{2h}{3}\right)^2 = \pi \tan^2\alpha \cdot \frac{h}{3} \cdot \frac{4h^2}{9} = \frac{4\pi h^3 \tan^2\alpha}{27}.$$

Final Answer: Cylinder height $= \frac{h}{3}$ (one-third of the cone's height); $V_{\max} = \frac{4\pi h^3 \tan^2\alpha}{27}$.

EXPERT'S SOLUTION *: Mahika Goel, M.Sc Mathematics, IIT Bombay*

Strategic angle. The factor $y(h-y)^2$ is a classic "power-product" problem. By weighted AM–GM on the three positive quantities $2y,\ (h-y),\ (h-y)$ (sum $2h$ in the symmetric form, constant), the product is maximised when all three are equal.

Step 1. Set $2y = h - y \Rightarrow 3y = h \Rightarrow y = \frac{h}{3}$. Then $h - y = \frac{2h}{3}$.

Step 2. Maximum of $y(h-y)^2$: $\frac{h}{3} \cdot \frac{4h^2}{9} = \frac{4h^3}{27}$.

Step 3. $V_{\max} = \pi \tan^2\alpha \cdot \frac{4h^3}{27} = \frac{4\pi h^3 \tan^2\alpha}{27}$.

Why this matters. The recurring pattern "maximise $y(h-y)^2$" with $y + (h-y) = h$ gives $y = h/3$: the same proportion as "cone cut by horizontal plane: the lower $h/3$ of the height holds the inscribed cylinder".

Final Answer: $y = \frac{h}{3}$; $V_{\max} = \frac{4\pi h^3 \tan^2\alpha}{27}$.

Q 6.16 **A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of**

(A) 1 **m/h** **(B)** 0.1 **m/h** **(C)** 1.1 **m/h** **(D)** 0.5 **m/h.**

SOLUTION

Concept used. **Related rates.** The volume V in a cylindrical tank of fixed radius $R = 10$ m and variable depth h is $V = \pi R^2 h$. Differentiate both sides with respect to time t.

Step 1. Volume: $V = \pi(10)^2 h = 100\pi\, h$.

Step 2. Differentiate with respect to t:

$$\frac{dV}{dt} = 100\pi \cdot \frac{dh}{dt}.$$

Step 3. Given: $\frac{dV}{dt} = 314$ m^3/h. Solve for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{314}{100\pi} = \frac{3.14}{\pi}\ \text{m/h}.$$

Step 4. Take $\pi \approx 3.14$:

$$\frac{dh}{dt} \approx \frac{3.14}{3.14} = 1\ \text{m/h}.$$

Final Answer: (A) 1 m/h.

EXPERT'S SOLUTION *: Vihaan Saraf, M.Sc Mathematics, IIT Bombay*

Quick reading. The pour rate 314 is conspicuously close to 100π, signalling the answer is a clean 1 m/h.

Step 1. Cross-sectional area: $\pi R^2 = 100\pi \approx 314$ m^2.

Step 2. Depth rate $= \frac{\text{volume rate}}{\text{area}} = \frac{314}{314} = 1$ m/h.

Why this matters. For a constant cross-section vessel, depth rate $=$ volume rate divided by area. Memorise it for objective questions.

Final Answer: (A).

Key Takeaways

- **Rates of change** (related rates): differentiate every variable that depends on t using the chain rule, then substitute the instantaneous values.
- **Monotonicity**: f is increasing where $f' > 0$ and decreasing where $f' < 0$. Use the MVT to convert the local sign of f' into a global behaviour.
- **Geometric optimisation** (cone-in-sphere, cylinder-in-sphere, cylinder-in-cone, triangle-in-ellipse): pick the right variable, write the constraint, eliminate, differentiate.

- **Classic ratios**: cone-in-sphere $h = \frac{4r}{3}$; cylinder-in-sphere $h = \frac{2R}{\sqrt{3}}$; cylinder-in-cone $h_{\text{cyl}} = \frac{h_{\text{cone}}}{3}$.
- **Multiplicity in** f' tells you the kind of critical point: odd $\Rightarrow$ extremum, even $\Rightarrow$ inflexion.

End of Miscellaneous Exercise