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Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

Chapter 7: Integrals

About this Chapter

Exercise 7.5 trains **Integration by Partial Fractions**. A proper rational function $\frac{P(x)}{Q(x)}$ (degree of P less than Q) decomposes into a sum of simpler fractions whose denominators are linear or quadratic. After decomposition, each piece integrates as a $\log$, a $\tan^{-1}$, or a power of $1/(x-a)$.

Topics covered: Distinct linear factors • Repeated linear factors • Irreducible quadratic factors • Improper-to-proper via long division

Quick Formula Sheet

Partial-fraction templates:

$$\frac{1}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$$

$$\frac{1}{(x-a)^2} \to \frac{A}{x-a} + \frac{B}{(x-a)^2}$$

$$\frac{1}{(x-a)(x^2+bx+c)} \to \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$$

Anti derivatives:

$$\int \frac{dx}{x-a} = \log|x-a| + C$$

$$\int \frac{dx}{(x-a)^2} = -\frac{1}{x-a} + C$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Exercise 7.5

Q 7.1 **Integrate** $\dfrac{x}{(x+1)(x+2)}$.

SOLUTION

Concept used. The denominator is a product of distinct linear factors, so the proper rational function decomposes as

$$\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

Multiply out and solve for A, B by equating coefficients or by substituting convenient x values.

Step 1. Multiply both sides by $(x+1)(x+2)$: $x = A(x+2) + B(x+1)$.

Step 2. Substitute $x = -1$: $-1 = A(1) + B(0) \Rightarrow A = -1$. Substitute $x = -2$: $-2 = A(0) + B(-1) \Rightarrow B = 2$.

Step 3. Decomposition:

$$\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}.$$

Step 4. Integrate term-by-term:

$$\int = -\log|x+1| + 2\log|x+2| + C = \log\left|\frac{(x+2)^2}{x+1}\right| + C.$$

Final Answer: $\log\left|\frac{(x+2)^2}{x+1}\right| + C$

☞ Cover-up rule

For distinct linear factors, to find A multiplying $1/(x-a)$, cover $(x-a)$ in the denominator and substitute $x = a$ in the rest.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. Distinct linear factors ⇒ two unknowns.

Step 1. $\frac{x}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{2}{x+2}$.

Step 2. Integrate: $-\log|x+1| + 2\log|x+2| + C$.

Final Answer: $2\log|x+2| - \log|x+1| + C$

Q 7.2 Integrate $\frac{1}{x^2-9}$.

SOLUTION

Concept used. $x^2 - 9 = (x-3)(x+3)$. Apply the standard form $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ with $a = 3$, or decompose into partial fractions directly.

Step 1. Apply the standard form with $a = 3$:

$$\int \frac{dx}{x^2-9} = \frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C.$$

Final Answer: $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C$

EXPERT'S SOLUTION *: Sneha Iyer, M.Sc Applied Mathematics, IIT Kanpur*

Alternative. Decompose directly: $\frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$.

Step 1. $1 = A(x+3) + B(x-3)$. $x = 3$: $1 = 6A \Rightarrow A = 1/6$. $x = -3$: $1 = -6B \Rightarrow B = -1/6$.

Step 2. Integrate: $\frac{1}{6}\log|x-3| - \frac{1}{6}\log|x+3| = \frac{1}{6}\log|(x-3)/(x+3)|$.

Final Answer: $\frac{1}{6}\log\left|\frac{x-3}{x+3}\right| + C$

Q 7.3 **Integrate** $\frac{3x-1}{(x-1)(x-2)(x-3)}$.

SOLUTION

Concept used. Three distinct linear factors:

$$\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Step 1. Multiply through: $3x - 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$.

Step 2. Substitute $x = 1$: $2 = A(-1)(-2) = 2A \Rightarrow A = 1$. Substitute $x = 2$: $5 = B(1)(-1) = -B \Rightarrow B = -5$. Substitute $x = 3$: $8 = C(2)(1) = 2C \Rightarrow C = 4$.

Step 3. Integrate term-by-term:

$$\int \frac{3x-1}{(x-1)(x-2)(x-3)}\,dx = \log|x-1| - 5\log|x-2| + 4\log|x-3| + C.$$

Final Answer: $\log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

EXPERT'S SOLUTION : *Karan Mehta, B.Tech Engineering Physics, IIT Bombay*

Quick reading. Three linear factors, three constants.

Step 1. Cover-up rule: $A = (3 \cdot 1 - 1)/[(1-2)(1-3)] = 2/2 = 1$. $B = 5/(1 \cdot -1) = -5$. $C = 8/(2 \cdot 1) = 4$.

Step 2. Integrate.

> **Final Answer:** $\log|x-1| - 5\log|x-2| + 4\log|x-3| + C$

Q 7.4 **Integrate** $\dfrac{x}{(x-1)(x-2)(x-3)}$.

SOLUTION

Concept used. Same template as Q3.

Step 1. $\dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$.

Step 2. Cover-up: $A = 1/[(1-2)(1-3)] = 1/2$. $B = 2/[(1)(-1)] = -2$. $C = 3/[(2)(1)] = 3/2$.

Step 3. Integrate:

$$\tfrac{1}{2}\log|x-1| - 2\log|x-2| + \tfrac{3}{2}\log|x-3| + C.$$

> **Final Answer:** $\dfrac{1}{2}\log|x-1| - 2\log|x-2| + \dfrac{3}{2}\log|x-3| + C$

EXPERT'S SOLUTION : *Pranav Patel, Ph.D Mathematics, IIT Delhi*

Quick reading. Cover-up rule.

Step 1. $A = 1/2$, $B = -2$, $C = 3/2$.

Step 2. Sum the logs.

> **Final Answer:** $\frac{1}{2}\log|x-1| - 2\log|x-2| + \frac{3}{2}\log|x-3| + C$

Q 7.5 **Integrate** $\dfrac{2x}{x^2+3x+2}$.

SOLUTION

Concept used. Factor: $x^2 + 3x + 2 = (x+1)(x+2)$. Distinct linear factors.

Step 1. $\dfrac{2x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2}$, so $2x = A(x+2) + B(x+1)$.

Step 2. $x = -1$: $-2 = A(1) \Rightarrow A = -2$. $x = -2$: $-4 = B(-1) \Rightarrow B = 4$.

Step 3. Integrate:

$$-2\log|x+1| + 4\log|x+2| + C = \log\left|\frac{(x+2)^4}{(x+1)^2}\right| + C.$$

Final Answer: $4\log|x+2| - 2\log|x+1| + C$

EXPERT'S SOLUTION : *Riya Gupta, M.Sc Mathematics, ISI Kolkata*

Quick reading. Factor, decompose, integrate.

Step 1. $(x+1)(x+2)$. $A = -2$, $B = 4$.

Step 2. $-2\log|x+1| + 4\log|x+2| + C$.

Final Answer: $4\log|x+2| - 2\log|x+1| + C$

Q 7.6 **Integrate** $\dfrac{1-x^2}{x(1-2x)}$.

SOLUTION

Concept used. The numerator has the same degree as the denominator, so first divide to make the rational function proper, then apply partial fractions.

Step 1. Multiply out denominator: $x(1-2x) = x - 2x^2$. Numerator $1 - x^2$ has degree 2; denominator also degree 2. Divide.

Step 2. Long division of $1 - x^2$ by $-2x^2 + x$:

$$\frac{1-x^2}{-2x^2+x} = \frac{1}{2} + \frac{1-x/2}{x(1-2x)}.$$

(Quotient $1/2$ comes from $1/(-2) = -1/2$ flipped because we keep the original denominator orientation $x(1-2x)$.)

Step 3. Decompose the remainder:

$$\frac{1-x/2}{x(1-2x)} = \frac{A}{x} + \frac{B}{1-2x}.$$

Multiply through: $1 - x/2 = A(1-2x) + Bx$. $x = 0$: $1 = A$. $x = 1/2$: $1 - 1/4 = B(1/2) \Rightarrow B = 3/2$.

Step 4. So

$$\frac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1}{x} + \frac{3/2}{1-2x}.$$

Step 5. Integrate:

$$\frac{x}{2} + \log|x| + \frac{3}{2}\cdot\left(-\tfrac{1}{2}\log|1-2x|\right) = \frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C.$$

Final Answer: $\frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$

EXPERT'S SOLUTION : *Yash Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. Improper $\Rightarrow$ divide first, then decompose the remainder.

Step 1. Divide: $\dfrac{1-x^2}{x(1-2x)} = \frac{1}{2} + \frac{1-x/2}{x(1-2x)}$.

Step 2. Decompose: $\dfrac{1-x/2}{x(1-2x)} = \frac{1}{x} + \frac{3/2}{1-2x}$.

Step 3. Integrate: $x/2 + \log|x| - (3/4)\log|1-2x| + C$.

Final Answer: $\frac{x}{2} + \log|x| - \frac{3}{4}\log|1-2x| + C$

Q 7.7 **Integrate** $\dfrac{x}{(x^2+1)(x-1)}$.

SOLUTION

Concept used. Mixed factor types: one linear $(x-1)$ and one irreducible quadratic (x^2+1). Decompose as

$$\frac{x}{(x^2+1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Step 1. Multiply through: $x = A(x^2+1) + (Bx+C)(x-1)$.

Step 2. Expand right side:
$Ax^2 + A + Bx^2 - Bx + Cx - C = (A+B)x^2 + (-B+C)x + (A-C)$.

Step 3. Equate coefficients with $x = 0 \cdot x^2 + 1 \cdot x + 0$: $A + B = 0$, $-B + C = 1$, $A - C = 0$.

Step 4. Solve: $A = C$ and $B = -A$. Substituting into the second: $A + A = 1 \Rightarrow A = 1/2$. So $A = C = 1/2$, $B = -1/2$.

Step 5. Decomposition:

$$\frac{x}{(x^2+1)(x-1)} = \frac{1/2}{x-1} + \frac{-x/2+1/2}{x^2+1} = \frac{1}{2(x-1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}.$$

Step 6. Integrate:

$$\int \frac{dx}{2(x-1)} = \tfrac{1}{2}\log|x-1|,$$
$$\int -\frac{x\,dx}{2(x^2+1)} = -\tfrac{1}{4}\log(x^2+1) \quad \text{(via } t = x^2+1\text{)},$$
$$\int \frac{dx}{2(x^2+1)} = \tfrac{1}{2}\tan^{-1}x.$$

Step 7. Combine:

$$\tfrac{1}{2}\log|x-1| - \tfrac{1}{4}\log(x^2+1) + \tfrac{1}{2}\tan^{-1}x + C.$$

Final Answer: $\frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C$

♥ Mixed-factor decomposition

Whenever the denominator mixes a linear factor with an irreducible quadratic, set up $\frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$. The quadratic piece splits further into a log (from the x in the numerator) and a $\tan^{-1}$ (from the constant).

EXPERT'S SOLUTION *: Aanya Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Linear + quadratic mix; solve via coefficient comparison.

Step 1. Decompose with A, B, C.

Step 2. Solve: $A = C = 1/2$, $B = -1/2$.

Step 3. Integrate each piece; combine.

Final Answer: $\frac{1}{2}\log|x-1| - \frac{1}{4}\log(x^2+1) + \frac{1}{2}\tan^{-1}x + C$

Q 7.8 **Integrate** $\dfrac{x}{(x-1)^2(x+2)}$.

SOLUTION

Concept used. Repeated linear factor: decompose as

$$\frac{x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}.$$

Step 1. Multiply through: $x = A(x-1)(x+2) + B(x+2) + C(x-1)^2$.

Step 2. $x = 1$: $1 = B(3) \Rightarrow B = 1/3$.

Step 3. $x = -2$: $-2 = C(-3)^2 = 9C \Rightarrow C = -2/9$.

Step 4. Comparing coefficients of x^2: on LHS 0, on RHS $A + C$. So $A + C = 0 \Rightarrow A = -C = 2/9$.

Step 5. Integrate:

$$\int = \tfrac{2}{9}\log|x-1| + \tfrac{1}{3}\cdot\left(-\tfrac{1}{x-1}\right) - \tfrac{2}{9}\log|x+2|.$$

Step 6. Simplify:

$$\tfrac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C.$$

Final Answer: $\dfrac{2}{9}\log\left|\dfrac{x-1}{x+2}\right| - \dfrac{1}{3(x-1)} + C$

EXPERT'S SOLUTION *: Diya Kapoor, M.Sc Mathematics, IIT Bombay*

Strategic angle. Repeated factor needs two terms; the squared term contributes $-1/(x-a)$ on integration.

Step 1. Set up $A/(x-1) + B/(x-1)^2 + C/(x+2)$.

Step 2. Cover-up for B ($x = 1 \Rightarrow B = 1/3$) and C ($x = -2 \Rightarrow C = -2/9$).

Step 3. Match x^2 coefficients: $A = 2/9$.

Step 4. Integrate.

Final Answer: $\frac{2}{9}\log\left|\frac{x-1}{x+2}\right| - \frac{1}{3(x-1)} + C$

Q 7.9 **Integrate** $\dfrac{3x+5}{x^3-x^2-x+1}$.

SOLUTION

Concept used. Factor the denominator by grouping, then apply partial fractions.

Step 1. Factor by grouping:

$$x^3 - x^2 - x + 1 = x^2(x-1) - (x-1)$$
$$= (x-1)(x^2-1) = (x-1)^2(x+1).$$

Step 2. Decompose:

$$\frac{3x+5}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}.$$

Step 3. Multiply through: $3x+5 = A(x-1)(x+1) + B(x+1) + C(x-1)^2$.

Step 4. $x=1$: $8 = 2B \Rightarrow B = 4$. $x=-1$: $2 = 4C \Rightarrow C = 1/2$. Coefficient of x^2: $0 = A + C \Rightarrow A = -1/2$.

Step 5. Integrate term-by-term and combine the logs:

$$-\tfrac{1}{2}\log|x-1| - \frac{4}{x-1} + \tfrac{1}{2}\log|x+1| = \tfrac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1}.$$

Final Answer: $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$

EXPERT'S SOLUTION *: Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Factor by grouping first; the result is a repeated factor case.

Step 1. $x^3 - x^2 - x + 1 = (x-1)^2(x+1)$.

Step 2. $A = -1/2$, $B = 4$, $C = 1/2$.

Step 3. Integrate, combine logs.

Final Answer: $\frac{1}{2}\log\left|\frac{x+1}{x-1}\right| - \frac{4}{x-1} + C$

Q 7.10 **Integrate** $\dfrac{2x-3}{(x^2-1)(2x+3)}$.

SOLUTION

Concept used. Factor $x^2-1=(x-1)(x+1)$. Three distinct linear factors.

Step 1. Decompose:

$$\frac{2x-3}{(x-1)(x+1)(2x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{2x+3}.$$

Step 2. Multiply through:
$2x-3 = A(x+1)(2x+3) + B(x-1)(2x+3) + C(x-1)(x+1)$.

Step 3. $x=1$: $-1 = A\cdot 2\cdot 5 = 10A \Rightarrow A=-1/10$. $x=-1$:
$-5 = B(-2)(1) = -2B \Rightarrow B = 5/2$. $x=-3/2$:
$-3-3 = C(-5/2)(-1/2) = (5/4)C \Rightarrow -6 = (5/4)C \Rightarrow C = -24/5$.

Step 4. Integrate (note $\int \frac{dx}{2x+3} = \frac{1}{2}\log|2x+3|$):

$$-\tfrac{1}{10}\log|x-1| + \tfrac{5}{2}\log|x+1| - \tfrac{12}{5}\log|2x+3| + C.$$

Final Answer: $-\dfrac{1}{10}\log|x-1| + \dfrac{5}{2}\log|x+1| - \dfrac{12}{5}\log|2x+3| + C$

EXPERT'S SOLUTION *: Tara Bhat, M.Sc Mathematics, ISI Kolkata*

Quick reading. Three linear factors; cover-up rule.

Step 1. Factor; set up $A/(x-1) + B/(x+1) + C/(2x+3)$.

Step 2. $A=-1/10$, $B=5/2$, $C=-24/5$.

Step 3. Anti derivative for $C/(2x+3)$: $(C/2)\log|2x+3|$.

Final Answer: $-\frac{1}{10}\log|x-1| + \frac{5}{2}\log|x+1| - \frac{12}{5}\log|2x+3| + C$

Q 7.11 **Integrate** $\dfrac{5x}{(x+1)(x^2-4)}$.

SOLUTION

Concept used. $x^2-4=(x-2)(x+2)$. Three distinct linear factors.

Step 1. $\dfrac{5x}{(x+1)(x-2)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x-2}+\dfrac{C}{x+2}$.

Step 2. Multiply through: $5x=A(x-2)(x+2)+B(x+1)(x+2)+C(x+1)(x-2)$.

Step 3. $x=-1$: $-5=A(-3)(1)=-3A\Rightarrow A=5/3$. $x=2$: $10=B(3)(4)=12B\Rightarrow B=5/6$. $x=-2$: $-10=C(-1)(-4)=4C\Rightarrow C=-5/2$.

Step 4. Integrate:

$$\tfrac{5}{3}\log|x+1|+\tfrac{5}{6}\log|x-2|-\tfrac{5}{2}\log|x+2|+C.$$

Final Answer: $\dfrac{5}{3}\log|x+1|+\dfrac{5}{6}\log|x-2|-\dfrac{5}{2}\log|x+2|+C$

EXPERT'S SOLUTION : *Krishna Rao, M.Sc Mathematics, IIT Bombay*

Quick reading. Three linear factors.

Step 1. $A=5/3$, $B=5/6$, $C=-5/2$.

Step 2. Integrate.

Final Answer: $\dfrac{5}{3}\log|x+1|+\dfrac{5}{6}\log|x-2|-\dfrac{5}{2}\log|x+2|+C$

Q 7.12 **Integrate** $\dfrac{x^3+x+1}{x^2-1}$.

SOLUTION

Concept used. Numerator degree $3>$ denominator degree 2. Divide first.

Step 1. Long division: x^3+x+1 divided by x^2-1. Quotient: x (since $x\cdot(x^2-1)=x^3-x$). Remainder: $(x^3+x+1)-(x^3-x)=2x+1$. So $\dfrac{x^3+x+1}{x^2-1}=x+\dfrac{2x+1}{x^2-1}$.

Step 2. Decompose the remainder: $\dfrac{2x+1}{(x-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}$.

$2x+1 = A(x+1) + B(x-1)$. $x=1$: $3 = 2A \Rightarrow A = 3/2$. $x=-1$: $-1 = -2B \Rightarrow B = 1/2$.

Step 3. Integrate:

$$\int = \int x\,dx + \tfrac{3}{2}\int \frac{dx}{x-1} + \tfrac{1}{2}\int \frac{dx}{x+1} = \tfrac{x^2}{2} + \tfrac{3}{2}\log|x-1| + \tfrac{1}{2}\log|x+1| + C.$$

Final Answer: $\dfrac{x^2}{2} + \dfrac{3}{2}\log|x-1| + \dfrac{1}{2}\log|x+1| + C$

EXPERT'S SOLUTION *: Aditi Verma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Improper rational $\Rightarrow$ divide, then partial-fraction the remainder.

Step 1. Quotient x; remainder $2x+1$.

Step 2. $\frac{2x+1}{x^2-1}$: $A = 3/2$, $B = 1/2$.

Step 3. Integrate.

Final Answer: $\frac{x^2}{2} + \frac{3}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C$

Q 7.13 Integrate $\dfrac{2}{(1-x)(1+x^2)}$.

SOLUTION

Concept used. Linear $\times$ irreducible quadratic.

Step 1. Decompose: $\dfrac{2}{(1-x)(1+x^2)} = \dfrac{A}{1-x} + \dfrac{Bx+C}{1+x^2}$. Multiply through: $2 = A(1+x^2) + (Bx+C)(1-x)$.

Step 2. Expand: $A + Ax^2 + Bx - Bx^2 + C - Cx = (A-B)x^2 + (B-C)x + (A+C)$.

Step 3. Match with $0\cdot x^2 + 0\cdot x + 2$: $A - B = 0$, $B - C = 0$, $A + C = 2$. So $A = B = C$, and $A + A = 2 \Rightarrow A = 1$. Hence $A = B = C = 1$.

Step 4. Decomposition:

$$\frac{2}{(1-x)(1+x^2)} = \frac{1}{1-x} + \frac{x+1}{1+x^2} = \frac{1}{1-x} + \frac{x}{1+x^2} + \frac{1}{1+x^2}.$$

Step 5. Integrate:

$$\int \frac{dx}{1-x} = -\log|1-x|,$$
$$\int \frac{x\,dx}{1+x^2} = \tfrac{1}{2}\log(1+x^2),$$
$$\int \frac{dx}{1+x^2} = \tan^{-1} x.$$

Step 6. Combine:

$$-\log|1-x| + \tfrac{1}{2}\log(1+x^2) + \tan^{-1} x + C.$$

Final Answer: $-\log|1-x| + \dfrac{1}{2}\log(1+x^2) + \tan^{-1} x + C$

EXPERT'S SOLUTION *: Rohit Desai, M.Sc Mathematics, IIT Bombay*

Quick reading. Linear + quadratic; three constants.

Step 1. $A = B = C = 1$.

Step 2. Integrate.

Final Answer: $-\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1} x + C$

Q 7.14 **Integrate** $\dfrac{3x-1}{(x+2)^2}$.

SOLUTION

Concept used. Single repeated factor.

Step 1. Write $3x - 1 = 3(x+2) - 7$ (so the numerator matches the structure $A(x+2) + B$): $3(x+2) - 7 = 3x + 6 - 7 = 3x - 1$. ✓

Step 2. Split:

$$\frac{3x-1}{(x+2)^2} = \frac{3(x+2)-7}{(x+2)^2} = \frac{3}{x+2} - \frac{7}{(x+2)^2}.$$

Step 3. Integrate:

$$3\log|x+2| - 7\cdot\left(-\frac{1}{x+2}\right) = 3\log|x+2| + \frac{7}{x+2} + C.$$

Final Answer: $3\log|x+2| + \dfrac{7}{x+2} + C$

EXPERT'S SOLUTION : *Aditya Kumar, Ph.D Mathematics, IIT Delhi*

Quick reading. Write numerator as $3(x+2)-7$.

Step 1. Split into $3/(x+2) - 7/(x+2)^2$.

Step 2. Integrate.

Final Answer: $3\log|x+2| + \dfrac{7}{x+2} + C$

Q 7.15 **Integrate** $\dfrac{1}{x^4-1}$.

SOLUTION

Concept used. Factor $x^4-1 = (x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$.

Step 1. Decompose:

$$\frac{1}{(x-1)(x+1)(x^2+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}.$$

Step 2. Multiply through:
$1 = A(x+1)(x^2+1) + B(x-1)(x^2+1) + (Cx+D)(x-1)(x+1)$.

Step 3. $x=1$: $1 = A(2)(2) = 4A \Rightarrow A = 1/4$. $x=-1$:
$1 = B(-2)(2) = -4B \Rightarrow B = -1/4$.

Step 4. Expand and compare. Coefficient of x^3:
$A+B+C = 1/4 - 1/4 + C = 0 \Rightarrow C = 0$. Coefficient of x^0:
$A - B - D = 1/4 + 1/4 - D = 1 \Rightarrow D = -1/2$.

Step 5. Decomposition:

$$\frac{1}{x^4-1} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x^2+1)}.$$

Step 6. Integrate and combine the logs:

$$\tfrac{1}{4}\log\left|\frac{x-1}{x+1}\right| - \tfrac{1}{2}\tan^{-1}x + C.$$

Final Answer: $\dfrac{1}{4}\log\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\tan^{-1}x + C$

EXPERT'S SOLUTION *: Ananya Bhat, M.Sc Mathematics, IIT Bombay*

Strategic angle. Difference of fourth powers factors as $(x^2-1)(x^2+1) = (x-1)(x+1)(x^2+1)$.

Step 1. Decompose with $A, B, Cx + D$.

Step 2. $A = 1/4$, $B = -1/4$, $C = 0$, $D = -1/2$.

Step 3. Integrate.

Final Answer: $\frac{1}{4}\log\left|\dfrac{x-1}{x+1}\right| - \frac{1}{2}\tan^{-1}x + C$

Q 7.16 **Integrate** $\dfrac{1}{x(x^n+1)}$ ***[Hint: multiply numerator and denominator by x^{n-1} and put $x^n = t$].***

SOLUTION

Concept used. Substitute $t = x^n$. Multiplying top and bottom by x^{n-1} gives a numerator $x^{n-1}\,dx$ which equals dt/n.

Step 1. Multiply numerator and denominator by x^{n-1}:

$$\frac{1}{x(x^n+1)} = \frac{x^{n-1}}{x^n(x^n+1)}.$$

Step 2. Put $t = x^n \Rightarrow dt = nx^{n-1}\,dx \Rightarrow x^{n-1}\,dx = dt/n$.

Step 3. Substitute:

$$\int \frac{x^{n-1}\,dx}{x^n(x^n+1)} = \tfrac{1}{n}\int \frac{dt}{t(t+1)}.$$

Step 4. Decompose: $\dfrac{1}{t(t+1)} = \dfrac{1}{t} - \dfrac{1}{t+1}$. Hence

$$\tfrac{1}{n}\left[\log|t| - \log|t+1|\right] = \tfrac{1}{n}\log\left|\frac{t}{t+1}\right| = \tfrac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C.$$

Final Answer: $\dfrac{1}{n}\log\left|\dfrac{x^n}{x^n+1}\right| + C$

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Apply the hint exactly; the substitution makes the decomposition trivial.

Step 1. Multiply top/bottom by x^{n-1}.

Step 2. $t = x^n$; $x^{n-1}\,dx = dt/n$.

Step 3. $\int dt/(t(t+1)) = \log|t/(t+1)|$.

Final Answer: $\frac{1}{n}\log\left|\frac{x^n}{x^n+1}\right| + C$

Q 7.17 **Integrate** $\frac{\cos x}{(1-\sin x)(2-\sin x)}$ ***[Hint:*** $\sin x = t$***].***

SOLUTION

Concept used. Substitute $t = \sin x$, $dt = \cos x\,dx$. The integrand becomes a rational function in t ready for partial fractions.

Step 1. $t = \sin x$; $dt = \cos x\,dx$.

Step 2. Integral becomes $\int \frac{dt}{(1-t)(2-t)}$.

Step 3. Decompose:

$$\frac{1}{(1-t)(2-t)} = \frac{A}{1-t} + \frac{B}{2-t}.$$

$1 = A(2-t) + B(1-t)$. $t = 1$: $A = 1$. $t = 2$: $B = -1$.

Step 4. Integrate:

$$\int \frac{dt}{1-t} - \int \frac{dt}{2-t} = -\log|1-t| + \log|2-t| = \log\left|\frac{2-t}{1-t}\right|.$$

Step 5. Back-substitute: $\log\left|\frac{2-\sin x}{1-\sin x}\right| + C$.

Final Answer: $\log\left|\frac{2-\sin x}{1-\sin x}\right| + C$

EXPERT'S SOLUTION : *Sneha Iyer, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. Substitute $\sin x = t$; partial fractions in t.

Step 1. $t = \sin x$.

Step 2. $A = 1$, $B = -1$.

Step 3. $\log|(2-t)/(1-t)|$.

Final Answer: $\log\left|\dfrac{2-\sin x}{1-\sin x}\right| + C$

Q 7.18 Integrate $\dfrac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)}$.

SOLUTION

Concept used. Substitute $y = x^2$ to make the rational function clearer; the numerator and denominator have the same degree in y, so divide first.

Step 1. Treat $y = x^2$: $\dfrac{(y+1)(y+2)}{(y+3)(y+4)} = \dfrac{y^2+3y+2}{y^2+7y+12}$.

Step 2. Divide: $\dfrac{y^2+3y+2}{y^2+7y+12} = 1 - \dfrac{4y+10}{y^2+7y+12}$.

Step 3. Decompose the remainder: $y^2+7y+12 = (y+3)(y+4)$, so $\dfrac{4y+10}{(y+3)(y+4)} = \dfrac{A}{y+3} + \dfrac{B}{y+4}$. $4y+10 = A(y+4) + B(y+3)$. $y=-3$: $-2 = A \Rightarrow A = -2$. $y = -4$: $-6 = -B \Rightarrow B = 6$.

Step 4. Substitute back $y = x^2$:

$$\frac{(x^2+1)(x^2+2)}{(x^2+3)(x^2+4)} = 1 + \frac{2}{x^2+3} - \frac{6}{x^2+4}.$$

(Note the sign: $1 - (-2/(y+3) + 6/(y+4)) = 1 + 2/(y+3) - 6/(y+4)$.)

Step 5. Integrate:

$$\int 1\,dx + 2\int \frac{dx}{x^2+3} - 6\int \frac{dx}{x^2+4} = x + \tfrac{2}{\sqrt{3}}\tan^{-1}\left(\tfrac{x}{\sqrt{3}}\right) - 6\cdot\tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right).$$

Step 6. Simplify:

$$x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\frac{x}{2}\right) + C.$$

Final Answer: $x + \dfrac{2}{\sqrt{3}}\tan^{-1}\left(\dfrac{x}{\sqrt{3}}\right) - 3\tan^{-1}\left(\dfrac{x}{2}\right) + C$

EXPERT'S SOLUTION : *Meera Banerjee, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Treat x^2 as a single variable y; decompose; substitute back.

Step 1. $y = x^2$. Divide to make proper. Decompose $4y + 10$ over $(y+3)(y+4)$: $A = -2, B = 6$.

Step 2. Substitute $y = x^2$, leading to $1 + 2/(x^2+3) - 6/(x^2+4)$.

Step 3. Integrate.

Final Answer: $x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x}{\sqrt{3}} - 3 \tan^{-1} \frac{x}{2} + C$

Q 7.19 **Integrate** $\frac{2x}{(x^2+1)(x^2+3)}$.

SOLUTION

Concept used. Substitute $y = x^2$, decompose in y, then substitute $t = x^2 + 1$ or similar for the log integrals.

Step 1. Use $y = x^2$: $\frac{2x}{(y+1)(y+3)} \cdot dx$. Note $2x\,dx = dy$.

Step 2. $\int \frac{dy}{(y+1)(y+3)}$. Decompose: $\frac{1}{(y+1)(y+3)} = \frac{A}{y+1} + \frac{B}{y+3}$, $1 = A(y+3) + B(y+1)$. $y = -1$: $1 = 2A \Rightarrow A = 1/2$. $y = -3$: $1 = -2B \Rightarrow B = -1/2$.

Step 3. Integrate and combine the logs:

$$\tfrac{1}{2} \log \left| \frac{y+1}{y+3} \right|.$$

Step 4. Substitute back $y = x^2$:

$$\tfrac{1}{2} \log \left| \frac{x^2+1}{x^2+3} \right| + C.$$

Final Answer: $\frac{1}{2} \log \left(\frac{x^2+1}{x^2+3} \right) + C$

EXPERT'S SOLUTION *: Priya Sharma, Ph.D Mathematics, IIT Delhi*

Quick reading. $2x\,dx = d(x^2)$. Decompose in $y = x^2$.

Step 1. $A = 1/2$, $B = -1/2$.

Step 2. $\frac{1}{2}\log|(x^2+1)/(x^2+3)|$.

Final Answer: $\dfrac{1}{2}\log\left(\dfrac{x^2+1}{x^2+3}\right) + C$

Q 7.20 **Integrate** $\dfrac{1}{x(x^4-1)}$.

SOLUTION

Concept used. Apply the hint of Q16: multiply by x^3/x^3, substitute $t = x^4$.

Step 1. Multiply top and bottom by x^3: $\dfrac{1}{x(x^4-1)} = \dfrac{x^3}{x^4(x^4-1)}$.

Step 2. Put $t = x^4 \Rightarrow dt = 4x^3\,dx \Rightarrow x^3\,dx = dt/4$.

Step 3. Substitute:

$$\int \frac{x^3\,dx}{x^4(x^4-1)} = \tfrac{1}{4}\int \frac{dt}{t(t-1)}.$$

Step 4. Decompose: $\dfrac{1}{t(t-1)} = -\dfrac{1}{t} + \dfrac{1}{t-1}$ (verify by combining the right-hand side).

Step 5. Integrate and combine:

$$\tfrac{1}{4}\log\left|\frac{t-1}{t}\right|.$$

Step 6. Back-substitute:

$$\tfrac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C.$$

Final Answer: $\dfrac{1}{4}\log\left|\dfrac{x^4-1}{x^4}\right| + C$

EXPERT'S SOLUTION *: Aanya Singh, M.Sc Mathematics, IIT Bombay*

Strategic angle. Multiply by x^3/x^3 to fit the $x^n = t$ substitution template.

Step 1. $t = x^4$; $x^3\,dx = dt/4$.

Step 2. $\frac{1}{4}\int dt/(t(t-1)) = \frac{1}{4}\log|(t-1)/t|$.

Final Answer: $\frac{1}{4}\log\left|\frac{x^4-1}{x^4}\right| + C$

Q 7.21 **Integrate** $\frac{1}{e^x-1}$ ***[Hint:*** $e^x = t$***].***

SOLUTION

Concept used. Substitute $t = e^x$; then $dt = e^x\,dx$, so $dx = dt/t$.

Step 1. $t = e^x$; $dx = dt/t$.

Step 2. Substitute:

$$\int \frac{dx}{e^x-1} = \int \frac{1}{t-1}\cdot\frac{dt}{t} = \int \frac{dt}{t(t-1)}.$$

Step 3. Decompose: $\frac{1}{t(t-1)} = -\frac{1}{t} + \frac{1}{t-1}$.

Step 4. Integrate:

$$-\log|t| + \log|t-1| = \log\left|\frac{t-1}{t}\right|.$$

Step 5. Back-substitute: $\log\left|\frac{e^x-1}{e^x}\right| = \log|e^x-1| - x + C$.

Final Answer: $\log|e^x-1| - x + C$

EXPERT'S SOLUTION *: Dev Chatterjee, B.Tech CSE, IIT Roorkee*

Quick reading. $e^x = t$; partial fractions in t.

Step 1. $t = e^x$.

Step 2. $\int dt/(t(t-1)) = \log|(t-1)/t|$.

Step 3. Back-substitute and simplify $-\log e^x = -x$.

Final Answer: $\log|e^x-1| - x + C$

Q 7.22 $\int \frac{x\,dx}{(x-1)(x-2)}$ **equals**

(A) $\log\left|\frac{(x-1)^2}{x-2}\right| + C$ **(B)** $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

(C) $\log\left|\frac{x-1}{x-2}\right|^2 + C$ **(D)** $\log|(x-1)(x-2)| + C$

SOLUTION

Concept used. Partial fractions.

Step 1. $\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$. Cover-up: $A = 1/(1-2) = -1$. $B = 2/(2-1) = 2$.

Step 2. Integrate: $-\log|x-1| + 2\log|x-2| = \log\left|\frac{(x-2)^2}{x-1}\right| + C$.

Step 3. Matches option (B).

Final Answer: Option (B): $\log\left|\frac{(x-2)^2}{x-1}\right| + C$

EXPERT'S SOLUTION *: Aditi Verma, M.Sc Mathematics, IIT Bombay*

Quick reading. Cover-up: $A = -1$, $B = 2$.

Step 1. Integrate: $\log|(x-2)^2/(x-1)|$.

Final Answer: Option (B)

Q 7.23 $\int \frac{dx}{x(x^2+1)}$ **equals**

(A) $\log|x| - \frac{1}{2}\log(x^2+1) + C$ **(B)** $\log|x| + \frac{1}{2}\log(x^2+1) + C$

(C) $-\log|x| + \frac{1}{2}\log(x^2+1) + C$ **(D)** $\frac{1}{2}\log|x| + \log(x^2+1) + C$

SOLUTION

Concept used. Decompose $\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$.

Step 1. Multiply through: $1 = A(x^2+1) + (Bx+C)x = (A+B)x^2 + Cx + A$.

Step 2. Equate coefficients: $A = 1$ (constant), $A + B = 0 \Rightarrow B = -1$, $C = 0$.

Step 3. Decomposition: $\frac{1}{x} - \frac{x}{x^2+1}$.

Step 4. Integrate:

$$\log|x| - \tfrac{1}{2}\log(x^2+1) + C.$$

Matches option (A).

Final Answer: Option (A): $\log|x| - \frac{1}{2}\log(x^2+1) + C$

EXPERT'S SOLUTION : *Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. $A = 1$, $B = -1$, $C = 0$.

Step 1. Decomposition.

Step 2. Integrate.

Final Answer: Option (A)

Key Takeaways

- Always check whether the rational function is proper. If degree of numerator $\geq$ degree of denominator, divide first.
- Distinct linear factors $\to A/(x-a) + B/(x-b) + \cdots$, one constant per factor.
- Repeated linear factor $(x-a)^k \to$ a sum $A_1/(x-a) + A_2/(x-a)^2 + \cdots + A_k/(x-a)^k$.
- Irreducible quadratic factor $x^2 + bx + c \to$ a single $(Bx + C)/(x^2 + bx + c)$ piece (splits further into a log and a $\tan^{-1}$).
- Cover-up rule: to find the constant attached to $1/(x-a)$ in the case of distinct linear factors, plug $x = a$ into the rest.

End of Exercise 7.5