

# Collegedunia NCERT Solutions

Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

## Chapter 7: Integrals

### About this Chapter

Exercise 7.2 is the heart of **Integration by Substitution**. You learn to spot a function whose derivative also occurs in the integrand (up to a constant), substitute  $t = u(x)$  so that  $dt = u'(x) dx$ , transform  $\int f(x) dx$  into a standard form  $\int g(t) dt$ , integrate, then back-substitute. The 39 problems here drill every recurring pattern: rational, surd, exponential, logarithmic and trigonometric substitutions.

**Topics covered:**  $u$ -substitution • Standard log-derivative form  $\int f'/f dx$  • Surds via  $\sqrt{ax+b} = t$  • Exponential and  $\tan^{-1}$  substitutions

#### Quick Formula Sheet

##### Substitution rule:

$$\int f(g(x)) g'(x) dx = \int f(t) dt, \text{ where } t = g(x)$$

##### Log derivative form:

$$\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C$$

##### Power form:

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C \quad (n \neq -1)$$

##### Useful substitutions:

$$\sqrt{ax+b} = t, \log x = t, e^x = t, \tan^{-1} x = t$$

### Exercise 7.2

**Q 7.1** Integrate  $\frac{2x}{1+x^2}$ .

#### SOLUTION

**Concept used.** **Integration by substitution.** If the integrand is of the form  $\frac{f'(x)}{f(x)}$ , substitute  $t = f(x)$  so that  $dt = f'(x) dx$ . Then

$$\int \frac{f'(x)}{f(x)} dx = \int \frac{dt}{t} = \log |t| + C = \log |f(x)| + C.$$

**Step 1.** Note that the numerator  $2x$  is exactly the derivative of the denominator  $1+x^2$ :

$$\frac{d}{dx}(1 + x^2) = 2x.$$

**Step 2.** Substitute  $t = 1 + x^2$  so that  $dt = 2x dx$ . The integral becomes

$$\int \frac{2x}{1 + x^2} dx = \int \frac{dt}{t}.$$

**Step 3.** Integrate:  $\int dt/t = \log |t|$ .

**Step 4.** Back-substitute  $t = 1 + x^2$  (which is always positive, so  $|t| = t$ ):  $\log(1 + x^2) + C$ .

**Final Answer:**  $\log(1 + x^2) + C$

### Log-derivative form

$\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C$ . Spotting this pattern saves enormous time.

**EXPERT'S SOLUTION** : Aarav Sharma, M.Sc Mathematics, IIT Bombay

**Quick reading.** The integrand has the form  $\frac{f'}{f}$  with  $f(x) = 1 + x^2$ . Read off the anti derivative directly.

**Step 1.**  $f(x) = 1 + x^2$ ,  $f'(x) = 2x$ .

**Step 2.** By the log-derivative rule,  $\int f'/f dx = \log |f| + C$ .

**Step 3.** Final:  $\log(1 + x^2) + C$ .

**Final Answer:**  $\log(1 + x^2) + C$

**Q 7.2** Integrate  $\frac{(\log x)^2}{x}$ .

### SOLUTION

**Concept used.** Substitution. Let  $t = \log x$  so that  $dt = dx/x$ . The integral then becomes  $\int t^2 dt$ , integrable by the power rule.

**Step 1.** Put  $t = \log x$ . Differentiating both sides:  $dt = \frac{1}{x} dx$ , i.e.  $\frac{dx}{x} = dt$ .

**Step 2.** Substitute:  $\int \frac{(\log x)^2}{x} dx = \int t^2 dt$ .

**Step 3.** Integrate using the power rule:  $\int t^2 dt = \frac{t^3}{3}$ .

**Step 4.** Back-substitute  $t = \log x$ :  $\frac{(\log x)^3}{3} + C$ .

**Final Answer:**  $\frac{(\log x)^3}{3} + C$

**EXPERT'S SOLUTION** : Sneha Iyer, M.Sc Applied Mathematics, IIT Kanpur

**Strategic angle.** The presence of  $1/x$  alongside a function of  $\log x$  is a fingerprint for substituting  $t = \log x$ .

**Step 1.**  $t = \log x \Rightarrow dt = dx/x$ .

**Step 2.** Integral becomes  $\int t^2 dt = t^3/3$ .

**Step 3.** Back-substitute:  $(\log x)^3/3 + C$ .

**Final Answer:**  $\frac{(\log x)^3}{3} + C$

**Q7.3** Integrate  $\frac{1}{x + x \log x}$ .

### SOLUTION

**Concept used.** Factor the denominator and use the substitution  $t = 1 + \log x$ .

**Step 1.** Factor:  $x + x \log x = x(1 + \log x)$ .

**Step 2.** Put  $t = 1 + \log x$ . Then  $dt = \frac{1}{x} dx$ , i.e.  $dx/x = dt$ .

**Step 3.** Substitute:

$$\int \frac{dx}{x(1 + \log x)} = \int \frac{1}{1 + \log x} \cdot \frac{dx}{x} = \int \frac{dt}{t}$$

**Step 4.** Integrate:  $\log |t| = \log |1 + \log x| + C$ .

**Final Answer:**  $\log |1 + \log x| + C$

**EXPERT'S SOLUTION** : *Karan Mehta, B.Tech Engineering Physics, IIT Bombay*

**Structural observation.** Pull out the  $x$  that is common to both denominator terms, then the  $1 + \log x$  that remains has derivative  $1/x$ , which is sitting outside.

**Step 1.** Factor denominator:  $x(1 + \log x)$ .

**Step 2.** Substitute  $t = 1 + \log x$ ;  $dt = dx/x$ .

**Step 3.** Integral becomes  $\int dt/t = \log |t| + C$ .

**Final Answer:**  $\log |1 + \log x| + C$

**Q 7.4** Integrate  $\sin x \sin(\cos x)$ .

**SOLUTION**

**Concept used.** Substitute  $t = \cos x$  so that  $dt = -\sin x dx$ , i.e.  $\sin x dx = -dt$ . The integral reduces to  $-\int \sin t dt = \cos t + C$ .

**Step 1.** Put  $t = \cos x$ ; then  $dt = -\sin x dx \Rightarrow \sin x dx = -dt$ .

**Step 2.** Substitute:

$$\int \sin x \sin(\cos x) dx = \int \sin(\cos x) \cdot \sin x dx = -\int \sin t dt.$$

**Step 3.** Integrate:  $-\int \sin t dt = -(-\cos t) = \cos t$ .

**Step 4.** Back-substitute:  $\cos(\cos x) + C$ .

**Final Answer:**  $\cos(\cos x) + C$

**EXPERT'S SOLUTION** : *Pranav Patel, Ph.D Mathematics, IIT Delhi*

**Quick reading.** The  $\sin x$  outside is the chain-rule companion to the inner  $\cos x$ .

**Step 1.**  $t = \cos x$ ;  $dt = -\sin x dx$ .

**Step 2.**  $\int \sin x \sin(\cos x) dx = -\int \sin t dt = \cos t = \cos(\cos x)$ .

**Step 3.** Add  $C$ .

**Final Answer:**  $\cos(\cos x) + C$

**Q 7.5** Integrate  $\sin(ax + b) \cos(ax + b)$ .

### SOLUTION

**Concept used.** Use the double-angle identity  $\sin \theta \cos \theta = \frac{1}{2} \sin 2\theta$  to collapse the product. Then apply the standard integral  $\int \sin(kx) dx = -\frac{1}{k} \cos(kx) + C$ .

**Step 1.** Rewrite:  $\sin(ax + b) \cos(ax + b) = \frac{1}{2} \sin(2(ax + b)) = \frac{1}{2} \sin(2ax + 2b)$ .

**Step 2.** Integrate:

$$\int \frac{1}{2} \sin(2ax + 2b) dx = \frac{1}{2} \cdot \left( -\frac{1}{2a} \cos(2ax + 2b) \right) = -\frac{\cos(2ax + 2b)}{4a}.$$

**Step 3.** Add the constant of integration.

**Final Answer:**  $-\frac{\cos(2ax + 2b)}{4a} + C$

### EXPERT'S SOLUTION : Riya Gupta, M.Sc Mathematics, ISI Kolkata

**Strategic angle.** Alternative: substitute  $t = \sin(ax + b)$ ,  $dt = a \cos(ax + b) dx$ .

**Step 1.**  $t = \sin(ax + b) \Rightarrow dt = a \cos(ax + b) dx$ , so  $\cos(ax + b) dx = dt/a$ .

**Step 2.** Integral becomes  $\int t \cdot \frac{dt}{a} = \frac{t^2}{2a} = \frac{\sin^2(ax + b)}{2a}$ .

**Step 3.** Both forms are equivalent:  $\sin^2 \theta/2 = (1 - \cos 2\theta)/4$ , differing by a constant absorbed into  $C$ .

**Final Answer:**  $\frac{\sin^2(ax + b)}{2a} + C$  (equivalent to  $-\frac{\cos(2ax+2b)}{4a} + C$ )

**Q 7.6** Integrate  $\sqrt{ax + b}$ .

### SOLUTION

**Concept used.** Substitute the inner linear function:  $t = ax + b$ ,  $dt = a dx$ ,  $dx = dt/a$ . Then apply the power rule  $\int t^{1/2} dt = \frac{2}{3} t^{3/2}$ .

**Step 1.** Put  $t = ax + b$ . Then  $dt = a dx \Rightarrow dx = \frac{dt}{a}$ .

**Step 2.** Substitute:

$$\int \sqrt{ax + b} dx = \int t^{1/2} \cdot \frac{dt}{a} = \frac{1}{a} \int t^{1/2} dt.$$

**Step 3.** Integrate:  $\int t^{1/2} dt = \frac{t^{3/2}}{3/2} = \frac{2}{3}t^{3/2}$ .

**Step 4.** Multiply by  $1/a$  and back-substitute:

$$\frac{1}{a} \cdot \frac{2}{3}t^{3/2} = \frac{2}{3a}(ax + b)^{3/2}.$$

**Final Answer:**  $\frac{2}{3a}(ax + b)^{3/2} + C$

**EXPERT'S SOLUTION** : Ankit Verma, M.Tech CS, IIT Madras

**Strategic angle.** For  $\int (ax + b)^n dx$  apply the chain-rule reversal: divide by  $a$  and use the power rule on  $(ax + b)$ .

**Step 1.** Apply  $\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)}$  with  $n = 1/2$ .

**Step 2.** Numerator  $(ax + b)^{3/2}$ ; denominator  $a \cdot \frac{3}{2} = \frac{3a}{2}$ .

**Step 3.** Final:  $\frac{(ax + b)^{3/2}}{3a/2} = \frac{2}{3a}(ax + b)^{3/2} + C$ .

**Final Answer:**  $\frac{2(ax + b)^{3/2}}{3a} + C$

**Q7.7** Integrate  $x\sqrt{x+2}$ .

**SOLUTION**

**Concept used.** Substitute  $t = x + 2$  so that the surd becomes a clean power  $t^{1/2}$  and  $x = t - 2$ ,  $dx = dt$ . Then integrate term by term.

**Step 1.** Put  $t = x + 2 \Rightarrow x = t - 2$  and  $dx = dt$ .

**Step 2.** Substitute:

$$\int x\sqrt{x+2} dx = \int (t-2)t^{1/2} dt = \int (t^{3/2} - 2t^{1/2}) dt.$$

**Step 3.** Integrate:

$$\int t^{3/2} dt = \frac{2}{5}t^{5/2}, \quad \int 2t^{1/2} dt = 2 \cdot \frac{2}{3}t^{3/2} = \frac{4}{3}t^{3/2}.$$

**Step 4.** Combine:  $\frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2}$ . Back-substitute  $t = x + 2$ :

$$\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C.$$

**Final Answer:**  $\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C$

**EXPERT'S SOLUTION** : Yash Joshi, M.Sc Mathematics, IIT Bombay

**Strategic angle.** When the integrand mixes a polynomial in  $x$  with a surd of  $x + c$ , substitute the contents of the surd. The polynomial then becomes a polynomial in  $t$ , which integrates routinely.

**Step 1.**  $t = x + 2$ ;  $x = t - 2$ ;  $dx = dt$ .

**Step 2.** Integrand becomes  $(t - 2)\sqrt{t} = t^{3/2} - 2t^{1/2}$ .

**Step 3.** Power-rule each: get  $\frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2}$ .

**Step 4.** Back-substitute.

**Final Answer:**  $\frac{2(x+2)^{5/2}}{5} - \frac{4(x+2)^{3/2}}{3} + C$

**Q 7.8** Integrate  $x\sqrt{1+2x^2}$ .

**SOLUTION**

**Concept used.** Substitute  $t = 1 + 2x^2$ . Then  $dt = 4x dx$ , so  $x dx = dt/4$ .

**Step 1.** Put  $t = 1 + 2x^2 \Rightarrow dt = 4x dx \Rightarrow x dx = \frac{dt}{4}$ .

**Step 2.** Substitute:

$$\int x\sqrt{1+2x^2} dx = \int \sqrt{t} \cdot \frac{dt}{4} = \frac{1}{4} \int t^{1/2} dt.$$

**Step 3.** Integrate:  $\frac{1}{4} \cdot \frac{2}{3}t^{3/2} = \frac{1}{6}t^{3/2}$ .

**Step 4.** Back-substitute:  $\frac{1}{6}(1+2x^2)^{3/2} + C$ .

**Final Answer:**  $\frac{1}{6}(1+2x^2)^{3/2} + C$

**EXPERT'S SOLUTION** : Aanya Singh, M.Sc Mathematics, IIT Bombay

**Quick reading.** The factor  $x$  outside the surd matches the derivative  $\frac{d}{dx}(1 + 2x^2) = 4x$ , up to the constant 4.

**Step 1.**  $t = 1 + 2x^2$ ;  $dt = 4x dx$ ;  $x dx = dt/4$ .

**Step 2.** Integral becomes  $\frac{1}{4} \int t^{1/2} dt = \frac{1}{6}t^{3/2}$ .

**Step 3.** Back-substitute.

**Final Answer:**  $\frac{(1 + 2x^2)^{3/2}}{6} + C$

**Q 7.9** Integrate  $(4x + 2)\sqrt{x^2 + x + 1}$ .**SOLUTION**

**Concept used.** Substitute  $t = x^2 + x + 1$ . Its derivative is  $2x + 1$ , and the factor  $4x + 2 = 2(2x + 1)$  matches up to a factor of 2.

**Step 1.** Put  $t = x^2 + x + 1$ . Then  $dt = (2x + 1) dx$ , so  $(2x + 1) dx = dt$ .

**Step 2.** Note  $4x + 2 = 2(2x + 1)$ , hence  $(4x + 2) dx = 2(2x + 1) dx = 2 dt$ .

**Step 3.** Substitute:

$$\int (4x + 2)\sqrt{x^2 + x + 1} dx = \int 2\sqrt{t} dt = 2 \int t^{1/2} dt.$$

**Step 4.** Integrate:  $2 \cdot \frac{2}{3}t^{3/2} = \frac{4}{3}t^{3/2}$ .

**Step 5.** Back-substitute:  $\frac{4}{3}(x^2 + x + 1)^{3/2} + C$ .

**Final Answer:**  $\frac{4}{3}(x^2 + x + 1)^{3/2} + C$

**EXPERT'S SOLUTION** : Diya Kapoor, M.Sc Mathematics, IIT Bombay

**Structural observation.** Look for an exact match between the outside polynomial and a constant multiple of  $\frac{d}{dx}$  of the inside expression.

**Step 1.**  $\frac{d}{dx}(x^2 + x + 1) = 2x + 1$ .

**Step 2.**  $4x + 2 = 2(2x + 1)$ .

**Step 3.** Hence  $\int (4x + 2)\sqrt{x^2 + x + 1} dx = 2 \int \sqrt{t} dt = \frac{4}{3}t^{3/2}$ .

**Final Answer:**  $\frac{4(x^2 + x + 1)^{3/2}}{3} + C$

**Q 7.10** Integrate  $\frac{1}{x - \sqrt{x}}$ .

**SOLUTION**

**Concept used.** Factor  $\sqrt{x}$  from the denominator, then substitute  $t = \sqrt{x}$ .

**Step 1.** Factor:  $x - \sqrt{x} = \sqrt{x}(\sqrt{x} - 1)$ .

**Step 2.** Put  $t = \sqrt{x}$ . Then  $x = t^2$  and  $dx = 2t dt$ .

**Step 3.** Substitute:

$$\int \frac{dx}{x - \sqrt{x}} = \int \frac{2t dt}{\sqrt{x}(\sqrt{x} - 1)} = \int \frac{2t dt}{t(t - 1)} = \int \frac{2 dt}{t - 1}.$$

**Step 4.** Integrate:  $\int \frac{2 dt}{t - 1} = 2 \log |t - 1|$ .

**Step 5.** Back-substitute:  $2 \log |\sqrt{x} - 1| + C$ .

**Final Answer:**  $2 \log |\sqrt{x} - 1| + C$

**EXPERT'S SOLUTION** : Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore

**Strategic angle.** Whenever the denominator mixes  $x$  and  $\sqrt{x}$ , substitute  $t = \sqrt{x}$  so that  $x$  becomes  $t^2$ .

**Step 1.**  $t = \sqrt{x}$ ;  $dx = 2t dt$ .

**Step 2.** Denominator:  $x - \sqrt{x} = t^2 - t = t(t - 1)$ .

**Step 3.** Integral:  $\int \frac{2t dt}{t(t - 1)} = 2 \int \frac{dt}{t - 1} = 2 \log |t - 1| = 2 \log |\sqrt{x} - 1|$ .

**Final Answer:**  $2 \log |\sqrt{x} - 1| + C$

**Q 7.11** Integrate  $\frac{x}{\sqrt{x+4}}$ ,  $x > 0$ .

## SOLUTION

**Concept used.** Substitute  $t^2 = x + 4$  (so the surd becomes  $t$ ) and rewrite  $x$  in terms of  $t$ :  $x = t^2 - 4$ .

**Step 1.** Put  $t = \sqrt{x + 4}$ , so  $t^2 = x + 4 \Rightarrow x = t^2 - 4$  and  $2t dt = dx$ .

**Step 2.** Substitute:

$$\int \frac{x dx}{\sqrt{x + 4}} = \int \frac{(t^2 - 4) \cdot 2t dt}{t} = \int 2(t^2 - 4) dt = 2 \int (t^2 - 4) dt.$$

**Step 3.** Integrate:  $2 \left( \frac{t^3}{3} - 4t \right) = \frac{2t^3}{3} - 8t$ .

**Step 4.** Back-substitute  $t = \sqrt{x + 4}$ :

$$\frac{2(x + 4)^{3/2}}{3} - 8\sqrt{x + 4} + C.$$

**Final Answer:**  $\frac{2(x + 4)^{3/2}}{3} - 8\sqrt{x + 4} + C$

## EXPERT'S SOLUTION : Tara Bhat, M.Sc Mathematics, ISI Kolkata

**Picture-first.** Replace the inside of the surd by  $t^2$  so the surd disappears.

**Step 1.**  $t^2 = x + 4 \Rightarrow x = t^2 - 4$ ,  $dx = 2t dt$ .

**Step 2.** Numerator  $x dx = (t^2 - 4) \cdot 2t dt$ ; denominator  $\sqrt{x + 4} = t$ .

**Step 3.**  $\int \frac{2t(t^2 - 4) dt}{t} = 2 \int (t^2 - 4) dt = \frac{2t^3}{3} - 8t$ .

**Step 4.** Back-substitute.

**Final Answer:**  $\frac{2(x + 4)^{3/2}}{3} - 8\sqrt{x + 4} + C$

**Q 7.12** Integrate  $(x^3 - 1)^{1/3} x^5$ .

## SOLUTION

**Concept used.** Substitute  $t = x^3 - 1$ . Then  $dt = 3x^2 dx$ . We need to express  $x^5 dx$  in terms of  $t$  and  $dt$ :  $x^5 = x^3 \cdot x^2 = (t + 1) \cdot x^2$ , and  $x^2 dx = dt/3$ .

**Step 1.** Put  $t = x^3 - 1$ . Then  $x^3 = t + 1$  and  $dt = 3x^2 dx \Rightarrow x^2 dx = dt/3$ .

**Step 2.** Rewrite  $x^5 dx = x^3 \cdot x^2 dx = (t + 1) \cdot \frac{dt}{3}$ .

**Step 3.** Substitute:

$$\int (x^3 - 1)^{1/3} x^5 dx = \int t^{1/3} \cdot (t + 1) \cdot \frac{dt}{3} = \frac{1}{3} \int (t^{4/3} + t^{1/3}) dt.$$

**Step 4.** Integrate:

$$\int t^{4/3} dt = \frac{t^{7/3}}{7/3} = \frac{3}{7} t^{7/3},$$

$$\int t^{1/3} dt = \frac{t^{4/3}}{4/3} = \frac{3}{4} t^{4/3}.$$

So the integral is  $\frac{1}{3} \left( \frac{3}{7} t^{7/3} + \frac{3}{4} t^{4/3} \right) = \frac{1}{7} t^{7/3} + \frac{1}{4} t^{4/3}$ .

**Step 5.** Back-substitute  $t = x^3 - 1$ :

$$\frac{1}{7} (x^3 - 1)^{7/3} + \frac{1}{4} (x^3 - 1)^{4/3} + C.$$

**Final Answer:**  $\frac{(x^3 - 1)^{7/3}}{7} + \frac{(x^3 - 1)^{4/3}}{4} + C$

**EXPERT'S SOLUTION** : Ishaan Reddy, B.Tech CSE, IIT Roorkee

**Strategic angle.** Substitute so that the surd  $(x^3 - 1)^{1/3}$  becomes a clean power. The leftover  $x^5$  splits as  $x^3 \cdot x^2$ ; both  $x^3$  and  $x^2 dx$  map cleanly to  $t$ -terms.

**Step 1.**  $t = x^3 - 1$ ;  $x^3 = t + 1$ ;  $x^2 dx = dt/3$ .

**Step 2.**  $\int (x^3 - 1)^{1/3} x^5 dx = \frac{1}{3} \int t^{1/3} (t + 1) dt = \frac{1}{3} \int (t^{4/3} + t^{1/3}) dt$ .

**Step 3.** Compute:  $\frac{1}{7} t^{7/3} + \frac{1}{4} t^{4/3}$ .

**Step 4.** Back-substitute.

**Final Answer:**  $\frac{(x^3 - 1)^{7/3}}{7} + \frac{(x^3 - 1)^{4/3}}{4} + C$

**Q 7.13** Integrate  $\frac{x^2}{(2 + 3x^3)^3}$ .

## SOLUTION

**Concept used.** Substitute  $t = 2 + 3x^3$  so  $dt = 9x^2 dx$ , i.e.  $x^2 dx = dt/9$ .

**Step 1.** Put  $t = 2 + 3x^3$ ;  $dt = 9x^2 dx$ ;  $x^2 dx = dt/9$ .

**Step 2.** Substitute:

$$\int \frac{x^2 dx}{(2 + 3x^3)^3} = \int \frac{1}{t^3} \cdot \frac{dt}{9} = \frac{1}{9} \int t^{-3} dt.$$

**Step 3.** Integrate:  $\int t^{-3} dt = \frac{t^{-2}}{-2} = -\frac{1}{2t^2}$ .

**Step 4.** Multiply by  $1/9$  and back-substitute:

$$\frac{1}{9} \cdot \left( -\frac{1}{2t^2} \right) = -\frac{1}{18(2 + 3x^3)^2}.$$

**Final Answer:**  $-\frac{1}{18(2 + 3x^3)^2} + C$

## EXPERT'S SOLUTION : Dev Chatterjee, B.Tech CSE, IIT Roorkee

**Quick reading.**  $x^2$  matches  $\frac{d}{dx}(2 + 3x^3)$  up to a  $1/9$  factor.

**Step 1.**  $t = 2 + 3x^3$ ;  $x^2 dx = dt/9$ .

**Step 2.**  $\int t^{-3} dt = -t^{-2}/2$ .

**Step 3.** Final:  $-\frac{1}{18(2 + 3x^3)^2} + C$ .

**Final Answer:**  $-\frac{1}{18(2 + 3x^3)^2} + C$

**Q7.14** Integrate  $\frac{1}{x(\log x)^m}$  for  $x > 0$ ,  $m \neq 1$ .

## SOLUTION

**Concept used.** Substitute  $t = \log x$  so  $dt = dx/x$ .

**Step 1.** Put  $t = \log x \Rightarrow dt = dx/x$ .

**Step 2.** Substitute:

$$\int \frac{dx}{x(\log x)^m} = \int \frac{1}{t^m} dt = \int t^{-m} dt.$$

**Step 3.** For  $m \neq 1$ , by the power rule with exponent  $-m$ :

$$\int t^{-m} dt = \frac{t^{-m+1}}{-m+1} = \frac{t^{1-m}}{1-m}.$$

**Step 4.** Back-substitute:

$$\frac{(\log x)^{1-m}}{1-m} + C.$$

**Final Answer:**  $\frac{(\log x)^{1-m}}{1-m} + C$  ( $m \neq 1$ )

🔍 **Why  $m \neq 1$ ?**

If  $m = 1$  the integrand is  $\frac{1}{x \log x}$  and the substitution gives  $\int dt/t = \log |t| = \log |\log x| + C$ , a different formula. The exclusion  $m \neq 1$  is necessary for the power-rule path.

**EXPERT'S SOLUTION** : Krishna Rao, M.Sc Mathematics, IIT Bombay

**Strategic angle.**  $1/x$  alongside a function of  $\log x$  is always a cue for  $t = \log x$ .

**Step 1.**  $t = \log x$ ;  $dt = dx/x$ .

**Step 2.**  $\int t^{-m} dt = \frac{t^{1-m}}{1-m}$  for  $m \neq 1$ .

**Step 3.** Back-substitute.

**Final Answer:**  $\frac{(\log x)^{1-m}}{1-m} + C$

**Q 7.15** Integrate  $\frac{x}{9 - 4x^2}$ .

**SOLUTION**

**Concept used.** The derivative of the denominator  $9 - 4x^2$  is  $-8x$ . Substitute  $t = 9 - 4x^2$  to convert into a log-derivative form.

**Step 1.** Put  $t = 9 - 4x^2$ . Then  $dt = -8x dx$ , so  $x dx = -dt/8$ .

**Step 2.** Substitute:

$$\int \frac{x dx}{9 - 4x^2} = \int \frac{1}{t} \cdot \left(-\frac{dt}{8}\right) = -\frac{1}{8} \int \frac{dt}{t}.$$

**Step 3.** Integrate:  $-\frac{1}{8} \log |t|$ .

**Step 4.** Back-substitute:  $-\frac{1}{8} \log |9 - 4x^2| + C$ .

**Final Answer:**  $-\frac{1}{8} \log |9 - 4x^2| + C$

**EXPERT'S SOLUTION** : Siddharth Pillai, M.Sc Mathematics, IIT Bombay

**Quick reading.** The factor  $x$  in the numerator is the derivative of  $9 - 4x^2$  divided by  $-8$ .

**Step 1.**  $t = 9 - 4x^2$ ;  $x dx = -dt/8$ .

**Step 2.**  $\int dt/t = \log |t|$ .

**Step 3.** Multiply by  $-1/8$  and back-substitute.

**Final Answer:**  $-\frac{1}{8} \log |9 - 4x^2| + C$

**Q7.16** Integrate  $e^{2x+3}$ .

**SOLUTION**

**Concept used.** Substitute  $t = 2x + 3$  so  $dt = 2 dx$ ,  $dx = dt/2$ . Then  $\int e^t dt = e^t$ .

**Step 1.** Put  $t = 2x + 3$ . Then  $dt = 2 dx \Rightarrow dx = dt/2$ .

**Step 2.** Substitute:

$$\int e^{2x+3} dx = \int e^t \cdot \frac{dt}{2} = \frac{1}{2} \int e^t dt = \frac{1}{2} e^t.$$

**Step 3.** Back-substitute:  $\frac{1}{2} e^{2x+3} + C$ .

**Final Answer:**  $\frac{1}{2} e^{2x+3} + C$

**EXPERT'S SOLUTION** : Aditya Kumar, Ph.D Mathematics, IIT Delhi

**Quick reading.**  $\int e^{ax+b} dx = e^{ax+b}/a + C$ .

**Step 1.** Apply with  $a = 2, b = 3$ .

**Step 2.** Final:  $\frac{e^{2x+3}}{2} + C$ .

**Final Answer:**  $\frac{e^{2x+3}}{2} + C$

**Q 7.17** Integrate  $\frac{x}{e^{x^2}}$ .

### SOLUTION

**Concept used.** Rewrite  $1/e^{x^2} = e^{-x^2}$ . Substitute  $t = -x^2$  so  $dt = -2x dx$ , i.e.  $x dx = -dt/2$ .

**Step 1.** Rewrite:  $\frac{x}{e^{x^2}} = xe^{-x^2}$ .

**Step 2.** Put  $t = -x^2 \Rightarrow dt = -2x dx \Rightarrow x dx = -dt/2$ .

**Step 3.** Substitute:

$$\int xe^{-x^2} dx = \int e^t \cdot \left(-\frac{dt}{2}\right) = -\frac{1}{2} \int e^t dt = -\frac{1}{2}e^t.$$

**Step 4.** Back-substitute:  $-\frac{1}{2}e^{-x^2} + C$ , equivalently  $-\frac{1}{2e^{x^2}} + C$ .

**Final Answer:**  $-\frac{1}{2}e^{-x^2} + C$

**EXPERT'S SOLUTION** : Neha Gupta, M.Sc Mathematics, ISI Kolkata

**Structural observation.** The exponent's derivative is  $-2x$ ; the integrand contains  $x$ , so substitute the exponent.

**Step 1.**  $t = -x^2$ ;  $x dx = -dt/2$ .

**Step 2.**  $\int e^t \cdot (-1/2) dt = -\frac{1}{2}e^t$ .

**Step 3.** Back-substitute:  $-\frac{1}{2}e^{-x^2} + C$ .

**Final Answer:**  $-\frac{1}{2}e^{-x^2} + C$

**Q 7.18** Integrate  $\frac{e^{\tan^{-1} x}}{1+x^2}$ .

### SOLUTION

**Concept used.** Recall  $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$ . Substitute  $t = \tan^{-1} x$  so  $dt = dx/(1+x^2)$ .

**Step 1.** Put  $t = \tan^{-1} x$ ;  $dt = \frac{dx}{1+x^2}$ .

**Step 2.** Substitute:

$$\int \frac{e^{\tan^{-1} x} dx}{1+x^2} = \int e^t dt = e^t.$$

**Step 3.** Back-substitute:  $e^{\tan^{-1} x} + C$ .

**Final Answer:**  $e^{\tan^{-1} x} + C$

**EXPERT'S SOLUTION** : Priya Sharma, Ph.D Mathematics, IIT Delhi

**Quick reading.** The denominator  $1/(1+x^2)$  is the derivative of  $\tan^{-1} x$ .

**Step 1.** Substitute  $t = \tan^{-1} x$ ;  $\int e^t dt = e^t$ .

**Step 2.** Back-substitute.

**Final Answer:**  $e^{\tan^{-1} x} + C$

**Q 7.19** Integrate  $\frac{e^{2x} - 1}{e^{2x} + 1}$ .

### SOLUTION

**Concept used.** Divide numerator and denominator by  $e^x$  to recognise the hyperbolic-tangent pattern, or substitute  $t = e^{2x} + 1$  so that  $dt = 2e^{2x} dx$ . Here we use a different clean approach: divide numerator and denominator by  $e^x$ .

**Step 1.** Divide top and bottom by  $e^x$ :

$$\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

**Step 2.** Notice  $\frac{d}{dx}(e^x + e^{-x}) = e^x - e^{-x}$  — exactly the numerator.

**Step 3.** Put  $t = e^x + e^{-x}$ ;  $dt = (e^x - e^{-x}) dx$ . Then

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int \frac{dt}{t} = \log |t| = \log(e^x + e^{-x}) + C,$$

where the absolute value is dropped since  $e^x + e^{-x} > 0$ .

**Final Answer:**  $\log(e^x + e^{-x}) + C$

**EXPERT'S SOLUTION** : Ananya Bhat, M.Sc Mathematics, IIT Bombay

**Structural observation.** Divide by  $e^x$  to expose the log-derivative pattern.

**Step 1.** Convert:  $\frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ .

**Step 2.** Numerator is  $\frac{d}{dx}$  denominator.

**Step 3.** Integral:  $\log(e^x + e^{-x}) + C$ .

**Final Answer:**  $\log(e^x + e^{-x}) + C$

**Q 7.20** Integrate  $\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}$ .

### SOLUTION

**Concept used.** Log-derivative form: the numerator is  $\frac{1}{2} \frac{d}{dx}$  of the denominator.

Substitute  $t = e^{2x} + e^{-2x}$ .

**Step 1.** Compute derivative of denominator:

$$\frac{d}{dx}(e^{2x} + e^{-2x}) = 2e^{2x} - 2e^{-2x} = 2(e^{2x} - e^{-2x}).$$

**Step 2.** Hence  $(e^{2x} - e^{-2x}) dx = \frac{dt}{2}$ , where  $t = e^{2x} + e^{-2x}$ .

**Step 3.** Substitute:

$$\int \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}} dx = \int \frac{1}{t} \cdot \frac{dt}{2} = \frac{1}{2} \log |t|.$$

**Step 4.** Back-substitute (denominator positive):

$$\frac{1}{2} \log(e^{2x} + e^{-2x}) + C.$$

$$\text{Final Answer: } \frac{1}{2} \log(e^{2x} + e^{-2x}) + C$$

**EXPERT'S SOLUTION** : Aditi Verma, M.Sc Mathematics, IIT Bombay

**Quick reading.** Numerator equals  $\frac{1}{2}$  times derivative of denominator.

**Step 1.**  $t = e^{2x} + e^{-2x}$ ;  $dt = 2(e^{2x} - e^{-2x}) dx$ .

**Step 2.**  $\int \frac{dt}{2t} = \frac{1}{2} \log |t|$ .

**Step 3.** Final:  $\frac{1}{2} \log(e^{2x} + e^{-2x}) + C$ .

$$\text{Final Answer: } \frac{1}{2} \log(e^{2x} + e^{-2x}) + C$$

**Q 7.21** Integrate  $\tan^2(2x - 3)$ .

#### SOLUTION

**Concept used.** Use the identity  $\tan^2 \theta = \sec^2 \theta - 1$ . Then integrate each term:  
 $\int \sec^2 \theta d\theta = \tan \theta$  and  $\int d\theta = \theta$ . Adjust for the inner linear function via  $1/k$  where the inner derivative is  $k$ .

**Step 1.** Use  $\tan^2(2x - 3) = \sec^2(2x - 3) - 1$ .

**Step 2.** Integrate term-by-term:

$$\int \sec^2(2x - 3) dx - \int 1 dx.$$

**Step 3.** For the first integral, substitute  $t = 2x - 3$ ,  $dt = 2 dx$ :

$$\int \sec^2(2x - 3) dx = \int \sec^2 t \frac{dt}{2} = \frac{1}{2} \tan t = \frac{1}{2} \tan(2x - 3).$$

**Step 4.** Combine:  $\frac{1}{2} \tan(2x - 3) - x + C$ .

$$\text{Final Answer: } \frac{1}{2} \tan(2x - 3) - x + C$$

**EXPERT'S SOLUTION** : Aanya Singh, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Convert  $\tan^2$  to  $\sec^2 - 1$  before integrating.

**Step 1.**  $\tan^2(2x - 3) = \sec^2(2x - 3) - 1$ .

**Step 2.** Anti derivative:  $\frac{1}{2} \tan(2x - 3) - x$ .

**Step 3.** Add  $C$ .

$$\text{Final Answer: } \frac{\tan(2x - 3)}{2} - x + C$$

**Q 7.22** Integrate  $\sec^2(7 - 4x)$ .**SOLUTION**

**Concept used.**  $\int \sec^2 t \, dt = \tan t + C$ . The inner function  $7 - 4x$  has derivative  $-4$ .

**Step 1.** Put  $t = 7 - 4x \Rightarrow dt = -4 \, dx \Rightarrow dx = -dt/4$ .

**Step 2.** Substitute:

$$\int \sec^2(7 - 4x) \, dx = \int \sec^2 t \cdot \left(-\frac{dt}{4}\right) = -\frac{1}{4} \int \sec^2 t \, dt.$$

**Step 3.** Integrate:  $-\frac{1}{4} \tan t$ .

**Step 4.** Back-substitute:  $-\frac{1}{4} \tan(7 - 4x) + C$ .

$$\text{Final Answer: } -\frac{1}{4} \tan(7 - 4x) + C$$

**EXPERT'S SOLUTION** : Dev Chatterjee, B.Tech CSE, IIT Roorkee

**Quick reading.** Standard form with inner factor  $-4$ .

**Step 1.** Anti derivative of  $\sec^2$  is  $\tan$ .

**Step 2.** Divide by inner derivative  $-4$ .

**Step 3.** Final:  $-\frac{\tan(7-4x)}{4} + C$ .

$$\text{Final Answer: } -\frac{\tan(7 - 4x)}{4} + C$$

**Q 7.23** Integrate  $\frac{\sin^{-1} x}{\sqrt{1-x^2}}$ .

### SOLUTION

**Concept used.**  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ . Substitute  $t = \sin^{-1} x$ .

**Step 1.** Put  $t = \sin^{-1} x$ ;  $dt = \frac{dx}{\sqrt{1-x^2}}$ .

**Step 2.** Substitute:

$$\int \frac{\sin^{-1} x}{\sqrt{1-x^2}} dx = \int t dt = \frac{t^2}{2}.$$

**Step 3.** Back-substitute:  $\frac{(\sin^{-1} x)^2}{2} + C$ .

**Final Answer:**  $\frac{(\sin^{-1} x)^2}{2} + C$

**EXPERT'S SOLUTION** : Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore

**Quick reading.** Pure log-/power-derivative form on  $\sin^{-1} x$ .

**Step 1.**  $t = \sin^{-1} x$ ;  $\int t dt = t^2/2$ .

**Step 2.** Back-substitute.

**Final Answer:**  $\frac{(\sin^{-1} x)^2}{2} + C$

**Q 7.24** Integrate  $\frac{2 \cos x - 3 \sin x}{6 \cos x + 4 \sin x}$ .

### SOLUTION

**Concept used.** For integrands of the form  $\frac{a \cos x + b \sin x}{c \cos x + d \sin x}$ , write the numerator as

$A \cdot (\text{denominator}) + B \cdot \frac{d}{dx}(\text{denominator})$  and split. The piece with the derivative integrates as a log; the other piece is just  $x$ .

**Step 1.** Denominator  $D(x) = 6 \cos x + 4 \sin x$ . Differentiate:

$$D'(x) = -6 \sin x + 4 \cos x = 4 \cos x - 6 \sin x.$$

**Step 2.** Express numerator  $N(x) = 2 \cos x - 3 \sin x$  as  $A D(x) + B D'(x)$ . Equate

coefficients of  $\cos x$  and  $\sin x$ :

$$6A + 4B = 2, \quad 4A - 6B = -3.$$

**Step 3.** Solve the linear system. From the first:  $3A + 2B = 1$ . Multiply the second by  $\frac{1}{2}$ :  $2A - 3B = -\frac{3}{2}$ . Multiply the first by 3, second by 2:

$$9A + 6B = 3, \quad 4A - 6B = -3.$$

Add:  $13A = 0 \Rightarrow A = 0$ . Then  $2B = 1 \Rightarrow B = 1/2$ .

**Step 4.** Substitute back:  $N(x) = 0 \cdot D(x) + \frac{1}{2}D'(x)$ , i.e.  
 $2 \cos x - 3 \sin x = \frac{1}{2}(4 \cos x - 6 \sin x)$ . (Verify:  $\frac{1}{2} \cdot 4 = 2\checkmark$ ,  $\frac{1}{2} \cdot (-6) = -3\checkmark$ .)

**Step 5.** Therefore

$$\int \frac{N(x)}{D(x)} dx = \frac{1}{2} \int \frac{D'(x)}{D(x)} dx = \frac{1}{2} \log |D(x)|.$$

**Step 6.** Back-substitute:  $\frac{1}{2} \log |6 \cos x + 4 \sin x| + C$ .

**Final Answer:**  $\frac{1}{2} \log |6 \cos x + 4 \sin x| + C$

#### ♥ The $A \cdot D + B \cdot D'$ trick

The decomposition  $N = AD + BD'$  is the standard method whenever the integrand is a ratio of two linear combinations of  $\sin x$  and  $\cos x$ . Always try this before panicking.

**EXPERT'S SOLUTION** : Meera Banerjee, Ph.D Pure Mathematics, IISc Bangalore

**Strategic angle.** Linear combination decomposition. Once  $N = AD + BD'$ , the integral splits into a constant times  $x$  plus a log term.

**Step 1.** Compute  $D' = 4 \cos x - 6 \sin x$ .

**Step 2.** Solve  $6A + 4B = 2$ ,  $4A - 6B = -3$ . Solution:  $A = 0$ ,  $B = 1/2$ .

**Step 3.** Hence  $\int N/D dx = \frac{1}{2} \int D'/D dx = \frac{1}{2} \log |D| + C$ .

**Final Answer:**  $\frac{1}{2} \log |6 \cos x + 4 \sin x| + C$

**Q 7.25** Integrate  $\frac{1}{\cos^2 x (1 - \tan x)^2}$ .

## SOLUTION

**Concept used.** Recognise  $1/\cos^2 x = \sec^2 x$ , which is the derivative of  $\tan x$ . Substitute  $t = 1 - \tan x$ .

**Step 1.** Rewrite:  $\frac{1}{\cos^2 x (1 - \tan x)^2} = \frac{\sec^2 x}{(1 - \tan x)^2}$ .

**Step 2.** Put  $t = 1 - \tan x$ . Then  $dt = -\sec^2 x dx$ , so  $\sec^2 x dx = -dt$ .

**Step 3.** Substitute:

$$\int \frac{\sec^2 x dx}{(1 - \tan x)^2} = \int \frac{-dt}{t^2} = -\int t^{-2} dt.$$

**Step 4.** Integrate:  $-\int t^{-2} dt = - \cdot \frac{t^{-1}}{-1} = \frac{1}{t}$ .

**Step 5.** Back-substitute:  $\frac{1}{1 - \tan x} + C$ .

**Final Answer:**  $\frac{1}{1 - \tan x} + C$

## EXPERT'S SOLUTION : Krishna Rao, M.Sc Mathematics, IIT Bombay

**Quick reading.**  $\sec^2 x dx$  pairs with  $\tan x$  as a substitution.

**Step 1.**  $t = 1 - \tan x$ ;  $\sec^2 x dx = -dt$ .

**Step 2.**  $\int -dt/t^2 = 1/t$ .

**Step 3.** Final:  $\frac{1}{1 - \tan x} + C$ .

**Final Answer:**  $\frac{1}{1 - \tan x} + C$

**Q 7.26** Integrate  $\frac{\cos \sqrt{x}}{\sqrt{x}}$ .

## SOLUTION

**Concept used.**  $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$ , so  $\frac{dx}{\sqrt{x}} = 2 dt$  when  $t = \sqrt{x}$ .

**Step 1.** Put  $t = \sqrt{x}$ . Then  $dt = \frac{dx}{2\sqrt{x}}$ , so  $\frac{dx}{\sqrt{x}} = 2 dt$ .

**Step 2.** Substitute:

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = \int \cos t \cdot 2 dt = 2 \int \cos t dt = 2 \sin t.$$

**Step 3.** Back-substitute:  $2 \sin \sqrt{x} + C$ .

**Final Answer:**  $2 \sin \sqrt{x} + C$

**EXPERT'S SOLUTION** : Aditya Kumar, Ph.D Mathematics, IIT Delhi

**Strategic angle.** The  $1/\sqrt{x}$  factor outside is exactly 2 times  $\frac{d}{dx}\sqrt{x}$ .

**Step 1.**  $t = \sqrt{x}$ ;  $dx/\sqrt{x} = 2 dt$ .

**Step 2.**  $\int 2 \cos t dt = 2 \sin t = 2 \sin \sqrt{x} + C$ .

**Final Answer:**  $2 \sin \sqrt{x} + C$

**Q 7.27** Integrate  $\sqrt{\sin 2x} \cos 2x$ .

**SOLUTION**

**Concept used.**  $\frac{d}{dx} \sin 2x = 2 \cos 2x$ . Substitute  $t = \sin 2x$ .

**Step 1.** Put  $t = \sin 2x \Rightarrow dt = 2 \cos 2x dx \Rightarrow \cos 2x dx = dt/2$ .

**Step 2.** Substitute:

$$\int \sqrt{\sin 2x} \cos 2x dx = \int \sqrt{t} \cdot \frac{dt}{2} = \frac{1}{2} \int t^{1/2} dt.$$

**Step 3.** Integrate:  $\frac{1}{2} \cdot \frac{2}{3} t^{3/2} = \frac{1}{3} t^{3/2}$ .

**Step 4.** Back-substitute:  $\frac{(\sin 2x)^{3/2}}{3} + C$ .

**Final Answer:**  $\frac{(\sin 2x)^{3/2}}{3} + C$

**EXPERT'S SOLUTION** : Sneha Iyer, M.Sc Applied Mathematics, IIT Kanpur

**Quick reading.**  $\cos 2x$  pairs with  $\sin 2x$  as a substitution.

**Step 1.**  $t = \sin 2x$ ;  $\cos 2x \, dx = dt/2$ .

**Step 2.**  $\int \frac{1}{2}t^{1/2} \, dt = \frac{1}{3}t^{3/2}$ .

**Final Answer:**  $\frac{(\sin 2x)^{3/2}}{3} + C$

**Q 7.28** Integrate  $\frac{\cos x}{\sqrt{1 + \sin x}}$ .

**SOLUTION**

**Concept used.**  $\frac{d}{dx}(1 + \sin x) = \cos x$ . Substitute  $t = 1 + \sin x$ .

**Step 1.** Put  $t = 1 + \sin x \Rightarrow dt = \cos x \, dx$ .

**Step 2.** Substitute:

$$\int \frac{\cos x \, dx}{\sqrt{1 + \sin x}} = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} \, dt.$$

**Step 3.** Integrate:  $\int t^{-1/2} \, dt = 2t^{1/2}$ .

**Step 4.** Back-substitute:  $2\sqrt{1 + \sin x} + C$ .

**Final Answer:**  $2\sqrt{1 + \sin x} + C$

**EXPERT'S SOLUTION** : Ankit Verma, M.Tech CS, IIT Madras

**Quick reading.** Numerator  $\cos x$  is the derivative of  $1 + \sin x$ .

**Step 1.**  $t = 1 + \sin x$ ;  $\cos x \, dx = dt$ .

**Step 2.**  $\int dt/\sqrt{t} = 2\sqrt{t}$ .

**Final Answer:**  $2\sqrt{1 + \sin x} + C$

**Q 7.29** Integrate  $\cot x \log \sin x$ .

## SOLUTION

**Concept used.**  $\cot x = \frac{\cos x}{\sin x}$  and  $\frac{d}{dx} \log \sin x = \cot x$ . Substitute  $t = \log \sin x$ .

**Step 1.** Put  $t = \log \sin x$ . Then  $dt = \frac{\cos x}{\sin x} dx = \cot x dx$ .

**Step 2.** Substitute:

$$\int \cot x \log \sin x dx = \int t dt = \frac{t^2}{2}.$$

**Step 3.** Back-substitute:  $\frac{(\log \sin x)^2}{2} + C$ .

**Final Answer:**  $\frac{(\log \sin x)^2}{2} + C$

## EXPERT'S SOLUTION : Ananya Bhat, M.Sc Mathematics, IIT Bombay

**Structural observation.** Derivative of  $\log \sin x$  is  $\cot x$ .

**Step 1.**  $t = \log \sin x$ ;  $dt = \cot x dx$ .

**Step 2.**  $\int t dt = t^2/2$ .

**Final Answer:**  $\frac{(\log \sin x)^2}{2} + C$

**Q 7.30** Integrate  $\frac{\sin x}{1 + \cos x}$ .

## SOLUTION

**Concept used.**  $\frac{d}{dx}(1 + \cos x) = -\sin x$ . Log-derivative form with a sign.

**Step 1.** Put  $t = 1 + \cos x \Rightarrow dt = -\sin x dx \Rightarrow \sin x dx = -dt$ .

**Step 2.** Substitute:

$$\int \frac{\sin x dx}{1 + \cos x} = \int \frac{-dt}{t} = -\log |t|.$$

**Step 3.** Back-substitute (with  $1 + \cos x \geq 0$  wherever defined):  $-\log |1 + \cos x| + C$ .

**Final Answer:**  $-\log |1 + \cos x| + C$

**EXPERT'S SOLUTION** : Aditi Verma, M.Sc Mathematics, IIT Bombay**Quick reading.** Negative log-derivative.

**Step 1.**  $t = 1 + \cos x$ ;  $\sin x \, dx = -dt$ .

**Step 2.**  $\int -dt/t = -\log |t|$ .

**Final Answer:**  $-\log |1 + \cos x| + C$

**Q 7.31** Integrate  $\frac{\sin x}{(1 + \cos x)^2}$ .

**SOLUTION****Concept used.** Substitute  $t = 1 + \cos x$ ; numerator becomes  $-dt$ .

**Step 1.** Put  $t = 1 + \cos x \Rightarrow dt = -\sin x \, dx \Rightarrow \sin x \, dx = -dt$ .

**Step 2.** Substitute:

$$\int \frac{\sin x \, dx}{(1 + \cos x)^2} = \int \frac{-dt}{t^2} = -\int t^{-2} \, dt.$$

**Step 3.** Integrate:  $-\int t^{-2} \, dt = - \cdot \frac{t^{-1}}{-1} = \frac{1}{t}$ .

**Step 4.** Back-substitute:  $\frac{1}{1 + \cos x} + C$ .

**Final Answer:**  $\frac{1}{1 + \cos x} + C$

**EXPERT'S SOLUTION** : Priya Sharma, Ph.D Mathematics, IIT Delhi**Quick reading.** Same substitution as Q30, different power.

**Step 1.**  $t = 1 + \cos x$ ;  $\sin x \, dx = -dt$ .

**Step 2.**  $\int -t^{-2} \, dt = 1/t$ .

**Final Answer:**  $\frac{1}{1 + \cos x} + C$

**Q 7.32** Integrate  $\frac{1}{1 + \cot x}$ .

## SOLUTION

**Concept used.** Convert  $\cot x = \cos x / \sin x$ , simplify, then either use the  $AD + BD'$  technique on the new integrand or combine with the conjugate.

**Step 1.** Convert:  $\frac{1}{1 + \cot x} = \frac{1}{1 + \cos x / \sin x} = \frac{\sin x}{\sin x + \cos x}$ .

**Step 2.** Apply the  $AD + BD'$  decomposition with  $D = \sin x + \cos x$  and  $D' = \cos x - \sin x$ . Write  $\sin x = A(\sin x + \cos x) + B(\cos x - \sin x)$ . Equate coefficients:

$$A - B = 1, \quad A + B = 0 \Rightarrow A = \frac{1}{2}, \quad B = -\frac{1}{2}.$$

**Step 3.** Therefore

$$\frac{\sin x}{\sin x + \cos x} = \frac{1}{2} - \frac{1}{2} \cdot \frac{\cos x - \sin x}{\sin x + \cos x}.$$

**Step 4.** Integrate term-by-term. The first piece gives  $x/2$ . For the second, substitute  $t = \sin x + \cos x$ ,  $dt = (\cos x - \sin x) dx$ :

$$-\frac{1}{2} \int \frac{dt}{t} = -\frac{1}{2} \log |t| = -\frac{1}{2} \log |\sin x + \cos x|.$$

**Step 5.** Combine:  $\frac{x}{2} - \frac{1}{2} \log |\sin x + \cos x| + C$ .

**Final Answer:**  $\frac{x}{2} - \frac{1}{2} \log |\sin x + \cos x| + C$

**EXPERT'S SOLUTION :** *Karan Mehta, B.Tech Engineering Physics, IIT Bombay*

**Strategic angle.** Convert  $\cot$  to ratio, then  $AD + BD'$ .

**Step 1.**  $\frac{1}{1 + \cot x} = \frac{\sin x}{\sin x + \cos x}$ .

**Step 2.**  $\sin x = \frac{1}{2}(\sin x + \cos x) - \frac{1}{2}(\cos x - \sin x)$ .

**Step 3.** Integrate:  $\frac{x}{2} - \frac{1}{2} \log |\sin x + \cos x| + C$ .

**Final Answer:**  $\frac{x}{2} - \frac{1}{2} \log |\sin x + \cos x| + C$

**Q 7.33** Integrate  $\frac{1}{1 - \tan x}$ .

**SOLUTION**

**Concept used.** Convert  $\tan x = \sin x / \cos x$  and use the  $AD + BD'$  decomposition.

**Step 1.** Convert:  $\frac{1}{1 - \tan x} = \frac{1}{1 - \sin x / \cos x} = \frac{\cos x}{\cos x - \sin x}$ .

**Step 2.** Take  $D = \cos x - \sin x$ ,  $D' = -\sin x - \cos x = -(\sin x + \cos x)$ . Write  $\cos x = AD + BD'$ :

$$A(\cos x - \sin x) - B(\sin x + \cos x) = (A - B)\cos x + (-A - B)\sin x.$$

Equate to  $\cos x$ :  $A - B = 1$  and  $-A - B = 0 \Rightarrow B = -A$ . Then

$$A - (-A) = 2A = 1 \Rightarrow A = 1/2, B = -1/2.$$

**Step 3.** Substitute:

$$\frac{\cos x}{\cos x - \sin x} = \frac{1}{2} - \frac{1}{2} \cdot \frac{-(\sin x + \cos x)}{\cos x - \sin x} = \frac{1}{2} + \frac{1}{2} \cdot \frac{\sin x + \cos x}{\cos x - \sin x}.$$

**Step 4.** Compute

$$\frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x = -(\sin x + \cos x).$$

Hence  $(\sin x + \cos x) / (\cos x - \sin x) = -D'/D$ , and

$$\int \frac{\sin x + \cos x}{\cos x - \sin x} dx = -\log |\cos x - \sin x|.$$

**Step 5.** Combine:

$$\int \frac{dx}{1 - \tan x} = \frac{1}{2}x - \frac{1}{2} \log |\cos x - \sin x| + C.$$

**Final Answer:**  $\frac{x}{2} - \frac{1}{2} \log |\cos x - \sin x| + C$

**EXPERT'S SOLUTION** : Tara Bhat, M.Sc Mathematics, ISI Kolkata

**Structural observation.** Same template as Q32 but with  $\cos x - \sin x$  in the denominator.

**Step 1.**  $\frac{1}{1 - \tan x} = \frac{\cos x}{\cos x - \sin x}$ .

**Step 2.**  $\cos x = \frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\cos x + \sin x)$ .

**Step 3.** Integrate:  $\frac{x}{2} - \frac{1}{2} \log |\cos x - \sin x| + C$ .

**Final Answer:**  $\frac{x}{2} - \frac{1}{2} \log |\cos x - \sin x| + C$

**Q 7.34** Integrate  $\frac{\sqrt{\tan x}}{\sin x \cos x}$ .

### SOLUTION

**Concept used.** Divide numerator and denominator by  $\cos^2 x$  to introduce  $\sec^2 x$ , then substitute  $t = \tan x$ .

**Step 1.** Divide top and bottom by  $\cos^2 x$ :

$$\frac{\sqrt{\tan x}}{\sin x \cos x} = \frac{\sqrt{\tan x} / \cos^2 x}{\sin x \cos x / \cos^2 x} = \frac{\sqrt{\tan x} \sec^2 x}{\tan x}$$

**Step 2.** Put  $t = \tan x$ . Then  $dt = \sec^2 x dx$  and  $\sqrt{\tan x} = \sqrt{t}$ .

**Step 3.** Substitute:

$$\int \frac{\sqrt{t}}{t} dt = \int t^{-1/2} dt = 2t^{1/2} = 2\sqrt{t}$$

**Step 4.** Back-substitute:  $2\sqrt{\tan x} + C$ .

**Final Answer:**  $2\sqrt{\tan x} + C$

### EXPERT'S SOLUTION : Yash Joshi, M.Sc Mathematics, IIT Bombay

**Strategic angle.** Multiply top and bottom by  $\sec^2 x$  to convert  $\sin x \cos x$  into  $\tan x$  form.

**Step 1.**  $\sin x \cos x = \tan x \cos^2 x = \tan x / \sec^2 x$ . So  $\frac{\sqrt{\tan x}}{\sin x \cos x} = \frac{\sqrt{\tan x} \sec^2 x}{\tan x}$ .

**Step 2.**  $t = \tan x$ ;  $dt = \sec^2 x dx$ .

**Step 3.**  $\int t^{-1/2} dt = 2\sqrt{t}$ .

**Final Answer:**  $2\sqrt{\tan x} + C$

**Q 7.35** Integrate  $\frac{(1 + \log x)^2}{x}$ .

### SOLUTION

**Concept used.** Substitute  $t = 1 + \log x$  so  $dt = dx/x$ .

**Step 1.** Put  $t = 1 + \log x \Rightarrow dt = dx/x$ .

**Step 2.** Substitute:

$$\int \frac{(1 + \log x)^2}{x} dx = \int t^2 dt = \frac{t^3}{3}.$$

**Step 3.** Back-substitute:  $\frac{(1 + \log x)^3}{3} + C$ .

**Final Answer:**  $\frac{(1 + \log x)^3}{3} + C$

**EXPERT'S SOLUTION** : Rohit Desai, M.Sc Mathematics, IIT Bombay

**Quick reading.**  $1/x$  is the derivative of  $\log x$ .

**Step 1.**  $t = 1 + \log x$ ;  $dt = dx/x$ .

**Step 2.**  $\int t^2 dt = t^3/3$ .

**Final Answer:**  $\frac{(1 + \log x)^3}{3} + C$

**Q 7.36** Integrate  $\frac{(x + 1)(x + \log x)^2}{x}$ .

**SOLUTION**

**Concept used.** Distribute the  $(x + 1)/x$  factor:  $\frac{x + 1}{x} = 1 + \frac{1}{x}$ , which is exactly  $\frac{d}{dx}(x + \log x)$ . Substitute  $t = x + \log x$ .

**Step 1.** Rewrite:  $\frac{(x + 1)(x + \log x)^2}{x} = \left(1 + \frac{1}{x}\right)(x + \log x)^2$ .

**Step 2.** Note  $\frac{d}{dx}(x + \log x) = 1 + \frac{1}{x}$ . Put  $t = x + \log x \Rightarrow dt = \left(1 + \frac{1}{x}\right) dx$ .

**Step 3.** Substitute:

$$\int \left(1 + \frac{1}{x}\right)(x + \log x)^2 dx = \int t^2 dt = \frac{t^3}{3}.$$

**Step 4.** Back-substitute:  $\frac{(x + \log x)^3}{3} + C$ .

**Final Answer:**  $\frac{(x + \log x)^3}{3} + C$

**EXPERT'S SOLUTION** : Aditya Kumar, Ph.D Mathematics, IIT Delhi

**Structural observation.** The factor  $(x + 1)/x = 1 + 1/x$  matches the derivative of  $x + \log x$ .

**Step 1.**  $t = x + \log x$ ;  $dt = (1 + 1/x) dx$ .

**Step 2.**  $\int t^2 dt = t^3/3$ .

**Final Answer:**  $\frac{(x + \log x)^3}{3} + C$

**Q 7.37** Integrate  $\frac{x^3 \sin(\tan^{-1} x^4)}{1 + x^8}$ .

### SOLUTION

**Concept used.** Two substitutions in sequence. First  $u = x^4$  to convert  $\tan^{-1} x^4$  into  $\tan^{-1} u$ , then  $t = \tan^{-1} u$ .

**Step 1.** Put  $u = x^4$ . Then  $du = 4x^3 dx$ , so  $x^3 dx = du/4$ , and  $x^8 = u^2$ . Substitute:

$$\int \frac{x^3 \sin(\tan^{-1} x^4)}{1 + x^8} dx = \frac{1}{4} \int \frac{\sin(\tan^{-1} u)}{1 + u^2} du.$$

**Step 2.** Now put  $t = \tan^{-1} u$ . Then  $dt = \frac{du}{1 + u^2}$ , and the integral becomes

$$\frac{1}{4} \int \sin t dt = -\frac{1}{4} \cos t.$$

**Step 3.** Back-substitute  $t = \tan^{-1} u = \tan^{-1} x^4$ :  $-\frac{1}{4} \cos(\tan^{-1} x^4) + C$ .

**Final Answer:**  $-\frac{1}{4} \cos(\tan^{-1} x^4) + C$

### ♥ Layered substitutions

Many NCERT integrals use two substitutions in sequence: the outer brings the integrand into a standard form; the inner finishes it. Plan the chain end-to-end before computing.

**EXPERT'S SOLUTION** : Meera Banerjee, Ph.D Pure Mathematics, IISc Bangalore

**Strategic angle.** Two nested substitutions:  $u = x^4$ , then  $t = \tan^{-1} u$ .

**Step 1.**  $u = x^4$ ;  $x^3 dx = du/4$ ;  $x^8 = u^2$ .

**Step 2.** Integral becomes  $\frac{1}{4} \int \frac{\sin(\tan^{-1} u)}{1 + u^2} du$ .

**Step 3.**  $t = \tan^{-1} u$ ;  $dt = du/(1 + u^2)$ . So integral is  $\frac{1}{4} \int \sin t dt = -\frac{1}{4} \cos t$ .

**Step 4.** Back-substitute.

**Final Answer:**  $-\frac{\cos(\tan^{-1} x^4)}{4} + C$

- Q 7.38**  $\int \frac{10x^9 + 10^x \log_e 10}{x^{10} + 10^x} dx$  equals
- (A)  $10^x - x^{10} + C$     (B)  $10^x + x^{10} + C$   
 (C)  $(10^x - x^{10})^{-1} + C$     (D)  $\log(10^x + x^{10}) + C$

### SOLUTION

**Concept used.** Differentiate the denominator and recognise the log-derivative form.

**Step 1.** Differentiate  $f(x) = x^{10} + 10^x$ :

$$f'(x) = 10x^9 + 10^x \log_e 10,$$

because  $\frac{d}{dx}(10^x) = 10^x \log_e 10$ .

**Step 2.** This matches the numerator exactly. The integrand is  $f'(x)/f(x)$ .

**Step 3.** By the log-derivative rule,  $\int f'/f dx = \log |f(x)| = \log(x^{10} + 10^x)$  (denominator positive for  $x > 0$ , and the question implicitly assumes that domain).

**Final Answer:** Option (D):  $\log(10^x + x^{10}) + C$

**EXPERT'S SOLUTION** : Rohit Desai, M.Sc Mathematics, IIT Bombay

**Quick reading.** Match the numerator with  $\frac{d}{dx}$  denominator.

**Step 1.**  $\frac{d}{dx}(x^{10} + 10^x) = 10x^9 + 10^x \log 10$ . Matches numerator.

**Step 2.** Log-derivative rule yields  $\log(x^{10} + 10^x) + C$ .

**Final Answer:** Option (D)

- Q 7.39**  $\int \frac{dx}{\sin^2 x \cos^2 x}$  equals  
 (A)  $\tan x + \cot x + C$  (B)  $\tan x - \cot x + C$   
 (C)  $\tan x \cot x + C$  (D)  $\tan x - \cot 2x + C$

### SOLUTION

**Concept used.** Use the identity  $\sin^2 x + \cos^2 x = 1$  to split the integrand into  $\sec^2 x + \csc^2 x$ .

**Step 1.** Write the numerator 1 as  $\sin^2 x + \cos^2 x$ :

$$\begin{aligned} \frac{1}{\sin^2 x \cos^2 x} &= \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} \\ &= \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \\ &= \sec^2 x + \csc^2 x. \end{aligned}$$

**Step 2.** Integrate:

$$\int \sec^2 x \, dx = \tan x, \quad \int \csc^2 x \, dx = -\cot x.$$

**Step 3.** Combine:  $\tan x + (-\cot x) = \tan x - \cot x + C$ . Matches option (B).

**Final Answer:** Option (B):  $\tan x - \cot x + C$

**EXPERT'S SOLUTION** : Aditi Verma, M.Sc Mathematics, IIT Bombay

**Structural observation.** Use  $1 = \sin^2 x + \cos^2 x$  in the numerator.

**Step 1.** Split:  $\frac{1}{\sin^2 x \cos^2 x} = \sec^2 x + \csc^2 x$ .

**Step 2.** Integrate:  $\tan x - \cot x + C$ .

**Final Answer:** Option (B)

### Key Takeaways

- Integration by substitution: pick  $t = u(x)$  where  $u(x)$  or its derivative appears in the integrand; then  $dt = u'(x) dx$  converts  $\int f(x) dx$  into a standard form in  $t$ .

- Log-derivative form:  $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C$  is the most common substitution pattern.
- Power-derivative form:  $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C$  for  $n \neq -1$ .
- Standard substitutions:  $t = \sqrt{x+a}$  for surds;  $t = \log x$  paired with  $1/x$ ;  $t = \tan^{-1} x$  paired with  $1/(1+x^2)$ ;  $t = e^x$  paired with  $e^x$ .
- Linear-combination trick: write  $a \cos x + b \sin x = AD(x) + BD'(x)$  where  $D = c \cos x + d \sin x$ ; split and integrate.

End of Exercise 7.2