Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 8: Application of Integrals

About this Chapter

The Miscellaneous Exercise consolidates the area-by-integration techniques developed in this chapter. You will compute areas under power curves, handle modulus and signed integrands, and apply the **vertical-strip** and **horizontal-strip** methods to trigonometric and odd-symmetry curves. Sign-handling around the x-axis ($|f(x)|$ versus $f(x)$) is the most common pitfall.

Topics covered: Area under Power Curves • Modulus Integrands • Trigonometric Curves over a Period • Signed vs Unsigned Area

Quick Formula Sheet

Area under $y = f(x)$ from $x = a$ to $x = b$ (curve above x-axis):

$$A = \int_a^b f(x)\,dx$$

When the curve dips below the x-axis use absolute value:

$$A = \int_a^b |f(x)|\,dx$$

Power-rule antiderivative:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$$

Cosine and sine integrals: $\int \cos x\,dx = \sin x + C$, $\int \sin x\,dx = -\cos x + C$

Modulus split: $|x+3| = x+3$ if $x \geq -3$, $-(x+3)$ if $x < -3$.

Miscellaneous Exercise on Chapter 8

Q 8.1 **Find the area under the given curves and given lines:**

(i) $y = x^2$**,** $x = 1$**,** $x = 2$ **and the** x**-axis.**

(ii) $y = x^4$**,** $x = 1$**,** $x = 5$ **and the** x**-axis.**

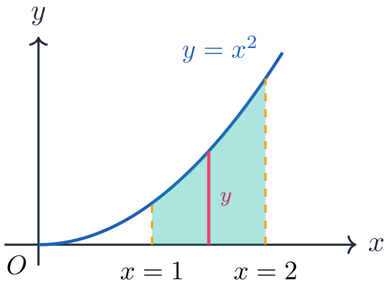
SOLUTION

Concept used. For a curve $y = f(x)$ that lies above the x-axis on $[a, b]$, the area between the curve, the x-axis and the vertical lines $x = a$, $x = b$ is the definite integral

$$A = \int_a^b f(x)\,dx.$$

Both parts of this question give a curve of the form $y = x^n$ ($n = 2$ or $n = 4$) on an interval where $x > 0$, so $y > 0$ throughout and the curve lies above the x-axis. The required antiderivative is the **power-rule antiderivative**

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$$



(i) Shaded region under $y = x^2$ between $x = 1$ and $x = 2$.

Part (i): $y = x^2$, between $x = 1$ and $x = 2$.

Step 1. Set up the definite integral:

$$A_1 = \int_1^2 x^2\,dx.$$

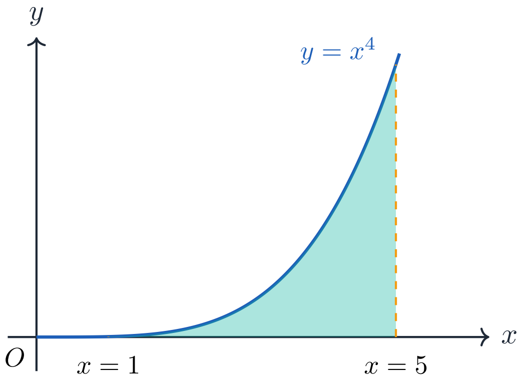
Step 2. Apply the power rule with $n = 2$:

$$A_1 = \left[\frac{x^3}{3}\right]_1^2.$$

Step 3. Substitute the limits:

$$A_1 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

> **Final Answer:** Area under $y = x^2$ from $x = 1$ to $x = 2$ is $\frac{7}{3}$ square units.



(scale: y-axis compressed)

(ii) Shaded region under $y = x^4$ between $x = 1$ and $x = 5$.

Part (ii): $y = x^4$, between $x = 1$ and $x = 5$.

Step 1. Set up the integral:

$$A_2 = \int_1^5 x^4 \, dx.$$

Step 2. Power rule with $n = 4$:

$$A_2 = \left[\frac{x^5}{5}\right]_1^5.$$

Step 3. Substitute the limits:

$$A_2 = \frac{5^5}{5} - \frac{1^5}{5} = \frac{3125}{5} - \frac{1}{5}.$$

Compute $\frac{3125}{5} = 625$, so

$$A_2 = 625 - \frac{1}{5} = \frac{3125 - 1}{5} = \frac{3124}{5}.$$

Final Answer: Area under $y = x^4$ from $x = 1$ to $x = 5$ is $\frac{3124}{5}$ square units.

☞ **Check sign at sight**

For any even power $y = x^{2k}$ $(k \in \mathbb{N})$, $y \geq 0$ for all real x, so the curve never dips below the x-axis. The plain integral $\int f(x)\,dx$ already gives the unsigned area; no modulus is needed.

EXPERT'S SOLUTION : *Vivaan Joshi, M.Sc Mathematics, IIT Kanpur*

Strategic angle. Both parts are vanilla applications of the power-rule antiderivative. The only thing to check is whether the curve stays above the x-axis on the integration interval — it does, in both parts, because $x^n \geq 0$ whenever n is even and x is real.

Concept used. For a non-negative function on $[a, b]$,

$$\text{Area beneath curve} = \int_a^b f(x)\,dx,$$

evaluated via the **Fundamental Theorem of Calculus**: if $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

Step 1. *Part (i): antiderivative of* x^2. $F(x) = \dfrac{x^3}{3}$. Apply FTC:

$$A_1 = F(2) - F(1) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

Step 2. *Part (ii): antiderivative of* x^4. $F(x) = \dfrac{x^5}{5}$.

$$A_2 = F(5) - F(1) = \frac{3125}{5} - \frac{1}{5} = \frac{3124}{5}.$$

Step 3. *Sanity check (i).* The bounding rectangle for the (i) region is $[1, 2] \times [0, 4]$, area 4. Our $\frac{7}{3} \approx 2.33$ is comfortably less than 4 ✓.

Step 4. *Sanity check (ii).* The bounding rectangle for the (ii) region is $[1, 5] \times [0, 625]$, area 2500. Our $\frac{3124}{5} = 624.8$ is well below 2500 ✓.

Why this matters. The power rule is the workhorse of every polynomial-area question — the algebra never gets harder than this. What students lose marks on is forgetting to use $|f(x)|$ when the curve crosses the axis. Always sketch first.

> **Final Answer:** $A_1 = \dfrac{7}{3}, \quad A_2 = \dfrac{3124}{5}.$

Q 8.2 Sketch the graph of $y = |x + 3|$ and evaluate $\int_{-6}^{0} |x + 3|\,dx$.

SOLUTION

Concept used. The **absolute-value function** $|u|$ is defined piecewise:

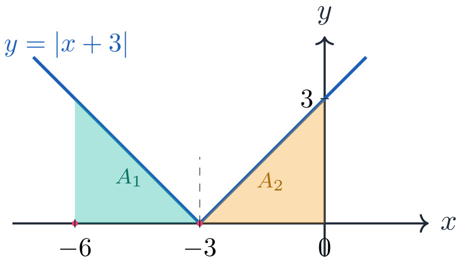
$$|u| = \begin{cases} u, & u \geq 0 \\ -u, & u < 0. \end{cases}$$

Applied to $u = x + 3$:

$$|x + 3| = \begin{cases} x + 3, & x \geq -3 \\ -(x + 3) = -x - 3, & x < -3. \end{cases}$$

So $y = |x+3|$ is a V-shaped graph with corner at $x = -3$, $y = 0$, made up of two straight half-lines of slope ± 1. Because the integrand changes formula at $x = -3$, we split the integral at that point:

$$\int_{-6}^{0} |x+3|\,dx = \int_{-6}^{-3} (-x-3)\,dx + \int_{-3}^{0} (x+3)\,dx.$$



Step 1. *Left piece,* $x \in [-6,-3]$. Here $x < -3$, so $|x+3| = -x-3$.

$$A_1 = \int_{-6}^{-3} (-x-3)\,dx.$$

The antiderivative is

$$\int (-x-3)\,dx = -\frac{x^2}{2} - 3x + C.$$

Applying the limits,

$$A_1 = \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3}.$$

Step 2. Substitute the upper limit $x = -3$:

$$-\frac{(-3)^2}{2} - 3(-3) = -\frac{9}{2} + 9 = \frac{-9+18}{2} = \frac{9}{2}.$$

Substitute the lower limit $x = -6$:

$$-\frac{(-6)^2}{2} - 3(-6) = -\frac{36}{2} + 18 = -18 + 18 = 0.$$

Therefore

$$A_1 = \frac{9}{2} - 0 = \frac{9}{2}.$$

Step 3. *Right piece,* $x \in [-3,0]$. Here $x \geq -3$, so $|x+3| = x+3$.

$$A_2 = \int_{-3}^{0} (x+3)\,dx.$$

Antiderivative: $\int (x+3)\,dx = \frac{x^2}{2} + 3x + C.$

$$A_2 = \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}.$$

Step 4. Substitute the upper limit $x = 0$:

$$\frac{0^2}{2} + 3(0) = 0.$$

Substitute the lower limit $x = -3$:

$$\frac{(-3)^2}{2} + 3(-3) = \frac{9}{2} - 9 = \frac{9-18}{2} = -\frac{9}{2}.$$

Therefore

$$A_2 = 0 - \left(-\frac{9}{2}\right) = \frac{9}{2}.$$

Step 5. Add the two pieces:

$$\int_{-6}^{0} |x+3|\,dx = A_1 + A_2 = \frac{9}{2} + \frac{9}{2} = 9.$$

Geometric check. The shaded region is two congruent right-angled triangles with legs 3 and 3. Each has area $\frac{1}{2}(3)(3) = \frac{9}{2}$; total $= 2 \cdot \frac{9}{2} = 9$ ✓.

Final Answer: $\int_{-6}^{0} |x+3|\,dx = 9$ square units.

✗ Common Mistake

A frequent error is to evaluate $\int_{-6}^{0}(x+3)\,dx$ without the modulus. That integral is $\left[\frac{x^2}{2} + 3x\right]_{-6}^{0} = 0 - (18 - 18) = 0$, which is the *signed* integral of $x + 3$ — not the geometric area of the region under $|x+3|$. The modulus must always be split at the zero of the inside expression.

EXPERT'S SOLUTION : *Tara Nair, M.Sc Applied Mathematics, IIT Kanpur*

Picture-first. Plot $y = |x+3|$: it is the graph of $y = |x|$ shifted three units to the left, so the vertex sits at $(-3, 0)$, and the two arms have slopes -1 (for $x < -3$) and $+1$ (for $x > -3$). On $[-6, 0]$, the graph is two straight segments meeting at $(-3, 0)$.

Concept used. Geometric area below a piecewise-linear graph and above the x-axis equals the sum of triangle (or trapezium) areas. Algebraically, $\int_a^b |u(x)|\,dx$ is split at every zero of $u(x)$ inside $[a, b]$.

Step 1. *Identify the zero of $u(x) = x + 3$ inside $[-6, 0]$.* $u = 0 \Rightarrow x = -3$. So split the integral at -3:

$$\int_{-6}^{0} |x+3|\,dx = \int_{-6}^{-3} -(x+3)\,dx + \int_{-3}^{0} (x+3)\,dx.$$

Step 2. *Recognise each piece as a triangle.* On $[-6, -3]$ the height varies linearly from

$|-6+3| = 3$ down to 0: right triangle with legs 3 (horizontal) and 3 (vertical), area $\frac{1}{2}(3)(3) = \frac{9}{2}$. On $[-3, 0]$ the height climbs linearly from 0 to $|0+3| = 3$: another right triangle, same legs, same area $\frac{9}{2}$.

Step 3. *Sum.* $\dfrac{9}{2} + \dfrac{9}{2} = 9$.

Step 4. *Algebraic verification.* $\displaystyle\int_{-6}^{-3} -(x+3)\,dx = \left[-\frac{x^2}{2} - 3x\right]_{-6}^{-3} = \frac{9}{2} - 0 = \frac{9}{2}$, and $\displaystyle\int_{-3}^{0} (x+3)\,dx = \left[\frac{x^2}{2} + 3x\right]_{-3}^{0} = 0 - \left(-\frac{9}{2}\right) = \frac{9}{2}$, sum $= 9$. Matches the geometric answer exactly.

Why this matters. For piecewise-linear integrands, the "triangle/trapezium" geometric reading is the fastest sanity check in the chapter. Set up the integral algebraically, then ask: would elementary geometry have given the same answer? When it does, you know your sign-handling is right.

Final Answer: Integral $= 9$ square units.

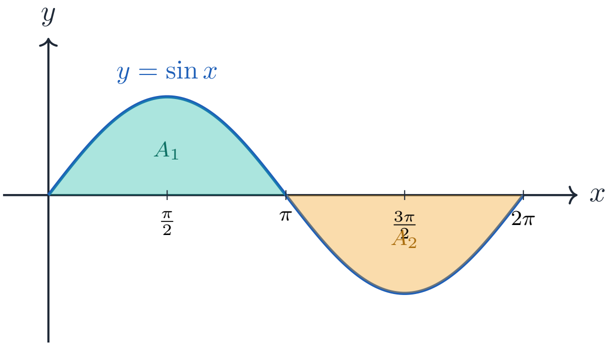
Q 8.3 **Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.**

SOLUTION

Concept used. The graph of $y = \sin x$ on $[0, 2\pi]$ has two arches: a positive arch on $[0, \pi]$ (lying above the x-axis) and a negative arch on $[\pi, 2\pi]$ (lying below the x-axis). The **geometric area** of the region between the curve and the x-axis is the sum of the unsigned areas of the two arches. We therefore use the absolute value of $\sin x$:

$$A = \int_0^{2\pi} |\sin x|\,dx = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx.$$

The required antiderivatives are $\displaystyle\int \sin x\,dx = -\cos x + C$.



Step 1. *First arch,* $x \in [0, \pi]$. Here $\sin x \geq 0$:

$$A_1 = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}.$$

Substitute:

$$A_1 = (-\cos \pi) - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2.$$

Step 2. *Second arch,* $x \in [\pi, 2\pi]$. Here $\sin x \leq 0$, so $|\sin x| = -\sin x$:

$$A_2 = \int_{\pi}^{2\pi} (-\sin x) \, dx = [\cos x]_{\pi}^{2\pi}.$$

Substitute:

$$A_2 = \cos 2\pi - \cos \pi = 1 - (-1) = 1 + 1 = 2.$$

Step 3. *Add the two pieces.*

$$A = A_1 + A_2 = 2 + 2 = 4.$$

Step 4. *Symmetry sanity check.* By symmetry of $|\sin x|$ on each half period, the two arches enclose equal areas, namely 2 each. The full integral over a period of $\sin$ gives the *signed* value 0; the modulus integral gives 4.

> **Final Answer:** Area bounded by $y = \sin x$ between $x = 0$ and $x = 2\pi$ is 4 square units.

♥ Signed vs unsigned

$\int_0^{2\pi} \sin x \, dx = [-\cos x]_0^{2\pi} = 0$, but the geometric area enclosed by the curve and the x-axis is 4. The difference is exactly what motivates the absolute-value formula $A = \int |f(x)| \, dx$: the integral cancels equal positive and negative contributions, but the picture does not.

EXPERT'S SOLUTION : *Krishna Verma, Ph.D Mathematics, IIT Delhi*

Strategic angle. The sine curve is point-symmetric about $(\pi, 0)$ on $[0, 2\pi]$: the right-hand arch is a mirror image of the left-hand arch reflected through that point. Therefore the unsigned area of the right arch equals that of the left arch.

Concept used. For any function f that changes sign at finitely many points $c_1 < c_2 < \cdots < c_k$ inside $[a, b]$, the unsigned area between $y = f(x)$ and the x-axis is

$$\int_a^b |f(x)| \, dx = \sum_{i=0}^{k} \left| \int_{c_i}^{c_{i+1}} f(x) \, dx \right|,$$

where $c_0 = a$ and $c_{k+1} = b$.

Step 1. *Find sign changes inside* $(0, 2\pi)$. $\sin x = 0$ at $x = \pi$ (the only interior zero on $(0, 2\pi)$). So split: $[0, \pi] \cup [\pi, 2\pi]$.

Step 2. *Compute the signed integrals.*

$$I_1 = \int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = 2,$$

$$I_2 = \int_{\pi}^{2\pi} \sin x \, dx = [-\cos x]_{\pi}^{2\pi} = -2.$$

Step 3. *Add the absolute values.* $|I_1| + |I_2| = 2 + 2 = 4$.

Step 4. *Cross-check by symmetry.* Each arch of $|\sin x|$ has the same area as the first arch — and that arch has area $\int_0^{\pi} \sin x \, dx = 2$. Two arches $\Rightarrow$ total 4.

Why this matters. Many CBSE questions hide a sign-change. Whenever you see an integrand that takes both signs on the interval, split at the zeros, integrate each piece, then add absolute values. The signed integral alone is for net displacement / net flux, not area.

Final Answer: $A = 4$ square units.

Q 8.4 **Area bounded by the curve $y = x^3$, the x-axis and the ordinates $x = -2$ and $x = 1$ is**

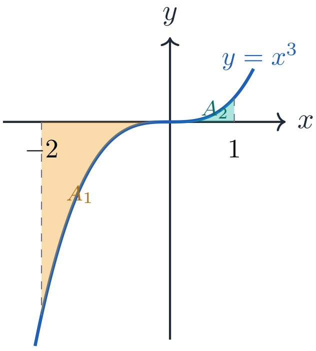
(A) -9 **(B)** $\frac{-15}{4}$ **(C)** $\frac{15}{4}$ **(D)** $\frac{17}{4}$

SOLUTION

Concept used. The cubic $y = x^3$ is an odd function: $y < 0$ for $x < 0$ and $y > 0$ for $x > 0$. So on the interval $[-2, 1]$ the curve dips below the x-axis on $[-2, 0]$ and rises above on $[0, 1]$. Geometric area is non-negative; therefore we use $|x^3|$:

$$A = \int_{-2}^{1} |x^3| \, dx = \int_{-2}^{0} (-x^3) \, dx + \int_0^1 x^3 \, dx.$$

Antiderivative needed: $\int x^3 \, dx = \frac{x^4}{4} + C$.



Step 1. *Left piece, $x \in [-2, 0]$, below the axis.*

$$A_1 = \int_{-2}^{0} |x^3|\,dx = \int_{-2}^{0} (-x^3)\,dx = -\left[\frac{x^4}{4}\right]_{-2}^{0}.$$

Substitute limits:

$$A_1 = -\left(\frac{0^4}{4} - \frac{(-2)^4}{4}\right) = -\left(0 - \frac{16}{4}\right) = -(-4) = 4.$$

Step 2. *Right piece, $x \in [0, 1]$, above the axis.*

$$A_2 = \int_{0}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4}.$$

Step 3. *Add.*

$$A = A_1 + A_2 = 4 + \frac{1}{4} = \frac{16+1}{4} = \frac{17}{4}.$$

Step 4. Compare with the options: $\frac{17}{4}$ matches option (D).

> **Final Answer:** Required area $= \dfrac{17}{4}$ square units. Correct option is **(D)**.

✗ Common Mistake

The trap option here is (B) $-\frac{15}{4}$, obtained by writing $\displaystyle\int_{-2}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-2}^{1} = \frac{1}{4} - \frac{16}{4} = -\frac{15}{4}$ without the modulus. That is the *signed* integral, not the geometric area. Area is never negative, so any negative option must be wrong on principle.

EXPERT'S SOLUTION *: Aditya Pillai, B.Tech CSE, IIT Roorkee*

Quick reading. x^3 is odd. On a symmetric interval like $[-1, 1]$ the signed integral would vanish, but $[-2, 1]$ is *not* symmetric, so we cannot use that shortcut. Sign-change at $x = 0$; split.

Concept used. Whenever a polynomial $f(x)$ changes sign at an interior point $c \in (a,b)$, $\int_a^b |f| = |\int_a^c f| + |\int_c^b f|$.

Step 1. *Find sign change.* $x^3 = 0 \Rightarrow x = 0$. So split at 0.

Step 2. *Compute the left piece.* Using $\displaystyle\int x^3\,dx = \frac{x^4}{4} + C$, we have

$$\int_{-2}^{0} x^3\,dx = \frac{0^4}{4} - \frac{(-2)^4}{4} = 0 - \frac{16}{4} = -4.$$

Step 3. *Compute the right piece.* $\displaystyle\int_0^1 x^3\,dx = \frac{1^4}{4} - \frac{0^4}{4} = \frac{1}{4} - 0 = \frac{1}{4}.$

Step 4. *Take absolute values and add.* $A = |-4| + |\frac{1}{4}| = 4 + \frac{1}{4} = \frac{17}{4}$.

Step 5. *Eliminate by inspection.* Options (A) and (B) are negative — impossible for area. Option (C) $\frac{15}{4}$ is the absolute value of the signed integral, a classic distractor. Only (D) $\frac{17}{4}$ matches both arches' areas.

Why this matters. For MCQs, knowing which distractor corresponds to "forgot the modulus" lets you spot the trap before doing the algebra. Always sketch the cubic mentally first.

> **Final Answer:** Option **(D)**: $A = \dfrac{17}{4}$.

Q 8.5 The area bounded by the curve $y = x\,|x|$, the x-axis and the ordinates $x = -1$ and $x = 1$ is given by

(A) 0 **(B)** $\frac{1}{3}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{4}{3}$

[Hint: $y = x^2$ if $x > 0$ and $y = -x^2$ if $x < 0$.]

SOLUTION

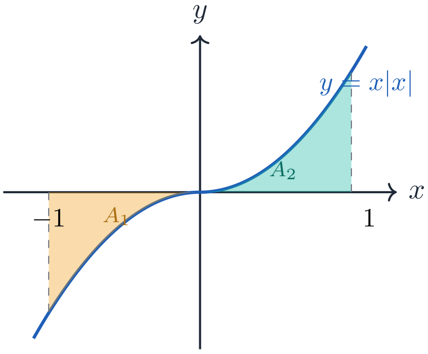
Concept used. The function $y = x|x|$ unpacks as

$$y = x|x| = \begin{cases} x \cdot x = x^2, & x \geq 0 \\ x \cdot (-x) = -x^2, & x < 0. \end{cases}$$

So on $[-1, 0]$ the curve is $y = -x^2$ (below the x-axis), and on $[0, 1]$ the curve is $y = x^2$ (above the x-axis). Geometric area uses the absolute value of the integrand:

$$A = \int_{-1}^{1} \big|\, x|x| \,\big|\,dx = \int_{-1}^{0} |-x^2|\,dx + \int_0^1 |x^2|\,dx = \int_{-1}^{0} x^2\,dx + \int_0^1 x^2\,dx,$$

because $x^2 \geq 0$ for all real x, so $|x^2| = x^2$ and $|-x^2| = x^2$.



Step 1. *Left piece,* $x \in [-1, 0]$. Here $y = -x^2 \le 0$, so $|y| = x^2$:

$$A_1 = \int_{-1}^{0} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{0} = \frac{0}{3} - \frac{(-1)^3}{3} = 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}.$$

Step 2. *Right piece,* $x \in [0, 1]$. Here $y = x^2 \ge 0$, so $|y| = x^2$:

$$A_2 = \int_{0}^{1} x^2\,dx = \left[\frac{x^3}{3}\right]_{0}^{1} = \frac{1}{3} - 0 = \frac{1}{3}.$$

Step 3. *Add.*

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Step 4. Compare with the options: $\frac{2}{3}$ matches option (C).

> **Final Answer:** Required area $= \dfrac{2}{3}$ square units. Correct option is **(C)**.

☞ Odd-function shortcut

$y = x|x|$ is an odd function (replace x by $-x$: $(-x)|-x| = -x|x|$). Its unsigned area on a symmetric interval $[-a, a]$ is therefore twice the area on $[0, a]$. Here $A = 2\int_0^1 x^2\,dx = 2 \cdot \frac{1}{3} = \frac{2}{3}$. The signed integral would be 0 — option (A) is the classic distractor.

EXPERT'S SOLUTION : *Ananya Kapoor, M.Sc Mathematics, IIT Bombay*

Structural observation. The graph of $y = x|x|$ is a smooth S-shape passing through the origin: a downward $-x^2$ branch on the left and an upward x^2 branch on the right. Because of the odd symmetry, the left and right pieces have equal unsigned area.

Concept used. For an odd function f (i.e. $f(-x) = -f(x)$), $\int_{-a}^{a} |f(x)|\,dx = 2\int_0^a |f(x)|\,dx$. This halves the work.

Step 1. *Verify oddness.* $f(-x) = (-x)|-x| = -x \cdot x = -x^2$ for $x \ge 0$; and $f(x) = x|x| = x \cdot x = x^2$ for $x \ge 0$. So $f(-x) = -f(x)$. Yes, odd.

Step 2. *Reduce.* $A = 2\int_0^1 |x^2|\,dx = 2\int_0^1 x^2\,dx$.

Step 3. *Integrate.* $\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$.

Step 4. *Conclude.* $A = 2 \cdot \frac{1}{3} = \frac{2}{3}$.

Why this matters. Recognising odd/even symmetry shaves a full integration step and protects against the "signed integral $= 0$" trap. The distractor option (A) is precisely what you would get by forgetting the modulus on an odd function over a symmetric interval.

> **Final Answer:** Option **(C)**: $A = \frac{2}{3}$.

Key Takeaways

- For any curve $y = f(x)$, geometric area on $[a, b]$ is $\displaystyle\int_a^b |f(x)|\,dx$, not just $\int_a^b f(x)\,dx$.
- Whenever f changes sign, split the integral at every zero of f inside $[a, b]$, then add the absolute values of the signed pieces.
- Power rule: $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$.
- Trigonometric antiderivatives: $\int \sin x = -\cos x + C$, $\int \cos x = \sin x + C$.
- For an odd function on a symmetric interval, $\int |f| = 2\int_0^a |f|$; the signed integral $\int_{-a}^a f$ is 0, which is the classic CBSE distractor.
- Always sketch the graph first — the picture tells you where the sign changes and where to split.

End of Miscellaneous Exercise on Chapter 8