



Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 9: Differential Equations

About this Chapter

Exercise 9.3 trains the **variable-separable** method. The idea: if $\frac{dy}{dx} = \frac{g(x)}{h(y)}$, rearrange to $h(y) dy = g(x) dx$ and integrate both sides. The first ten questions ask for general solutions; the next ones add initial conditions to fix the constant; the last group are real-world models (curves through a given point, balloon inflation, compound interest, bacterial growth).

Topics covered: Separating the variables • General solution • Particular solution • Modelling: growth, decay, compound interest

Quick Formula Sheet

Separation:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)} \implies \int h(y) dy = \int g(x) dx + C.$$

Standard integrals:

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C,$$
$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C.$$

Growth/decay:

$$\frac{dN}{dt} = kN \implies N(t) = N_0 e^{kt}.$$

Exercise 9.3

Questions 1–10: find the general solution. Questions 11–14: find a particular solution. Questions 15–22: word problems. Question 23: MCQ.

Q9.1 $\frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$.

SOLUTION

Concept used. Variables separable: an equation $\frac{dy}{dx} = f(x)$ (depending on x only) can be solved by direct integration $y = \int f(x) dx + C$. We simplify the RHS using the half-angle identities $1 - \cos x = 2 \sin^2(x/2)$ and $1 + \cos x = 2 \cos^2(x/2)$.

Step 1. Apply the half-angle identities:

$$\frac{1 - \cos x}{1 + \cos x} = \frac{2 \sin^2(x/2)}{2 \cos^2(x/2)} = \tan^2(x/2).$$

Step 2. Use the identity $\tan^2 \theta = \sec^2 \theta - 1$:

$$\frac{dy}{dx} = \sec^2(x/2) - 1.$$

Step 3. Integrate both sides with respect to x . Recall $\int \sec^2(x/2) dx = 2 \tan(x/2)$:

$$y = 2 \tan(x/2) - x + C.$$

Final Answer: $y = 2 \tan(x/2) - x + C$.

EXPERT'S SOLUTION : Aarav Sharma, M.Sc Mathematics, IIT Bombay

Strategic angle. Whenever you see $\frac{1 \pm \cos x}{1 \mp \cos x}$, reach for the half-angle identities first; they collapse the fraction to a single trig power.

Step 1. $\frac{1 - \cos x}{1 + \cos x} = \tan^2(x/2)$.

Step 2. $\tan^2(x/2) = \sec^2(x/2) - 1$, so $dy = (\sec^2(x/2) - 1) dx$.

Step 3. Integrate: $\int \sec^2(x/2) dx = 2 \tan(x/2)$; $\int dx = x$.

Step 4. Hence $y = 2 \tan(x/2) - x + C$.

Final Answer: $y = 2 \tan(x/2) - x + C$.

Half-angle identities

$$1 - \cos x = 2 \sin^2(x/2), \quad 1 + \cos x = 2 \cos^2(x/2), \quad \sin x = 2 \sin(x/2) \cos(x/2).$$

Q 9.2 $\frac{dy}{dx} = \sqrt{4 - y^2} \quad (-2 < y < 2).$

SOLUTION

Concept used. Variable-separable: move the y -dependence to the LHS and the x -dependence to the RHS, then integrate. Use $\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}(y/a) + C$.

Step 1. Separate the variables. The RHS depends on y only and the LHS, after moving the $\sqrt{4 - y^2}$ across, becomes $\frac{dy}{\sqrt{4 - y^2}} = dx$.

Step 2. Integrate both sides:

$$\int \frac{dy}{\sqrt{4 - y^2}} = \int dx.$$

Step 3. Using $\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}(y/a)$ with $a = 2$:

$$\sin^{-1}\left(\frac{y}{2}\right) = x + C.$$

Step 4. Solve for y if convenient: $y = 2 \sin(x + C)$.

Final Answer: $\sin^{-1}(y/2) = x + C \Leftrightarrow y = 2 \sin(x + C)$.

EXPERT'S SOLUTION : Sneha Iyer; M.Sc Applied Mathematics, IIT Kanpur

Quick reading. The integrand $1/\sqrt{4 - y^2}$ is a textbook \sin^{-1} form with $a = 2$.

Step 1. Separate: $\frac{dy}{\sqrt{4 - y^2}} = dx$.

Step 2. Integrate LHS as $\sin^{-1}(y/2)$, RHS as x .

Step 3. Add an arbitrary constant: $\sin^{-1}(y/2) = x + C$, so $y = 2 \sin(x + C)$.

Final Answer: $y = 2 \sin(x + C)$.

Q 9.3 $\frac{dy}{dx} + y = 1$ ($y \neq 1$).

SOLUTION

Concept used. Rearrange to make variables separable. The single arbitrary constant is the constant of integration; absorb signs into it freely.

Step 1. Rewrite: $\frac{dy}{dx} = 1 - y$, so

$$\frac{dy}{1 - y} = dx.$$

Step 2. Integrate both sides:

$$\int \frac{dy}{1-y} = \int dx \implies -\log |1-y| = x + C_1.$$

Step 3. Multiply by -1 and rename constants: $\log |1-y| = -x - C_1$. Exponentiate:

$$|1-y| = e^{-x-C_1} = e^{-C_1} \cdot e^{-x}.$$

Step 4. Drop the absolute value by allowing $C = \pm e^{-C_1}$ (any nonzero real). Then

$$1-y = Ce^{-x} \implies y = 1 - Ce^{-x}.$$

Final Answer: $y = 1 - Ce^{-x}$, $C \in \mathbb{R}$.

EXPERT'S SOLUTION : Arjun Patel, M.Tech CS, IIT Madras

Strategic angle. A first-order linear DE in disguise. Either separate variables (here) or use an integrating factor; both yield the same one-parameter family.

Step 1. $dy = (1-y) dx$, i.e. $\frac{dy}{1-y} = dx$.

Step 2. $-\log |1-y| = x + C_1$, so $\log |1-y| = -x - C_1$.

Step 3. Exponentiate and absorb signs: $1-y = Ce^{-x}$, hence $y = 1 - Ce^{-x}$.

Final Answer: $y = 1 - Ce^{-x}$.

✗ Sign of the log

$\int \frac{dy}{1-y} = -\log |1-y| + C$, with a minus sign (because the derivative of $-\log |1-y|$ is $1/(1-y)$). Forgetting this minus is the most common slip on this question.

Q 9.4 $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0.$

SOLUTION

Concept used. Separate variables by dividing by $\tan x \cdot \tan y$.

Step 1. Divide both sides of the equation by $\tan x \cdot \tan y$ (assuming neither is zero):

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0.$$

Step 2. Each integrand has the form $\frac{f'(u)}{f(u)} du$ with $f(u) = \tan u$. Hence

$$\int \frac{\sec^2 x}{\tan x} dx = \log |\tan x|, \quad \int \frac{\sec^2 y}{\tan y} dy = \log |\tan y|.$$

Step 3. Add a constant of integration:

$$\log |\tan x| + \log |\tan y| = \log C,$$

which gives $\log |\tan x \cdot \tan y| = \log C$, i.e. $\tan x \tan y = C$ (allowing both signs through C).

Final Answer: $\tan x \cdot \tan y = C$.

EXPERT'S SOLUTION : Priya Gupta, Ph.D Mathematics, IIT Delhi

Quick reading. $\sec^2 u du / \tan u$ is $d(\log \tan u)$. So the equation is $d(\log \tan x) + d(\log \tan y) = 0$, integrating to $\log \tan x + \log \tan y = \text{const}$.

Step 1. Divide by $\tan x \tan y$: $\frac{\sec^2 x dx}{\tan x} + \frac{\sec^2 y dy}{\tan y} = 0$.

Step 2. Integrate: $\log |\tan x| + \log |\tan y| = \log C$.

Step 3. Combine: $\tan x \tan y = C$.

Final Answer: $\tan x \tan y = C$.

Q 9.5 $(e^x + e^{-x}) dy - (e^x - e^{-x}) dx = 0$.

SOLUTION

Concept used. Separate; recognise that the numerator $e^x - e^{-x}$ is the derivative of the denominator $e^x + e^{-x}$.

Step 1. Rearrange: $(e^x + e^{-x}) dy = (e^x - e^{-x}) dx$, so

$$dy = \frac{e^x - e^{-x}}{e^x + e^{-x}} dx.$$

Step 2. Observe that $(d/dx)(e^x + e^{-x}) = e^x - e^{-x}$. The RHS is therefore of the form du/u with $u = e^x + e^{-x}$.

Step 3. Integrate:

$$y = \log |e^x + e^{-x}| + C = \log(e^x + e^{-x}) + C,$$

(the modulus is unnecessary since $e^x + e^{-x} > 0$).

Final Answer: $y = \log(e^x + e^{-x}) + C$.

EXPERT'S SOLUTION : Vivaan Mehta, B.Tech CSE, IIT Roorkee

Structural observation. $\frac{f'(x)}{f(x)} dx$ always integrates to $\log |f(x)| + C$. Spotting this form saves a substitution.

Step 1. Numerator is $f'(x)$ where $f(x) = e^x + e^{-x}$.

Step 2. Therefore $dy = d(\log f(x))$, so $y = \log(e^x + e^{-x}) + C$.

Final Answer: $y = \log(e^x + e^{-x}) + C$.

Exam Tip

The pattern $\int \frac{f'(x)}{f(x)} dx = \log |f(x)| + C$ appears in almost every separable problem with trigonometric or exponential ratios. Train the eye to spot it.

Q 9.6 $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$.

SOLUTION

Concept used. The RHS factors as a product of a function of x and a function of y , the textbook signal for separability.

Step 1. Separate:

$$\frac{dy}{1 + y^2} = (1 + x^2) dx.$$

Step 2. Integrate both sides. LHS uses $\int dy/(1 + y^2) = \tan^{-1} y$; RHS is straightforward:

$$\tan^{-1} y = x + \frac{x^3}{3} + C.$$

Final Answer: $\tan^{-1} y = x + \frac{x^3}{3} + C$.

EXPERT'S SOLUTION : Aanya Bhat, Ph.D Pure Mathematics, IISc Bangalore

Quick reading. A product of two single-variable factors is the cleanest separable form.

Step 1. $\frac{dy}{1+y^2} = (1+x^2) dx.$

Step 2. Integrate LHS to $\tan^{-1} y$, RHS to $x + x^3/3$.

Step 3. Equate: $\tan^{-1} y = x + x^3/3 + C.$

Final Answer: $\tan^{-1} y = x + x^3/3 + C.$

Q9.7 $y \log y dx - x dy = 0.$

SOLUTION

Concept used. Separate variables and recognise $\frac{dy}{y \log y}$ as the derivative of $\log |\log y|$.

Step 1. Rearrange:

$$y \log y dx = x dy \implies \frac{dx}{x} = \frac{dy}{y \log y}.$$

Step 2. Compute the RHS integral via the substitution $u = \log y$, $du = dy/y$:

$$\int \frac{dy}{y \log y} = \int \frac{du}{u} = \log |u| + C_1 = \log |\log y| + C_1.$$

Step 3. LHS gives $\log |x| + C_2$. Combine constants:

$$\log |x| = \log |\log y| + C.$$

Step 4. Exponentiate (absorbing constants into C):

$$x = C \log y.$$

Final Answer: $x = C \log y.$

EXPERT'S SOLUTION : Rohit Singh, M.Sc Mathematics, IIT Bombay

Strategic angle. Whenever you see $\frac{dy}{y \log y}$, substitute $u = \log y$ – the integral becomes $\int du/u$, which is $\log |u|$.

Step 1. Cross-divide: $\frac{dy}{y \log y} = \frac{dx}{x}$.

Step 2. Integrate both sides: $\log |\log y| = \log |x| + C_1$.

Step 3. Exponentiate: $\log y = Cx$, giving $x = \frac{1}{C} \log y$, which we rewrite as $x = C' \log y$ (absorbing $1/C$ into C').

Final Answer: $x = C \log y$.

Q 9.8 $x^5 \frac{dy}{dx} = -y^5$.

SOLUTION

Concept used. Variable-separable. Use the power rule $\int x^n dx = \frac{x^{n+1}}{n+1}$ for $n = -5$.

Step 1. Separate:

$$\frac{dy}{y^5} = -\frac{dx}{x^5} \implies y^{-5} dy = -x^{-5} dx.$$

Step 2. Integrate:

$$\int y^{-5} dy = \frac{y^{-4}}{-4}, \quad \int -x^{-5} dx = -\frac{x^{-4}}{-4} = \frac{x^{-4}}{4}.$$

Step 3. Hence

$$-\frac{1}{4y^4} = \frac{1}{4x^4} + C_1.$$

Step 4. Multiply both sides by -4 :

$$\frac{1}{y^4} = -\frac{1}{x^4} - 4C_1.$$

Rename $-4C_1 = C$:

$$\frac{1}{y^4} + \frac{1}{x^4} = C, \quad \text{i.e. } x^{-4} + y^{-4} = C.$$

Final Answer: $x^{-4} + y^{-4} = C$.

EXPERT'S SOLUTION : Aditi Joshi, M.Sc Mathematics, ISI Kolkata

Quick reading. Symmetric form: terms in $x^{-5} dx$ and $y^{-5} dy$ add to zero, integrate to $x^{-4} + y^{-4} = \text{const.}$

Step 1. Move terms: $y^{-5} dy + x^{-5} dx = 0.$

Step 2. Integrate: $\frac{y^{-4}}{-4} + \frac{x^{-4}}{-4} = C_1.$

Step 3. Multiply by -4 : $y^{-4} + x^{-4} = C.$

Final Answer: $x^{-4} + y^{-4} = C.$

Q 9.9 $\frac{dy}{dx} = \sin^{-1} x.$

SOLUTION

Concept used. Direct integration: $y = \int \sin^{-1} x dx + C.$ Apply integration by parts with $u = \sin^{-1} x, dv = dx.$

Step 1. Take $u = \sin^{-1} x \implies du = \frac{dx}{\sqrt{1-x^2}},$ and $dv = dx \implies v = x.$

Step 2. Integration by parts $\int u dv = uv - \int v du:$

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx.$$

Step 3. For the remaining integral, let $t = 1 - x^2, dt = -2x dx:$

$$\int \frac{x}{\sqrt{1-x^2}} dx = -\frac{1}{2} \int \frac{dt}{\sqrt{t}} = -\frac{1}{2} \cdot 2\sqrt{t} = -\sqrt{1-x^2}.$$

Step 4. Combine:

$$y = x \sin^{-1} x - (-\sqrt{1-x^2}) + C = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

Final Answer: $y = x \sin^{-1} x + \sqrt{1-x^2} + C.$

EXPERT'S SOLUTION : Yash Rao, M.Sc Mathematics, IIT Bombay

Strategic angle. Integration by parts is the standard tool for inverse-trig integrands. Take the inverse-trig function as $u,$ and dx as $dv.$

Step 1. By parts: $\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x \, dx}{\sqrt{1-x^2}}$.

Step 2. Substitution $t = 1 - x^2$ handles the remaining integral; result $\sqrt{1-x^2}$.

Step 3. $y = x \sin^{-1} x + \sqrt{1-x^2} + C$.

Final Answer: $y = x \sin^{-1} x + \sqrt{1-x^2} + C$.

Integration by parts

$\int u \, dv = uv - \int v \, du$. Choose u as the function that simplifies on differentiation (here $\sin^{-1} x$ becomes a simpler algebraic expression).

Q 9.10 $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$.

SOLUTION

Concept used. Separate: divide by $\tan y \cdot (1 - e^x)$.

Step 1. Divide both sides by $\tan y \cdot (1 - e^x)$:

$$\frac{e^x}{1 - e^x} \, dx + \frac{\sec^2 y}{\tan y} \, dy = 0.$$

Step 2. For the first integrand, let $u = 1 - e^x$, so $du = -e^x \, dx$:

$$\int \frac{e^x}{1 - e^x} \, dx = - \int \frac{du}{u} = -\log |u| = -\log |1 - e^x|.$$

Step 3. For the second, $\int \frac{\sec^2 y}{\tan y} \, dy = \log |\tan y|$.

Step 4. Add and equate to a constant:

$$-\log |1 - e^x| + \log |\tan y| = \log C \implies \log \frac{|\tan y|}{|1 - e^x|} = \log C.$$

Step 5. Hence

$$\tan y = C(1 - e^x).$$

Final Answer: $\tan y = C(1 - e^x)$.

EXPERT'S SOLUTION : Ananya Kapoor, M.Tech Applied Mathematics, IIT Delhi

Quick reading. Both fractions are $f'(u)/f(u)$ forms after a sign flip: $e^x/(1 - e^x)$ comes from $-\log(1 - e^x)$; $\sec^2 y/\tan y$ comes from $\log \tan y$.

Step 1. Separate and integrate.

Step 2. $-\log|1 - e^x| + \log|\tan y| = \log C$.

Step 3. Solve: $\tan y = C(1 - e^x)$.

Final Answer: $\tan y = C(1 - e^x)$.

Q 9.11 $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$; $y = 1$ when $x = 0$.

SOLUTION

Concept used. Separate, integrate, and use the initial condition $y(0) = 1$ to fix the constant. Partial-fraction the rational function on the RHS.

Step 1. Factor the denominator: $x^3 + x^2 + x + 1 = x^2(x + 1) + (x + 1) = (x + 1)(x^2 + 1)$.

Step 2. Separate:

$$dy = \frac{2x^2 + x}{(x + 1)(x^2 + 1)} dx.$$

Step 3. Partial fractions: assume

$$\frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}.$$

Multiplying by $(x + 1)(x^2 + 1)$:

$$2x^2 + x = A(x^2 + 1) + (Bx + C)(x + 1).$$

Step 4. Set $x = -1$: $2 - 1 = A(2) + 0 \implies A = 1/2$. Compare x^2 coefficients: $2 = A + B$, so $B = 3/2$. Compare constants: $0 = A + C$, so $C = -1/2$.

Step 5. Therefore

$$\int \frac{2x^2 + x}{(x + 1)(x^2 + 1)} dx = \frac{1}{2} \log|x + 1| + \frac{3}{2} \cdot \frac{1}{2} \log(x^2 + 1) - \frac{1}{2} \tan^{-1} x.$$

That is,

$$y = \frac{1}{2} \log|x + 1| + \frac{3}{4} \log(x^2 + 1) - \frac{1}{2} \tan^{-1} x + C_0.$$

Step 6. Use $y(0) = 1$: $1 = \frac{1}{2} \log 1 + \frac{3}{4} \log 1 - \frac{1}{2} \cdot 0 + C_0 = C_0$. So $C_0 = 1$.

$$\text{Final Answer: } y = \frac{1}{2} \log|x+1| + \frac{3}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1} x + 1.$$

EXPERT'S SOLUTION : Dev Nair, Ph.D Mathematics, IIT Delhi

Strategic angle. Factor the cubic by grouping, then use partial fractions with $\frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ (since x^2+1 is an irreducible quadratic).

Step 1. $x^3 + x^2 + x + 1 = (x+1)(x^2+1)$.

Step 2. Coefficients via comparison: $A = 1/2, B = 3/2, C = -1/2$.

Step 3. Integrate the three pieces:

$$\frac{1}{2} \log|x+1|, \frac{3}{4} \log(x^2+1), -\frac{1}{2} \tan^{-1} x.$$

Step 4. Apply $y(0) = 1$ to fix the constant: $C_0 = 1$.

$$\text{Final Answer: } y = \frac{1}{2} \log|x+1| + \frac{3}{4} \log(x^2+1) - \frac{1}{2} \tan^{-1} x + 1.$$

Exam Tip

For partial fractions over irreducible quadratics x^2+a^2 , $\frac{Bx}{x^2+a^2}$ integrates to $\frac{B}{2} \log(x^2+a^2)$, and $\frac{C}{x^2+a^2}$ integrates to $\frac{C}{a} \tan^{-1}(x/a)$. Memorise these.

Q 9.12 $x(x^2-1) \frac{dy}{dx} = 1$; $y = 0$ when $x = 2$.

SOLUTION

Concept used. Variable-separable; partial fractions on $\frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)}$.

Step 1. Separate:

$$dy = \frac{dx}{x(x-1)(x+1)}.$$

Step 2. Partial fractions: assume

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Step 3. Cover-up at $x = 0$: $A = 1/[(0-1)(0+1)] = -1$. At $x = 1$: $B = 1/(1 \cdot 2) = 1/2$.

At $x = -1$: $C = 1/[(-1)(-2)] = 1/2$.

Step 4. Integrate:

$$y = -\log|x| + \frac{1}{2}\log|x-1| + \frac{1}{2}\log|x+1| + C_0.$$

Combine: $y = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| + C_0.$

Step 5. Apply $y(2) = 0$: $0 = \frac{1}{2}\log\frac{3}{4} + C_0$, so $C_0 = -\frac{1}{2}\log\frac{3}{4} = \frac{1}{2}\log\frac{4}{3}$.

Final Answer: $y = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| + \frac{1}{2}\log\frac{4}{3}$, i.e. $y = \frac{1}{2}\log\left|\frac{4(x^2-1)}{3x^2}\right|.$

EXPERT'S SOLUTION : Krishna Mehta, M.Sc Mathematics, ISI Kolkata

Strategic angle. The factorisation $x(x^2-1) = x(x-1)(x+1)$ makes the cover-up method trivial. Combine the resulting logs into a single fraction.

Step 1. Partial fractions: $-\frac{1}{x} + \frac{1/2}{x-1} + \frac{1/2}{x+1}$.

Step 2. Integrate to $-\log|x| + \frac{1}{2}\log|x^2-1| = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right|.$

Step 3. Use $y(2) = 0$: $\frac{1}{2}\log(3/4) + C_0 = 0$, so $C_0 = \frac{1}{2}\log(4/3)$.

Final Answer: $y = \frac{1}{2}\log\left|\frac{4(x^2-1)}{3x^2}\right|.$

Q 9.13 $\cos\left(\frac{dy}{dx}\right) = a$ ($a \in \mathbb{R}$); $y = 1$ when $x = 0$.

SOLUTION

Concept used. Solve algebraically for $\frac{dy}{dx}$, then integrate.

Step 1. From $\cos(dy/dx) = a$, take inverse cosine:

$$\frac{dy}{dx} = \cos^{-1} a.$$

(Here $\cos^{-1} a$ is a constant; we pick its principal value.)

Step 2. Integrate:

$$y = (\cos^{-1} a)x + C.$$

Step 3. Apply $y(0) = 1$:

$$1 = 0 + C \implies C = 1.$$

Step 4. Hence $y = x \cos^{-1} a + 1$. Solving for $\cos^{-1} a$: $\cos^{-1} a = \frac{y-1}{x}$, so

$$a = \cos\left(\frac{y-1}{x}\right).$$

Final Answer: $y = x \cos^{-1} a + 1$, or equivalently $\cos \frac{y-1}{x} = a$.

EXPERT'S SOLUTION : *Ishita Banerjee, Ph.D Mathematics, IIT Delhi*

Quick reading. Once $\frac{dy}{dx}$ is a constant, the solution is a straight line. The initial condition pins down its y -intercept.

Step 1. $\frac{dy}{dx} = \cos^{-1} a$, a constant.

Step 2. Integrate: $y = (\cos^{-1} a)x + C$.

Step 3. $y(0) = 1 \implies C = 1$.

Final Answer: $y = x \cos^{-1} a + 1$.

Q 9.14 $\frac{dy}{dx} = y \tan x$; $y = 1$ when $x = 0$.

SOLUTION

Concept used. Variables separable. Use $\int \tan x \, dx = -\log |\cos x| = \log |\sec x|$.

Step 1. Separate:

$$\frac{dy}{y} = \tan x \, dx.$$

Step 2. Integrate:

$$\log |y| = -\log |\cos x| + C_1 = \log |\sec x| + C_1.$$

Step 3. Exponentiate: $|y| = e^{C_1} \cdot |\sec x|$, i.e. $y = C \sec x$ (with $C = \pm e^{C_1}$).

Step 4. Apply $y(0) = 1$: $1 = C \sec 0 = C \cdot 1 \implies C = 1$.

Final Answer: $y = \sec x$.

EXPERT'S SOLUTION : Sanya Verma, M.Sc Mathematics, IIT Bombay

Quick reading. A first-order linear/separable equation with $P = -\tan x$; the integrating factor or direct separation both give $y = C \sec x$.

Step 1. $dy/y = \tan x dx$.

Step 2. $\log |y| = \log |\sec x| + C_1$, so $y = C \sec x$.

Step 3. $y(0) = 1$ forces $C = 1$.

Final Answer: $y = \sec x$.

Q 9.15 Find the equation of a curve passing through the point $(0, 0)$ whose differential equation is $y' = e^x \sin x$.

SOLUTION

Concept used. The slope $\frac{dy}{dx} = e^x \sin x$ depends only on x ; integrate. The point $(0, 0)$ fixes the constant.

Step 1. Integrate the RHS by parts. With $I = \int e^x \sin x dx$, use integration by parts twice.

First pass: $u_1 = \sin x$, $dv_1 = e^x dx \Rightarrow du_1 = \cos x dx$, $v_1 = e^x$:

$$I = e^x \sin x - \int e^x \cos x dx.$$

Second pass: in the new integral, $u_2 = \cos x$, $dv_2 = e^x dx$:

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx = e^x \cos x + I.$$

Step 2. Substitute back: $I = e^x \sin x - (e^x \cos x + I) = e^x \sin x - e^x \cos x - I$.

Step 3. Solve for I : $2I = e^x(\sin x - \cos x)$, so

$$I = \frac{e^x(\sin x - \cos x)}{2}.$$

Step 4. Therefore $y = \frac{e^x(\sin x - \cos x)}{2} + C$.

Step 5. Curve passes through $(0, 0)$:

$$0 = \frac{e^0(\sin 0 - \cos 0)}{2} + C = \frac{1 \cdot (0 - 1)}{2} + C = -\frac{1}{2} + C, \text{ so } C = \frac{1}{2}.$$

Step 6. Hence $y = \frac{e^x(\sin x - \cos x) + 1}{2}$, or equivalently $2y - 1 = e^x(\sin x - \cos x)$.

Final Answer: $2y - 1 = e^x(\sin x - \cos x)$.

EXPERT'S SOLUTION : Aditya Chatterjee, M.Sc Mathematics, IIT Madras

Strategic angle. $\int e^x \sin x \, dx$ and $\int e^x \cos x \, dx$ are the classic “cyclic” integration-by-parts pair. Two passes return to the original integral with a sign flip; solve algebraically.

Step 1. Compute $I = \int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x)$.

Step 2. Hence $y = \frac{1}{2}e^x(\sin x - \cos x) + C$.

Step 3. $y(0) = 0 \Rightarrow -\frac{1}{2} + C = 0 \Rightarrow C = \frac{1}{2}$.

Final Answer: $2y - 1 = e^x(\sin x - \cos x)$.

Q9.16 For the DE $xy \frac{dy}{dx} = (x+2)(y+2)$, find the solution curve passing through $(1, -1)$.

SOLUTION

Concept used. Rewrite the equation so y -functions sit with dy and x -functions with dx .

Step 1. Separate:

$$\frac{y}{y+2} dy = \frac{x+2}{x} dx.$$

Step 2. Rewrite each side by splitting:

$$\frac{y}{y+2} = 1 - \frac{2}{y+2}, \quad \frac{x+2}{x} = 1 + \frac{2}{x}.$$

Step 3. Integrate:

$$\int \left(1 - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx,$$

giving

$$y - 2 \log |y+2| = x + 2 \log |x| + C_0.$$

Step 4. Apply $(x, y) = (1, -1)$: $-1 - 2 \log |1| = 1 + 2 \log |1| + C_0$, i.e. $-1 = 1 + C_0$, so $C_0 = -2$.

Final Answer: $y - x = 2 \log |x| + 2 \log |y+2| - 2$, i.e. $y - x = 2 \log |x(y+2)| - 2$.

EXPERT'S SOLUTION : Ishaan Desai, B.Tech Engineering Physics, IIT Bombay

Strategic angle. The trick is to split $y/(y+2)$ and $(x+2)/x$ into a polynomial part plus a rational part; the integrals become elementary.

Step 1. Separate and rewrite as above.

Step 2. Integrate to $y - 2 \log |y + 2| = x + 2 \log |x| + C_0$.

Step 3. Apply $(1, -1)$: $-1 - 0 = 1 + 0 + C_0 \Rightarrow C_0 = -2$.

Step 4. Curve: $y - x + 2 = 2 \log |x(y + 2)|$.

Final Answer: $y - x + 2 = 2 \log |x(y + 2)|$.

Q 9.17 Find the equation of a curve passing through $(0, -2)$ given that at any point (x, y) on the curve, the product of the slope of its tangent and the y -coordinate equals the x -coordinate.

SOLUTION

Concept used. Translate the verbal condition into a DE, then separate and integrate.

Step 1. "Slope of tangent $\cdot y$ -coordinate = x -coordinate" means $\frac{dy}{dx} \cdot y = x$, i.e.

$$y \, dy = x \, dx.$$

Step 2. Integrate:

$$\int y \, dy = \int x \, dx \implies \frac{y^2}{2} = \frac{x^2}{2} + C_1.$$

Step 3. Multiply by 2 and rename $2C_1 = C$: $y^2 - x^2 = C$.

Step 4. Apply $(0, -2)$: $(-2)^2 - 0^2 = C \implies C = 4$.

Final Answer: $y^2 - x^2 = 4$.

EXPERT'S SOLUTION : Ananya Kapoor, M.Sc Mathematics, IIT Bombay

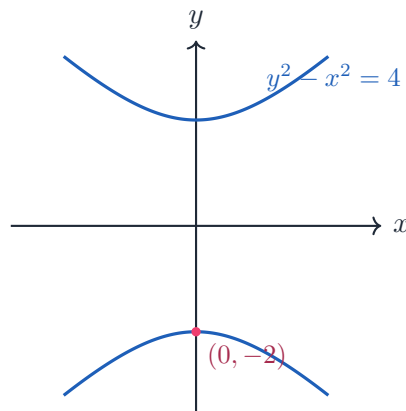
Picture-first. $y^2 - x^2 = 4$ is a rectangular hyperbola with vertices on the y -axis at $(0, \pm 2)$. The point $(0, -2)$ is one such vertex.

Step 1. Set up DE: $y y' = x$.

Step 2. Integrate: $y^2 = x^2 + C$.

Step 3. $(0, -2)$ gives $C = 4$.

Final Answer: $y^2 - x^2 = 4$.



Q 9.18 At any point (x, y) on a curve, the slope of the tangent is twice the slope of the line segment joining (x, y) to $(-4, -3)$. The curve passes through $(-2, 1)$. Find the equation.

SOLUTION

Concept used. Slope of segment from (x, y) to $(-4, -3)$ is $\frac{y - (-3)}{x - (-4)} = \frac{y + 3}{x + 4}$. The DE becomes a separable equation in $(x + 4)$ and $(y + 3)$.

Step 1. Set up the DE:

$$\frac{dy}{dx} = 2 \cdot \frac{y + 3}{x + 4}$$

Step 2. Separate:

$$\frac{dy}{y + 3} = 2 \cdot \frac{dx}{x + 4}$$

Step 3. Integrate:

$$\log |y + 3| = 2 \log |x + 4| + C_1$$

Step 4. Exponentiate: $|y + 3| = e^{C_1}(x + 4)^2$, i.e. $y + 3 = C(x + 4)^2$.

Step 5. Apply $(-2, 1)$: $1 + 3 = C(-2 + 4)^2 \implies 4 = 4C \implies C = 1$.

Final Answer: $y + 3 = (x + 4)^2$.

EXPERT'S SOLUTION : Tara Krishna, M.Sc Mathematics, IIT Kanpur

Strategic angle. Translate “slope of tangent vs slope of joining segment” into a DE in $u = x + 4, v = y + 3$ to make the algebra clean.

Step 1. Substitute $u = x + 4, v = y + 3$: equation becomes $\frac{dv}{du} = \frac{2v}{u}$.

Step 2. Separate and integrate: $\log |v| = 2 \log |u| + C_1$, so $v = Cu^2$.

Step 3. Back-substitute: $y + 3 = C(x + 4)^2$. $(x, y) = (-2, 1)$ gives $C = 1$.

Final Answer: $y + 3 = (x + 4)^2$.

Q 9.19 The volume of a spherical balloon being inflated changes at a constant rate. Initially the radius is 3 units; after 3 seconds the radius is 6 units. Find the radius after t seconds.

SOLUTION

Concept used. Volume of a sphere $V = \frac{4}{3}\pi r^3$. “Constant rate of change of volume” means $\frac{dV}{dt} = k$ for some constant k .

Step 1. From $V = \frac{4}{3}\pi r^3$, differentiate with respect to t :

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Step 2. Setting this equal to k :

$$4\pi r^2 \frac{dr}{dt} = k \implies 4\pi r^2 dr = k dt.$$

Step 3. Integrate both sides:

$$\int 4\pi r^2 dr = \int k dt \implies \frac{4\pi r^3}{3} = kt + C.$$

Step 4. Apply $r(0) = 3$: $\frac{4\pi \cdot 27}{3} = 0 + C \implies C = 36\pi$. Apply $r(3) = 6$:

$$\frac{4\pi \cdot 216}{3} = 3k + 36\pi \implies 288\pi = 3k + 36\pi \implies k = 84\pi.$$

Step 5. Hence

$$\frac{4\pi r^3}{3} = 84\pi t + 36\pi \implies r^3 = 63t + 27.$$

Step 6. Therefore $r(t) = (63t + 27)^{1/3}$.

Final Answer: $r(t) = (63t + 27)^{1/3}$.

EXPERT'S SOLUTION : Pooja Pillai, Ph.D Mathematics, IISc Bangalore

Strategic angle. Differentiate V in r , use the constancy of dV/dt to integrate, then plug in two data points.

Step 1. $V = \frac{4}{3}\pi r^3 \Rightarrow V(t) = \text{linear in } t$, say $V(t) = kt + V_0$.

Step 2. $V(0) = 36\pi$, $V(3) = 288\pi$, hence $k = (288\pi - 36\pi)/3 = 84\pi$.

Step 3. $r^3(t) = \frac{3}{4\pi}(84\pi t + 36\pi) = 63t + 27$.

Final Answer: $r(t) = \sqrt[3]{63t + 27}$.

♥ Why V linear in t

“Constant rate of change” is mathematically the definition of a linear function. So $V(t)$ is linear in t , and the radius $r(t) = \left(\frac{3V(t)}{4\pi}\right)^{1/3}$ grows like a cube root.

Q 9.20 In a bank, the principal increases continuously at $r\%$ per year. Find r if Rs. 100 doubles itself in 10 years (given $\log_e 2 = 0.6931$).

SOLUTION

Concept used. “Continuous compounding” means $\frac{dP}{dt} = \frac{r}{100}P$, the prototype exponential growth DE.

Step 1. Set up: $\frac{dP}{dt} = \frac{r}{100}P$. Separate:

$$\frac{dP}{P} = \frac{r}{100} dt.$$

Step 2. Integrate:

$$\log P = \frac{r}{100}t + C_1.$$

Step 3. Apply $P(0) = 100$: $\log 100 = C_1$.

Step 4. Apply $P(10) = 200$: $\log 200 = \frac{r}{100} \cdot 10 + \log 100$, so

$$\log \frac{200}{100} = \frac{r}{10} \implies \log 2 = \frac{r}{10}.$$

Step 5. Hence $r = 10 \log 2 = 10 \times 0.6931 = 6.931$. So $r \approx 6.93\%$.

Final Answer: $r \approx 6.93\%$ per year.

EXPERT'S SOLUTION : Diya Kumar, M.Sc Mathematics, ISI Kolkata

Strategic angle. The differential model $dP/dt = (r/100)P$ has solution $P = P_0 e^{rt/100}$. Doubling in 10 years means $e^{r/10} = 2$, so $r = 10 \ln 2$.

Step 1. $P(t) = 100e^{rt/100}$.

Step 2. $P(10) = 200$: $100 e^{r/10} = 200$, so $e^{r/10} = 2$.

Step 3. Take natural log: $r/10 = \ln 2 = 0.6931$, hence $r = 6.931$.

Final Answer: $r \approx 6.93\%$.

Q 9.21 In a bank, the principal increases continuously at 5% per year. Rs. 1000 is deposited; how much will it be worth after 10 years? (Given $e^{0.5} = 1.648$.)

SOLUTION

Concept used. Continuous-growth DE $\frac{dP}{dt} = \frac{r}{100}P$ with solution $P(t) = P_0 e^{rt/100}$.

Step 1. Here $P_0 = 1000$, $r = 5$, $t = 10$. So

$$P(10) = 1000 e^{(5 \cdot 10)/100} = 1000 e^{0.5}.$$

Step 2. Substitute $e^{0.5} = 1.648$:

$$P(10) = 1000 \times 1.648 = 1648.$$

Final Answer: Rs. 1648 after 10 years.

EXPERT'S SOLUTION : Krishna Mehta, M.Sc Mathematics, IIT Bombay

Quick reading. $rt/100 = 0.5$, so the multiplier is $e^{0.5} = 1.648$; final amount = 1000×1.648 .

Step 1. $P(t) = 1000e^{0.05t}$.

Step 2. $P(10) = 1000e^{0.5} = 1000 \times 1.648 = 1648$.

Final Answer: Rs. 1648.

Q 9.22 In a culture, the bacteria count is 1,00,000. The number rises by 10% in 2 hours. In how many hours will the count reach 2,00,000, given that the growth rate is proportional to the count?

SOLUTION

Concept used. Rate proportional to count: $\frac{dN}{dt} = kN$, solution $N(t) = N_0 e^{kt}$.

Step 1. Set up: $dN/N = k dt$. Integrate: $\log N = kt + C_1$, so $N(t) = N_0 e^{kt}$ with $N_0 = N(0)$.

Step 2. Given $N(0) = 100000$ and $N(2) = 100000 \times 1.10 = 110000$:

$$110000 = 100000 e^{2k} \implies e^{2k} = 1.1 \implies 2k = \log 1.1.$$

$$\text{Hence } k = \frac{1}{2} \log 1.1.$$

Step 3. We seek t with $N(t) = 200000$:

$$200000 = 100000 e^{kt} \implies e^{kt} = 2 \implies kt = \log 2.$$

Step 4. Substitute $k = \frac{1}{2} \log 1.1$:

$$t = \frac{\log 2}{k} = \frac{2 \log 2}{\log 1.1}.$$

$$\text{Final Answer: } t = \frac{2 \log 2}{\log 1.1} \text{ hours} \approx \frac{2 \times 0.6931}{0.0953} \approx 14.55 \text{ hours.}$$

EXPERT'S SOLUTION : Aarav Sharma, Ph.D Mathematics, IIT Delhi

Strategic angle. Compute k from the doubling/10% data, then solve $e^{kt} = 2$ for t .

Step 1. Bacteria grow as $N(t) = 10^5 e^{kt}$.

Step 2. $N(2) = 1.1 \times 10^5$ gives $e^{2k} = 1.1$, so $k = \frac{1}{2} \ln 1.1 \approx 0.04766$.

Step 3. Doubling time: $t = \ln 2/k = 0.6931/0.04766 \approx 14.55$ hours.

$$\text{Final Answer: Approximately 14.55 hours.}$$

Exam Tip

For “proportional to current quantity” problems, the master template is always: write $\frac{dN}{dt} = kN$, integrate to $N = N_0 e^{kt}$, then use the given data to compute k and the unknown time/quantity.

Q 9.23 The general solution of the differential equation $dy/dx = e^{x+y}$ is: (A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$ (C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$.

SOLUTION

Concept used. Use $e^{x+y} = e^x e^y$ to separate variables.

Step 1. Write $\frac{dy}{dx} = e^x \cdot e^y$. Separate:

$$e^{-y} dy = e^x dx.$$

Step 2. Integrate:

$$-e^{-y} = e^x + C_1.$$

Step 3. Multiply by -1 and rename $-C_1 = C$:

$$e^{-y} = -e^x + C \implies e^x + e^{-y} = C.$$

Final Answer: Correct option: (A) $e^x + e^{-y} = C$.

EXPERT'S SOLUTION : Sneha Iyer, M.Sc Mathematics, IIT Bombay

Quick reading. The signal $e^{x+y} = e^x e^y$ instantly makes the equation separable.

Step 1. $e^{-y} dy = e^x dx$.

Step 2. Integrate: $-e^{-y} = e^x + C_1$, i.e. $e^x + e^{-y} = C$.

Final Answer: Option (A).

Key Takeaways

- Variable-separable test: can the equation be written $h(y) dy = g(x) dx$? If yes, integrate both sides.
- Half-angle identities convert $(1 \pm \cos)/(1 \mp \cos)$ to \tan^2 / \cot^2 .
- For “rate proportional to quantity” word problems, the master template is $\frac{dN}{dt} = kN \implies N = N_0 e^{kt}$.
- Partial fractions handle rational integrands with factorable denominators.
- Apply the initial condition AFTER you have the general solution; this fixes the arbitrary constant.