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Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

Chapter 7: Integrals

About this Chapter

Exercise 7.4 introduces the six **special-form integrals** that arise from quadratics in the denominator: $1/(x^2 \pm a^2)$, $1/(a^2 - x^2)$, $1/\sqrt{a^2 - x^2}$, $1/\sqrt{x^2 \pm a^2}$, plus their generalisations with $ax^2 + bx + c$ via **completing the square**. For linear-in-numerator-over-quadratic cases, you split the numerator as $A \cdot (2ax + b) + B$.

Topics covered: Standard quadratic integrals • Completing the square • $\tan^{-1}, \sin^{-1}$ forms • Linear-over-quadratic splitting

Quick Formula Sheet

Algebraic forms:

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left|\frac{a+x}{a-x}\right| + C$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\frac{x}{a} + C$$

Surd forms:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

Exercise 7.4

Q 7.1 **Integrate** $\dfrac{3x^2}{x^6 + 1}$.

SOLUTION

Concept used. Substitute $t = x^3$ so $dt = 3x^2\,dx$, matching the numerator. Then apply the standard integral $\int \dfrac{dt}{t^2 + 1} = \tan^{-1} t + C$.

Step 1. Note $x^6 = (x^3)^2$ and the numerator $3x^2$ is the derivative of x^3.

Step 2. Put $t = x^3 \Rightarrow dt = 3x^2\,dx$.

Step 3. Substitute:

$$\int \frac{3x^2\,dx}{x^6+1} = \int \frac{dt}{t^2+1} = \tan^{-1} t.$$

Step 4. Back-substitute: $\tan^{-1}(x^3) + C$.

Final Answer: $\tan^{-1}(x^3) + C$

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. $x^6 + 1 = (x^3)^2 + 1$; numerator matches derivative of x^3.

Step 1. $t = x^3$; $dt = 3x^2\,dx$.

Step 2. $\int dt/(t^2+1) = \tan^{-1} t$.

Final Answer: $\tan^{-1}(x^3) + C$

Q 7.2 **Integrate** $\dfrac{1}{\sqrt{1+4x^2}}$.

SOLUTION

Concept used. $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \log|x + \sqrt{x^2+a^2}| + C$. Rescale x so that the squared term has coefficient 1.

Step 1. Write $1 + 4x^2 = 1 + (2x)^2$.

Step 2. Put $u = 2x$; $du = 2\,dx$, $dx = du/2$.

Step 3. Substitute:

$$\int \frac{dx}{\sqrt{1+4x^2}} = \int \frac{du/2}{\sqrt{1+u^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2+1}}.$$

Step 4. Apply standard form with $a = 1$:

$$\tfrac{1}{2}\log|u + \sqrt{u^2+1}| = \tfrac{1}{2}\log|2x + \sqrt{4x^2+1}| + C.$$

Final Answer: $\dfrac{1}{2}\log\left|2x + \sqrt{4x^2+1}\right| + C$

EXPERT'S SOLUTION : *Sneha Iyer, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. Rescale $2x = u$ so the integrand fits the standard $1/\sqrt{u^2+1}$ form.

Step 1. $u = 2x$; $dx = du/2$.

Step 2. $\int du/\sqrt{u^2+1} = \log|u + \sqrt{u^2+1}|$.

Step 3. Back-substitute and divide by 2.

Final Answer: $\frac{1}{2}\log|2x + \sqrt{1+4x^2}| + C$

Q 7.3 **Integrate** $\dfrac{1}{\sqrt{(2-x)^2+1}}$.

SOLUTION

Concept used. Substitute the inner linear function so the integrand becomes $1/\sqrt{u^2+1}$.

Step 1. Put $u = 2 - x$; $du = -dx \Rightarrow dx = -du$.

Step 2. Substitute:

$$\int \frac{dx}{\sqrt{(2-x)^2+1}} = \int \frac{-du}{\sqrt{u^2+1}} = -\log|u + \sqrt{u^2+1}|.$$

Step 3. Back-substitute:

$$-\log\left|(2-x) + \sqrt{(2-x)^2+1}\right| + C.$$

Final Answer: $-\log\left|(2-x) + \sqrt{(2-x)^2+1}\right| + C$

EXPERT'S SOLUTION : *Karan Mehta, B.Tech Engineering Physics, IIT Bombay*

Quick reading. Linear inner $\Rightarrow$ single substitution.

Step 1. $u = 2 - x$; $dx = -du$.

Step 2. Standard log form with $a = 1$ and a sign flip.

Final Answer: $-\log\left|(2-x) + \sqrt{(2-x)^2+1}\right| + C$

Q 7.4 **Integrate** $\dfrac{1}{\sqrt{9-25x^2}}$.

SOLUTION

Concept used. $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C$. Rescale to fit this form.

Step 1. Write $9 - 25x^2 = 9 - (5x)^2 = 3^2 - (5x)^2$.

Step 2. Put $u = 5x$; $du = 5\,dx \Rightarrow dx = du/5$.

Step 3. Substitute:

$$\int \frac{dx}{\sqrt{9-25x^2}} = \int \frac{du/5}{\sqrt{9-u^2}} = \tfrac{1}{5}\int \frac{du}{\sqrt{3^2-u^2}}.$$

Step 4. Apply $\int \dfrac{du}{\sqrt{a^2-u^2}} = \sin^{-1}(u/a)$ with $a = 3$:

$$\tfrac{1}{5}\sin^{-1}(u/3) = \tfrac{1}{5}\sin^{-1}(5x/3) + C.$$

Final Answer: $\dfrac{1}{5}\sin^{-1}\left(\dfrac{5x}{3}\right) + C$

EXPERT'S SOLUTION *: Pranav Patel, Ph.D Mathematics, IIT Delhi*

Quick reading. Factor 5 inside the surd, scale, $\sin^{-1}$ form.

Step 1. $9 - 25x^2 = 3^2 - (5x)^2$.

Step 2. $u = 5x$, $dx = du/5$.

Step 3. $\int du/\sqrt{9-u^2} = \sin^{-1}(u/3)$; multiply by $1/5$.

Final Answer: $\dfrac{1}{5}\sin^{-1}\left(\dfrac{5x}{3}\right) + C$

Q 7.5 **Integrate** $\dfrac{3x}{1+2x^4}$.

SOLUTION

Concept used. Substitute $u = \sqrt{2}\,x^2$ to convert into $1/(1+u^2)$ form. Alternatively, substitute $t = x^2$, which simplifies the polynomial degrees first.

Step 1. Put $t = x^2 \Rightarrow dt = 2x\,dx \Rightarrow x\,dx = dt/2$.

Step 2. Substitute:

$$\int \frac{3x\,dx}{1+2x^4} = \int \frac{3\cdot dt/2}{1+2t^2} = \tfrac{3}{2}\int \frac{dt}{1+2t^2}.$$

Step 3. Factor inside: $1+2t^2 = 2(t^2+1/2)$. Put $u = t\sqrt{2}$, $du = \sqrt{2}\,dt$, $dt = du/\sqrt{2}$:

$$\tfrac{3}{2}\int \frac{dt}{1+2t^2} = \tfrac{3}{2}\int \frac{du/\sqrt{2}}{1+u^2} = \tfrac{3}{2\sqrt{2}}\int \frac{du}{1+u^2} = \tfrac{3}{2\sqrt{2}}\tan^{-1} u.$$

Step 4. Back-substitute $u = t\sqrt{2} = x^2\sqrt{2}$:

$$\frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}\,x^2\right) + C.$$

Final Answer: $\frac{3}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}\,x^2\right) + C$

EXPERT'S SOLUTION : *Riya Gupta, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Substitute $t = x^2$ first to compress the degree from 4 to 2.

Step 1. $t = x^2$; $x\,dx = dt/2$.

Step 2. $\tfrac{3}{2}\int dt/(1+2t^2) = \tfrac{3}{2\sqrt{2}}\tan^{-1}(t\sqrt{2})$.

Step 3. Back-substitute.

Final Answer: $\frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,x^2) + C$

Q 7.6 Integrate $\dfrac{x^2}{1-x^6}$.

SOLUTION

Concept used. Recognise $1-x^6 = 1-(x^3)^2 = (1-x^3)(1+x^3)$, but the neater path is to substitute $t = x^3$ so the integrand becomes $\dfrac{1}{1-t^2}$, a standard form.

Step 1. Put $t = x^3 \Rightarrow dt = 3x^2\,dx \Rightarrow x^2\,dx = dt/3$.

Step 2. Substitute:

$$\int \frac{x^2\,dx}{1-x^6} = \int \frac{dt/3}{1-t^2} = \tfrac{1}{3}\int \frac{dt}{1-t^2}.$$

Step 3. Apply the standard integral $\int \dfrac{dt}{a^2 - t^2} = \dfrac{1}{2a} \log\left|\dfrac{a+t}{a-t}\right|$ with $a = 1$:

$$\int \frac{dt}{1-t^2} = \tfrac{1}{2} \log\left|\frac{1+t}{1-t}\right|.$$

Step 4. Multiply by $1/3$ and back-substitute $t = x^3$:

$$\frac{1}{6} \log\left|\frac{1+x^3}{1-x^3}\right| + C.$$

> **Final Answer:** $\dfrac{1}{6} \log\left|\dfrac{1+x^3}{1-x^3}\right| + C$

EXPERT'S SOLUTION *: Yash Joshi, M.Sc Mathematics, IIT Bombay*

Quick reading. $x^6 = (x^3)^2$, numerator x^2 matches derivative.

Step 1. $t = x^3$; $x^2\,dx = dt/3$.

Step 2. $\int dt/(1-t^2) = \frac{1}{2} \log|(1+t)/(1-t)|$.

> **Final Answer:** $\dfrac{1}{6} \log\left|\dfrac{1+x^3}{1-x^3}\right| + C$

Q 7.7 **Integrate** $\dfrac{x-1}{\sqrt{x^2-1}}$.

SOLUTION

Concept used. Split the numerator: $\dfrac{x-1}{\sqrt{x^2-1}} = \dfrac{x}{\sqrt{x^2-1}} - \dfrac{1}{\sqrt{x^2-1}}$. The first is a substitution ($t = x^2 - 1$); the second is the standard surd integral.

Step 1. Split: $\dfrac{x-1}{\sqrt{x^2-1}} = \dfrac{x}{\sqrt{x^2-1}} - \dfrac{1}{\sqrt{x^2-1}}$.

Step 2. For the first piece, put $t = x^2 - 1 \Rightarrow dt = 2x\,dx$, $x\,dx = dt/2$:

$$\int \frac{x\,dx}{\sqrt{x^2-1}} = \int \frac{dt/2}{\sqrt{t}} = \tfrac{1}{2} \cdot 2\sqrt{t} = \sqrt{t} = \sqrt{x^2-1}.$$

Step 3. For the second, apply $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log|x + \sqrt{x^2 - a^2}|$ with $a = 1$:

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \log|x + \sqrt{x^2 - 1}|.$$

Step 4. Combine: $\sqrt{x^2 - 1} - \log|x + \sqrt{x^2 - 1}| + C$.

Final Answer: $\sqrt{x^2 - 1} - \log|x + \sqrt{x^2 - 1}| + C$

EXPERT'S SOLUTION *: Aanya Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Split a linear numerator into a derivative-of-radicand piece plus a constant; each integrates by its own standard form.

Step 1. Split into $x/\sqrt{x^2 - 1}$ and $1/\sqrt{x^2 - 1}$.

Step 2. Anti derivatives: $\sqrt{x^2 - 1}$ and $\log|x + \sqrt{x^2 - 1}|$.

Step 3. Subtract.

Final Answer: $\sqrt{x^2 - 1} - \log|x + \sqrt{x^2 - 1}| + C$

Q 7.8 **Integrate** $\frac{x^2}{x^6 + a^6}$.

SOLUTION

Concept used. Substitute $t = x^3$ to compress.

Step 1. $t = x^3$; $x^2\,dx = dt/3$.

Step 2. Denominator: $x^6 + a^6 = t^2 + (a^3)^2$. Put $b = a^3$.

Step 3. Substitute:

$$\int \frac{x^2\,dx}{x^6 + a^6} = \frac{1}{3}\int \frac{dt}{t^2 + b^2}.$$

Step 4. Apply $\int \frac{dt}{t^2 + b^2} = \frac{1}{b}\tan^{-1}(t/b)$:

$$\frac{1}{3} \cdot \frac{1}{b}\tan^{-1}(t/b) = \frac{1}{3a^3}\tan^{-1}\left(\frac{x^3}{a^3}\right) + C.$$

Final Answer: $\dfrac{1}{3a^3}\tan^{-1}\left(\dfrac{x^3}{a^3}\right)+C$

EXPERT'S SOLUTION *: Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. $x^6=(x^3)^2$, $a^6=(a^3)^2$.

Step 1. $t=x^3$; integral becomes $\frac{1}{3}\int dt/(t^2+(a^3)^2)$.

Step 2. $\tan^{-1}$ form.

Final Answer: $\dfrac{1}{3a^3}\tan^{-1}(x^3/a^3)+C$

Q 7.9 **Integrate** $\dfrac{\sec^2 x}{\tan^2 x+4}$.

SOLUTION

Concept used. $\sec^2 x\,dx=d(\tan x)$. Substitute $t=\tan x$.

Step 1. Put $t=\tan x$; $dt=\sec^2 x\,dx$.

Step 2. Substitute:

$$\int\frac{\sec^2 x\,dx}{\tan^2 x+4}=\int\frac{dt}{t^2+4}=\int\frac{dt}{t^2+2^2}.$$

Step 3. Apply $\tan^{-1}$ form with $a=2$:

$$\tfrac{1}{2}\tan^{-1}(t/2).$$

Step 4. Back-substitute: $\frac{1}{2}\tan^{-1}(\tan x/2)+C$.

Final Answer: $\dfrac{1}{2}\tan^{-1}\left(\dfrac{\tan x}{2}\right)+C$

EXPERT'S SOLUTION *: Tara Bhat, M.Sc Mathematics, ISI Kolkata*

Quick reading. $\sec^2 x\,dx=d(\tan x)$.

Step 1. $t=\tan x$; integral becomes $\int dt/(t^2+4)$.

Step 2. Anti derivative: $\frac{1}{2}\tan^{-1}(t/2)$.

Final Answer: $\frac{1}{2}\tan^{-1}\left(\frac{\tan x}{2}\right) + C$

Q 7.10 **Integrate** $\frac{1}{x^2+2x+2}$.

SOLUTION

Concept used. **Complete the square**: write $x^2+2x+2 = (x+1)^2+1$. Then apply $\int \frac{dx}{u^2+a^2} = \frac{1}{a}\tan^{-1}(u/a)$.

Step 1. Complete the square: $x^2+2x+2 = (x^2+2x+1)+1 = (x+1)^2+1$.

Step 2. Put $u = x+1$; $du = dx$.

Step 3. Substitute:

$$\int \frac{dx}{x^2+2x+2} = \int \frac{du}{u^2+1} = \tan^{-1} u.$$

Step 4. Back-substitute: $\tan^{-1}(x+1)+C$.

Final Answer: $\tan^{-1}(x+1)+C$

☞ **Completing the square**

For ax^2+bx+c: write $a\left[(x+b/2a)^2+(c/a-b^2/4a^2)\right]$. This converts the denominator into a sum or difference of squares, ready for the standard integrals.

EXPERT'S SOLUTION *: Ishaan Reddy, B.Tech CSE, IIT Roorkee*

Quick reading. Complete the square; $\tan^{-1}$ form.

Step 1. $(x+1)^2+1$.

Step 2. $\tan^{-1}(x+1)$.

Final Answer: $\tan^{-1}(x+1)+C$

Q 7.11 **Integrate** $\frac{1}{9x^2+6x+5}$.

SOLUTION

Concept used. Complete the square inside the quadratic; the leading coefficient 9 pulls outside the bracket.

Step 1. Factor out 9 from the first two terms: $9x^2 + 6x + 5 = 9(x^2 + \frac{2}{3}x) + 5$.

Step 2. Complete the square in the bracket: $x^2 + \frac{2}{3}x = (x + 1/3)^2 - 1/9$.

Step 3. Therefore
$9x^2 + 6x + 5 = 9\big[(x + 1/3)^2 - 1/9\big] + 5 = 9(x + 1/3)^2 - 1 + 5 = 9(x + 1/3)^2 + 4$.
Equivalently $9x^2 + 6x + 5 = (3x + 1)^2 + 4$.

Step 4. Put $u = 3x + 1$; $du = 3\,dx \Rightarrow dx = du/3$.

$$\int \frac{dx}{(3x+1)^2 + 4} = \frac{1}{3}\int \frac{du}{u^2 + 2^2} = \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}(u/2) = \frac{1}{6}\tan^{-1}\left(\frac{3x+1}{2}\right) + C.$$

Final Answer: $\dfrac{1}{6}\tan^{-1}\left(\dfrac{3x+1}{2}\right) + C$

EXPERT'S SOLUTION *: Dev Chatterjee, B.Tech CSE, IIT Roorkee*

Strategic angle. Complete the square; rescale.

Step 1. $9x^2 + 6x + 5 = (3x + 1)^2 + 4$.

Step 2. $u = 3x + 1$; $dx = du/3$.

Step 3. Final: $\frac{1}{6}\tan^{-1}((3x + 1)/2) + C$.

Final Answer: $\dfrac{1}{6}\tan^{-1}\left(\dfrac{3x+1}{2}\right) + C$

Q 7.12 **Integrate** $\dfrac{1}{\sqrt{7 - 6x - x^2}}$.

SOLUTION

Concept used. Complete the square; the result is of the form $a^2 - u^2$, so apply $\sin^{-1}$.

Step 1. Rewrite: $7 - 6x - x^2 = -(x^2 + 6x - 7) = -[(x + 3)^2 - 16] = 16 - (x + 3)^2$.

Step 2. Put $u = x + 3$; $du = dx$.

Step 3. Substitute:

$$\int \frac{dx}{\sqrt{16-(x+3)^2}} = \int \frac{du}{\sqrt{4^2-u^2}} = \sin^{-1}(u/4).$$

Step 4. Back-substitute: $\sin^{-1}\left(\frac{x+3}{4}\right) + C.$

Final Answer: $\sin^{-1}\left(\frac{x+3}{4}\right) + C$

EXPERT'S SOLUTION *: Krishna Rao, M.Sc Mathematics, IIT Bombay*

Strategic angle. Negate the bracket and complete the square; $\sin^{-1}$ form.

Step 1. $7-6x-x^2 = 16-(x+3)^2.$

Step 2. $\sin^{-1}((x+3)/4).$

Final Answer: $\sin^{-1}\left(\frac{x+3}{4}\right) + C$

Q 7.13 **Integrate** $\frac{1}{\sqrt{(x-1)(x-2)}}.$

SOLUTION

Concept used. Expand and complete the square.

Step 1. Expand: $(x-1)(x-2) = x^2-3x+2.$

Step 2. Complete the square: $x^2-3x+2 = (x-3/2)^2 - 9/4 + 2 = (x-3/2)^2 - 1/4.$

Step 3. Put $u = x - 3/2$; $du = dx$. Then

$$\int \frac{dx}{\sqrt{(x-3/2)^2-(1/2)^2}} = \int \frac{du}{\sqrt{u^2-(1/2)^2}}.$$

Step 4. Apply $\int \frac{du}{\sqrt{u^2-a^2}} = \log|u+\sqrt{u^2-a^2}|$ with $a = 1/2$:

$$\log\left|(x-3/2) + \sqrt{(x-3/2)^2 - 1/4}\right| + C.$$

Equivalently, $\log\left|(x-3/2) + \sqrt{x^2-3x+2}\right| + C.$

Final Answer: $\log\left|(x-3/2)+\sqrt{x^2-3x+2}\right|+C$

EXPERT'S SOLUTION *: Aditya Kumar, Ph.D Mathematics, IIT Delhi*

Quick reading. Expand the product; complete the square; standard log form.

Step 1. $(x-1)(x-2)=(x-3/2)^2-1/4$.

Step 2. $\log|(x-3/2)+\sqrt{x^2-3x+2}|$.

Final Answer: $\log\left|(x-3/2)+\sqrt{x^2-3x+2}\right|+C$

Q 7.14 **Integrate** $\dfrac{1}{\sqrt{8+3x-x^2}}$.

SOLUTION

Concept used. Complete the square; result is a^2-u^2, apply $\sin^{-1}$.

Step 1. $8+3x-x^2=-(x^2-3x-8)=-[(x-3/2)^2-9/4-8]=$
$-(x-3/2)^2+41/4=\frac{41}{4}-(x-3/2)^2$.

Step 2. Put $u=x-3/2$.

$$\int\frac{du}{\sqrt{(\sqrt{41}/2)^2-u^2}}=\sin^{-1}\left(\frac{u}{\sqrt{41}/2}\right)=\sin^{-1}\left(\frac{2u}{\sqrt{41}}\right).$$

Step 3. Back-substitute: $\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$.

Final Answer: $\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$

EXPERT'S SOLUTION *: Neha Gupta, M.Sc Mathematics, ISI Kolkata*

Quick reading. Complete the square; $\sin^{-1}$ form.

Step 1. $8+3x-x^2=41/4-(x-3/2)^2$.

Step 2. Anti derivative: $\sin^{-1}((2x-3)/\sqrt{41})$.

Final Answer: $\sin^{-1}\left(\dfrac{2x-3}{\sqrt{41}}\right)+C$

Q 7.15 **Integrate** $\dfrac{1}{\sqrt{(x-a)(x-b)}}$.

SOLUTION

Concept used. Expand and complete the square as in Q13.

Step 1. Expand: $(x-a)(x-b)=x^2-(a+b)x+ab$.

Step 2. Complete the square:

$$x^2-(a+b)x+ab=\left(x-\frac{a+b}{2}\right)^2-\frac{(a+b)^2}{4}+ab=\left(x-\frac{a+b}{2}\right)^2-\frac{(a-b)^2}{4}.$$

Step 3. Put $u=x-(a+b)/2$. Apply standard log form:

$$\log\left|u+\sqrt{u^2-\left(\tfrac{a-b}{2}\right)^2}\right|+C=\log\left|x-\tfrac{a+b}{2}+\sqrt{(x-a)(x-b)}\right|+C.$$

Final Answer: $\log\left|x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)}\right|+C$

EXPERT'S SOLUTION *: Priya Sharma, Ph.D Mathematics, IIT Delhi*

Quick reading. Generalised Q13.

Step 1. Centre: $(a+b)/2$; gap: $(a-b)/2$.

Step 2. Anti derivative: $\log|x-(a+b)/2+\sqrt{(x-a)(x-b)}|$.

Final Answer: $\log\left|x-\frac{a+b}{2}+\sqrt{(x-a)(x-b)}\right|+C$

Q 7.16 **Integrate** $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$.

SOLUTION

Concept used. Linear-over-square-root strategy. Write the numerator as $A \cdot \frac{d}{dx}(\text{radicand}) + B$, then integrate the two parts separately.

Step 1. Differentiate the radicand: $\frac{d}{dx}(2x^2 + x - 3) = 4x + 1$.

Step 2. The numerator already equals the derivative. Substitute $t = 2x^2 + x - 3$, $dt = (4x + 1)\,dx$:

$$\int \frac{(4x+1)\,dx}{\sqrt{2x^2 + x - 3}} = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t}.$$

Step 3. Back-substitute: $2\sqrt{2x^2 + x - 3} + C$.

Final Answer: $2\sqrt{2x^2 + x - 3} + C$

♥ **Numerator-equals-derivative**

Whenever the numerator is the derivative of the radicand, the integral collapses to $2\sqrt{\text{radicand}}$ in one step. Always differentiate the radicand first.

EXPERT'S SOLUTION *: Aditi Verma, M.Sc Mathematics, IIT Bombay*

Quick reading. Numerator = derivative of radicand.

Step 1. $t = 2x^2 + x - 3$; $dt = (4x + 1)\,dx$.

Step 2. $\int dt/\sqrt{t} = 2\sqrt{t}$.

Final Answer: $2\sqrt{2x^2 + x - 3} + C$

Q 7.17 Integrate $\frac{x+2}{\sqrt{x^2-1}}$.

SOLUTION

Concept used. Split as in Q7: $\frac{x+2}{\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}} + \frac{2}{\sqrt{x^2-1}}$.

Step 1. First piece: substitute $t = x^2 - 1$, $x\,dx = dt/2$. Then $\int x/\sqrt{x^2-1}\,dx = \sqrt{x^2-1}$.

Step 2. Second piece: $\int \frac{2\,dx}{\sqrt{x^2-1}} = 2\log|x + \sqrt{x^2-1}|$.

Step 3. Combine:

$$\sqrt{x^2-1} + 2\log|x + \sqrt{x^2-1}| + C.$$

Final Answer: $\sqrt{x^2-1}+2\log|x+\sqrt{x^2-1}|+C$

EXPERT'S SOLUTION : *Aditi Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Linear-over-surd: $x+2=x+2\cdot 1$; pieces split.

Step 1. $x/\sqrt{x^2-1}\to\sqrt{x^2-1}$.

Step 2. $2/\sqrt{x^2-1}\to 2\log|x+\sqrt{x^2-1}|$.

Final Answer: $\sqrt{x^2-1}+2\log|x+\sqrt{x^2-1}|+C$

Q 7.18 **Integrate** $\dfrac{5x-2}{1+2x+3x^2}$.

SOLUTION

Concept used. Write the numerator as $A\cdot\dfrac{d}{dx}(\text{denominator})+B$. Split into a log-derivative piece (yields $\log$) and a constant-over-quadratic piece (yields $\tan^{-1}$ after completing the square).

Step 1. Differentiate denominator: $\dfrac{d}{dx}(1+2x+3x^2)=2+6x$.

Step 2. Write $5x-2=A(2+6x)+B$. Equate coefficients: $6A=5\Rightarrow A=5/6$. Constant: $2A+B=-2\Rightarrow B=-2-2\cdot 5/6=-2-5/3=-\frac{11}{3}$. So $5x-2=\frac{5}{6}(6x+2)-\frac{11}{3}$.

Step 3. Split:

$$\int\frac{5x-2}{3x^2+2x+1}\,dx=\frac{5}{6}\int\frac{6x+2}{3x^2+2x+1}\,dx-\frac{11}{3}\int\frac{dx}{3x^2+2x+1}.$$

Step 4. First integral: log-derivative gives $\frac{5}{6}\log|3x^2+2x+1|$.

Step 5. Second integral: complete the square.
$3x^2+2x+1=3\left[x^2+\frac{2}{3}x\right]+1=3\left[(x+1/3)^2-1/9\right]+1=3(x+1/3)^2+2/3$.
Hence

$$\int\frac{dx}{3(x+1/3)^2+2/3}=\frac{1}{3}\int\frac{dx}{(x+1/3)^2+2/9}.$$

With $a^2=2/9$, $a=\sqrt{2}/3$:

$$\frac{1}{3}\cdot\frac{1}{a}\tan^{-1}\left(\frac{x+1/3}{a}\right)=\frac{1}{3}\cdot\frac{3}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right)=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right).$$

Step 6. Multiply by $-11/3$ and combine:

$$\tfrac{5}{6}\log|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C.$$

Final Answer: $\dfrac{5}{6}\log|3x^2+2x+1| - \dfrac{11}{3\sqrt{2}}\tan^{-1}\left(\dfrac{3x+1}{\sqrt{2}}\right) + C$

EXPERT'S SOLUTION : *Ananya Bhat, M.Sc Mathematics, IIT Bombay*

Strategic angle. Decompose numerator $= A\cdot D' + B$; integrate the two pieces independently.

Step 1. $D' = 6x+2$. $5x-2 = \tfrac{5}{6}(6x+2) - 11/3$.

Step 2. First: $\tfrac{5}{6}\log|D|$.

Step 3. Second: complete the square to $3(x+1/3)^2 + 2/3$. Standard $\tan^{-1}$ form.

Step 4. Combine and add C.

Final Answer: $\dfrac{5}{6}\log|3x^2+2x+1| - \dfrac{11}{3\sqrt{2}}\tan^{-1}\left(\dfrac{3x+1}{\sqrt{2}}\right) + C$

Q 7.19 Integrate $\dfrac{6x+7}{\sqrt{(x-5)(x-4)}}$.

SOLUTION

Concept used. Expand the radicand and use the $A\cdot D' + B$ split.

Step 1. Expand: $(x-5)(x-4) = x^2-9x+20$. Derivative: $2x-9$.

Step 2. Write $6x+7 = A(2x-9)+B$. Compare x-coeff: $2A = 6 \Rightarrow A = 3$. Constant: $-9A+B = 7 \Rightarrow B = 7+27 = 34$.

Step 3. Split:

$$\int \frac{6x+7}{\sqrt{x^2-9x+20}}\,dx = 3\int \frac{2x-9}{\sqrt{x^2-9x+20}}\,dx + 34\int \frac{dx}{\sqrt{x^2-9x+20}}.$$

Step 4. First piece: substitute $t = x^2-9x+20$, $dt = (2x-9)\,dx$:

$$3\int \frac{dt}{\sqrt{t}} = 6\sqrt{t} = 6\sqrt{x^2-9x+20}.$$

Step 5. Second piece: complete the square.
$x^2 - 9x + 20 = (x - 9/2)^2 - 81/4 + 20 = (x - 9/2)^2 - 1/4$. Standard log form gives

$$34\log\left|(x - 9/2) + \sqrt{x^2 - 9x + 20}\right|.$$

Step 6. Combine:

$$6\sqrt{x^2 - 9x + 20} + 34\log\left|(x - 9/2) + \sqrt{x^2 - 9x + 20}\right| + C.$$

Final Answer: $6\sqrt{x^2 - 9x + 20} + 34\log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + C$

EXPERT'S SOLUTION : *Rohit Desai, M.Sc Mathematics, IIT Bombay*

Quick reading. Same template as Q18 but with a surd.

Step 1. $D = x^2 - 9x + 20$; $D' = 2x - 9$.

Step 2. $6x + 7 = 3D' + 34$.

Step 3. First piece: $6\sqrt{D}$.

Step 4. Second piece: $34\log|x - 9/2 + \sqrt{D}|$.

Final Answer: $6\sqrt{x^2 - 9x + 20} + 34\log\left|x - \frac{9}{2} + \sqrt{x^2 - 9x + 20}\right| + C$

Q 7.20 **Integrate** $\dfrac{x+2}{\sqrt{4x - x^2}}$.

SOLUTION

Concept used. $A \cdot D' + B$ split, where $D = 4x - x^2$, $D' = 4 - 2x$.

Step 1. Write $x + 2 = A(4 - 2x) + B$. Coefficient of x: $-2A = 1 \Rightarrow A = -1/2$.
Constant: $4A + B = 2 \Rightarrow B = 2 - 4(-1/2) = 4$.

Step 2. Split:

$$\int \frac{x+2}{\sqrt{4x - x^2}}\,dx = -\tfrac{1}{2}\int \frac{4 - 2x}{\sqrt{4x - x^2}}\,dx + 4\int \frac{dx}{\sqrt{4x - x^2}}.$$

Step 3. First piece: $t = 4x - x^2$, $dt = (4 - 2x)\,dx$:

$$-\tfrac{1}{2}\int \frac{dt}{\sqrt{t}} = -\sqrt{t} = -\sqrt{4x - x^2}.$$

Step 4. Second piece: complete the square.
$4x - x^2 = -(x^2 - 4x) = -[(x-2)^2 - 4] = 4 - (x-2)^2$. Hence

$$4\int \frac{dx}{\sqrt{4-(x-2)^2}} = 4\sin^{-1}\left(\frac{x-2}{2}\right).$$

Step 5. Combine:

$$-\sqrt{4x - x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C.$$

Final Answer: $-\sqrt{4x - x^2} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$

EXPERT'S SOLUTION *: Meera Banerjee, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. Linear-over-surd with $a^2 - u^2$ structure inside.

Step 1. $D = 4x - x^2 = 4 - (x-2)^2$.

Step 2. $x + 2 = -\frac{1}{2}(4 - 2x) + 4$.

Step 3. First: $-\sqrt{4x - x^2}$. Second: $4\sin^{-1}((x-2)/2)$.

Final Answer: $4\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x - x^2} + C$

Q 7.21 **Integrate** $\dfrac{x+2}{\sqrt{x^2+2x+3}}$.

SOLUTION

Concept used. $A \cdot D' + B$ split with $D = x^2 + 2x + 3$, $D' = 2x + 2$.

Step 1. $x + 2 = A(2x+2) + B$. Coeff of x: $2A = 1 \Rightarrow A = 1/2$. Constant: $2A + B = 2 \Rightarrow B = 1$.

Step 2. Split:

$$\int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \int \frac{dx}{\sqrt{x^2+2x+3}}.$$

Step 3. First piece: $t = x^2 + 2x + 3$, $dt = (2x+2)\,dx$:

$$\frac{1}{2}\int \frac{dt}{\sqrt{t}} = \sqrt{t} = \sqrt{x^2+2x+3}.$$

Step 4. Second piece: $x^2 + 2x + 3 = (x+1)^2 + 2$.

$$\int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}} = \log\left|(x+1) + \sqrt{x^2+2x+3}\right|.$$

Step 5. Combine:

$$\sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C.$$

> **Final Answer:** $\sqrt{x^2+2x+3} + \log\left|(x+1) + \sqrt{x^2+2x+3}\right| + C$

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Same decomposition as Q19/20 but with $x^2 + a^2$ form inside the surd.

Step 1. $D = x^2 + 2x + 3 = (x+1)^2 + 2$.

Step 2. $x + 2 = \frac{1}{2}(2x+2) + 1$.

Step 3. First: $\sqrt{D}$. Second: $\log|x + 1 + \sqrt{D}|$.

> **Final Answer:** $\sqrt{x^2+2x+3} + \log|x + 1 + \sqrt{x^2+2x+3}| + C$

Q 7.22 **Integrate** $\dfrac{x+3}{x^2-2x-5}$.

SOLUTION

Concept used. $A \cdot D' + B$ split. $D = x^2 - 2x - 5$, $D' = 2x - 2$.

Step 1. $x + 3 = A(2x-2) + B$. Coeff: $2A = 1 \Rightarrow A = 1/2$. Constant: $-2A + B = 3 \Rightarrow B = 4$.

Step 2. Split:

$$\int \frac{x+3}{x^2-2x-5}\,dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}\,dx + 4\int \frac{dx}{x^2-2x-5}.$$

Step 3. First: $\frac{1}{2}\log|x^2 - 2x - 5|$.

Step 4. Second: $x^2 - 2x - 5 = (x-1)^2 - 6$.

$$4\int \frac{du}{u^2 - (\sqrt{6})^2} = 4 \cdot \frac{1}{2\sqrt{6}} \log\left|\frac{u-\sqrt{6}}{u+\sqrt{6}}\right| = \frac{2}{\sqrt{6}} \log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|.$$

Step 5. Combine:

$$\tfrac{1}{2}\log|x^2-2x-5|+\frac{2}{\sqrt{6}}\log\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C.$$

Final Answer: $\dfrac{1}{2}\log|x^2-2x-5|+\dfrac{2}{\sqrt{6}}\log\left|\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$

EXPERT'S SOLUTION *: Sneha Iyer, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Linear-over-quadratic, $D=(x-1)^2-6$, hence u^2-a^2 form.

Step 1. $x+3=\tfrac{1}{2}(2x-2)+4$.

Step 2. $\tfrac{1}{2}\log|D|+\tfrac{2}{\sqrt{6}}\log|(x-1-\sqrt{6})/(x-1+\sqrt{6})|+C$.

Final Answer: $\dfrac{1}{2}\log|x^2-2x-5|+\dfrac{2}{\sqrt{6}}\log\left|\dfrac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$

Q 7.23 **Integrate** $\dfrac{5x+3}{\sqrt{x^2+4x+10}}$.

SOLUTION

Concept used. $A\cdot D'+B$ split with $D=x^2+4x+10$, $D'=2x+4$.

Step 1. $5x+3=A(2x+4)+B$. Coeff: $2A=5\Rightarrow A=5/2$. Constant: $4A+B=3\Rightarrow B=3-10=-7$.

Step 2. Split:

$$\int\frac{5x+3}{\sqrt{x^2+4x+10}}\,dx=\tfrac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}\,dx-7\int\frac{dx}{\sqrt{x^2+4x+10}}.$$

Step 3. First: $\tfrac{5}{2}\cdot 2\sqrt{x^2+4x+10}=5\sqrt{x^2+4x+10}$.

Step 4. Second: $x^2+4x+10=(x+2)^2+6$. Standard log form:

$$-7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|.$$

Step 5. Combine:

$$5\sqrt{x^2+4x+10}-7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C.$$

> **Final Answer:** $5\sqrt{x^2+4x+10} - 7\log\left|(x+2)+\sqrt{x^2+4x+10}\right| + C$

EXPERT'S SOLUTION *: Aanya Singh, M.Sc Mathematics, IIT Bombay*

Quick reading. Linear-over-surd $x^2+4x+10 = (x+2)^2+6$.

Step 1. $5x+3 = \frac{5}{2}(2x+4) - 7$.

Step 2. First $\to 5\sqrt{D}$; second $\to -7\log|x+2+\sqrt{D}|$.

> **Final Answer:** $5\sqrt{x^2+4x+10} - 7\log|x+2+\sqrt{x^2+4x+10}| + C$

Q 7.24 $\displaystyle\int \frac{dx}{x^2+2x+2}$ **equals**

(A) $x\tan^{-1}(x+1)+C$ **(B)** $\tan^{-1}(x+1)+C$

(C) $(x+1)\tan^{-1}x+C$ **(D)** $\tan^{-1}x+C$

SOLUTION

Concept used. Complete the square. (Same calculation as Q10.)

Step 1. $x^2+2x+2 = (x+1)^2+1$.

Step 2. $\int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1)+C$. Matches option (B).

> **Final Answer:** Option (B): $\tan^{-1}(x+1)+C$

EXPERT'S SOLUTION *: Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. Complete the square; $\tan^{-1}$ form.

Step 1. $(x+1)^2+1$.

Step 2. $\tan^{-1}(x+1)$.

> **Final Answer:** Option (B)

Q 7.25 $\displaystyle\int \frac{dx}{\sqrt{9x-4x^2}}$ **equals**

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ **(B)** $\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C$
(C) $\frac{1}{3}\sin^{-1}\left(\frac{9x-8}{8}\right)+C$ **(D)** $\frac{1}{2}\sin^{-1}\left(\frac{9x-8}{9}\right)+C$

SOLUTION

Concept used. Complete the square in the radicand; rescale to $\sin^{-1}$.

Step 1. $9x-4x^2=-4(x^2-\frac{9}{4}x)=-4\left[(x-9/8)^2-81/64\right]=-4(x-9/8)^2+81/16=\frac{81}{16}-4(x-9/8)^2$. Equivalently $9x-4x^2=\left(\frac{9}{4}\right)^2-\left(2x-\frac{9}{4}\right)^2$ — but let us use the factored form directly: $9x-4x^2=(9/4)^2-(2x-9/4)^2$ after multiplying inside the bracket by 4. More cleanly: pull out 4 and write $4x^2-9x=4(x-9/8)^2-81/16$, hence $9x-4x^2=81/16-4(x-9/8)^2$.

Step 2. Put $u=2(x-9/8)=2x-9/4$. Then $du=2\,dx$, $dx=du/2$, and the radicand becomes $81/16-u^2=(9/4)^2-u^2$.

Step 3. Substitute:

$$\int\frac{dx}{\sqrt{81/16-u^2}}=\frac{1}{2}\int\frac{du}{\sqrt{(9/4)^2-u^2}}=\frac{1}{2}\sin^{-1}\left(\frac{u}{9/4}\right)=\frac{1}{2}\sin^{-1}\left(\frac{4u}{9}\right).$$

Step 4. Substitute back $u=2x-9/4$, so $4u/9=(8x-9)/9$:

$$\frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right)+C.$$

Matches option (B).

Final Answer: Option (B): $\dfrac{1}{2}\sin^{-1}\left(\dfrac{8x-9}{9}\right)+C$

EXPERT'S SOLUTION : *Tara Bhat, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Pull out 4, complete the square, rescale to $\sin^{-1}$.

Step 1. $9x-4x^2=81/16-4(x-9/8)^2$.

Step 2. $u=2(x-9/8)=2x-9/4$.

Step 3. $\int dx/\sqrt{81/16-u^2}=\frac{1}{2}\sin^{-1}((8x-9)/9)$.

Final Answer: Option (B)

Key Takeaways

- Six standard quadratic-form integrals: $1/(x^2\pm a^2)$, $1/(a^2-x^2)$ on the algebraic side;

$1/\sqrt{x^2 \pm a^2}$, $1/\sqrt{a^2 - x^2}$ on the surd side.

- For $ax^2 + bx + c$ denominators, complete the square first: write the quadratic as $a[(x + b/2a)^2 + (c/a - b^2/4a^2)]$.
- For linear-over-quadratic or linear-over-surd cases, decompose the numerator as $A \cdot D'(x) + B$ to split the integral into a log/surd piece plus a standard quadratic piece.
- $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$ is a special case that saves work whenever the numerator equals the derivative of the radicand.

End of Exercise 7.4