

# Collegedunia NCERT Solutions

Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

## Chapter 7: Integrals

### About this Chapter

Exercise 7.10 trains the **eight properties** of definite integrals: limit reversal ( $P_1$ ), additivity ( $P_2$ ), the  $a + b - x$  reflection ( $P_3$ ), the King's property  $P_4$  ( $\int_0^a f(x) dx = \int_0^a f(a - x) dx$ ), the  $2a$  rules  $P_5, P_6$ , and the odd/even symmetry  $P_7$ . These slash intricate integrals to a few lines.

**Topics covered:**  $P_1$ – $P_7$  properties of definite integrals • King's property • Even/odd integrand • Periodic-function integrals • Modulus integrands

#### Quick Formula Sheet

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx \quad (\text{King's, } P_4)$$

$$\int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a - x) dx \quad (P_5)$$

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx & f \text{ even} \\ 0 & f \text{ odd} \end{cases} \quad (P_7)$$

### Exercise 7.10

**Q 7.1** Evaluate  $\int_0^{\pi/2} \cos^2 x dx$ .

#### SOLUTION

**Concept used.** Apply  $P_4$ :  $\int_0^a f(x) dx = \int_0^a f(a - x) dx$  with  $a = \pi/2$ . Adding the original and the reflected forms produces a clean cancellation.

**Step 1.** Let  $I = \int_0^{\pi/2} \cos^2 x dx$ . By  $P_4$ :

$$I = \int_0^{\pi/2} \cos^2(\pi/2 - x) dx = \int_0^{\pi/2} \sin^2 x dx.$$

**Step 2.** Add the two forms of  $I$ :

$$2I = \int_0^{\pi/2} (\cos^2 x + \sin^2 x) dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}.$$

**Step 3.** Solve:  $I = \frac{\pi}{4}$ .

**Final Answer:**  $\pi/4$

 **King's property**

$\int_0^a f(x) dx = \int_0^a f(a-x) dx$ . Replace  $x$  by  $a-x$  and integrate again from 0 to  $a$ . Add the two equations to expose Pythagorean-like cancellations.

**EXPERT'S SOLUTION** : Aanchal Roy, M.Sc Mathematics, IIT Delhi

**Direct computation.** Use  $\cos^2 x = \frac{1+\cos 2x}{2}$ :

$$\int_0^{\pi/2} \cos^2 x dx = \frac{1}{2} \left[ x + \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} = \pi/4.$$

**Step 1.** Power-reducing identity.

**Step 2.** Integrate and evaluate.

**Final Answer:**  $\pi/4$

**Q 7.2** Evaluate  $\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ .

**SOLUTION**

**Concept used.** The hallmark King's-property integral. Replace  $x$  by  $\pi/2 - x$  and add.

**Step 1.** Let  $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx$ .

**Step 2.** Apply  $P_4$  with  $a = \pi/2$ . Since  $\sin(\pi/2 - x) = \cos x$  and  $\cos(\pi/2 - x) = \sin x$ :

$$I = \int_0^{\pi/2} \frac{\cos x}{\cos x + \sin x} dx.$$

**Step 3.** Add the two forms:

$$2I = \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x} dx = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}.$$

**Step 4.** Solve:  $I = \frac{\pi}{4}$ .

**Final Answer:**  $\pi/4$

### ♥ Why pairing up works

Any integrand of the form  $\frac{u(x)}{u(x) + u(\pi/2 - x)}$  on  $[0, \pi/2]$  pairs with itself under King's to give  $2I = \int_0^{\pi/2} 1 dx = \pi/2$ , so  $I = \pi/4$  regardless of  $u$ .

**EXPERT'S SOLUTION** : Rohit Bansal, M.Sc Mathematics, IIT Bombay

**Pattern memo.** Any  $\frac{\sin^n x}{\sin^n x + \cos^n x}$ -type integrand on  $[0, \pi/2]$  evaluates to  $\pi/4$  by this exact pairing.

**Step 1.** Replace  $x \rightarrow \pi/2 - x$ .

**Step 2.** Add; cancellation gives  $2I = \pi/2$ .

**Final Answer:**  $\pi/4$

**Q7.3** Evaluate  $\int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$ .

### SOLUTION

**Concept used.** Same pairing as Q2 — generalises to any positive power  $n$ .

**Step 1.** Let  $I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx$ .

**Step 2.** Apply  $P_4$  with  $a = \pi/2$ :

$$I = \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx.$$

**Step 3.** Add:

$$2I = \int_0^{\pi/2} \frac{\sin^{3/2} x + \cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx = \frac{\pi}{2}.$$

**Step 4.** Solve:  $I = \frac{\pi}{4}$ .

**Final Answer:**  $\pi/4$

**EXPERT'S SOLUTION** : Snehal Mishra, M.Sc Mathematics, IIT Madras

**No surprise.** Power  $n = 3/2$  doesn't change the pairing logic.

**Step 1.** Pair via  $P_4$ .

**Step 2.** Add; result  $\pi/4$ .

**Final Answer:**  $\pi/4$

**Q 7.4** Evaluate  $\int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$ .

**SOLUTION**

**Concept used.** Same pairing logic; numerator and denominator swap under King's.

**Step 1.** Let  $I = \int_0^{\pi/2} \frac{\cos^5 x}{\sin^5 x + \cos^5 x} dx$ .

**Step 2.** Apply  $P_4$ :

$$I = \int_0^{\pi/2} \frac{\sin^5 x}{\cos^5 x + \sin^5 x} dx.$$

**Step 3.** Add:

$$2I = \int_0^{\pi/2} 1 dx = \frac{\pi}{2}.$$

**Step 4.** Therefore  $I = \frac{\pi}{4}$ .

**Final Answer:**  $\pi/4$

**EXPERT'S SOLUTION** : Yashika Bhandari, M.Sc Mathematics, IIT Guwahati

**Same family.** Replace 5 with any  $n > 0$  — answer stays  $\pi/4$ .

**Step 1.** Pair, add, divide.

**Final Answer:**  $\pi/4$

**Q 7.5** Evaluate  $\int_{-5}^5 |x + 2| dx$ .

## SOLUTION

**Concept used.** Split the absolute-value integrand at the sign-change point  $x = -2$ , then integrate each piece separately.

**Step 1.** For  $x < -2$ ,  $|x + 2| = -(x + 2)$ . For  $x \geq -2$ ,  $|x + 2| = x + 2$ .

**Step 2.** Split:

$$I = \int_{-5}^{-2} -(x + 2) dx + \int_{-2}^5 (x + 2) dx.$$

**Step 3.** First piece:  $\int_{-5}^{-2} -(x + 2) dx = -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2}$ . At  $-2$ :  $\frac{4}{2} + 2(-2) = 2 - 4 = -2$ .

At  $-5$ :  $\frac{25}{2} - 10 = \frac{5}{2}$ . Difference  $(-2) - (5/2) = -9/2$ . Multiply by  $-1$ :  $9/2$ .

**Step 4.** Second piece:  $\int_{-2}^5 (x + 2) dx = \left[\frac{x^2}{2} + 2x\right]_{-2}^5$ . At  $5$ :  $\frac{25}{2} + 10 = \frac{45}{2}$ . At  $-2$ :  $-2$ .

Difference  $(45/2) - (-2) = 49/2$ .

**Step 5.** Sum:  $I = \frac{9}{2} + \frac{49}{2} = \frac{58}{2} = 29$ .

**Final Answer:** 29

 **Geometric view**

$|x + 2|$  on  $[-5, 5]$  is two right triangles meeting at  $x = -2$ : one with base 3, height 3; the other with base 7, height 7. Areas  $9/2$  and  $49/2$ . Total 29.

**EXPERT'S SOLUTION** : Devansh Tandon, M.Sc Mathematics, IIT Kanpur

**Triangle areas.** Two triangles, areas  $\frac{1}{2} \cdot 3 \cdot 3 = 4.5$  and  $\frac{1}{2} \cdot 7 \cdot 7 = 24.5$ , sum 29.

**Step 1.** Identify the corner.

**Step 2.** Compute each triangular area.

**Step 3.** Add: 29.

**Final Answer:** 29

**Q 7.6** Evaluate  $\int_2^8 |x - 5| dx$ .

## SOLUTION

**Concept used.** Same absolute-value split at  $x = 5$ .

**Step 1.** For  $x < 5$ :  $|x - 5| = -(x - 5) = 5 - x$ . For  $x \geq 5$ :  $|x - 5| = x - 5$ .

**Step 2.** Split:

$$I = \int_2^5 (5 - x) dx + \int_5^8 (x - 5) dx.$$

**Step 3.** First piece:  $[5x - \frac{x^2}{2}]_2^5 = (25 - \frac{25}{2}) - (10 - 2) = \frac{25}{2} - 8 = \frac{9}{2}$ .

**Step 4.** Second piece:  $[\frac{x^2}{2} - 5x]_5^8 = (32 - 40) - (\frac{25}{2} - 25) = -8 - (-\frac{25}{2}) = -8 + \frac{25}{2} = \frac{9}{2}$ .

**Step 5.** Sum:  $I = \frac{9}{2} + \frac{9}{2} = 9$ .

**Final Answer:** 9

## EXPERT'S SOLUTION : Pranay Sehgal, M.Sc Mathematics, IIT Bombay

**Geometric.** Two triangles, each base 3 and height 3, total area  $2 \cdot \frac{1}{2} \cdot 3 \cdot 3 = 9$ .

**Step 1.** Identify corner at  $x = 5$ .

**Step 2.** Both halves are symmetric triangles.

**Step 3.** Sum = 9.

**Final Answer:** 9

**Q 7.7** Evaluate  $\int_0^1 x(1-x)^n dx$ .

## SOLUTION

**Concept used.** Apply  $P_4$  with  $a = 1$ : replace  $x$  by  $1 - x$ .

**Step 1.** By  $P_4$ :

$$I = \int_0^1 (1-x)(1-(1-x))^n dx = \int_0^1 (1-x)x^n dx.$$

**Step 2.** Expand:  $(1-x)x^n = x^n - x^{n+1}$ .

**Step 3.** Integrate:

$$I = \int_0^1 x^n dx - \int_0^1 x^{n+1} dx = \frac{1}{n+1} - \frac{1}{n+2}.$$

**Step 4.** Combine:  $\frac{(n+2) - (n+1)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}$ .

**Final Answer:**  $\frac{1}{(n+1)(n+2)}$

**EXPERT'S SOLUTION** : Sanya Khurana, M.Sc Mathematics, IIT Delhi

**Beta function.** This is

$B(2, n+1) = \Gamma(2)\Gamma(n+1)/\Gamma(n+3) = 1 \cdot n!/(n+2)! = 1/[(n+1)(n+2)]$ .  $P_4$  avoids the Gamma function and reaches the answer in three lines.

**Step 1.** Apply  $P_4$ .

**Step 2.** Expand and integrate.

**Step 3.** Combine fractions.

**Final Answer:**  $\frac{1}{(n+1)(n+2)}$

**Q 7.8** Evaluate  $\int_0^{\pi/4} \log(1 + \tan x) dx$ .

### SOLUTION

**Concept used.** Famous  $P_4$  identity. Apply on  $[0, \pi/4]$  replacing  $x$  by  $\pi/4 - x$ , then use the addition formula for tan.

**Step 1.** By  $P_4$  with  $a = \pi/4$ :

$$I = \int_0^{\pi/4} \log(1 + \tan(\pi/4 - x)) dx.$$

**Step 2.** Tangent subtraction:  $\tan(\pi/4 - x) = \frac{1 - \tan x}{1 + \tan x}$ . So

$$\begin{aligned} 1 + \tan(\pi/4 - x) &= 1 + \frac{1 - \tan x}{1 + \tan x} \\ &= \frac{(1 + \tan x) + (1 - \tan x)}{1 + \tan x} = \frac{2}{1 + \tan x}. \end{aligned}$$

**Step 3.** Take logs:

$$\log(1 + \tan(\pi/4 - x)) = \log 2 - \log(1 + \tan x).$$

**Step 4.** Therefore the King's version gives:

$$I = \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx = \frac{\pi \log 2}{4} - I.$$

**Step 5.** Solve:  $2I = \frac{\pi \log 2}{4}$ , so  $I = \frac{\pi \log 2}{8}$ .

**Final Answer:**  $\frac{\pi \log 2}{8}$

### Exam Tip

$\int_0^{\pi/4} \log(1 + \tan x) dx = \frac{\pi}{8} \log 2$ . A classic JEE-style identity worth memorising — King's collapses it in five lines.

**EXPERT'S SOLUTION** : Vihaan Trivedi, M.Sc Mathematics, IIT Madras

**Two-line outline.** (i) Replace  $x \rightarrow \pi/4 - x$  inside the log. (ii) Use  $\tan(\pi/4 - x) = (1 - \tan x)/(1 + \tan x)$  to get  $\log 2 - \log(1 + \tan x)$ . Add to  $I$ .

**Step 1.** Apply  $P_4$  and tangent subtraction.

**Step 2.**  $2I = (\pi/4) \log 2$ .

**Step 3.**  $I = \pi \log 2/8$ .

**Final Answer:**  $\frac{\pi \log 2}{8}$

**Q 7.9** Evaluate  $\int_0^2 x\sqrt{2-x} dx$ .

### SOLUTION

**Concept used.** Apply  $P_4$  with  $a = 2$ , then expand.

**Step 1.** By  $P_4$ :

$$I = \int_0^2 (2-x)\sqrt{2-(2-x)} dx = \int_0^2 (2-x)\sqrt{x} dx.$$

**Step 2.** Expand:  $(2-x)\sqrt{x} = 2\sqrt{x} - x^{3/2} = 2x^{1/2} - x^{3/2}$ .

**Step 3.** Integrate term-by-term:

$$I = 2 \cdot \frac{2x^{3/2}}{3} \Big|_0^2 - \frac{2x^{5/2}}{5} \Big|_0^2 = \frac{4(2)^{3/2}}{3} - \frac{2(2)^{5/2}}{5}.$$

**Step 4.** Compute the radicals:  $2^{3/2} = 2\sqrt{2}$  and  $2^{5/2} = 4\sqrt{2}$ . So

$$I = \frac{4 \cdot 2\sqrt{2}}{3} - \frac{2 \cdot 4\sqrt{2}}{5} = \frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5} = 8\sqrt{2} \left( \frac{1}{3} - \frac{1}{5} \right) = 8\sqrt{2} \cdot \frac{2}{15} = \frac{16\sqrt{2}}{15}.$$

**Final Answer:**  $\frac{16\sqrt{2}}{15}$

**EXPERT'S SOLUTION** : Suhas Reddy, M.Sc Mathematics, IIT Hyderabad

**Why  $P_4$  helps.** The integrand  $\sqrt{2-x}$  is awkward; after  $P_4$  it becomes  $\sqrt{x}$ , a power. Then linearity finishes the job.

**Step 1.**  $P_4$  converts radical to  $\sqrt{x}$ .

**Step 2.** Expand and integrate.

**Step 3.** Tidy:  $16\sqrt{2}/15$ .

**Final Answer:**  $\frac{16\sqrt{2}}{15}$

**Q 7.10** Evaluate  $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x) dx$ .

### SOLUTION

**Concept used.** Use  $\sin 2x = 2 \sin x \cos x$  to telescope the log expression. Then combine with the standard  $\int_0^{\pi/2} \log \sin x dx = -\frac{\pi}{2} \log 2$ .

**Step 1.** Expand  $\log \sin 2x = \log 2 + \log \sin x + \log \cos x$ . So

$$\begin{aligned} 2 \log \sin x - \log \sin 2x &= 2 \log \sin x - \log 2 - \log \sin x - \log \cos x \\ &= \log \sin x - \log \cos x - \log 2. \end{aligned}$$

**Step 2.** Use the symmetry  $\int_0^{\pi/2} \log \sin x dx = \int_0^{\pi/2} \log \cos x dx$  (by  $P_4$ ). So

$$\int_0^{\pi/2} [\log \sin x - \log \cos x] dx = 0.$$

**Step 3.** Remaining:  $-\int_0^{\pi/2} \log 2 \, dx = -\frac{\pi}{2} \log 2$ .

**Step 4.** Therefore  $I = -\frac{\pi}{2} \log 2$ .

**Final Answer:**  $-\frac{\pi}{2} \log 2$

### ♥ Classic identity

$\int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2$ . Used in many derivations including the Gauss multiplication formula for  $\Gamma$ .

**EXPERT'S SOLUTION** : Ananya Goswami, M.Sc Mathematics, IIT BHU

**Telescoping.** The  $\log 2 + \log \sin x + \log \cos x$  split is the engine. Symmetry kills the  $\log \cos x$  term against  $\log \sin x$ .

**Step 1.** Expand  $\log \sin 2x$ .

**Step 2.** Symmetric pair cancels.

**Step 3.** Constant  $-\log 2$  over  $[0, \pi/2]$  gives  $-\frac{\pi}{2} \log 2$ .

**Final Answer:**  $-\frac{\pi}{2} \log 2$

**Q 7.11** Evaluate  $\int_{-\pi/2}^{\pi/2} \sin^2 x \, dx$ .

### SOLUTION

**Concept used.**  $\sin^2 x$  is an even function (since  $\sin(-x) = -\sin x$  squares to  $\sin^2 x$ ), so  $P_7(i)$  applies:  $\int_{-a}^a f \, dx = 2 \int_0^a f \, dx$ .

**Step 1.** Check parity:  $\sin^2(-x) = \sin^2 x$ . Even.

**Step 2.** Apply  $P_7$ :  $I = 2 \int_0^{\pi/2} \sin^2 x \, dx$ .

**Step 3.** Use  $\sin^2 x = \frac{1 - \cos 2x}{2}$ :

$$\int_0^{\pi/2} \sin^2 x \, dx = \frac{1}{2} \left[ x - \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}.$$

**Step 4.** Multiply by 2:  $I = \frac{\pi}{2}$ .

**Final Answer:**  $\pi/2$

**EXPERT'S SOLUTION** : Shaurya Pillai, M.Sc Mathematics, IIT Kharagpur

**Average-value.**  $\sin^2 x$  averages  $1/2$  over a  $\pi$ -interval; length here is  $\pi$ ; product  $\pi/2$ .

**Step 1.** Recognise even integrand and full period.

**Step 2.** Mean value  $\times$  length.

**Final Answer:**  $\pi/2$

**Q 7.12** Evaluate  $\int_0^\pi \frac{x dx}{1 + \sin x}$ .

### SOLUTION

**Concept used.**  $P_4$  with  $a = \pi$ : replace  $x$  by  $\pi - x$ . Since  $\sin(\pi - x) = \sin x$ , the denominator is unchanged; the  $x$  in the numerator changes.

**Step 1.** By  $P_4$ :

$$I = \int_0^\pi \frac{(\pi - x) dx}{1 + \sin x}.$$

**Step 2.** Add to original:

$$2I = \int_0^\pi \frac{x + (\pi - x)}{1 + \sin x} dx = \pi \int_0^\pi \frac{dx}{1 + \sin x}.$$

**Step 3.** Multiply numerator and denominator inside by  $1 - \sin x$  (and use  $1 - \sin^2 x = \cos^2 x$ ):

$$\frac{1}{1 + \sin x} = \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x.$$

**Step 4.** Integrate:  $\int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x$ .

**Step 5.** Evaluate on  $[0, \pi]$ . At  $\pi$ :  $\tan \pi - \sec \pi = 0 - (-1) = 1$ . At  $0$ :  $0 - 1 = -1$ .  
Difference:  $1 - (-1) = 2$ .

**Step 6.** Therefore  $2I = \pi \cdot 2 = 2\pi$ , so  $I = \pi$ .

**Final Answer:**  $\pi$

**EXPERT'S SOLUTION** : Aditya Banerjee, M.Sc Mathematics, IIT Roorkee

**Trig manipulation tip.** Rationalising  $1/(1 + \sin x)$  by multiplying by  $(1 - \sin x)/(1 - \sin x)$  turns it into  $\sec^2 - \sec \tan$  — both elementary antiderivatives.

**Step 1.** Pair via  $P_4$ .

**Step 2.** Rationalise the denominator.

**Step 3.** Evaluate;  $I = \pi$ .

**Final Answer:**  $\pi$

**Q 7.13** Evaluate  $\int_{-\pi/2}^{\pi/2} \sin^7 x \, dx$ .

**SOLUTION**

**Concept used.**  $\sin^7 x$  is odd:  $\sin^7(-x) = (-\sin x)^7 = -\sin^7 x$ . Apply  $P_7$ (ii) ( $\int_{-a}^a$  of an odd function is 0).

**Step 1.** Check parity:  $f(-x) = -f(x)$ . ✓

**Step 2.** Apply  $P_7$ (ii):  $I = 0$ .

**Final Answer:** 0

**EXPERT'S SOLUTION** : Niharika Dey, M.Sc Mathematics, IIT Kanpur

**Odd power, odd function.** Any odd power of  $\sin x$  inherits the oddness.

**Step 1.** Parity check.

**Step 2.** Conclude.

**Final Answer:** 0

**Q 7.14** Evaluate  $\int_0^{2\pi} \cos^5 x \, dx$ .

## SOLUTION

**Concept used.**  $\cos x$  has period  $2\pi$  but  $\cos^5 x$  over a full period  $[0, 2\pi]$  splits at  $x = \pi$  via

$P_5$ :  $\int_0^{2\pi} f(x) dx = \int_0^\pi f(x) dx + \int_0^\pi f(2\pi - x) dx$ . Since  $\cos(2\pi - x) = \cos x$ , we get

$\int_0^{2\pi} \cos^5 x dx = 2 \int_0^\pi \cos^5 x dx$ . Then on  $[0, \pi]$ ,  $P_5$  again with  $a = \pi/2$ :

$\cos(\pi - x) = -\cos x$ , so  $\cos^5(\pi - x) = -\cos^5 x$ . Therefore

$$\int_0^\pi \cos^5 x dx = \int_0^{\pi/2} \cos^5 x dx + \int_0^{\pi/2} (-\cos^5 x) dx = 0.$$

**Step 1.** First split:  $\int_0^{2\pi} \cos^5 x dx = 2 \int_0^\pi \cos^5 x dx$  (since  $\cos$  is symmetric about  $\pi$ ).

**Step 2.** Second split ( $P_5$  on  $[0, \pi]$ ):  $\int_0^\pi \cos^5 x dx = \int_0^{\pi/2} \cos^5 x dx - \int_0^{\pi/2} \cos^5 x dx = 0$ .

**Step 3.** Therefore  $I = 2 \cdot 0 = 0$ .

**Final Answer:** 0

## EXPERT'S SOLUTION : Maitri Khurana, M.Sc Mathematics, IIT Bombay

**Symmetry-only argument.**  $\cos^5 x$  is symmetric about  $x = \pi/2$  with opposite signs on the two halves of  $[0, \pi]$ ; over  $[0, 2\pi]$  another symmetry doubles this — both pieces cancel.

**Step 1.** Sketch the graph:  $\cos^5 x$  is positive on  $(-\pi/2, \pi/2)$ , negative on  $(\pi/2, 3\pi/2)$ , positive again on  $(3\pi/2, 2\pi)$  — equal magnitudes, opposite signs.

**Step 2.** Net area zero.

**Final Answer:** 0

**Q 7.15** Evaluate  $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ .

## SOLUTION

**Concept used.**  $P_4$  with  $a = \pi/2$ : replace  $x$  by  $\pi/2 - x$ . Since  $\sin(\pi/2 - x) = \cos x$  and  $\cos(\pi/2 - x) = \sin x$ , the numerator changes sign and the denominator stays the same.

**Step 1.** By  $P_4$ :

$$I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} dx = -I.$$

**Step 2.** So  $2I = 0$ , hence  $I = 0$ .

**Final Answer:** 0

**EXPERT'S SOLUTION** : *Karan Bhat, M.Sc Mathematics, IIT Guwahati*

**Reading.** The numerator is antisymmetric under  $x \rightarrow \pi/2 - x$ ; the denominator is symmetric. Antisymmetric / symmetric  $\rightarrow$  integral zero on a symmetric interval.

**Step 1.** Apply  $P_4$ ; integrand picks up a minus sign.

**Step 2.**  $I = -I \Rightarrow I = 0$ .

**Final Answer:** 0

**Q 7.16** Evaluate  $\int_0^\pi \log(1 + \cos x) dx$ .

**SOLUTION**

**Concept used.** Use  $1 + \cos x = 2 \cos^2(x/2)$ ; the log splits into a constant and  $\log \cos(x/2)$ . Then change variable  $u = x/2$  and use the classical  $\int_0^{\pi/2} \log \cos u du = -\frac{\pi}{2} \log 2$ .

**Step 1.** Half-angle:  $1 + \cos x = 2 \cos^2(x/2)$ .

$$\log(1 + \cos x) = \log 2 + 2 \log |\cos(x/2)| = \log 2 + 2 \log \cos(x/2)$$

(since  $\cos(x/2) > 0$  for  $x \in (0, \pi)$ ).

**Step 2.** Integrate the first piece:  $\int_0^\pi \log 2 dx = \pi \log 2$ .

**Step 3.** Integrate the second piece via  $u = x/2$ ,  $du = dx/2$ , limits 0 to  $\pi/2$ :

$$\int_0^\pi 2 \log \cos(x/2) dx = 4 \int_0^{\pi/2} \log \cos u du = 4 \cdot \left(-\frac{\pi}{2} \log 2\right) = -2\pi \log 2.$$

**Step 4.** Sum:  $I = \pi \log 2 - 2\pi \log 2 = -\pi \log 2$ .

**Final Answer:**  $-\pi \log 2$

**EXPERT'S SOLUTION** : *Yuvika Saluja, M.Sc Mathematics, IIT Hyderabad*

**Tools assembled.** Half-angle + change of variable + classical  $\log \cos$  identity.

**Step 1.** Apply  $1 + \cos x = 2 \cos^2(x/2)$ .

**Step 2.** Split log and integrate piece-by-piece.

**Step 3.** Use  $\int_0^{\pi/2} \log \cos u du = -\frac{\pi}{2} \log 2$ .

**Final Answer:**  $-\pi \log 2$

**Q 7.17** Evaluate  $\int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ .

### SOLUTION

**Concept used.** King's-property pairing on  $[0, a]$ .

**Step 1.** Let  $I = \int_0^a \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ .

**Step 2.** By  $P_4$ :

$$I = \int_0^a \frac{\sqrt{a-x}}{\sqrt{a-x} + \sqrt{x}} dx.$$

**Step 3.** Add the two:

$$2I = \int_0^a \frac{\sqrt{x} + \sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx = \int_0^a 1 dx = a.$$

**Step 4.** Solve:  $I = \frac{a}{2}$ .

**Final Answer:**  $a/2$

**EXPERT'S SOLUTION** : *Tanmay Kabir, M.Sc Mathematics, IIT Delhi*

**Same pairing.** Same logic as Q2 with a square-root flavour.

**Step 1.** Pair via  $P_4$ , add, divide.

**Final Answer:**  $a/2$

**Q 7.18** Evaluate  $\int_0^4 |x - 1| dx$ .

### SOLUTION

**Concept used.** Split at the breakpoint  $x = 1$ .

**Step 1.** For  $0 \leq x \leq 1$ :  $|x - 1| = 1 - x$ . For  $1 \leq x \leq 4$ :  $|x - 1| = x - 1$ .

**Step 2.** Split:

$$I = \int_0^1 (1-x) dx + \int_1^4 (x-1) dx.$$

**Step 3.** First piece:  $[x - x^2/2]_0^1 = (1 - 1/2) - 0 = 1/2$ .

**Step 4.** Second piece:  $[x^2/2 - x]_1^4 = (8 - 4) - (1/2 - 1) = 4 - (-1/2) = 9/2$ .

**Step 5.** Sum:  $I = 1/2 + 9/2 = 5$ .

**Final Answer:** 5

**EXPERT'S SOLUTION** : Ashvika Pandya, M.Sc Mathematics, IIT Madras

**Triangles.** Triangle of base 1, height 1 → area 1/2. Triangle of base 3, height 3 → area 9/2. Total 5.

**Step 1.** Two triangles.

**Step 2.** Add: 5.

**Final Answer:** 5

**Q 7.19** Show that  $\int_0^a f(x)g(x) dx = 2 \int_0^a f(x) dx$ , if  $f$  and  $g$  are defined as  $f(x) = f(a-x)$  and  $g(x) + g(a-x) = 4$ .

**SOLUTION**

**Concept used.** Use  $P_4$  on the product  $fg$ , then exploit the two given conditions.

**Step 1.** Let  $J = \int_0^a f(x)g(x) dx$ .

**Step 2.** By  $P_4$ :

$$J = \int_0^a f(a-x)g(a-x) dx = \int_0^a f(x)g(a-x) dx \quad (\text{since } f(a-x) = f(x)).$$

**Step 3.** Add the two forms:

$$2J = \int_0^a f(x)[g(x) + g(a-x)] dx = \int_0^a f(x) \cdot 4 dx = 4 \int_0^a f(x) dx.$$

**Step 4.** Therefore  $J = 2 \int_0^a f(x) dx$ , as required.

**Final Answer:** Hence proved.

**EXPERT'S SOLUTION** : Aanya Bhalla, M.Sc Mathematics, IIT Roorkee

**Compact.**  $f$  is symmetric about  $a/2$ ;  $g + g(a - \cdot) = 4$ , so the King's trick adds to  $4 \int f$ .

**Step 1.** Apply  $P_4$ .

**Step 2.** Use  $g + g(a - x) = 4$ .

**Step 3.** Divide by 2.

**Final Answer:** Hence proved.

**Q 7.20** Choose the correct answer.

The value of the integral  $\int (x^3 + x \cos x + \tan^5 x + 1) dx$  from  $-\pi/2$  to  $\pi/2$  is

(A) 0 (B) 2 (C)  $\pi$  (D) 1.

**SOLUTION**

**Concept used.** Split into pieces; identify odd vs even.

**Step 1.**  $x^3$  is odd;  $x \cos x$  is odd  $\times$  even = odd;  $\tan^5 x$  is odd; 1 is even.

**Step 2.** Odd parts integrate to 0 on  $[-a, a]$  ( $P_7$ (ii)). Constant 1 over  $[-\pi/2, \pi/2]$  gives  $\pi/2 - (-\pi/2) = \pi$ .

**Step 3.** Total:  $I = 0 + 0 + 0 + \pi = \pi$ . Matches (C).

**Final Answer:** Option (C):  $\pi$

**EXPERT'S SOLUTION** : Mohit Khanna, M.Sc Mathematics, IIT Bombay

**Read it off.** Three of the four pieces are odd; only the constant 1 survives.

**Step 1.** Constant 1 over interval of length  $\pi \rightarrow \pi$ .

**Final Answer:** Option (C)

**Q 7.21** Choose the correct answer.

The value of  $\int_0^{\pi/2} \log[(4 + 3 \sin x)/(4 + 3 \cos x)] dx$  is

(A) 2 (B)  $3/4$  (C) 0 (D)  $-2$ .

**SOLUTION**

**Concept used.**  $P_4$  with  $a = \pi/2$  flips  $\sin x \leftrightarrow \cos x$ , so the integrand picks up a minus sign.

**Step 1.** Let  $I = \int_0^{\pi/2} \log\left(\frac{4 + 3 \sin x}{4 + 3 \cos x}\right) dx$ .

**Step 2.** Apply  $P_4$ :

$$I = \int_0^{\pi/2} \log\left(\frac{4 + 3 \cos x}{4 + 3 \sin x}\right) dx = -I.$$

**Step 3.** So  $2I = 0$ ,  $I = 0$ . Matches (C).

**Final Answer:** Option (C): 0

**EXPERT'S SOLUTION** : Aaryan Bahl, M.Sc Mathematics, IIT Kanpur

**Symmetric ratio.** A ratio that flips into its reciprocal under  $P_4$  produces a log of its reciprocal — a sign flip — and the integral cancels itself.

**Step 1.**  $\log(p/q) \rightarrow \log(q/p) = -\log(p/q)$ .

**Step 2.**  $I = -I \Rightarrow I = 0$ .

**Final Answer:** Option (C)

**Key Takeaways**

- **King's ( $P_4$ ):**  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ . Adds with the original; cancellation is the workhorse on  $[0, \pi/2]$ ,  $[0, \pi]$ ,  $[0, a]$ .
- **Odd/even ( $P_7$ ):** on  $[-a, a]$ , odd  $\rightarrow 0$ , even  $\rightarrow 2 \int_0^a$ . Always check parity of each summand separately.
- **Absolute value integrals:** always split at the sign-change point(s).
- Symmetric-ratio log integrals (Q21-style) collapse to 0 by  $P_4$ .
- Half-angle identities turn  $\log(1 \pm \cos x)$  and  $\log(1 + \sin x)$  into  $\log \cos(x/2)$  /  $\log \sin(x/2)$  pieces solvable via classical identities.

End of Exercise 7.10