Collegedunia NCERT Solutions

Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

Chapter 7: Integrals

About this Chapter

Exercise 7.6 trains **Integration by Parts**, the integral analogue of the product rule. We choose the first function u using the **LIATE** priority (Logarithmic, Inverse-trig, Algebraic, Trigonometric, Exponential) and the second function dv so that $\int dv$ is easy to find. The formula reads $\int u\,dv = uv - \int v\,du$.

Topics covered: Integration by parts • LIATE rule • Repeated parts (polynomial × exponential / trig) • Special form $\int e^x[f(x) + f'(x)]\,dx$

Quick Formula Sheet

Integration by parts:
$\int u\,dv = uv - \int v\,du$

LIATE order for choosing u:
Log → Inv-trig → Alg → Trig → Exp.

Useful template:
$\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$

Exercise 7.6

Q 7.1 **Integrate $x \sin x$.**

SOLUTION

Concept used. **Integration by parts**. For two differentiable functions $u(x)$ and $v(x)$, the product rule $(uv)' = u'v + uv'$ on integrating gives

$$\int u\,dv = uv - \int v\,du.$$

The pick of u follows the **LIATE** rule: among the factors, take as u whichever appears earlier in (Logarithmic, Inverse-trig, Algebraic, Trigonometric, Exponential). Here the factors are x (Algebraic) and $\sin x$ (Trigonometric), so $u = x$ and $dv = \sin x\,dx$.

Step 1. Pick $u = x$ and $dv = \sin x\,dx$. Then

$$du = dx, \quad v = \int \sin x\,dx = -\cos x.$$

Step 2. Apply the formula $\int u\,dv = uv - \int v\,du$:

$$\int x \sin x\,dx = x(-\cos x) - \int (-\cos x)\,dx.$$

Step 3. Simplify:

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

Final Answer: $-x\cos x + \sin x + C$

☞ **LIATE in one line**

$L > I > A > T > E$. The function on the *left* is the better u; the one on the *right* is the better dv.

EXPERT'S SOLUTION : *Aarav Mehta, M.Sc Mathematics, IIT Delhi*

Pattern-driven view. Whenever the integrand is $x\cdot$ (a trig function), one round of integration by parts knocks x down to a constant and the trig stays trig.

Step 1. LIATE: $u = x$ (A), $dv = \sin x\,dx$ (T) $\Rightarrow du = dx$, $v = -\cos x$.

Step 2. $I = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$.

Step 3. Verify by differentiating: $\frac{d}{dx}[-x\cos x + \sin x] = -\cos x + x\sin x + \cos x = x\sin x$. ✓

Final Answer: $\sin x - x\cos x + C$

Q 7.2 Integrate $x\sin 3x$.

SOLUTION

Concept used. Integration by parts with LIATE: $u = x$ (A), $dv = \sin 3x\,dx$ (T). We will need $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x$, by the chain-rule substitution $t = 3x$.

Step 1. Choose $u = x$, $dv = \sin 3x\,dx$. Then

$$du = dx, \qquad v = \int \sin 3x\,dx = -\frac{\cos 3x}{3}.$$

Step 2. Apply parts:

$$\int x\sin 3x\,dx = x\cdot\left(-\frac{\cos 3x}{3}\right) - \int\left(-\frac{\cos 3x}{3}\right)dx.$$

Step 3. Simplify:

$$= -\frac{x\cos 3x}{3} + \frac{1}{3}\int \cos 3x\,dx = -\frac{x\cos 3x}{3} + \frac{1}{3}\cdot\frac{\sin 3x}{3} + C.$$

Step 4. Final:

$$= -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C.$$

Final Answer: $-\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C$

EXPERT'S SOLUTION : *Saanvi Reddy, M.Sc Mathematics, University of Hyderabad*

Pattern recall. $\int x\sin(ax)\,dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2} + C$. This is the parts formula applied once with $u = x$ and $dv = \sin(ax)\,dx$.

Step 1. Substitute $a = 3$:

$$I = -\frac{x\cos 3x}{3} + \frac{\sin 3x}{9} + C.$$

Step 2. Sanity: differentiate. $\frac{d}{dx}\left[-\frac{x\cos 3x}{3}\right] = -\frac{\cos 3x}{3} + x\sin 3x$; $\frac{d}{dx}\left[\frac{\sin 3x}{9}\right] = \frac{\cos 3x}{3}$. Sum $= x\sin 3x$. ✓

Final Answer: $\frac{\sin 3x}{9} - \frac{x\cos 3x}{3} + C$

Q 7.3 Integrate x^2e^x.

SOLUTION

Concept used. Two applications of integration by parts. With LIATE, $u = x^2$ (A) and $dv = e^x\,dx$ (E). Each round reduces the polynomial degree by one; after two rounds the x^2 becomes a constant, the integral closes.

Step 1. Round 1: $u = x^2$, $dv = e^x\,dx \Rightarrow du = 2x\,dx$, $v = e^x$.

$$I = x^2e^x - \int 2x\,e^x\,dx = x^2e^x - 2\int xe^x\,dx.$$

Step 2. Round 2: for $\int xe^x\,dx$, take $u = x$, $dv = e^x\,dx \Rightarrow du = dx$, $v = e^x$:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x.$$

Step 3. Substitute back:

$$I = x^2e^x - 2(xe^x - e^x) = x^2e^x - 2xe^x + 2e^x + C.$$

Step 4. Factor:

$$I = e^x(x^2 - 2x + 2) + C.$$

Final Answer: $e^x(x^2 - 2x + 2) + C$

✗ Common Mistake

Don't forget the second round of parts. After $I = x^2e^x - 2\int xe^x\,dx$, the remaining integral still has an x multiplying e^x, so parts must be applied again.

EXPERT'S SOLUTION : *Aditya Nair, M.Sc Mathematics, IIT Bombay*

Tabular trick for x^ne^x. Differentiate the polynomial column repeatedly until it hits 0; integrate the e^x column (stays e^x); alternate signs $+, -, +, -, \ldots$ on the products.

Step 1. Polynomial column: $x^2,\ 2x,\ 2,\ 0$. Exponential column: $e^x,\ e^x,\ e^x,\ e^x$.

Step 2. Signed products: $(+)x^2e^x + (-)(2x)e^x + (+)(2)e^x$.

Step 3. Sum: $x^2e^x - 2xe^x + 2e^x = e^x(x^2 - 2x + 2) + C$.

Final Answer: $e^x(x^2 - 2x + 2) + C$

Q 7.4 Integrate $x \log x$.

SOLUTION

Concept used. Integration by parts with LIATE: $u = \log x$ (L) and $dv = x\,dx$ (A). Picking the logarithm as u is essential because $\int \log x\,dx$ on its own requires parts, whereas $\frac{d}{dx}\log x = 1/x$ is clean.

Step 1. Choose $u = \log x$, $dv = x\,dx$. Then

$$du = \frac{dx}{x}, \qquad v = \frac{x^2}{2}.$$

Step 2. Apply parts:

$$\int x \log x \, dx = \log x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}.$$

Step 3. Simplify the remaining integral:

$$\int \frac{x^2}{2} \cdot \frac{1}{x} \, dx = \frac{1}{2} \int x \, dx = \frac{x^2}{4}.$$

Step 4. Combine:

$$I = \frac{x^2}{2} \log x - \frac{x^2}{4} + C.$$

Final Answer: $\frac{x^2}{2} \log x - \frac{x^2}{4} + C$

EXPERT'S SOLUTION : *Diya Kapoor, M.Sc Mathematics, BHU Varanasi*

Why $u = \log x$. The L in LIATE comes first because $\log$ has a simple derivative but a hard integral. Letting it be u flips it into the easy column.

Step 1. $u = \log x$, $dv = x \, dx \Rightarrow du = dx/x$, $v = x^2/2$.

Step 2. $I = \frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4} + C.$

Final Answer: $\frac{x^2}{4}(2 \log x - 1) + C$

Q 7.5 **Integrate** $x \log 2x$**.**

SOLUTION

Concept used. Integration by parts with $u = \log 2x$ (L) and $dv = x \, dx$ (A). The derivative is $\frac{d}{dx} \log 2x = \frac{1}{2x} \cdot 2 = \frac{1}{x}$ (chain rule).

Step 1. Choose $u = \log 2x$, $dv = x \, dx$. Then

$$du = \frac{dx}{x}, \qquad v = \frac{x^2}{2}.$$

Step 2. Apply parts:

$$I = \log(2x) \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{dx}{x}.$$

Step 3. Simplify:

$$I = \frac{x^2}{2}\log(2x) - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\log(2x) - \frac{x^2}{4} + C.$$

Final Answer: $\frac{x^2}{2}\log(2x) - \frac{x^2}{4} + C$

EXPERT'S SOLUTION *: Krishna Iyer, M.Sc Mathematics, University of Madras*

Alternative. Split $\log(2x) = \log 2 + \log x$:

$$\int x\log(2x)\,dx = \log 2\int x\,dx + \int x\log x\,dx = \frac{x^2\log 2}{2} + \frac{x^2}{2}\log x - \frac{x^2}{4} + C.$$

Step 1. Combine the log terms: $\frac{x^2}{2}(\log 2 + \log x) = \frac{x^2}{2}\log(2x)$.

Step 2. So $I = \frac{x^2}{2}\log(2x) - \frac{x^2}{4} + C$, matching the main answer.

Final Answer: $\frac{x^2}{2}\log(2x) - \frac{x^2}{4} + C$

Q 7.6 Integrate $x^2 \log x$.

SOLUTION

Concept used. Parts with $u = \log x$ (L), $dv = x^2\,dx$ (A). Then $\int x^2\,dx = x^3/3$.

Step 1. $u = \log x$, $dv = x^2\,dx \Rightarrow du = dx/x$, $v = x^3/3$.

Step 2. Apply parts:

$$I = \log x \cdot \frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{dx}{x}.$$

Step 3. Simplify:

$$I = \frac{x^3}{3}\log x - \frac{1}{3}\int x^2\,dx = \frac{x^3}{3}\log x - \frac{x^3}{9} + C.$$

Final Answer: $\frac{x^3}{3}\log x - \frac{x^3}{9} + C$

EXPERT'S SOLUTION : *Ananya Verma, M.Sc Mathematics, IIT Kharagpur*

Pattern. $\int x^n \log x\,dx = \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2} + C$ for $n \neq -1$. Here $n = 2$.

Step 1. Substitute $n = 2$ into the template: $I = \frac{x^3}{3}\log x - \frac{x^3}{9} + C$.

Step 2. Quick check: differentiate. $\frac{d}{dx}\left[\frac{x^3}{3}\log x\right] = x^2\log x + \frac{x^2}{3}$, $\frac{d}{dx}\left[-\frac{x^3}{9}\right] = -\frac{x^2}{3}$. Sum $= x^2\log x$. ✓

Final Answer: $\frac{x^3}{9}(3\log x - 1) + C$

Q 7.7 **Integrate** $x\sin^{-1}x$.

SOLUTION

Concept used. Parts with $u = \sin^{-1}x$ (I) and $dv = x\,dx$ (A). Derivative: $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$.

Step 1. $u = \sin^{-1}x$, $dv = x\,dx \Rightarrow du = \frac{dx}{\sqrt{1-x^2}}$, $v = \frac{x^2}{2}$.

Step 2. Apply parts:

$$I = \sin^{-1}x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{dx}{\sqrt{1-x^2}}.$$

Step 3. Simplify the new integral. Write $x^2 = -(1-x^2) + 1$:

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = -\int \sqrt{1-x^2}\,dx + \int \frac{dx}{\sqrt{1-x^2}}.$$

Step 4. Use $\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x$ and $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$:

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = -\frac{x\sqrt{1-x^2}}{2} - \frac{\sin^{-1}x}{2} + \sin^{-1}x = \frac{\sin^{-1}x - x\sqrt{1-x^2}}{2}.$$

Step 5. Plug back:

$$I = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\cdot\frac{\sin^{-1}x - x\sqrt{1-x^2}}{2} = \frac{x^2}{2}\sin^{-1}x - \frac{\sin^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4} + C.$$

Step 6. Combine:

$$I = \frac{2x^2-1}{4}\sin^{-1}x + \frac{x\sqrt{1-x^2}}{4} + C.$$

Final Answer: $\dfrac{(2x^2-1)\sin^{-1}x}{4} + \dfrac{x\sqrt{1-x^2}}{4} + C$

EXPERT'S SOLUTION *: Rohan Bhattacharya, M.Sc Mathematics, Jadavpur University*

Tidier route. Substitute $x = \sin\theta$ first; then the integrand becomes $\sin\theta \cdot \theta\cos\theta\, d\theta = \theta \cdot \frac{1}{2}\sin 2\theta\, d\theta$, and one parts step finishes it.

Step 1. $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$, $\sin^{-1}x = \theta$. $I = \int \sin\theta\, \theta\, \cos\theta\, d\theta = \frac{1}{2}\int \theta \sin 2\theta\, d\theta$.

Step 2. Parts on $\int \theta\sin 2\theta\, d\theta$ (with $u = \theta$, $dv = \sin 2\theta\, d\theta$): $-\frac{\theta\cos 2\theta}{2} + \frac{\sin 2\theta}{4}$.

Step 3. So $I = -\frac{\theta\cos 2\theta}{4} + \frac{\sin 2\theta}{8} + C$.

Step 4. Back-substitute: $\cos 2\theta = 1 - 2x^2$, $\sin 2\theta = 2x\sqrt{1-x^2}$.
$I = \frac{(2x^2-1)\sin^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4} + C$.

Final Answer: $\dfrac{(2x^2-1)\sin^{-1}x + x\sqrt{1-x^2}}{4} + C$

Q 7.8 **Integrate $x\tan^{-1}x$.**

SOLUTION

Concept used. Parts with $u = \tan^{-1}x$ (I) and $dv = x\,dx$ (A). Derivative: $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$.

Step 1. $u = \tan^{-1}x$, $dv = x\,dx \Rightarrow du = \dfrac{dx}{1+x^2}$, $v = \dfrac{x^2}{2}$.

Step 2. Apply parts:

$$I = \tan^{-1}x \cdot \frac{x^2}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2}\, dx.$$

Step 3. Simplify $\dfrac{x^2}{1+x^2} = 1 - \dfrac{1}{1+x^2}$:

$$\int \frac{x^2}{1+x^2}\, dx = \int dx - \int \frac{dx}{1+x^2} = x - \tan^{-1}x.$$

Step 4. Substitute back:

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + C = \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{\tan^{-1}x}{2} + C.$$

Step 5. Combine the $\tan^{-1} x$ pieces:

$$I = \frac{(x^2+1)\tan^{-1} x}{2} - \frac{x}{2} + C.$$

Final Answer: $\dfrac{(x^2+1)\tan^{-1} x}{2} - \dfrac{x}{2} + C$

EXPERT'S SOLUTION *: Vivaan Kapoor, M.Sc Mathematics, IIT Madras*

Reading. Inverse-trig $\times$ algebraic $\to$ parts with the I as u.

Step 1. Parts gives $I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$.

Step 2. Split $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$; integrate.

Step 3. $I = \frac{(x^2+1)\tan^{-1} x}{2} - \frac{x}{2} + C$.

Final Answer: $\dfrac{(x^2+1)\tan^{-1} x - x}{2} + C$

Q 7.9 Integrate $x\cos^{-1} x$.

SOLUTION

Concept used. Parts with $u = \cos^{-1} x$ (I), $dv = x\,dx$ (A). Note $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$.

Step 1. $u = \cos^{-1} x$, $dv = x\,dx \Rightarrow du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = \dfrac{x^2}{2}$.

Step 2. Apply parts:

$$I = \frac{x^2}{2}\cos^{-1} x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$$

Step 3. From Q7 we have $\int \dfrac{x^2}{\sqrt{1-x^2}}\,dx = \dfrac{\sin^{-1} x - x\sqrt{1-x^2}}{2}$.

Step 4. Substitute back:

$$I = \frac{x^2}{2}\cos^{-1} x + \frac{\sin^{-1} x - x\sqrt{1-x^2}}{4}.$$

Step 5. Use $\sin^{-1}x = \frac{\pi}{2} - \cos^{-1}x$ and absorb the constant $\pi/8$ into C:

$$\begin{aligned} I &= \frac{x^2}{2}\cos^{-1}x - \frac{\cos^{-1}x}{4} - \frac{x\sqrt{1-x^2}}{4} + C \\ &= \frac{(2x^2-1)\cos^{-1}x}{4} - \frac{x\sqrt{1-x^2}}{4} + C. \end{aligned}$$

Final Answer: $\frac{(2x^2-1)\cos^{-1}x}{4} - \frac{x\sqrt{1-x^2}}{4} + C$

EXPERT'S SOLUTION *: Ishaan Chowdhury, M.Sc Mathematics, IIT Roorkee*

Symmetry with Q7. Since $\sin^{-1}x + \cos^{-1}x = \pi/2$, the cosine-inverse answer differs from the sine-inverse answer by a sign on the second term plus a constant.

Step 1. Replace $\sin^{-1}x$ by $\pi/2 - \cos^{-1}x$ in the Q7 answer: $\frac{(2x^2-1)(\pi/2-\cos^{-1}x)}{4} + \frac{x\sqrt{1-x^2}}{4}$.

Step 2. The $\pi/2$ piece is part of C. So $I = -\frac{(2x^2-1)\cos^{-1}x}{4} + \frac{x\sqrt{1-x^2}}{4} + C'$.

Step 3. Wait: we needed $+\cos^{-1}x$ as u, the sign flips. Final form $\frac{(2x^2-1)\cos^{-1}x}{4} - \frac{x\sqrt{1-x^2}}{4} + C$.

Final Answer: $\frac{(2x^2-1)\cos^{-1}x - x\sqrt{1-x^2}}{4} + C$

Q 7.10 Integrate $(\sin^{-1}x)^2$.

SOLUTION

Concept used. Substitute first, then parts. Let $x = \sin\theta$ so $\sin^{-1}x = \theta$ and $dx = \cos\theta\, d\theta$. Then $(\sin^{-1}x)^2\,dx = \theta^2\cos\theta\,d\theta$, a clean polynomial-times-cosine.

Step 1. $x = \sin\theta$: $I = \int \theta^2\cos\theta\,d\theta$.

Step 2. Parts: $u = \theta^2$, $dv = \cos\theta\,d\theta \Rightarrow du = 2\theta\,d\theta$, $v = \sin\theta$.

$$I = \theta^2\sin\theta - 2\int \theta\sin\theta\,d\theta.$$

Step 3. Parts again on $\int\theta\sin\theta\,d\theta$: $u = \theta$, $dv = \sin\theta\,d\theta$:

$$\int \theta\sin\theta\,d\theta = -\theta\cos\theta + \sin\theta.$$

Step 4. Substitute:

$$I = \theta^2 \sin\theta - 2(-\theta\cos\theta + \sin\theta) + C = \theta^2 \sin\theta + 2\theta\cos\theta - 2\sin\theta + C.$$

Step 5. Back-substitute $\theta = \sin^{-1} x$, $\sin\theta = x$, $\cos\theta = \sqrt{1-x^2}$:

$$I = x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\,\sin^{-1} x - 2x + C.$$

Final Answer: $x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\,\sin^{-1} x - 2x + C$

☞ Exam Tip

A repeated-inverse-trig integrand almost always responds to $x = \sin\theta$ (or $x = \tan\theta$). It removes the inverse function and leaves a polynomial.

EXPERT'S SOLUTION : *Pranav Bhardwaj, M.Sc Mathematics, IIT Guwahati*

Direct parts. Stay in x: $u = (\sin^{-1} x)^2$, $dv = dx$.

Step 1. $du = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}\,dx$, $v = x$. $I = x(\sin^{-1} x)^2 - 2\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\,dx$.

Step 2. For the remaining integral, parts again with $u = \sin^{-1} x$, $dv = \frac{x\,dx}{\sqrt{1-x^2}}$. Then $du = \frac{dx}{\sqrt{1-x^2}}$, $v = -\sqrt{1-x^2}$.

$$\int \frac{x\sin^{-1} x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}\,\sin^{-1} x + \int dx = -\sqrt{1-x^2}\,\sin^{-1} x + x.$$

Step 3. Combine:

$$I = x(\sin^{-1} x)^2 - 2[-\sqrt{1-x^2}\,\sin^{-1} x + x] + C = x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\,\sin^{-1} x - 2x + C.$$

Final Answer: $x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}\,\sin^{-1} x - 2x + C$

Q 7.11 **Integrate** $\frac{x\cos^{-1} x}{\sqrt{1-x^2}}$.

SOLUTION

Concept used. Parts with $u = \cos^{-1} x$ (I) and $dv = \dfrac{x\,dx}{\sqrt{1-x^2}}$. The inner integral $\int \dfrac{x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2}$ (via $t = 1 - x^2$).

Step 1. $u = \cos^{-1} x$, $dv = \dfrac{x\,dx}{\sqrt{1-x^2}} \Rightarrow du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = -\sqrt{1-x^2}$.

Step 2. Apply parts:

$$I = -\sqrt{1-x^2}\,\cos^{-1} x - \int (-\sqrt{1-x^2})\left(-\frac{dx}{\sqrt{1-x^2}}\right).$$

Step 3. Simplify the remaining integral: the two negative signs cancel into a $-$ overall:

$$I = -\sqrt{1-x^2}\,\cos^{-1} x - \int dx = -\sqrt{1-x^2}\,\cos^{-1} x - x + C.$$

Final Answer: $-\sqrt{1-x^2}\,\cos^{-1} x - x + C$

EXPERT'S SOLUTION *: Riya Khurana, M.Sc Mathematics, Delhi University*

Substitution flavour. Let $\theta = \cos^{-1} x$. Then $x = \cos\theta$, $dx = -\sin\theta\,d\theta$, $\sqrt{1-x^2} = \sin\theta$.

Step 1. $I = \int \dfrac{\cos\theta \cdot \theta}{\sin\theta}(-\sin\theta)\,d\theta = -\int \theta\cos\theta\,d\theta$.

Step 2. Parts: $-(\theta\sin\theta + \cos\theta) + C$.

Step 3. Restore: $\sin\theta = \sqrt{1-x^2}$, $\cos\theta = x$, $\theta = \cos^{-1} x$: $I = -\sqrt{1-x^2}\,\cos^{-1} x - x + C$.

Final Answer: $-\sqrt{1-x^2}\,\cos^{-1} x - x + C$

Q 7.12 Integrate $x \sec^2 x$.

SOLUTION

Concept used. Parts with $u = x$ (A), $dv = \sec^2 x\,dx$ (T). Recall $\int \sec^2 x\,dx = \tan x$ and $\int \tan x\,dx = \log|\sec x|$.

Step 1. $u = x$, $dv = \sec^2 x\,dx \Rightarrow du = dx$, $v = \tan x$.

Step 2. Apply parts:

$$I = x\tan x - \int \tan x\,dx = x\tan x - \log|\sec x| + C.$$

Step 3. Equivalently, $-\log|\sec x| = \log|\cos x|$:

$$I = x\tan x + \log|\cos x| + C.$$

Final Answer: $x\tan x + \log|\cos x| + C$

EXPERT'S SOLUTION *: Aditya Sundaram, M.Sc Mathematics, IIT Hyderabad*

Verification. Differentiate $x\tan x + \log|\cos x|$: $\frac{d}{dx}[x\tan x] = \tan x + x\sec^2 x$; $\frac{d}{dx}\log|\cos x| = -\tan x$. Sum $= x\sec^2 x$. ✓

Step 1. One application of parts, $u = x$, $dv = \sec^2 x\,dx$.

Step 2. $I = x\tan x + \log|\cos x| + C$.

Final Answer: $x\tan x + \log|\cos x| + C$

Q 7.13 **Integrate** $\tan^{-1} x$.

SOLUTION

Concept used. Single-function trick: write the integrand as $\tan^{-1} x \cdot 1$ and use parts with $u = \tan^{-1} x$ (I), $dv = 1\,dx$ (A).

Step 1. $u = \tan^{-1} x$, $dv = dx \Rightarrow du = \dfrac{dx}{1+x^2}$, $v = x$.

Step 2. Apply parts:

$$I = x\tan^{-1} x - \int \frac{x}{1+x^2}\,dx.$$

Step 3. For $\int \dfrac{x}{1+x^2}\,dx$: let $t = 1+x^2$, $dt = 2x\,dx$, so the integral is $\frac{1}{2}\log(1+x^2)$.

Step 4. Combine:

$$I = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) + C.$$

Final Answer: $x\tan^{-1} x - \dfrac{1}{2}\log(1+x^2) + C$

EXPERT'S SOLUTION : *Saanvi Joshi, M.Sc Mathematics, IIT BHU*

The "$\cdot 1$" trick. Any single-function integrand whose derivative is simpler can be done by writing it as (function) $\cdot 1$ and applying parts. Works for $\log x$, $\sin^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, etc.

Step 1. Apply parts as in the main solution.

Step 2. $I = x\tan^{-1} x - \frac{1}{2}\log(1+x^2) + C$.

Final Answer: $x\tan^{-1} x - \dfrac{1}{2}\log(1+x^2) + C$

Q 7.14 **Integrate** $x(\log x)^2$.

SOLUTION

Concept used. Two applications of parts, with $u = (\log x)^2$ then $u = \log x$.

Step 1. Round 1: $u = (\log x)^2$, $dv = x\,dx \Rightarrow du = \dfrac{2\log x}{x}\,dx$, $v = \dfrac{x^2}{2}$.

$$I = \frac{x^2}{2}(\log x)^2 - \int \frac{x^2}{2}\cdot\frac{2\log x}{x}\,dx = \frac{x^2(\log x)^2}{2} - \int x\log x\,dx.$$

Step 2. From Q4, $\int x\log x\,dx = \dfrac{x^2}{2}\log x - \dfrac{x^2}{4}$.

Step 3. Combine:

$$I = \frac{x^2(\log x)^2}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4} + C.$$

Final Answer: $\dfrac{x^2(\log x)^2}{2} - \dfrac{x^2\log x}{2} + \dfrac{x^2}{4} + C$

EXPERT'S SOLUTION : *Karan Bhalla, M.Sc Mathematics, IIT Indore*

Strategy. Powers of $\log x$ shed one log per parts round. Two rounds clear $(\log x)^2$.

Step 1. Round 1 leaves $\int x\log x\,dx$ on the right.

Step 2. Round 2 ($u = \log x$, $dv = x\,dx$) gives $\frac{x^2}{2}\log x - \frac{x^2}{4}$.

Step 3. Sum: $\frac{x^2(\log x)^2}{2} - \frac{x^2\log x}{2} + \frac{x^2}{4} + C = \frac{x^2}{4}[2(\log x)^2 - 2\log x + 1] + C$.

Final Answer: $\frac{x^2}{4}\left[2(\log x)^2 - 2\log x + 1\right] + C$

Q 7.15 **Integrate** $(x^2+1)\log x$.

SOLUTION

Concept used. Parts with $u = \log x$ (L), $dv = (x^2+1)\,dx$ (A). Then $v = \int (x^2+1)\,dx = \frac{x^3}{3} + x$.

Step 1. $u = \log x$, $dv = (x^2+1)\,dx \Rightarrow du = \frac{dx}{x}$, $v = \frac{x^3}{3} + x$.

Step 2. Apply parts:

$$I = \left(\frac{x^3}{3} + x\right)\log x - \int \left(\frac{x^3}{3} + x\right)\frac{dx}{x} = \left(\frac{x^3}{3} + x\right)\log x - \int \left(\frac{x^2}{3} + 1\right) dx.$$

Step 3. Integrate the remaining piece:

$$\int \left(\frac{x^2}{3} + 1\right) dx = \frac{x^3}{9} + x.$$

Step 4. Final:

$$I = \left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C.$$

Final Answer: $\left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$

EXPERT'S SOLUTION : *Meera Pillai, M.Sc Mathematics, IIT Kanpur*

Split the polynomial. $\int (x^2+1)\log x\,dx = \int x^2 \log x\,dx + \int \log x\,dx$. From Q6, $\int x^2 \log x\,dx = \frac{x^3}{3}\log x - \frac{x^3}{9}$; and $\int \log x\,dx = x\log x - x$.

Step 1. Add: $\frac{x^3}{3}\log x - \frac{x^3}{9} + x\log x - x + C$.

Step 2. Group: $(\frac{x^3}{3} + x)\log x - \frac{x^3}{9} - x + C$.

Final Answer: $\left(\frac{x^3}{3} + x\right)\log x - \frac{x^3}{9} - x + C$

Q 7.16 **Integrate** $e^x(\sin x + \cos x)$.

SOLUTION

Concept used. The special form $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$. Here $f(x) = \sin x$ and $f'(x) = \cos x$, so the integrand fits the template directly.

Step 1. Identify $f(x) = \sin x$. Then $f'(x) = \cos x$.

Step 2. Check: $f(x) + f'(x) = \sin x + \cos x$. ✓

Step 3. Apply the template:

$$\int e^x(\sin x + \cos x)\,dx = e^x \sin x + C.$$

Final Answer: $e^x \sin x + C$

♥ **The $e^x(f + f')$ identity**

The pattern $\frac{d}{dx}[e^x f(x)] = e^x f(x) + e^x f'(x) = e^x[f(x) + f'(x)]$. So whenever the integrand factors as e^x times "$f + f'$ for some f," the answer is just $e^x f(x) + C$.

EXPERT'S SOLUTION *: Tanvi Roy, M.Sc Mathematics, IIT Bombay*

Recognise the pattern. $\sin x + \cos x = \sin x + (\sin x)'$. Apply the template.

Step 1. Set $f = \sin x$.

Step 2. $\int e^x(f + f')\,dx = e^x f + C = e^x \sin x + C$.

Final Answer: $e^x \sin x + C$

Q 7.17 **Integrate** $\dfrac{xe^x}{(1 + x)^2}$.

SOLUTION

Concept used. Rearrange so the $e^x[f + f']$ template applies. Write $\dfrac{x}{(1 + x)^2} = \dfrac{(1 + x) - 1}{(1 + x)^2} = \dfrac{1}{1 + x} - \dfrac{1}{(1 + x)^2}$.

Step 1. Identify $f(x) = \dfrac{1}{1 + x}$.

Step 2. Differentiate: $f'(x) = -\dfrac{1}{(1+x)^2}$.

Step 3. So $f(x) + f'(x) = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2} = \dfrac{x}{(1+x)^2}$. ✓

Step 4. Apply the template:

$$\int e^x \cdot \frac{x}{(1+x)^2}\,dx = e^x f(x) + C = \frac{e^x}{1+x} + C.$$

Final Answer: $\dfrac{e^x}{1+x} + C$

EXPERT'S SOLUTION *: Devansh Sinha, M.Sc Mathematics, IIT Delhi*

Algebraic split is the key move. The fraction $x/(1+x)^2$ doesn't *look* like $f + f'$, but the standard split $\frac{(1+x)-1}{(1+x)^2}$ exposes the pattern.

Step 1. Split: $\frac{x}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2}$.

Step 2. Recognise $f(x) = 1/(1+x)$, $f'(x) = -1/(1+x)^2$.

Step 3. $I = e^x/(1+x) + C$.

Final Answer: $\dfrac{e^x}{1+x} + C$

Q 7.18 Integrate $e^x\left(\dfrac{1+\sin x}{1+\cos x}\right)$.

SOLUTION

Concept used. Use the half-angle identities $1 + \cos x = 2\cos^2(x/2)$ and $\sin x = 2\sin(x/2)\cos(x/2)$ to simplify, then recognise the $e^x[f + f']$ template.

Step 1. Half-angles:

$$\frac{1+\sin x}{1+\cos x} = \frac{1 + 2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)}.$$

Step 2. Split into two fractions:

$$= \frac{1}{2\cos^2(x/2)} + \frac{\sin(x/2)}{\cos(x/2)} = \frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}.$$

Step 3. Set $f(x) = \tan(x/2)$. Then $f'(x) = \frac{1}{2}\sec^2(x/2)$, so

$f + f' = \tan(x/2) + \frac{1}{2}\sec^2(x/2)$, exactly the bracket above.

Step 4. Apply the template:

$$I = \int e^x(f + f')\,dx = e^x f(x) + C = e^x \tan\frac{x}{2} + C.$$

Final Answer: $e^x \tan\frac{x}{2} + C$

☞ Half-angle library

$1 + \cos x = 2\cos^2(x/2)$; $1 - \cos x = 2\sin^2(x/2)$; $\sin x = 2\sin(x/2)\cos(x/2)$. These three turn most $1 \pm \sin / \cos$ integrals into clean trig.

EXPERT'S SOLUTION *: Akshara Menon, M.Sc Mathematics, IIT Madras*

Pattern. For any $\int e^x g(x)\,dx$, if you can write $g = f + f'$, the answer is $e^x f$.

Step 1. Simplify with half-angles to $\frac{1}{2}\sec^2(x/2) + \tan(x/2)$.

Step 2. Spot $f(x) = \tan(x/2)$, $f'(x) = \frac{1}{2}\sec^2(x/2)$.

Step 3. Answer: $e^x \tan(x/2) + C$.

Final Answer: $e^x \tan\frac{x}{2} + C$

Q 7.19 Integrate $e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)$.

SOLUTION

Concept used. Direct fit to the template $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.

Step 1. Set $f(x) = \frac{1}{x}$.

Step 2. Differentiate: $f'(x) = -\frac{1}{x^2}$.

Step 3. Then $f + f' = \frac{1}{x} - \frac{1}{x^2}$, exactly the bracket. ✓

Step 4. Apply the template:

$$I = e^x \cdot \frac{1}{x} + C = \frac{e^x}{x} + C.$$

Final Answer: $\dfrac{e^x}{x} + C$

EXPERT'S SOLUTION : *Pooja Subramaniam, M.Sc Mathematics, IIT Kanpur*

One-line. $f = 1/x$, $f' = -1/x^2$, so the answer is $e^x \cdot (1/x) + C$.

Step 1. Recognise template.

Step 2. Read off f.

Step 3. Write $e^x f$.

Final Answer: $\dfrac{e^x}{x} + C$

Q 7.20 **Integrate** $\dfrac{(x-3)\,e^x}{(x-1)^3}$.

SOLUTION

Concept used. Algebraic split to expose the $e^x[f + f']$ template. Write $x - 3 = (x-1) - 2$:

$$\frac{x-3}{(x-1)^3} = \frac{(x-1)-2}{(x-1)^3} = \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}.$$

Step 1. Identify $f(x) = \dfrac{1}{(x-1)^2}$.

Step 2. Differentiate: $f'(x) = -\dfrac{2}{(x-1)^3}$.

Step 3. Then $f + f' = \dfrac{1}{(x-1)^2} - \dfrac{2}{(x-1)^3}$, matching the bracket.

Step 4. Apply the template:

$$I = e^x \cdot \frac{1}{(x-1)^2} + C = \frac{e^x}{(x-1)^2} + C.$$

Final Answer: $\dfrac{e^x}{(x-1)^2} + C$

EXPERT'S SOLUTION : *Yash Tiwari, M.Sc Mathematics, IIT Roorkee*

Why the split. When you see $e^x \cdot \dfrac{\text{linear}}{\text{cubic}}$, try shifting the numerator by the denominator's root so the fraction breaks into "function plus its derivative."

Step 1. Split numerator: $x - 3 = (x-1) - 2$.

Step 2. Recognise $f = 1/(x-1)^2$, $f' = -2/(x-1)^3$.

Step 3. Answer: $e^x/(x-1)^2 + C$.

Final Answer: $\dfrac{e^x}{(x-1)^2} + C$

Q 7.21 Integrate $e^{2x} \sin x$.

SOLUTION

Concept used. Two applications of parts produce a copy of the original integral on the right-hand side; solve algebraically.

Step 1. Let $I = \int e^{2x} \sin x\, dx$. Parts: $u = \sin x$, $dv = e^{2x}\, dx \Rightarrow du = \cos x\, dx$, $v = \frac{1}{2}e^{2x}$:

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2} \int e^{2x} \cos x\, dx.$$

Step 2. Parts again on $\int e^{2x} \cos x\, dx$: $u = \cos x$, $dv = e^{2x}\, dx$:

$$\int e^{2x} \cos x\, dx = \frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x\, dx = \frac{e^{2x} \cos x}{2} + \frac{I}{2}.$$

Step 3. Substitute back:

$$I = \frac{e^{2x} \sin x}{2} - \frac{1}{2}\left(\frac{e^{2x} \cos x}{2} + \frac{I}{2}\right).$$

Step 4. Expand and collect I:

$$I = \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{I}{4},$$

$$I + \frac{I}{4} = \frac{2e^{2x} \sin x - e^{2x} \cos x}{4}, \quad \frac{5I}{4} = \frac{e^{2x}(2 \sin x - \cos x)}{4}.$$

Step 5. Solve: $I = \dfrac{e^{2x}(2 \sin x - \cos x)}{5} + C$.

Final Answer: $\dfrac{e^{2x}(2 \sin x - \cos x)}{5} + C$

EXPERT'S SOLUTION : *Sahana Kulkarni, M.Sc Mathematics, IIT Bombay*

Template. $\int e^{ax}\sin(bx)\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$. Here $a=2$, $b=1$.

Step 1. Plug in $a=2$, $b=1$: $I = \dfrac{e^{2x}(2\sin x - \cos x)}{4+1} + C$.

Step 2. Simplify: $I = \dfrac{e^{2x}(2\sin x - \cos x)}{5} + C$.

Step 3. Verify by differentiating: derivative equals $e^{2x}\sin x$ after one expansion.

> **Final Answer:** $\dfrac{e^{2x}(2\sin x - \cos x)}{5} + C$

Q 7.22 **Integrate** $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$.

SOLUTION

Concept used. Trig substitution $x=\tan\theta$ collapses the inverse: from $2\tan\theta/(1+\tan^2\theta) = \sin 2\theta$, the integrand reduces to 2θ.

Step 1. Put $x=\tan\theta \Rightarrow dx = \sec^2\theta\,d\theta$, $\theta=\tan^{-1}x$. Also $\dfrac{2x}{1+x^2} = \sin 2\theta$, so $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\theta$ (in the principal-branch range).

Step 2. The integrand in θ: $2\theta\cdot\sec^2\theta\,d\theta$. Hence $I = 2\int\theta\sec^2\theta\,d\theta$.

Step 3. Parts: $u=\theta$, $dv=\sec^2\theta\,d\theta \Rightarrow du = d\theta$, $v=\tan\theta$.

$$\int \theta\sec^2\theta\,d\theta = \theta\tan\theta - \int\tan\theta\,d\theta = \theta\tan\theta - \log|\sec\theta|.$$

Step 4. Back-substitute $\theta=\tan^{-1}x$, $\tan\theta = x$, $\sec^2\theta = 1+x^2$ so $\log|\sec\theta| = \frac{1}{2}\log(1+x^2)$.

Step 5. Multiply by 2:

$$I = 2\left[x\tan^{-1}x - \tfrac{1}{2}\log(1+x^2)\right] + C = 2x\tan^{-1}x - \log(1+x^2) + C.$$

> **Final Answer:** $2x\tan^{-1}x - \log(1+x^2) + C$

☞ **Exam Tip**

$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ for $|x| \le 1$. Spot it and the integrand reduces to $2\tan^{-1}x$ - a Q13 cousin.

EXPERT'S SOLUTION *: Reyansh Goswami, M.Sc Mathematics, IIT Guwahati*

Shortcut. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = 2\tan^{-1}x$ (for principal-value range). So $I = 2\int \tan^{-1}x\,dx$.

Step 1. From Q13, $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$.

Step 2. Multiply by 2: $I = 2x\tan^{-1}x - \log(1+x^2) + C$.

Final Answer: $2x\tan^{-1}x - \log(1+x^2) + C$

Q 7.23 $\int x^2 e^{x^3}\,dx$ **equals**

(A) $\frac{1}{3}e^{x^3} + C$ (B) $\frac{1}{3}e^{x^2} + C$ (C) $\frac{1}{2}e^{x^3} + C$ (D) $\frac{1}{2}e^{x^2} + C$

SOLUTION

Concept used. Substitution $t = x^3$ gives $dt = 3x^2\,dx$, exposing $\int e^t\,dt/3$.

Step 1. Let $t = x^3$. Then $dt = 3x^2\,dx \Rightarrow x^2\,dx = \dfrac{dt}{3}$.

Step 2. Substitute:

$$\int x^2 e^{x^3}\,dx = \int e^t \cdot \frac{dt}{3} = \frac{1}{3}e^t + C = \frac{1}{3}e^{x^3} + C.$$

Step 3. Matches option (A).

Final Answer: Option (A): $\frac{1}{3}e^{x^3} + C$

EXPERT'S SOLUTION *: Niharika Saxena, M.Sc Mathematics, IIT Roorkee*

Pattern. Whenever you see $e^{g(x)}$ multiplied by a constant multiple of $g'(x)$, the answer is $e^{g(x)}$ divided by that constant.

Step 1. Here $g(x) = x^3$, $g'(x) = 3x^2$. Integrand $= \frac{1}{3}g'(x)e^{g(x)}$.

Step 2. Answer: $\frac{1}{3}e^{x^3} + C$.

Final Answer: Option (A)

Q 7.24 $\int e^x \sec x(1+\tan x)\,dx$ **equals**

(A) $e^x \cos x + C$ **(B)** $e^x \sec x + C$ **(C)** $e^x \sin x + C$ **(D)** $e^x \tan x + C$

SOLUTION

Concept used. The $e^x[f(x)+f'(x)]$ template. Expand the bracket:

$$\sec x(1+\tan x) = \sec x + \sec x \tan x.$$

Recall $\frac{d}{dx}\sec x = \sec x \tan x$.

Step 1. Set $f(x) = \sec x$. Then $f'(x) = \sec x \tan x$.

Step 2. Check: $f + f' = \sec x + \sec x \tan x = \sec x(1+\tan x)$. ✓

Step 3. Apply template:

$$\int e^x \sec x(1+\tan x)\,dx = e^x \sec x + C.$$

Step 4. Matches option (B).

> **Final Answer:** Option (B): $e^x \sec x + C$

EXPERT'S SOLUTION *: Aryan Patil, M.Sc Mathematics, IISER Pune*

Reading. Spot $\sec x + (\sec x)' = \sec x(1+\tan x)$. The template fires.

Step 1. $f = \sec x$.

Step 2. Answer: $e^x f + C = e^x \sec x + C$.

> **Final Answer:** Option (B)

Key Takeaways

- **Integration by parts:** $\int u\,dv = uv - \int v\,du$. Choose u via LIATE: Log → Inv-trig → Algebraic → Trig → Exp.
- Single-function integrals ($\log x$, $\tan^{-1} x$, $\sin^{-1} x$) yield to parts by writing them as (function) $\cdot\, 1$, then $dv = 1\,dx$.
- Polynomial $\times e^x$ / trig: each parts round drops the polynomial degree by one; x^n needs n rounds.
- Self-referential integrals (like $\int e^{2x}\sin x\,dx$): two parts rounds produce a copy of I; solve algebraically.

- **Template:** $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$. Always try to write the bracket as "function plus its derivative" before reaching for parts.

End of Exercise 7.6