Collegedunia NCERT Solutions

Step-by-step solutions for the 2026-27 NCERT (Latest Edition), Class 12 Mathematics

Chapter 7: Integrals

About this Chapter

The Miscellaneous Exercise mixes every technique from the chapter: substitution, partial fractions, integration by parts, the $e^x[f+f']$ template, the eight properties of definite integrals, and trigonometric identities. The questions test *which* technique to reach for as much as the technique itself.

Topics covered: Mixed techniques • Tricky substitutions • Partial fractions in disguise • Definite-integral properties • Modulus-of-trig integrals

Quick Formula Sheet

Strategy quick-pick:
Rational $\to$ partial fractions; $\sqrt{Q}$ + quadratic $\to$ complete the square;
poly $\times$ trans $\to$ parts (LIATE);
$e^x[f+f'] \to$ template; definite + symmetric $\to P_4/P_7$.
$\int e^x[f(x)+f'(x)]\,dx = e^x f(x) + C$

Miscellaneous Exercise

Q 7.1 **Integrate** $\dfrac{1}{x-x^3}$.

SOLUTION

Concept used. Factor and apply partial fractions. $x - x^3 = x(1-x^2) = x(1-x)(1+x)$.

Step 1. Decomposition: $\dfrac{1}{x(1-x)(1+x)} = \dfrac{A}{x} + \dfrac{B}{1-x} + \dfrac{C}{1+x}$.

Step 2. Multiply by denominator: $1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x)$. At $x=0$: $1 = A\cdot 1\cdot 1 = A \Rightarrow A = 1$. At $x=1$: $1 = B\cdot 1\cdot 2 = 2B \Rightarrow B = 1/2$. At $x=-1$: $1 = C(-1)(2) = -2C \Rightarrow C = -1/2$.

Step 3. Integrate:

$$I = \int \left[\frac{1}{x} + \frac{1/2}{1-x} + \frac{-1/2}{1+x}\right] dx = \log|x| - \tfrac{1}{2}\log|1-x| - \tfrac{1}{2}\log|1+x| + C.$$

Step 4. Combine logs:

$$I = \log|x| - \tfrac{1}{2}\log|1-x^2| + C = \tfrac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C.$$

Final Answer: $\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$

EXPERT'S SOLUTION *: Aanya Pathak, M.Sc Mathematics, IIT Bombay*

Shortcut. Multiply top and bottom by x: $\dfrac{1}{x-x^3} = \dfrac{x}{x^2-x^4} = \dfrac{x}{x^2(1-x^2)}$. Then substitute $u = x^2$, $du = 2x\,dx$.

Step 1. $I = \frac{1}{2}\int \dfrac{du}{u(1-u)} = \frac{1}{2}\int\left(\frac{1}{u} + \frac{1}{1-u}\right) du = \frac{1}{2}\log\left|\dfrac{u}{1-u}\right| + C.$

Step 2. Restore $u = x^2$: $\frac{1}{2}\log|x^2/(1-x^2)| + C$.

Final Answer: $\frac{1}{2}\log\left|\frac{x^2}{1-x^2}\right| + C$

Q 7.2 Integrate $\dfrac{1}{\sqrt{x+a}+\sqrt{x+b}}$.

SOLUTION

Concept used. Rationalise by multiplying by the conjugate over itself.

Step 1. Multiply by $\dfrac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$:

$$\frac{1}{\sqrt{x+a}+\sqrt{x+b}} \cdot \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} = \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)} = \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}.$$

Step 2. Integrate each piece: $\int \sqrt{x+a}\,dx = \dfrac{2(x+a)^{3/2}}{3}$, similarly for $\sqrt{x+b}$.

Step 3. Combine:

$$I = \frac{1}{a-b}\left[\frac{2(x+a)^{3/2}}{3} - \frac{2(x+b)^{3/2}}{3}\right] + C = \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C.$$

Final Answer: $\frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$

EXPERT'S SOLUTION : *Arnav Joshi, M.Sc Mathematics, IIT Delhi*

Reading. Rationalise to remove the sum of radicals; integrate each $\sqrt{\ }$ piece separately.

Step 1. Conjugate trick.

Step 2. Linear power rule on each piece.

Final Answer: $\frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + C$

Q 7.3 **Integrate** $\frac{1}{x\sqrt{ax-x^2}}$ **[Hint: Put** $x = \frac{a}{t}$**].**

SOLUTION

Concept used. Substitution per the hint. Let $x = a/t$, so $dx = -a/t^2\,dt$. Then $ax - x^2 = a(a/t) - a^2/t^2 = \frac{a^2 t - a^2}{t^2} = \frac{a^2(t-1)}{t^2}$, so $\sqrt{ax-x^2} = \frac{a}{|t|}\sqrt{t-1}$ (assume $a > 0$, $t > 0$).

Step 1. $x\sqrt{ax-x^2} = \frac{a}{t} \cdot \frac{a\sqrt{t-1}}{t} = \frac{a^2\sqrt{t-1}}{t^2}$.

Step 2. $\frac{dx}{x\sqrt{ax-x^2}} = \frac{-a/t^2\,dt}{a^2\sqrt{t-1}/t^2} = -\frac{dt}{a\sqrt{t-1}}$.

Step 3. Integrate:

$$I = -\frac{1}{a}\int \frac{dt}{\sqrt{t-1}} = -\frac{1}{a} \cdot 2\sqrt{t-1} + C = -\frac{2\sqrt{t-1}}{a} + C.$$

Step 4. Back-substitute $t = a/x$:

$$I = -\frac{2}{a}\sqrt{\frac{a}{x} - 1} + C = -\frac{2}{a}\sqrt{\frac{a-x}{x}} + C.$$

Final Answer: $-\dfrac{2}{a}\sqrt{\dfrac{a-x}{x}}+C$

EXPERT'S SOLUTION *: Hardik Mehta, M.Sc Mathematics, IIT Roorkee*

Why $x = a/t$. The substitution turns the awkward $\sqrt{ax-x^2}$ into $\sqrt{t-1}$ — a single power.

Step 1. Substitute $x = a/t$.

Step 2. Integrand reduces to $-1/(a\sqrt{t-1})$.

Step 3. Integrate and restore.

Final Answer: $-\dfrac{2}{a}\sqrt{\dfrac{a-x}{x}}+C$

Q 7.4 Integrate $\dfrac{1}{x^2(x^4+1)^{3/4}}$.

SOLUTION

Concept used. Factor x^4 from inside the bracket. $(x^4+1)^{3/4} = x^3(1+1/x^4)^{3/4}$ (for $x > 0$). So the integrand is

$$\frac{1}{x^2 \cdot x^3(1+1/x^4)^{3/4}} = \frac{1}{x^5(1+1/x^4)^{3/4}}.$$

Substitute $t = 1 + 1/x^4$, $dt = -4/x^5\,dx$, so $dx/x^5 = -dt/4$.

Step 1. Transform:

$$I = \int \frac{-dt/4}{t^{3/4}} = -\frac{1}{4}\int t^{-3/4}\,dt = -\frac{1}{4}\cdot 4t^{1/4} + C = -t^{1/4} + C.$$

Step 2. Back-substitute:

$$I = -(1+1/x^4)^{1/4} + C = -\frac{(x^4+1)^{1/4}}{x} + C.$$

Final Answer: $-\dfrac{(x^4+1)^{1/4}}{x}+C$

EXPERT'S SOLUTION : *Mira Choudhary, M.Sc Mathematics, IIT Madras*

Key move. Bringing x^4 inside the bracket creates the $1 + 1/x^4$ form whose derivative naturally matches $1/x^5$.

Step 1. Factor inside the radical.

Step 2. Substitute $t = 1 + 1/x^4$.

Step 3. Integrate $t^{-3/4}$; restore.

Final Answer: $-\dfrac{(x^4+1)^{1/4}}{x} + C$

Q 7.5 **Integrate** $\dfrac{1}{x^{1/2} + x^{1/3}}$ **[Hint: put $x = t^6$].**

SOLUTION

Concept used. Pick $t = x^{1/6}$ so $x = t^6$ and $dx = 6t^5\,dt$. Then $x^{1/2} = t^3$ and $x^{1/3} = t^2$, eliminating fractional exponents.

Step 1. Sub: $\dfrac{dx}{x^{1/2}+x^{1/3}} = \dfrac{6t^5\,dt}{t^3+t^2} = \dfrac{6t^5\,dt}{t^2(t+1)} = \dfrac{6t^3\,dt}{t+1}$.

Step 2. Divide t^3 by $t+1$: $t^3 = (t+1)(t^2-t+1) - 1$, so $\dfrac{t^3}{t+1} = t^2 - t + 1 - \dfrac{1}{t+1}$.

Step 3. Integrate:

$$I = 6\int\left(t^2 - t + 1 - \frac{1}{t+1}\right)dt = 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \log|t+1|\right] + C.$$

Step 4. Restore $t = x^{1/6}$:

$$I = 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\log|x^{1/6}+1| + C.$$

Final Answer: $2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log\left|x^{1/6}+1\right| + C$

EXPERT'S SOLUTION : *Sneha Vikram, M.Sc Mathematics, IIT Bombay*

LCM trick. For an integrand with $x^{1/2}$ and $x^{1/3}$, set $t = x^{1/\text{lcm}(2,3)} = x^{1/6}$. Every fractional exponent collapses to an integer.

Step 1. Sub $x = t^6$.

Step 2. Reduce rational; long-divide.

Step 3. Integrate; restore.

Final Answer: $2\sqrt{x} - 3x^{1/3} + 6x^{1/6} - 6\log\left|x^{1/6} + 1\right| + C$

Q 7.6 **Integrate** $\dfrac{5x}{(x+1)(x^2+9)}$.

SOLUTION

Concept used. Partial fractions with a linear and an irreducible quadratic factor. Write

$$\frac{5x}{(x+1)(x^2+9)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+9}.$$

Step 1. Multiply: $5x = A(x^2+9) + (Bx+C)(x+1) = (A+B)x^2 + (B+C)x + (9A+C)$.

Step 2. Match coefficients: $A+B=0$; $B+C=5$; $9A+C=0$.

Step 3. Solve: from the first, $B=-A$; substituting into the second, $C=5+A$. Third: $9A+5+A=0 \Rightarrow 10A=-5 \Rightarrow A=-1/2$. Then $B=1/2$, $C=9/2$.

Step 4. Decomposition:

$$\frac{5x}{(x+1)(x^2+9)} = \frac{-1/2}{x+1} + \frac{(1/2)x+9/2}{x^2+9}.$$

Step 5. Integrate piece-wise:

$$\int \frac{-1/2}{x+1}\,dx = -\tfrac{1}{2}\log|x+1|;$$

$$\int \frac{(1/2)x}{x^2+9}\,dx = \tfrac{1}{4}\log(x^2+9);$$

$$\int \frac{9/2}{x^2+9}\,dx = \tfrac{9}{2}\cdot\tfrac{1}{3}\tan^{-1}(x/3) = \tfrac{3}{2}\tan^{-1}(x/3).$$

Step 6. Combine:

$$I = -\tfrac{1}{2}\log|x+1| + \tfrac{1}{4}\log(x^2+9) + \tfrac{3}{2}\tan^{-1}\frac{x}{3} + C.$$

Final Answer: $-\tfrac{1}{2}\log|x+1| + \tfrac{1}{4}\log(x^2+9) + \tfrac{3}{2}\tan^{-1}\dfrac{x}{3} + C$

EXPERT'S SOLUTION : *Vivaan Saluja, M.Sc Mathematics, IIT Hyderabad*

Tidy. Three pieces: a $-\log|x+1|/2$, a $\log(x^2+9)/4$, and a $\tan^{-1}$.

Step 1. Setup partial fractions.

Step 2. Solve A, B, C.

Step 3. Integrate the three standard pieces.

Final Answer: $-\frac{1}{2}\log|x+1| + \frac{1}{4}\log(x^2+9) + \frac{3}{2}\tan^{-1}\frac{x}{3} + C$

Q 7.7 **Integrate** $\dfrac{\sin x}{\sin(x-a)}$.

SOLUTION

Concept used. Write $x = (x-a)+a$ and expand $\sin x = \sin((x-a)+a)$ using the sum formula.

Step 1. Expand: $\sin x = \sin(x-a)\cos a + \cos(x-a)\sin a$.

Step 2. Divide by $\sin(x-a)$:

$$\frac{\sin x}{\sin(x-a)} = \cos a + \sin a \cdot \cot(x-a).$$

Step 3. Integrate: $\int \cos a\, dx = x\cos a$; $\int \cot(x-a)\, dx = \log|\sin(x-a)|$.

Step 4. Combine:

$$I = x\cos a + \sin a \cdot \log|\sin(x-a)| + C.$$

Final Answer: $x\cos a + \sin a \cdot \log|\sin(x-a)| + C$

EXPERT'S SOLUTION : *Shaurya Verma, M.Sc Mathematics, IIT Kanpur*

Sum-formula trick. Whenever the numerator and denominator differ by a constant phase, write the numerator as the denominator-angle plus that constant.

Step 1. Apply $\sin$ sum formula.

Step 2. Split into $\cos a$ and $\sin a \cdot \cot$.

Step 3. Integrate each.

Final Answer: $x \cos a + \sin a \cdot \log|\sin(x-a)| + C$

Q 7.8 Integrate $\dfrac{e^{5\log x} - e^{4\log x}}{e^{3\log x} - e^{2\log x}}$.

SOLUTION

Concept used. $e^{n\log x} = x^n$. Simplify before integrating.

Step 1. Convert: numerator $= x^5 - x^4 = x^4(x-1)$; denominator $= x^3 - x^2 = x^2(x-1)$.

Step 2. Cancel $(x-1)$: ratio $= x^4/x^2 = x^2$.

Step 3. Integrate: $I = \int x^2\,dx = \dfrac{x^3}{3} + C$.

Final Answer: $x^3/3 + C$

EXPERT'S SOLUTION *: Aakanksha Bhaskar, M.Sc Mathematics, IIT Delhi*

The trick is reading $e^{n\log x}$. It is just x^n; no integration is hard once you collapse the disguise.

Step 1. Rewrite as x^n.

Step 2. Cancel common factor.

Final Answer: $x^3/3 + C$

Q 7.9 Integrate $\dfrac{\cos x}{\sqrt{4 - \sin^2 x}}$.

SOLUTION

Concept used. Substitution $t = \sin x$ converts to $\int \dfrac{dt}{\sqrt{4-t^2}} = \sin^{-1}(t/2) + C$.

Step 1. $t = \sin x \Rightarrow dt = \cos x\,dx$.

Step 2. Transform:

$$I = \int \frac{dt}{\sqrt{4-t^2}} = \sin^{-1}\frac{t}{2} + C.$$

Step 3. Restore: $I = \sin^{-1}\frac{\sin x}{2} + C$.

Final Answer: $\sin^{-1}\frac{\sin x}{2} + C$

EXPERT'S SOLUTION *: Adithi Krishnan, M.Sc Mathematics, IIT Madras*

Reading. $\cos x$ on top with $\sin x$ inside the radical $\to t = \sin x$.

Step 1. Sub.

Step 2. Standard $\sin^{-1}$.

Final Answer: $\sin^{-1}\frac{\sin x}{2} + C$

Q 7.10 **Integrate** $\frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x \cos^2 x}$.

SOLUTION

Concept used. Factor as a difference of squares (twice) and use $\sin^2 + \cos^2 = 1$; the denominator equals $\sin^4 + \cos^4$.

Step 1. Note $\sin^4 + \cos^4 = (\sin^2 + \cos^2)^2 - 2\sin^2\cos^2 = 1 - 2\sin^2\cos^2$. So the denominator equals $\sin^4 x + \cos^4 x$.

Step 2. Difference of squares: $\sin^8 - \cos^8 = (\sin^4 - \cos^4)(\sin^4 + \cos^4)$.

Step 3. Cancel the common factor $\sin^4 + \cos^4$ (positive):

$$\text{Integrand} = \sin^4 x - \cos^4 x = (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) = -\cos 2x.$$

(Used $\sin^2 - \cos^2 = -\cos 2x$ and $\sin^2 + \cos^2 = 1$.)

Step 4. Integrate: $\int -\cos 2x\, dx = -\frac{\sin 2x}{2} + C$.

Final Answer: $-\frac{\sin 2x}{2} + C$

EXPERT'S SOLUTION : *Aryaman Bhardwaj, M.Sc Mathematics, IIT Roorkee*

Algebra-first. Trig identities collapse the eighth-power monsters into a single $\cos 2x$.

Step 1. Difference-of-squares twice.

Step 2. Reach $-\cos 2x$.

Step 3. Integrate.

Final Answer: $-\dfrac{\sin 2x}{2} + C$

Q 7.11 **Integrate** $\dfrac{1}{\cos(x+a)\cos(x+b)}$.

SOLUTION

Concept used. Multiply and divide by $\sin(a-b)$, then use $\sin((x+a)-(x+b)) = \sin(a-b)$ to break up the integrand.

Step 1. Expand
$\sin(a-b) = \sin((x+a)-(x+b)) = \sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)$.

Step 2. Divide by $\cos(x+a)\cos(x+b)$:

$$\frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} = \tan(x+a) - \tan(x+b).$$

Step 3. So

$$\frac{1}{\cos(x+a)\cos(x+b)} = \frac{1}{\sin(a-b)}\big[\tan(x+a) - \tan(x+b)\big].$$

Step 4. Integrate: $\int \tan u\, du = \log|\sec u|$. So

$$I = \frac{1}{\sin(a-b)}[\log|\sec(x+a)| - \log|\sec(x+b)|] + C.$$

Combining the logs:

$$I = \frac{1}{\sin(a-b)} \log\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + C.$$

Final Answer: $\dfrac{1}{\sin(a-b)} \log\left|\dfrac{\cos(x+b)}{\cos(x+a)}\right| + C$

EXPERT'S SOLUTION : *Devansh Pillai, M.Sc Mathematics, IIT Guwahati*

The $\sin(a-b)$ trick. Whenever two cosine factors with shifted arguments appear, multiply-and-divide by $\sin(a-b)$ to expose $\tan$ differences.

Step 1. Insert $\sin(a-b)/\sin(a-b)$.

Step 2. Expand and simplify.

Step 3. Integrate two tangents.

Final Answer: $\dfrac{1}{\sin(a-b)}\log\left|\dfrac{\cos(x+b)}{\cos(x+a)}\right| + C$

Q 7.12 Integrate $\dfrac{x^3}{\sqrt{1-x^8}}$.

SOLUTION

Concept used. Substitute $t = x^4$, $dt = 4x^3\,dx$.

Step 1. Transform: $1 - x^8 = 1 - (x^4)^2 = 1 - t^2$. And $x^3\,dx = dt/4$.

Step 2. Integrate:

$$I = \int \frac{dt/4}{\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1} t + C.$$

Step 3. Restore: $I = \dfrac{1}{4}\sin^{-1}(x^4) + C$.

Final Answer: $\dfrac{1}{4}\sin^{-1}(x^4) + C$

EXPERT'S SOLUTION : *Reyansh Verma, M.Sc Mathematics, IIT Bombay*

Quick. x^3 on top, x^8 inside the radical: hit it with $t = x^4$.

Step 1. Sub; standard $\sin^{-1}$.

Final Answer: $\dfrac{1}{4}\sin^{-1}(x^4) + C$

Q 7.13 Integrate $\dfrac{e^x}{(1+e^x)(2+e^x)}$.

SOLUTION

Concept used. Substitute $t = e^x$, $dt = e^x\,dx$. Then partial fractions.

Step 1. Transform: $\dfrac{e^x\,dx}{(1+e^x)(2+e^x)} = \dfrac{dt}{(1+t)(2+t)}$.

Step 2. Partial fractions: $\dfrac{1}{(1+t)(2+t)} = \dfrac{1}{1+t} - \dfrac{1}{2+t}$.

Step 3. Integrate:

$$I = \int \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt = \log|1+t| - \log|2+t| + C = \log\left|\frac{1+t}{2+t}\right| + C.$$

Step 4. Restore $t = e^x$:

$$I = \log\left|\frac{1+e^x}{2+e^x}\right| + C.$$

Final Answer: $\log\left|\dfrac{1+e^x}{2+e^x}\right| + C$

EXPERT'S SOLUTION : *Maitri Tripathi, M.Sc Mathematics, IIT Madras*

Spot it. $e^x\,dx$ pairs with $t = e^x$ instantly.

Step 1. Sub.

Step 2. Two-pole partial fractions.

Step 3. Log ratio.

Final Answer: $\log\left|\dfrac{1+e^x}{2+e^x}\right| + C$

Q 7.14 **Integrate** $\dfrac{1}{(x^2+1)(x^2+4)}$.

SOLUTION

Concept used. Partial fractions on two irreducible quadratics.

Step 1. Write $\dfrac{1}{(x^2+1)(x^2+4)} = \dfrac{A}{x^2+1} + \dfrac{B}{x^2+4}$.

Step 2. Multiply: $1 = A(x^2+4) + B(x^2+1) = (A+B)x^2 + (4A+B)$.

Step 3. Match: $A + B = 0$ and $4A + B = 1 \Rightarrow 3A = 1 \Rightarrow A = 1/3$, $B = -1/3$.

Step 4. Integrate:

$$I = \frac{1}{3}\int \frac{dx}{x^2+1} - \frac{1}{3}\int \frac{dx}{x^2+4} = \frac{1}{3}\tan^{-1}x - \frac{1}{3}\cdot\frac{1}{2}\tan^{-1}(x/2) + C.$$

Step 5. Tidy:

$$I = \frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C.$$

Final Answer: $\frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$

EXPERT'S SOLUTION *: Pranav Khurana, M.Sc Mathematics, IIT Hyderabad*

Use $1/(x^2+a^2)$ template. Both poles are pure imaginary; both pieces become $\tan^{-1}$.

Step 1. Match coefficients.

Step 2. Two $\tan^{-1}$ pieces.

Final Answer: $\frac{1}{3}\tan^{-1}x - \frac{1}{6}\tan^{-1}\frac{x}{2} + C$

Q 7.15 Integrate $\cos^3 x \cdot e^{\log \sin x}$.

SOLUTION

Concept used. $e^{\log \sin x} = \sin x$. So the integrand simplifies to $\sin x \cdot \cos^3 x$, a textbook $t = \cos x$ substitution.

Step 1. Simplify: integrand $= \sin x \cos^3 x$.

Step 2. Substitute $t = \cos x$, $dt = -\sin x\, dx$, so $\sin x\, dx = -dt$.

Step 3. Transform:

$$I = \int t^3 \cdot (-dt) = -\frac{t^4}{4} + C.$$

Step 4. Restore: $I = -\frac{\cos^4 x}{4} + C$.

Final Answer: $-\frac{\cos^4 x}{4} + C$

EXPERT'S SOLUTION : *Ojas Sehgal, M.Sc Mathematics, IIT Kanpur*

Disguise. $e^{\log \sin x} = \sin x$, plain and simple.

Step 1. Unmask.

Step 2. Sub $t = \cos x$.

Step 3. Integrate.

Final Answer: $-\dfrac{\cos^4 x}{4} + C$

Q 7.16 **Integrate** $e^{3\log x}(x^4+1)^{-1}$.

SOLUTION

Concept used. $e^{3\log x} = x^3$. So the integrand is $\dfrac{x^3}{x^4+1}$.

Step 1. Substitute $t = x^4 + 1$, $dt = 4x^3\,dx$, so $x^3\,dx = dt/4$.

Step 2. Transform: $I = \int \dfrac{dt/4}{t} = \dfrac{1}{4}\log|t| + C = \dfrac{1}{4}\log(x^4+1) + C$.

Final Answer: $\dfrac{1}{4}\log(x^4+1) + C$

EXPERT'S SOLUTION : *Aaradhya Bose, M.Sc Mathematics, IIT Bombay*

Unmask. $e^{3\log x} = x^3$, then derivative-of-denominator pattern.

Step 1. Simplify.

Step 2. Sub; log.

Final Answer: $\dfrac{1}{4}\log(x^4+1) + C$

Q 7.17 **Integrate** $f'(ax+b)\left[f(ax+b)\right]^n$.

SOLUTION

Concept used. Generalised power rule with a substitution. Let $u = f(ax+b)$, then $du = f'(ax+b)\cdot a\,dx$, so $f'(ax+b)\,dx = du/a$.

Step 1. Transform: $f'(ax+b)\,[f(ax+b)]^n\,dx = u^n \cdot \dfrac{du}{a}$.

Step 2. Integrate: $\dfrac{1}{a}\int u^n\,du = \dfrac{u^{n+1}}{a(n+1)} + C$.

Step 3. Restore: $I = \dfrac{[f(ax+b)]^{n+1}}{a(n+1)} + C$.

Final Answer: $\dfrac{[f(ax+b)]^{n+1}}{a(n+1)} + C$

EXPERT'S SOLUTION : *Karan Vasudev, M.Sc Mathematics, IIT Roorkee*

General template. Whenever an integrand is "g' multiplied by a power of g," the answer is $g^{n+1}/(n+1)$ — chain rule run backwards.

Step 1. Set $u = f(ax+b)$.

Step 2. Power rule on u.

Final Answer: $\dfrac{[f(ax+b)]^{n+1}}{a(n+1)} + C$

Q 7.18 **Integrate** $\dfrac{1}{\sqrt{\sin^3 x \sin(x+\alpha)}}$.

SOLUTION

Concept used. Factor and use $\sin(x+\alpha) = \sin x\cos\alpha + \cos x \sin\alpha$, then substitute $t = \cot x + \cos\alpha/\sin x$ or, more compactly, $t = \cos\alpha + \cot x\,\sin\alpha$ — a manipulation that turns the integrand into $-dt/(2\sqrt{t}\sin\alpha)$. The standard NCERT solution uses $t = \dfrac{\sin(x+\alpha)}{\sin x}$.

Step 1. Note $\dfrac{\sin(x+\alpha)}{\sin x} = \cos\alpha + \cot x \sin\alpha$. So let $t = \cos\alpha + \cot x\sin\alpha$. Then

$$\frac{dt}{dx} = -\operatorname{cosec}^2 x \sin\alpha, \quad dt = -\frac{\sin\alpha}{\sin^2 x}\,dx.$$

Step 2. Rewrite the integrand. $\sin^3 x\sin(x+\alpha) = \sin^4 x \cdot \dfrac{\sin(x+\alpha)}{\sin x} = \sin^4 x \cdot t$. So

$$\sqrt{\sin^3 x \sin(x+\alpha)} = \sin^2 x\sqrt{t}.$$

Step 3. Transform:

$$\frac{dx}{\sin^2 x\sqrt{t}} = \frac{1}{\sqrt{t}} \cdot \frac{dx}{\sin^2 x} = -\frac{1}{\sin\alpha} \cdot \frac{dt}{\sqrt{t}}.$$

Step 4. Integrate:

$$I = -\frac{1}{\sin\alpha}\int \frac{dt}{\sqrt{t}} = -\frac{2\sqrt{t}}{\sin\alpha} + C.$$

Step 5. Restore:

$$I = -\frac{2}{\sin\alpha}\sqrt{\cos\alpha + \cot x \sin\alpha} + C = -\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C.$$

Final Answer: $-\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$

EXPERT'S SOLUTION *: Aditi Karan, M.Sc Mathematics, IIT Bombay*

Why the substitution. $\sin(x+\alpha)/\sin x$ is the natural ratio; its derivative turns the awkward radical into a $1/\sqrt{t}$ integrand.

Step 1. Set $t = \sin(x+\alpha)/\sin x$.

Step 2. Compute dt.

Step 3. Integrate $1/\sqrt{t}$ pattern.

Final Answer: $-\frac{2}{\sin\alpha}\sqrt{\frac{\sin(x+\alpha)}{\sin x}} + C$

Q 7.19 Integrate $\sqrt{\frac{1-x}{1+x}}$.

SOLUTION

Concept used. Trig substitution $x = \cos\theta$ removes the radical via the half-angle identity $\sqrt{(1-\cos\theta)/(1+\cos\theta)} = \tan(\theta/2)$.

Step 1. Substitute $x = \cos\theta$, $dx = -\sin\theta\, d\theta$, $\theta = \cos^{-1} x$.

Step 2. Integrand: $\sqrt{(1-\cos\theta)/(1+\cos\theta)} = \tan(\theta/2)$.

Step 3. Transform: $I = \int \tan(\theta/2) \cdot (-\sin\theta)\, d\theta$. Use $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$:

$$\tan(\theta/2)\sin\theta = \frac{\sin(\theta/2)}{\cos(\theta/2)} \cdot 2\sin(\theta/2)\cos(\theta/2) = 2\sin^2(\theta/2) = 1 - \cos\theta.$$

Step 4. So $I = -\int (1-\cos\theta)\, d\theta = -\theta + \sin\theta + C$.

Step 5. Restore: $\theta = \cos^{-1} x$, $\sin\theta = \sqrt{1-x^2}$:

$$I = \sqrt{1-x^2} - \cos^{-1} x + C.$$

Final Answer: $\sqrt{1-x^2} - \cos^{-1} x + C$

EXPERT'S SOLUTION *: Diya Singh, M.Sc Mathematics, IIT Madras*

Rationalisation alternative. Multiply top and bottom by $\sqrt{1-x}$: $\sqrt{(1-x)/(1+x)} = (1-x)/\sqrt{1-x^2}$. Then split: $\int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\, dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$.

Step 1. Rationalise.

Step 2. Two standard integrals.

Step 3. Combine: $\sqrt{1-x^2} + \sin^{-1} x + C$ (equivalent to the boxed form, up to a $\pi/2$ constant).

Final Answer: $\sqrt{1-x^2} + \sin^{-1} x + C$

Q 7.20 **Integrate** $\frac{2+\sin 2x}{1+\cos 2x} e^x$.

SOLUTION

Concept used. Use $1 + \cos 2x = 2\cos^2 x$ and $\sin 2x = 2\sin x \cos x$, then recognise the $e^x[f + f']$ template.

Step 1. Simplify:

$$\frac{2+\sin 2x}{1+\cos 2x} = \frac{2 + 2\sin x\cos x}{2\cos^2 x} = \frac{1}{\cos^2 x} + \frac{\sin x}{\cos x} = \sec^2 x + \tan x.$$

Step 2. Set $f(x) = \tan x$. Then $f'(x) = \sec^2 x$, so $f + f' = \tan x + \sec^2 x$. ✓

Step 3. Apply template: $I = e^x f(x) + C = e^x \tan x + C$.

Final Answer: $e^x \tan x + C$

EXPERT'S SOLUTION *: Anvi Rao, M.Sc Mathematics, IIT Delhi*

Reading. The $1 + \cos 2x \to 2\cos^2 x$ identity opens the integrand into $\sec^2 x + \tan x$ — exactly $f' + f$ for $f = \tan x$.

Step 1. Simplify with double-angle identities.

Step 2. Spot template.

Final Answer: $e^x \tan x + C$

Q 7.21 **Integrate** $\dfrac{x^2 + x + 1}{(x+1)^2(x+2)}$.

SOLUTION

Concept used. Partial fractions with a repeated linear factor.

$$\frac{x^2 + x + 1}{(x+1)^2(x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x+2}.$$

Step 1. Multiply: $x^2 + x + 1 = A(x+1)(x+2) + B(x+2) + C(x+1)^2$.

Step 2. $x = -1$: $1 - 1 + 1 = 1 = B(1) \Rightarrow B = 1$.

Step 3. $x = -2$: $4 - 2 + 1 = 3 = C(1) \Rightarrow C = 3$.

Step 4. Expand and match x^2 coefficient: $A + C = 1 \Rightarrow A = 1 - 3 = -2$.

Step 5. Decomposition:

$$\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{3}{x+2}.$$

Step 6. Integrate:

$$I = -2\log|x+1| - \frac{1}{x+1} + 3\log|x+2| + C.$$

Final Answer: $3\log|x+2| - 2\log|x+1| - \dfrac{1}{x+1} + C$

EXPERT'S SOLUTION *: Ishita Bhalla, M.Sc Mathematics, IIT Bombay*

Repeated factor recipe. $1/(x+1)^2$ contributes both an $A/(x+1)$ and a $B/(x+1)^2$ piece; solve B first by plugging $x=-1$ (only B term survives), then chase A via x^2 coefficient.

Step 1. Solve B, C by direct plug-in.

Step 2. Solve A via leading-coefficient match.

Step 3. Integrate.

Final Answer: $3\log|x+2| - 2\log|x+1| - \dfrac{1}{x+1} + C$

Q 7.22 **Integrate** $\tan^{-1}\sqrt{\dfrac{1-x}{1+x}}$.

SOLUTION

Concept used. Substitute $x=\cos\theta$. Use the identity $\sqrt{(1-\cos\theta)/(1+\cos\theta)} = \tan(\theta/2)$.

Step 1. Let $x = \cos\theta \Rightarrow dx = -\sin\theta\, d\theta$, $\theta = \cos^{-1} x \in [0,\pi]$.

Step 2. Integrand: $\tan^{-1}(\tan(\theta/2)) = \theta/2$.

Step 3. Transform: $I = \int \frac{\theta}{2}(-\sin\theta)\, d\theta = -\frac{1}{2}\int \theta\sin\theta\, d\theta$.

Step 4. Parts: $\int \theta\sin\theta\, d\theta = -\theta\cos\theta + \sin\theta$ (from Ex 7.6).

Step 5. So $I = -\frac{1}{2}\left[-\theta\cos\theta + \sin\theta\right] + C = \dfrac{\theta\cos\theta - \sin\theta}{2} + C$.

Step 6. Restore: $\theta = \cos^{-1}x$, $\cos\theta = x$, $\sin\theta = \sqrt{1-x^2}$:

$$I = \frac{x\cos^{-1}x - \sqrt{1-x^2}}{2} + C.$$

Step 7. Tidier form: $I = \frac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + C$.

Final Answer: $\dfrac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + C$

EXPERT'S SOLUTION : *Krishnan Mahesh, M.Sc Mathematics, IIT Madras*

Hidden simplification. Inverse trig of $\tan(\theta/2)$ is just $\theta/2$ in the principal range; the brutal-looking integrand collapses to $\theta \sin\theta/2$.

Step 1. Sub $x = \cos\theta$.

Step 2. Integrand becomes $\theta/2$; multiply by $-\sin\theta\, d\theta$.

Step 3. One parts step.

Final Answer: $\frac{1}{2}\left[x\cos^{-1}x - \sqrt{1-x^2}\right] + C$

Q 7.23 **Integrate** $\dfrac{\sqrt{x^2+1}\,[\log(x^2+1) - 2\log x]}{x^4}$.

SOLUTION

Concept used. Simplify the log: $\log(x^2+1) - 2\log x = \log\dfrac{x^2+1}{x^2} = \log(1+1/x^2)$. Then substitute $t = 1 + 1/x^2$, $dt = -2/x^3\,dx$.

Step 1. Algebra: $\sqrt{x^2+1}/x^4 = \dfrac{1}{x^3}\sqrt{(x^2+1)/x^2} = \dfrac{1}{x^3}\sqrt{1+1/x^2}$.

Step 2. So the integrand is $\dfrac{1}{x^3}\sqrt{1+1/x^2}\,\log(1+1/x^2)$.

Step 3. Let $t = 1+1/x^2$. Then $dt = -2/x^3\,dx$, so $dx/x^3 = -dt/2$.

Step 4. Transform: $I = \int\sqrt{t}\log t\cdot(-dt/2) = -\dfrac{1}{2}\int\sqrt{t}\log t\,dt$.

Step 5. Parts on $\int t^{1/2}\log t\,dt$: $u = \log t$, $dv = t^{1/2}\,dt \Rightarrow du = dt/t$, $v = \frac{2t^{3/2}}{3}$.

$$\int t^{1/2}\log t\,dt = \tfrac{2t^{3/2}}{3}\log t - \tfrac{2}{3}\int t^{1/2}\,dt = \tfrac{2t^{3/2}}{3}\log t - \tfrac{4t^{3/2}}{9}.$$

Step 6. Multiply by $-1/2$:

$$I = -\frac{t^{3/2}}{3}\log t + \frac{2t^{3/2}}{9} + C = \frac{t^{3/2}}{9}\left[2 - 3\log t\right] + C.$$

Step 7. Restore $t = 1+1/x^2 = (x^2+1)/x^2$, so $t^{3/2} = (x^2+1)^{3/2}/x^3$:

$$I = \frac{(x^2+1)^{3/2}}{9x^3}\left[2 - 3\log\frac{x^2+1}{x^2}\right] + C.$$

Final Answer: $-\dfrac{(x^2+1)^{3/2}}{3x^3}\log\dfrac{x^2+1}{x^2} + \dfrac{2(x^2+1)^{3/2}}{9x^3} + C$

EXPERT'S SOLUTION : *Tarini Khanna, M.Sc Mathematics, IIT Kanpur*

Two collapses. (i) The log piece collapses by log-laws to $\log(1 + 1/x^2)$. (ii) The $\sqrt{x^2 + 1}/x^4$ piece is the matching dt for $t = 1 + 1/x^2$. Parts finishes the job.

Step 1. Combine logs.

Step 2. Substitute t.

Step 3. Parts on $\int t^{1/2} \log t \, dt$.

Final Answer: $\dfrac{(x^2+1)^{3/2}}{9x^3}\left[2 - 3\log\dfrac{x^2+1}{x^2}\right] + C$

Q 7.24 **Evaluate** $\displaystyle\int_{\pi/2}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x}\right) dx$.

SOLUTION

Concept used. Half-angle: $1 - \cos x = 2\sin^2(x/2)$ and $1 - \sin x = 1 - 2\sin(x/2)\cos(x/2) = (\sin(x/2) - \cos(x/2))^2 \cdot$ — no, simpler: split into two fractions.

Step 1. Half-angle: $\dfrac{1 - \sin x}{1 - \cos x} = \dfrac{1 - 2\sin(x/2)\cos(x/2)}{2\sin^2(x/2)} = \dfrac{1}{2\sin^2(x/2)} - \dfrac{\cos(x/2)}{\sin(x/2)} = \frac{1}{2}\operatorname{cosec}^2(x/2) - \cot(x/2)$.

Step 2. Set $f(x) = -\cot(x/2)$. Then $f'(x) = \frac{1}{2}\operatorname{cosec}^2(x/2)$, so $f(x) + f'(x) = \frac{1}{2}\operatorname{cosec}^2(x/2) - \cot(x/2)$, matching the bracket.

Step 3. Apply template: $\int e^x(f + f')\,dx = e^x f + C = -e^x \cot(x/2)$.

Step 4. Evaluate from $\pi/2$ to π:
$-e^{\pi}\cot(\pi/2) - \left[-e^{\pi/2}\cot(\pi/4)\right] = -e^{\pi}\cdot 0 + e^{\pi/2}\cdot 1 = e^{\pi/2}$.

Final Answer: $e^{\pi/2}$

EXPERT'S SOLUTION : *Soham Iyer, M.Sc Mathematics, IIT Bombay*

Spot. Half-angle split + $e^x(f + f')$ template = clean answer.

Step 1. Half-angle simplification.

Step 2. $f = -\cot(x/2)$.

Step 3. Boundary values: $\cot(\pi/2) = 0$, $\cot(\pi/4) = 1$.

Final Answer: $e^{\pi/2}$

Q 7.25 **Evaluate** $\displaystyle\int_0^{\pi/4} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\, dx$.

SOLUTION

Concept used. Divide numerator and denominator by $\cos^4 x$, then substitute $t = \tan^2 x$.

Step 1. Divide top and bottom by $\cos^4 x$:

$$\frac{\sin x \cos x / \cos^4 x}{1 + \sin^4 x / \cos^4 x} = \frac{\tan x \sec^2 x}{1 + \tan^4 x}.$$

Step 2. Let $t = \tan^2 x$, $dt = 2 \tan x \sec^2 x\, dx$, so $\tan x \sec^2 x\, dx = dt/2$. Limits: $x = 0 \to t = 0$; $x = \pi/4 \to t = 1$.

Step 3. Transform:

$$I = \int_0^1 \frac{dt/2}{1 + t^2} = \frac{1}{2}\left[\tan^{-1} t\right]_0^1 = \frac{1}{2} \cdot \frac{\pi}{4} = \frac{\pi}{8}.$$

Final Answer: $\pi/8$

EXPERT'S SOLUTION : *Trisha Banerjee, M.Sc Mathematics, IIT Kharagpur*

Why $t = \tan^2 x$**.** The denominator $\cos^4 x + \sin^4 x$ becomes $1 + \tan^4 x$ after the divide; numerator becomes $\tan x \sec^2 x$, which is exactly $\frac{1}{2} d(\tan^2 x)$.

Step 1. Divide by $\cos^4 x$.

Step 2. Sub $t = \tan^2 x$.

Step 3. $\tan^{-1}$ on $[0, 1]$ gives $\pi/4$; halved gives $\pi/8$.

Final Answer: $\pi/8$

Q 7.26 **Evaluate** $\displaystyle\int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos^2 x + 4 \sin^2 x}$.

SOLUTION

Concept used. Divide top and bottom by $\cos^2 x$.

Step 1. Rewrite the denominator using $\cos^2 + \sin^2 = 1$:
$\cos^2 x + 4\sin^2 x = (\cos^2 x + \sin^2 x) + 3\sin^2 x = 1 + 3\sin^2 x$.

Step 2. Use $\cos^2 x = 1 - \sin^2 x$ in the numerator. The integrand becomes $\dfrac{1-\sin^2 x}{1+3\sin^2 x}$.

Step 3. Split: $\dfrac{1-\sin^2 x}{1+3\sin^2 x} = -\dfrac{1}{3} + \dfrac{4/3}{1+3\sin^2 x}$. (Verify: $-\frac{1}{3}(1+3\sin^2 x) + \frac{4}{3} = -\frac{1}{3} - \sin^2 x + \frac{4}{3} = 1 - \sin^2 x$. ✓)

Step 4. First piece: $-\frac{1}{3}\int_0^{\pi/2} dx = -\frac{\pi}{6}$.

Step 5. Second piece: $\frac{4}{3}\int_0^{\pi/2} \dfrac{dx}{1+3\sin^2 x}$. Substitute $u = \tan x$, $du = \sec^2 x\, dx$, $\sin^2 x = u^2/(1+u^2)$, $dx = du/(1+u^2)$. Then $1 + 3\sin^2 x = (1+u^2+3u^2)/(1+u^2) = (1+4u^2)/(1+u^2)$. So

$$\frac{dx}{1+3\sin^2 x} = \frac{du/(1+u^2)}{(1+4u^2)/(1+u^2)} = \frac{du}{1+4u^2}.$$

Limits: $x = 0 \to u = 0$; $x = \pi/2 \to u = \infty$.

Step 6. Integrate: $\frac{4}{3}\int_0^{\infty} \dfrac{du}{1+4u^2} = \frac{4}{3}\cdot\frac{1}{2}\tan^{-1}(2u)\Big|_0^{\infty} = \frac{2}{3}\cdot\frac{\pi}{2} = \frac{\pi}{3}$.

Step 7. Sum: $I = -\frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{6}$.

Final Answer: $\pi/6$

EXPERT'S SOLUTION *: Anvi Saluja, M.Sc Mathematics, IIT Delhi*

$u = \tan x$ **universal trick.** On $[0, \pi/2]$, u ranges over $[0, \infty)$ and any rational of $\sin^2, \cos^2$ becomes a rational of u^2.

Step 1. Convert to $1 + 4u^2$ form.

Step 2. $\tan^{-1}(2u)$ on $[0, \infty)$ gives $\pi/2$.

Step 3. Combine with the constant piece to get $\pi/6$.

Final Answer: $\pi/6$

Q 7.27 **Evaluate** $\displaystyle\int_{\pi/6}^{\pi/3} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\, dx.$

SOLUTION

Concept used. Rewrite $\sin 2x = 1 - (\sin x - \cos x)^2$ and substitute $u = \sin x - \cos x$.

Step 1. Identity: $(\sin x - \cos x)^2 = \sin^2 x - 2\sin x \cos x + \cos^2 x = 1 - \sin 2x$, so $\sin 2x = 1 - u^2$ where $u = \sin x - \cos x$.

Step 2. Differentiate: $du = (\cos x + \sin x)\,dx$. So $(\sin x + \cos x)\,dx = du$.

Step 3. Transform:

$$I = \int \frac{du}{\sqrt{1-u^2}}.$$

Step 4. Limits: $x = \pi/6 \to u = 1/2 - \sqrt{3}/2 = (1-\sqrt{3})/2$;
$x = \pi/3 \to u = \sqrt{3}/2 - 1/2 = (\sqrt{3}-1)/2$.

Step 5. Antiderivative $\sin^{-1} u$:

$$I = \sin^{-1}\frac{\sqrt{3}-1}{2} - \sin^{-1}\frac{1-\sqrt{3}}{2} = 2\sin^{-1}\frac{\sqrt{3}-1}{2}.$$

(Used $\sin^{-1}(-v) = -\sin^{-1} v$.)

Final Answer: $2\sin^{-1}\frac{\sqrt{3}-1}{2}$

EXPERT'S SOLUTION *: Adithri Krishnan, M.Sc Mathematics, IIT Madras*

Reading. The hidden identity $\sin 2x = 1 - (\sin x - \cos x)^2$ pairs with the numerator $\sin x + \cos x$ — exactly $d(\sin x - \cos x)$.

Step 1. Pair numerator $\leftrightarrow du$.

Step 2. Identify $1 - u^2$.

Step 3. $\sin^{-1}$ on the symmetric interval.

Final Answer: $2\sin^{-1}\frac{\sqrt{3}-1}{2}$

Q 7.28 **Evaluate** $\displaystyle\int_0^1 \frac{dx}{\sqrt{1+x} - \sqrt{x}}$.

SOLUTION

Concept used. Rationalise.

Step 1. Multiply by conjugate:

$$\frac{1}{\sqrt{1+x}-\sqrt{x}}\cdot\frac{\sqrt{1+x}+\sqrt{x}}{\sqrt{1+x}+\sqrt{x}}=\frac{\sqrt{1+x}+\sqrt{x}}{(1+x)-x}=\sqrt{1+x}+\sqrt{x}.$$

Step 2. Integrate piece by piece:

$$I=\int_0^1\sqrt{1+x}\,dx+\int_0^1\sqrt{x}\,dx.$$

Each is a power-rule integral with the substitution shift:

$$I=\left[\frac{2(1+x)^{3/2}}{3}\right]_0^1+\left[\frac{2x^{3/2}}{3}\right]_0^1.$$

Step 3. Evaluate: $\frac{2}{3}(2\sqrt{2}-1)+\frac{2}{3}(1)=\frac{4\sqrt{2}-2+2}{3}=\frac{4\sqrt{2}}{3}$.

Final Answer: $\frac{4\sqrt{2}}{3}$

EXPERT'S SOLUTION : *Madhav Rao, M.Sc Mathematics, IIT Hyderabad*

Conjugate. The denominator difference of square roots is the textbook conjugate setup.

Step 1. Rationalise.

Step 2. Sum of two $\sqrt{\cdot}$ integrals.

Final Answer: $\frac{4\sqrt{2}}{3}$

Q 7.29 Evaluate $\displaystyle\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}\,dx$.

SOLUTION

Concept used. Substitute $u=\sin x-\cos x$ (so $du=(\cos x+\sin x)\,dx$). Then $\sin 2x=1-u^2$, and the denominator becomes $9+16(1-u^2)=25-16u^2$.

Step 1. Transform:

$$I = \int \frac{du}{25 - 16u^2}.$$

Step 2. Limits: $x = 0 \to u = -1$; $x = \pi/4 \to u = 0$.

Step 3. Factor: $25 - 16u^2 = (5 - 4u)(5 + 4u)$. Use $\int \frac{du}{a^2 - b^2u^2} = \frac{1}{2ab} \log\left|\frac{a + bu}{a - bu}\right|$ with $a = 5$, $b = 4$:

$$I = \frac{1}{40} \log\left|\frac{5 + 4u}{5 - 4u}\right|\Bigg|_{-1}^{0}.$$

Step 4. At $u = 0$: $\log 1 = 0$. At $u = -1$: $\log|(5 - 4)/(5 + 4)| = \log(1/9) = -\log 9$.

Step 5. Subtract: $I = \frac{1}{40}[0 - (-\log 9)] = \frac{\log 9}{40} = \frac{2 \log 3}{40} = \frac{\log 3}{20}$.

Final Answer: $\frac{\log 3}{20}$

EXPERT'S SOLUTION *: Sneha Vasudev, M.Sc Mathematics, IIT Bombay*

Same family as Q27. Numerator and identity pair to give $du/(\text{quadratic in } u)$.

Step 1. Sub $u = \sin x - \cos x$.

Step 2. Apply $\int du/(a^2 - b^2u^2)$ formula.

Step 3. Plug limits, simplify.

Final Answer: $\frac{\log 3}{20}$

Q 7.30 **Evaluate** $\int_0^{\pi/2} \sin 2x \tan^{-1}(\sin x)\, dx$.

SOLUTION

Concept used. Substitute $t = \sin x$, $dt = \cos x\, dx$. Note $\sin 2x = 2 \sin x \cos x = 2t \cos x$, so $\sin 2x\, dx = 2t\, dt$.

Step 1. Transform: $I = \int_0^1 2t \tan^{-1} t\, dt$.

Step 2. Parts: $u = \tan^{-1} t$, $dv = 2t\, dt \Rightarrow du = dt/(1 + t^2)$, $v = t^2$.

$$\int 2t \tan^{-1} t\, dt = t^2 \tan^{-1} t - \int \frac{t^2}{1 + t^2}\, dt.$$

Step 3. $\int \dfrac{t^2}{1+t^2}\,dt = \int(1 - \frac{1}{1+t^2})\,dt = t - \tan^{-1} t$.

Step 4. So antiderivative is $t^2 \tan^{-1} t - t + \tan^{-1} t = (t^2+1)\tan^{-1} t - t$.

Step 5. Evaluate at $t = 1$: $(2)\cdot \pi/4 - 1 = \pi/2 - 1$. At $t = 0$: 0.

Step 6. $I = \pi/2 - 1$.

Final Answer: $\dfrac{\pi}{2} - 1$

EXPERT'S SOLUTION *: Bhuvi Patel, M.Sc Mathematics, IIT Roorkee*

Reading. $\sin 2x = 2\sin x\cos x$, so $\sin x = t$ converts $\sin 2x\,dx$ to $2t\,dt$. Then parts on $2t\tan^{-1} t$.

Step 1. Sub.

Step 2. Parts.

Step 3. Evaluate to $\pi/2 - 1$.

Final Answer: $\pi/2 - 1$

Q 7.31 **Evaluate** $\displaystyle\int_1^4 \big[|x-1| + |x-2| + |x-3|\big]\,dx$.

SOLUTION

Concept used. Three modulus integrands; split each at its breakpoint, integrate piecewise, then sum.

Step 1. On $[1,4]$: $|x-1| = x-1$ throughout (since $x \geq 1$). $|x-2| = 2-x$ for $x \in [1,2]$; $= x-2$ for $x \in [2,4]$. $|x-3| = 3-x$ for $x \in [1,3]$; $= x-3$ for $x \in [3,4]$.

Step 2. $\int_1^4 |x-1|\,dx = \int_1^4 (x-1)\,dx = \big[(x-1)^2/2\big]_1^4 = 9/2$.

Step 3. $\int_1^4 |x-2|\,dx = \int_1^2 (2-x)\,dx + \int_2^4 (x-2)\,dx = 1/2 + 2 = 5/2$.

Step 4. $\int_1^4 |x-3|\,dx = \int_1^3 (3-x)\,dx + \int_3^4 (x-3)\,dx = 2 + 1/2 = 5/2$.

Step 5. Sum: $I = 9/2 + 5/2 + 5/2 = 19/2$.

Final Answer: $19/2$

EXPERT'S SOLUTION : *Vivaan Bhasin, M.Sc Mathematics, IIT Bombay*

Per-modulus split. Each modulus has its own breakpoint; integrate piecewise.

Step 1. Modulus 1: pure positive piece.

Step 2. Modulus 2: split at $x = 2$.

Step 3. Modulus 3: split at $x = 3$.

Step 4. Sum the three.

> **Final Answer:** $19/2$

Q 7.32 **Prove that** $\int_1^3 \frac{dx}{x^2(x+1)} = \frac{2}{3} + \log\frac{2}{3}$.

SOLUTION

Concept used. Partial fractions with repeated linear factor.

Step 1. Decompose: $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$. Multiply: $1 = Ax(x+1) + B(x+1) + Cx^2$. At $x = 0$: $1 = B \Rightarrow B = 1$. At $x = -1$: $1 = C \Rightarrow C = 1$. Coefficient of x^2: $0 = A + C \Rightarrow A = -1$.

Step 2. Integrate:

$$I = -\log|x| - \frac{1}{x} + \log|x+1|.$$

Step 3. Evaluate at $x = 3$: $-\log 3 - 1/3 + \log 4$.
At $x = 1$: $-\log 1 - 1 + \log 2 = -1 + \log 2$.

Step 4. Difference: $(-\log 3 - 1/3 + \log 4) - (-1 + \log 2) =$
$-\log 3 + \log 4 - \log 2 + 1 - 1/3 = \log(4/(3 \cdot 2)) + 2/3 = \log(2/3) + 2/3$.

Step 5. This is the desired result.

> **Final Answer:** $\frac{2}{3} + \log\frac{2}{3}$ QED.

EXPERT'S SOLUTION : *Yash Bose, M.Sc Mathematics, IIT Madras*

Bookkeeping. The $-1/x$ piece comes from the repeated factor $1/x^2$.

Step 1. PF gives $A = -1$, $B = 1$, $C = 1$.

Step 2. Integrate.

Step 3. Combine endpoint values to $2/3 + \log(2/3)$.

Final Answer: Hence proved.

Q 7.33 **Prove that $\int_0^1 xe^x\,dx = 1$.**

SOLUTION

Concept used. Integration by parts with $u = x$, $dv = e^x\,dx$.

Step 1. $u = x \Rightarrow du = dx$. $dv = e^x\,dx \Rightarrow v = e^x$.

Step 2. $\int xe^x\,dx = xe^x - \int e^x\,dx = (x-1)e^x$.

Step 3. Evaluate: $F(1) - F(0) = (1-1)e - (-1)(1) = 0 + 1 = 1$.

Final Answer: 1 QED.

EXPERT'S SOLUTION : *Anaya Sehgal, M.Sc Mathematics, IIT Delhi*

Reading. One parts step; clean endpoints because $(x-1)\big|_1 = 0$.

Step 1. Parts.

Step 2. Evaluate.

Final Answer: Hence proved.

Q 7.34 **Prove that $\int_{-1}^1 x^{17}\cos^4 x\,dx = 0$.**

SOLUTION

Concept used. Parity. x^{17} is odd; $\cos^4 x$ is even; product is odd. Integral of an odd function over $[-a, a]$ is 0.

Step 1. Define $f(x) = x^{17}\cos^4 x$. Check:
$f(-x) = (-x)^{17}\cos^4(-x) = -x^{17}\cos^4 x = -f(x)$. Odd.

Step 2. By P_7(ii): $\int_{-1}^1 f(x)\,dx = 0$.

Final Answer: 0 QED.

EXPERT'S SOLUTION : *Ishaan Khurana, M.Sc Mathematics, IIT Kanpur*

Parity is decisive. No need to compute anything.

Step 1. Identify odd $\times$ even $=$ odd.

Step 2. Apply P_7.

Final Answer: Hence proved.

Q 7.35 **Prove that** $\displaystyle\int_0^{\pi/2} \sin^3 x\, dx = \frac{2}{3}$.

SOLUTION

Concept used. Write $\sin^3 x = \sin x(1 - \cos^2 x)$ and substitute $t = \cos x$.

Step 1. $\sin^3 x = \sin x - \sin x \cos^2 x$.

Step 2. $\int \sin x\, dx = -\cos x$. For the second piece, $t = \cos x$, $dt = -\sin x\, dx$:
$\int \sin x \cos^2 x\, dx = -\int t^2\, dt = -t^3/3 = -\cos^3 x/3$.

Step 3. Antiderivative: $-\cos x + \cos^3 x/3$.

Step 4. Evaluate on $[0, \pi/2]$: at $\pi/2$: $0 + 0 = 0$. At 0: $-1 + 1/3 = -2/3$. Difference: $0 - (-2/3) = 2/3$.

Final Answer: $2/3$ QED.

EXPERT'S SOLUTION : *Pranav Khurana, M.Sc Mathematics, IIT Hyderabad*

Standard reduction. Odd power of $\sin$ on $[0, \pi/2]$ yields a rational.

Step 1. Split with $\sin^2 = 1 - \cos^2$.

Step 2. Sub $t = \cos x$.

Final Answer: $2/3$.

Q 7.36 **Prove that** $\int_0^{\pi/4} 2\tan^3 x\,dx = 1 - \log 2$.

SOLUTION

Concept used. $\tan^3 x = \tan x(\sec^2 x - 1) = \tan x \sec^2 x - \tan x$.

Step 1. Split: $2\tan^3 x = 2\tan x \sec^2 x - 2\tan x$.

Step 2. $\int 2\tan x \sec^2 x\,dx$: let $u = \tan x$, $du = \sec^2 x\,dx$; integral $= u^2 = \tan^2 x$.

Step 3. $\int 2\tan x\,dx = -2\log|\cos x| = 2\log|\sec x|$.

Step 4. Antiderivative: $\tan^2 x - 2\log|\sec x|$.

Step 5. Evaluate at $\pi/4$: $1 - 2\log\sqrt{2} = 1 - \log 2$. At 0: $0 - 0 = 0$.

Step 6. Difference: $1 - \log 2$.

Final Answer: $1 - \log 2$ QED.

EXPERT'S SOLUTION *: Shaurya Krishnan, M.Sc Mathematics, IIT Bombay*

Identity. $\tan^3 = \tan\sec^2 - \tan$; both pieces integrate cleanly.

Step 1. Use the identity.

Step 2. Integrate each piece.

Step 3. Evaluate to $1 - \log 2$.

Final Answer: $1 - \log 2$.

Q 7.37 **Prove that** $\int_0^{1} \sin^{-1} x\,dx = \frac{\pi}{2} - 1$.

SOLUTION

Concept used. Parts with $u = \sin^{-1} x$, $dv = dx$.

Step 1. $du = dx/\sqrt{1-x^2}$, $v = x$.

Step 2. $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}} = x\sin^{-1} x + \sqrt{1-x^2}$.

Step 3. Evaluate at 1: $1 \cdot \pi/2 + 0 = \pi/2$. At 0: $0 + 1 = 1$.

Step 4. Difference: $\pi/2 - 1$.

Final Answer: $\pi/2 - 1$ QED.

EXPERT'S SOLUTION *: Aaradhya Mehta, M.Sc Mathematics, IIT Madras*

Single-function-with-1 trick. $\sin^{-1} x \cdot 1\, dx$ is the textbook setup.

Step 1. Parts.

Step 2. Evaluate.

Final Answer: $\pi/2 - 1$.

Q 7.38 $\int \frac{dx}{e^x + e^{-x}}$ **equals**

(A) $\tan^{-1}(e^x) + C$ **(B)** $\tan^{-1}(e^{-x}) + C$ **(C)** $\log(e^x - e^{-x}) + C$ **(D)** $\log(e^x + e^{-x}) + C$

SOLUTION

Concept used. Multiply top and bottom by e^x.

Step 1. $\frac{1}{e^x + e^{-x}} = \frac{e^x}{e^{2x} + 1}$.

Step 2. Let $t = e^x$, $dt = e^x\, dx$: $I = \int \frac{dt}{1 + t^2} = \tan^{-1} t + C = \tan^{-1}(e^x) + C$.

Step 3. Matches (A).

Final Answer: Option (A): $\tan^{-1}(e^x) + C$

EXPERT'S SOLUTION *: Aaron Singh, M.Sc Mathematics, IIT Roorkee*

Reading. Sum of e^x and e^{-x} on the bottom is the Hyperbolic-cosine setup; the $\arctan(e^x)$ antiderivative is the standard answer.

Step 1. Multiply by e^x/e^x.

Step 2. Sub $t = e^x$.

Final Answer: Option (A)

Q 7.39 $\int \dfrac{\cos 2x}{(\sin x + \cos x)^2}\, dx$ **equals**

(A) $\dfrac{-1}{\sin x + \cos x} + C$ **(B)** $\log|\sin x + \cos x| + C$ **(C)** $\log|\sin x - \cos x| + C$ **(D)** $\dfrac{1}{(\sin x + \cos x)^2}$

SOLUTION

Concept used. $\cos 2x = \cos^2 x - \sin^2 x = (\cos x - \sin x)(\cos x + \sin x)$.

Step 1. Cancel one factor of $(\sin x + \cos x)$: $\dfrac{\cos 2x}{(\sin x + \cos x)^2} = \dfrac{\cos x - \sin x}{\sin x + \cos x}$.

Step 2. Note $\dfrac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$, which is the numerator.

Step 3. Apply $\int \dfrac{D'(x)}{D(x)}\, dx = \log|D(x)| + C$: $I = \log|\sin x + \cos x| + C$. Matches (B).

> **Final Answer:** Option (B)

EXPERT'S SOLUTION *: Mahek Pillai, M.Sc Mathematics, IIT Bombay*

Reading. Numerator equals the derivative of one factor in the denominator → log.

Step 1. Spot the factoring.

Step 2. Log antiderivative.

> **Final Answer:** Option (B)

Q 7.40 **If** $f(a+b-x) = f(x)$**, then** $\int_a^b x f(x)\, dx$ **is equal to**

(A) $\dfrac{a+b}{2} \int_a^b f(b-x)\, dx$ **(B)** $\dfrac{a+b}{2} \int_a^b f(b+x)\, dx$ **(C)** $\dfrac{b-a}{2} \int_a^b f(x)\, dx$ **(D)** $\dfrac{a+b}{2} \int_a^b f(x)\, dx$

SOLUTION

Concept used. Property P_3: $\int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx$, applied to $x f(x)$.

Step 1. Let $I = \int_a^b x f(x)\, dx$. By P_3:
$I = \int_a^b (a+b-x) f(a+b-x)\, dx = \int_a^b (a+b-x) f(x)\, dx$, using $f(a+b-x) = f(x)$.

Step 2. Add the two forms: $2I = \int_a^b [x + (a+b-x)] f(x)\, dx = (a+b) \int_a^b f(x)\, dx$.

Step 3. Solve: $I = \dfrac{a+b}{2} \int_a^b f(x)\, dx$. Matches (D).

Final Answer: Option (D)

EXPERT'S SOLUTION : *Karthik Sehgal, M.Sc Mathematics, IIT Madras*

Generalises Q12. Same averaging trick — pairing x with $a+b-x$ gives the average $(a+b)/2$.

Step 1. Apply P_3.

Step 2. Symmetry of f.

Step 3. Constant $(a+b)/2$ slides out.

Final Answer: Option (D)

Key Takeaways

- Read the integrand first: rational $\to$ PF; $\sqrt{Q} \to$ complete square or trig sub; poly $\times$ trans $\to$ parts; $e^x \cdot (\dots) \to$ try $f + f'$ template.
- For definite integrals, ALWAYS check parity (P_7) and reflective symmetry (P_3, P_4) before committing to a brute-force antiderivative.
- Conjugates rationalise sum-of-roots denominators. Half-angle identities collapse $1 \pm \cos x$ and $1 \pm \sin x$.
- Power $e^{n \log x} = x^n$ is a common disguise — unmask before reaching for fancy techniques.
- Identity $\sin 2x = 1 - (\sin x - \cos x)^2$ is the key for Q27 / Q29: turns $(\sin x + \cos x)\,dx$ into a perfect du for $u = \sin x - \cos x$.

End of Miscellaneous Exercise