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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 8: Application of Integrals

About this Chapter

In this chapter we use the definite integral to compute the area of a region bounded by curves and lines. The chapter builds on the Fundamental Theorem of Calculus from Class 12th Mathematics Chapter 7 and develops two complementary techniques: the **vertical-strip method** ($A = \int_a^b y\,dx$) and the **horizontal-strip method** ($A = \int_c^d x\,dy$). The standard forms covered are circles, parabolas and ellipses, with symmetry exploited wherever possible.

Topics covered: Area as a Definite Integral • Vertical & Horizontal Strips • Symmetry of Standard Curves • Area under an Ellipse, Circle, Parabola

Quick Formula Sheet

Area between curve $y = f(x)$, the x-axis and the lines $x = a$, $x = b$:
$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx$$

Area between curve $x = g(y)$, the y-axis and the lines $y = c$, $y = d$:
$$A = \int_c^d x\,dy = \int_c^d g(y)\,dy$$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:
$A = \pi ab$ (total area)

Circle $x^2 + y^2 = a^2$:
$A = \pi a^2$ (total area)

Useful antiderivative:
$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$$

Exercise 8.1

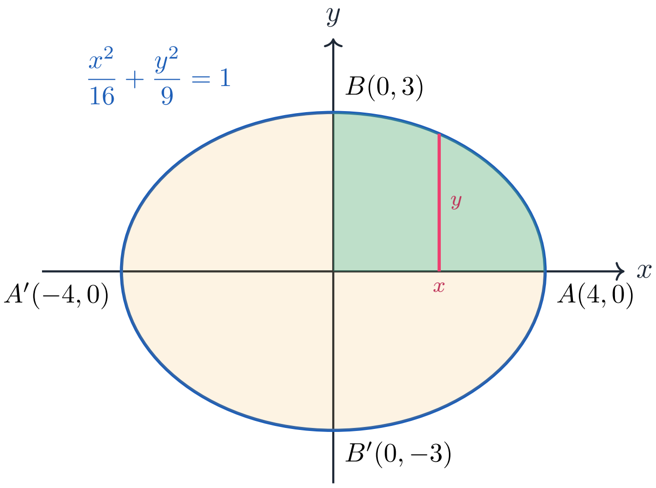
Q 8.1 **Find the area of the region bounded by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.**

SOLUTION

Concept used. An **ellipse** in standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is symmetric about both coordinate axes. Therefore the total area enclosed by the curve is four times the area of the portion lying in the first quadrant. The first-quadrant area is computed by the **vertical-strip method**: imagine a thin vertical strip of width dx and height y; its elementary area is $dA = y\,dx$, and summing these strips gives

$$A_{\text{Q1}} = \int_0^a y\,dx.$$

For the present curve, comparing with the standard form gives $a^2 = 16$, so $a = 4$, and $b^2 = 9$, so $b = 3$.



Step 1. Solve the ellipse equation for y in the first quadrant. Starting from

$$\frac{x^2}{16} + \frac{y^2}{9} = 1,$$

rearrange to isolate y^2:

$$\frac{y^2}{9} = 1 - \frac{x^2}{16} = \frac{16 - x^2}{16}.$$

Multiply both sides by 9:

$$y^2 = \frac{9\,(16 - x^2)}{16}.$$

Take the positive square root (first quadrant means $y \geq 0$):

$$y = \frac{3}{4}\sqrt{16 - x^2}.$$

Step 2. Write the area of the ellipse using symmetry.

$$A = 4\int_0^4 y\,dx = 4\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx.$$

The constant $4 \times \frac{3}{4} = 3$ comes out:

$$A = 3\int_0^4 \sqrt{16 - x^2}\,dx.$$

Step 3. Evaluate using the standard antiderivative $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$. Here $a = 4$:

$$\int_0^4 \sqrt{16 - x^2}\, dx = \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_0^4.$$

Step 4. Substitute the upper limit $x = 4$:

$$\frac{4}{2}\sqrt{16 - 16} + 8\sin^{-1}(1) = 2 \cdot 0 + 8 \cdot \frac{\pi}{2} = 4\pi.$$

Substitute the lower limit $x = 0$:

$$\frac{0}{2}\sqrt{16 - 0} + 8\sin^{-1}(0) = 0 + 0 = 0.$$

Hence

$$\int_0^4 \sqrt{16 - x^2}\, dx = 4\pi - 0 = 4\pi.$$

Step 5. Multiply by the factor 3 pulled out in Step 2:

$$A = 3 \times 4\pi = 12\pi \text{ square units.}$$

Sanity check using $A = \pi ab$ with $a = 4$, $b = 3$: $A = \pi(4)(3) = 12\pi$ ✓.

☞ **Standard antiderivative**

$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$. Derived by the substitution $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$.

Final Answer: Area of the region bounded by the ellipse $= 12\pi$ square units.

☞ Exam Tip

For any ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in standard form, the total enclosed area is simply πab. CBSE almost always expects the full integral derivation in long-answer questions, but the shortcut is gold for MCQs and verification.

EXPERT'S SOLUTION *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

Picture-first. Read the equation at sight: the semi-axes are $a = \sqrt{16} = 4$ along the x-direction and $b = \sqrt{9} = 3$ along the y-direction. The ellipse sits symmetrically about both axes, so we only need to integrate over one quadrant and multiply by 4.

Concept used. An ellipse in standard form has two mutually perpendicular axes of symmetry (the coordinate axes themselves), so the four quadrants contribute equally to the total area. The first-quadrant area is $A_{\text{Q1}} = \int_0^a y\,dx$, and the total area is $A = 4A_{\text{Q1}}$.

Step 1. *Solve for y in the upper half.* From the ellipse equation

$$y^2 = 9\left(1 - \frac{x^2}{16}\right) = \frac{9}{16}(16 - x^2) \implies y = \frac{3}{4}\sqrt{16 - x^2}.$$

Step 2. *Set up the symmetric integral.* The total area is

$$A = 4\int_0^4 \frac{3}{4}\sqrt{16 - x^2}\,dx = 3\int_0^4 \sqrt{16 - x^2}\,dx.$$

Step 3. *Trigonometric substitution.* Put $x = 4\sin\theta$, so $dx = 4\cos\theta\,d\theta$ and $\sqrt{16 - x^2} = 4\cos\theta$. Limits change: $x = 0 \Rightarrow \theta = 0$; $x = 4 \Rightarrow \theta = \frac{\pi}{2}$.

$$\int_0^4 \sqrt{16 - x^2}\,dx = \int_0^{\pi/2} (4\cos\theta)(4\cos\theta)\,d\theta = 16\int_0^{\pi/2} \cos^2\theta\,d\theta.$$

Step 4. *Half-angle identity.* Use $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$:

$$16\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta = 8\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}.$$

At $\theta = \frac{\pi}{2}$: $\frac{\pi}{2} + \frac{\sin\pi}{2} = \frac{\pi}{2}$. At $\theta = 0$: $0 + 0 = 0$. So the integral equals $8 \cdot \frac{\pi}{2} = 4\pi$.

Step 5. *Multiply by 3.* $A = 3 \times 4\pi = 12\pi$.

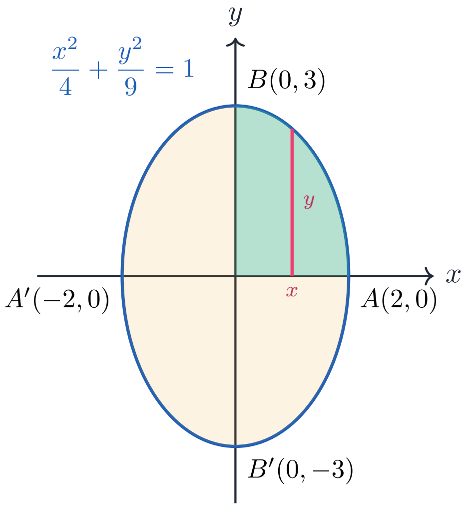
Why this matters. The same workflow — solve for the non-strip variable, exploit symmetry, evaluate the standard integral $\int \sqrt{a^2 - x^2}\,dx$ — handles every circle / ellipse area question in this chapter. Memorise the structure, not just the answer.

Final Answer: $A = 12\pi$ square units.

Q 8.2 Find the area of the region bounded by the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

SOLUTION

Concept used. Same as Q1: the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is symmetric about both axes, so the total area equals four times the first-quadrant area. Comparing with the standard form, $a^2 = 4$ gives $a = 2$, and $b^2 = 9$ gives $b = 3$. Here the *minor* axis is along the x-direction (length $2a = 4$) and the *major* axis is along the y-direction (length $2b = 6$) — the ellipse is taller than it is wide.



Step 1. Solve the ellipse equation for y in the first quadrant. From $\frac{x^2}{4}+\frac{y^2}{9}=1$:

$$\frac{y^2}{9}=1-\frac{x^2}{4}=\frac{4-x^2}{4},$$
$$y^2=\frac{9(4-x^2)}{4},$$
$$y=\frac{3}{2}\sqrt{4-x^2} \quad \text{(positive root, first quadrant)}.$$

Step 2. Apply symmetry. The required area is

$$A=4\int_0^2 y\,dx=4\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx=6\int_0^2 \sqrt{4-x^2}\,dx.$$

Step 3. Evaluate using $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$ with $a=2$:

$$\int_0^2\sqrt{4-x^2}\,dx=\left[\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}\right]_0^2.$$

Step 4. Upper limit $x=2$:

$$\frac{2}{2}\sqrt{4-4}+2\sin^{-1}(1)=1\cdot 0+2\cdot\frac{\pi}{2}=\pi.$$

Lower limit $x=0$:

$$\frac{0}{2}\sqrt{4-0}+2\sin^{-1}(0)=0+0=0.$$

So $\int_0^2\sqrt{4-x^2}\,dx=\pi$.

Step 5. Multiply by 6:

$$A=6\times\pi=6\pi \text{ square units.}$$

Sanity check using $A=\pi ab$: $A=\pi(2)(3)=6\pi$ ✓.

Final Answer: Area of the region bounded by the ellipse $=6\pi$ square units.

✗ Common Mistake

Do not assume a is always the horizontal semi-axis. In the standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the symbol a is just the denominator under x^2. Here $a = 2 < b = 3$, so the ellipse is *taller* than wide. The product πab is unaffected, but make sure your sketch is right.

EXPERT'S SOLUTION *: Diya Reddy, Ph.D Pure Mathematics, IISc Bangalore*

Picture-first. A look at $\frac{x^2}{4} + \frac{y^2}{9} = 1$ tells me the curve passes through $(\pm 2, 0)$ and $(0, \pm 3)$. The ellipse is taller than wide, so I might equally have used horizontal strips (x in terms of y) — but the formula and the symmetry argument work identically.

Concept used. For a standard-form ellipse, the four quadrant pieces are congruent under reflection in the axes; the total area is therefore four times the first-quadrant area. The first-quadrant area can be found with vertical or horizontal strips.

Step 1. *Horizontal-strip set-up.* Solving for x in the first quadrant: from $\frac{x^2}{4} + \frac{y^2}{9} = 1$,

$$x^2 = 4\left(1 - \frac{y^2}{9}\right) = \frac{4(9 - y^2)}{9} \implies x = \frac{2}{3}\sqrt{9 - y^2}.$$

Step 2. *Symmetric area integral.* Total area

$$A = 4\int_0^3 x\,dy = 4\int_0^3 \frac{2}{3}\sqrt{9 - y^2}\,dy = \frac{8}{3}\int_0^3 \sqrt{9 - y^2}\,dy.$$

Step 3. *Substitution.* Put $y = 3\sin\phi$, $dy = 3\cos\phi\,d\phi$; $\sqrt{9 - y^2} = 3\cos\phi$. Limits: $y = 0 \to \phi = 0$, $y = 3 \to \phi = \frac{\pi}{2}$.

$$\int_0^3 \sqrt{9 - y^2}\,dy = \int_0^{\pi/2} (3\cos\phi)(3\cos\phi)\,d\phi = 9\int_0^{\pi/2} \cos^2\phi\,d\phi.$$

Step 4. *Half-angle.* $\cos^2\phi = \frac{1 + \cos 2\phi}{2}$:

$$9 \cdot \frac{1}{2}\left[\phi + \frac{\sin 2\phi}{2}\right]_0^{\pi/2} = \frac{9}{2}\left[\frac{\pi}{2} + 0 - 0 - 0\right] = \frac{9\pi}{4}.$$

Step 5. *Multiply by $\frac{8}{3}$.*

$$A = \frac{8}{3} \cdot \frac{9\pi}{4} = \frac{72\pi}{12} = 6\pi.$$

Why this matters. Vertical strips and horizontal strips must give the same area, by construction. Switching between them is a powerful sanity check and often makes the algebra cleaner when one axis offers an easier substitution.

Final Answer: $A = 6\pi$ square units.

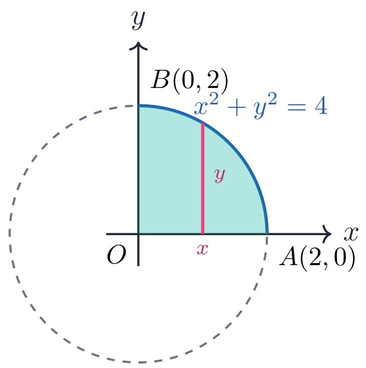
Q 8.3 **Area lying in the first quadrant and bounded by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $x = 2$ is**

(A) π **(B)** $\frac{\pi}{2}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{4}$

SOLUTION

Concept used. A circle of radius r centred at the origin encloses a total area of πr^2. For $x^2 + y^2 = 4$ we have $r^2 = 4$, i.e. $r = 2$. The region in the *first quadrant* bounded by the circle and the lines $x = 0$ (the y-axis) and $x = 2$ (a vertical tangent to the circle on the right) is exactly the quarter-disk in the first quadrant. We compute it directly by the vertical-strip method:

$$A = \int_0^2 y\,dx, \quad y = \sqrt{4 - x^2}\ (y \geq 0).$$



Step 1. Solve the circle equation for y. In the first quadrant $y \geq 0$, so

$$x^2 + y^2 = 4 \implies y^2 = 4 - x^2 \implies y = \sqrt{4 - x^2}.$$

Step 2. Set up the area integral. The region is bounded on the left by $x = 0$ and on the right by $x = 2$:

$$A = \int_0^2 \sqrt{4 - x^2}\,dx.$$

Step 3. Apply the standard antiderivative $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$ with $a = 2$:

$$A = \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2.$$

Step 4. Upper limit $x = 2$:

$$\frac{2}{2}\sqrt{4 - 4} + 2\sin^{-1}(1) = 0 + 2 \cdot \frac{\pi}{2} = \pi.$$

Lower limit $x = 0$:

$$\frac{0}{2}\sqrt{4 - 0} + 2\sin^{-1}(0) = 0 + 0 = 0.$$

Hence $A = \pi - 0 = \pi$.

Step 5. Sanity check: a circle of radius 2 has total area $\pi(2)^2 = 4\pi$. A quarter of that is π ✓. This matches option (A).

Final Answer: Area $= \pi$ square units. The correct option is **(A)**.

♥ Quarter of the disk

The lines $x = 0$ and $x = 2$ are precisely the left-most x-value of the first-quadrant arc and its right-most x-value. The region they carve out together with the arc is therefore the whole first-quadrant disk — there is no "inside slice" or "outside slice". Recognising this lets you write down $A = \frac{1}{4}\pi r^2 = \pi$ at sight.

EXPERT'S SOLUTION *: Kavya Banerjee, M.Sc Mathematics, ISI Kolkata*

Quick reading. The line $x = 2$ touches the circle of radius 2 at exactly one point, $(2, 0)$. The line $x = 0$ is the y-axis, which meets the circle at $(0, \pm 2)$. So the region trapped between the curves $x = 0$, $x = 2$, $y = 0$ (which is implied by "first quadrant") and the arc of $x^2 + y^2 = 4$ is the entire quarter-disk in the first quadrant.

Concept used. The disk $\{(x, y) : x^2 + y^2 \leq r^2\}$ is symmetric under reflection in both axes, so its four quadrants have equal area. Total disk area is πr^2; each quadrant therefore has area $\frac{1}{4}\pi r^2$.

Step 1. *Identify the region as a quarter disk.* First-quadrant arc of $x^2 + y^2 = 4$ together with the x-axis segment $[0, 2]$ and the y-axis segment $[0, 2]$ traces the boundary of one quadrant of the disk.

Step 2. *Use the area formula.* Quarter-disk area

$$A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(2)^2 = \frac{4\pi}{4} = \pi.$$

Step 3. *Cross-check by integration.*

$$A = \int_0^2 \sqrt{4 - x^2}\, dx = \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = \pi.$$

Both methods agree.

Why this matters. In MCQs, recognising a region as a "quarter-disk", "half-ellipse", or "30° circular sector" collapses pages of integration into a single multiplication. Always sketch first.

Final Answer: Option **(A)**: $A = \pi$.

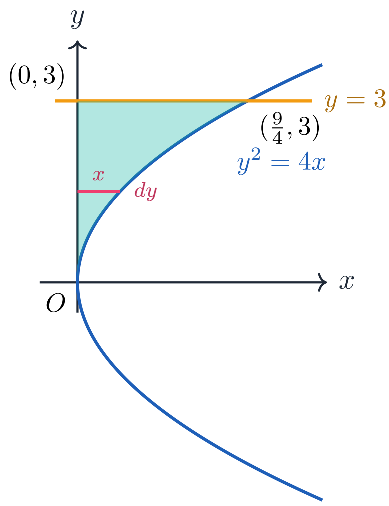
Q 8.4 **Area of the region bounded by the curve $y^2 = 4x$, the y-axis and the line $y = 3$ is**

(A) 2 **(B)** $\frac{9}{4}$ **(C)** $\frac{9}{3}$ **(D)** $\frac{9}{2}$

SOLUTION

Concept used. The curve $y^2 = 4x$ is a right-opening parabola with vertex at the origin. Comparing with the standard form $y^2 = 4ax$, we read $4a = 4$, so $a = 1$. To find the area bounded by this parabola, the y-axis ($x = 0$) and the horizontal line $y = 3$, horizontal strips are the natural choice: take a thin strip of width dy and length x (the x-coordinate of the parabola at that y). The area formula is

$$A = \int_0^3 x\,dy, \quad \text{with } x = \frac{y^2}{4} \text{ from } y^2 = 4x.$$



Step 1. Solve the parabola equation for x:

$$y^2 = 4x \implies x = \frac{y^2}{4}.$$

Step 2. Determine the integration limits. The region is bounded below by the vertex (at $y = 0$, where the parabola meets the y-axis) and above by the line $y = 3$. So y runs from 0 to 3:

$$A = \int_0^3 x\,dy = \int_0^3 \frac{y^2}{4}\,dy.$$

Step 3. Pull the constant out and integrate:

$$A = \frac{1}{4}\int_0^3 y^2\,dy = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3.$$

Step 4. Substitute limits:

$$A = \frac{1}{4}\left(\frac{3^3}{3} - \frac{0^3}{3}\right) = \frac{1}{4}\left(\frac{27}{3} - 0\right) = \frac{1}{4}\cdot 9 = \frac{9}{4}.$$

Step 5. Compare with the options. The value $\frac{9}{4}$ matches option (B).

> **Final Answer:** Area $= \dfrac{9}{4}$ square units. The correct option is **(B)**.

✗ Common Mistake

A common slip is to use *vertical* strips ($\int y\,dx$) for this region. With vertical strips the upper boundary of the region changes at $x = \frac{9}{4}$ (from the parabola to the line $y = 3$), which forces an awkward split-integral. The y-axis and the line $y = 3$ are horizontal/vertical lines parallel to the axes; horizontal strips give a clean one-shot integral.

EXPERT'S SOLUTION : *Pranav Sharma, M.Tech CS, IIT Madras*

Strategic angle. The bounding lines $x = 0$ and $y = 3$ are parallel to the coordinate axes, and the parabola opens to the right. Picture a thin horizontal slab of thickness dy at height y: its left end sits on the y-axis ($x = 0$), its right end on the parabola ($x = y^2/4$). Length of slab $= \frac{y^2}{4} - 0 = \frac{y^2}{4}$.

Concept used. For a region whose left boundary is the y-axis and whose right boundary is a curve $x = g(y)$, between horizontal lines $y = c$ and $y = d$,

$$A = \int_c^d g(y)\,dy.$$

This is the horizontal-strip version of the fundamental area integral.

Step 1. *Boundaries.* Bottom $y = 0$ (the parabola meets the y-axis at the origin), top $y = 3$ (given), left $x = 0$, right $x = \frac{y^2}{4}$.

Step 2. *Set up.*

$$A = \int_0^3 \left(\frac{y^2}{4} - 0\right) dy = \frac{1}{4}\int_0^3 y^2\,dy.$$

Step 3. *Antiderivative.* $\displaystyle\int y^2\,dy = \frac{y^3}{3}$. Hence

$$A = \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{4}\left(\frac{27}{3} - 0\right) = \frac{1}{4}(9) = \frac{9}{4}.$$

Step 4. *Verify by a sketch-check.* The region is a thin "sliver" that widens with y; its area is positive and less than the bounding rectangle $\{0 \le x \le 9/4,\ 0 \le y \le 3\}$ whose area is $\frac{9}{4} \times 3 = \frac{27}{4} = 6.75$. Our value 2.25 is well inside that bound ✓.

Why this matters. The choice between $\int y\,dx$ and $\int x\,dy$ is rarely a matter of taste: the bounding curves dictate which one keeps the integrand a single function. Pick the strip direction that gives a single-formula integrand over the whole interval.

Final Answer: Option **(B)**: $A = \dfrac{9}{4}$.

Key Takeaways

- Area under a curve $y = f(x)$ between the ordinates $x = a$ and $x = b$ is $A = \displaystyle\int_a^b y\,dx$. The horizontal version is $A = \displaystyle\int_c^d x\,dy$.
- For a standard-form ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, the total enclosed area is πab. Always derive it for long-answer questions, quote it for MCQs.
- For a circle $x^2 + y^2 = a^2$, total area $= \pi a^2$. Any quadrant or sector area follows from symmetry.
- For a parabola $y^2 = 4ax$, switching to horizontal strips (treating $x = g(y)$) usually makes the algebra cleaner.
- Always sketch the region first — the diagram tells you which strip direction to choose and whether a split-integral is needed.
- Master integral: $\displaystyle\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$

End of Exercise 8.1