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Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 9: Differential Equations

About this Chapter

A function $F(x, y)$ is **homogeneous of degree n** if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for every nonzero λ. A differential equation $\frac{dy}{dx} = F(x, y)$ is called **homogeneous** when F is homogeneous of degree zero. The substitution $y = vx$ (or $x = vy$) reduces every such equation to a variable-separable equation in v and x (or v and y).

Topics covered: Homogeneous function • Homogeneous DE • Substitution $y = vx$ • Particular solution

Quick Formula Sheet

Homogeneity test: $F(\lambda x, \lambda y) = \lambda^n F(x, y)$ for all $\lambda \neq 0$.

Reduction substitution: $y = vx \implies \frac{dy}{dx} = v + x\frac{dv}{dx}$.

Form $dx/dy = h(x/y)$: use $x = vy$ instead.

Exercise 9.4

Questions 1–10: show the DE is homogeneous and solve. Questions 11–15: find a particular solution. Questions 16–17: MCQs.

Q 9.1 $(x^2 + xy)\,dy = (x^2 + y^2)\,dx$.

SOLUTION

Concept used. Write the equation as $\frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$. Replacing (x, y) by $(\lambda x, \lambda y)$ multiplies both numerator and denominator by λ^2, so $F(\lambda x, \lambda y) = F(x, y) = \lambda^0 F(x, y)$. Hence F is homogeneous of degree zero and the DE is a homogeneous DE. Substitute $y = vx$ so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Step 1. Rewrite the RHS as a function of $v = y/x$:

$$\frac{x^2+y^2}{x^2+xy} = \frac{1+v^2}{1+v} \quad \text{(divide top and bottom by } x^2\text{)}.$$

Step 2. Substitute:

$$v + x\frac{dv}{dx} = \frac{1+v^2}{1+v}.$$

Step 3. Isolate $x\dfrac{dv}{dx}$:

$$x\frac{dv}{dx} = \frac{1+v^2}{1+v} - v = \frac{(1+v^2)-v(1+v)}{1+v} = \frac{1-v}{1+v}.$$

Step 4. Separate variables:

$$\frac{1+v}{1-v}\,dv = \frac{dx}{x}.$$

Step 5. Simplify the LHS by long division: $\dfrac{1+v}{1-v} = -1 + \dfrac{2}{1-v}$. Integrate:

$$\int\left(-1+\frac{2}{1-v}\right)dv = \int\frac{dx}{x} \implies -v - 2\log|1-v| = \log|x| + C_1.$$

Step 6. Substitute back $v = y/x$:

$$-\frac{y}{x} - 2\log\left|1-\frac{y}{x}\right| = \log|x| + C_1.$$

Combine: $\log|x| + 2\log\left|\dfrac{x-y}{x}\right| = -\dfrac{y}{x} - C_1$, hence

$$\log\left|x\cdot\frac{(x-y)^2}{x^2}\right| = -\frac{y}{x} - C_1 \implies \log\left|\frac{(x-y)^2}{x}\right| = -\frac{y}{x} - C_1.$$

Exponentiating and absorbing constants:

$$(x-y)^2 = C\,x\,e^{-y/x}.$$

Final Answer: $(x-y)^2 = C\,x\,e^{-y/x}$.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. Substitute $y = vx$ early; the equation becomes separable in v and x. All algebraic simplification happens on the v-side.

Step 1. $\dfrac{dy}{dx} = \dfrac{1+v^2}{1+v}$ with $v = y/x$.

Step 2. $v + xv' = \dfrac{1+v^2}{1+v}$, giving $xv' = \dfrac{1-v}{1+v}$.

Step 3. Separate, then integrate $\int \left(-1 + \frac{2}{1-v}\right) dv = \int dx/x$.

Step 4. Back-substitute and exponentiate to get $(x-y)^2 = Cxe^{-y/x}$.

> **Final Answer:** $(x-y)^2 = Cxe^{-y/x}$.

☞ **Homogeneous DE template**

$\dfrac{dy}{dx} = F(y/x) \xrightarrow{y=vx} x\dfrac{dv}{dx} = F(v) - v$. Always separable thereafter.

Q 9.2 $y' = \dfrac{x+y}{x}$.

SOLUTION

Concept used. Simplify the RHS to $1 + y/x$, a function of y/x alone, confirming homogeneity of degree zero.

Step 1. $\dfrac{x+y}{x} = 1 + \dfrac{y}{x}$. So $F(x,y) = 1 + y/x$, homogeneous of degree zero.

Step 2. Substitute $y = vx$, $y' = v + xv'$:

$$v + xv' = 1 + v \implies xv' = 1 \implies dv = \frac{dx}{x}.$$

Step 3. Integrate: $v = \log|x| + C$.

Step 4. Back-substitute: $\dfrac{y}{x} = \log|x| + C \implies y = x\log|x| + Cx$.

> **Final Answer:** $y = x\log|x| + Cx$.

EXPERT'S SOLUTION *: Vivaan Mehta, M.Sc Applied Mathematics, IIT Kanpur*

Quick reading. The DE is so simple after the substitution that you can read off the answer: $v' = 1/x$, so $v = \log|x| + C$.

Step 1. Let $v = y/x$; then $v + xv' = 1 + v$, so $xv' = 1$.

Step 2. Integrate: $v = \log|x| + C$.

Step 3. Multiply by x: $y = x\log|x| + Cx$.

Final Answer: $y = x\log|x| + Cx$.

Q 9.3 $(x-y)\,dy - (x+y)\,dx = 0$.

SOLUTION

Concept used. Rewrite as $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$. Both numerator and denominator are homogeneous of degree 1, so the ratio is homogeneous of degree zero.

Step 1. Divide numerator and denominator by x:

$$\frac{dy}{dx} = \frac{1+y/x}{1-y/x}.$$

Step 2. Substitute $y = vx$:

$$v + xv' = \frac{1+v}{1-v} \implies xv' = \frac{1+v}{1-v} - v = \frac{(1+v)-v(1-v)}{1-v} = \frac{1+v^2}{1-v}.$$

Step 3. Separate:

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}.$$

Step 4. Split the LHS:

$$\frac{1}{1+v^2}\,dv - \frac{v}{1+v^2}\,dv = \frac{dx}{x}.$$

Step 5. Integrate:

$$\tan^{-1}v - \frac{1}{2}\log(1+v^2) = \log|x| + C_1.$$

Step 6. Substitute back $v = y/x$:

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\log\left(1+\frac{y^2}{x^2}\right) = \log|x| + C_1.$$

Combine the logs:

$$\tan^{-1}\frac{y}{x} = \log|x| + \frac{1}{2}\log\frac{x^2+y^2}{x^2} + C_1 = \frac{1}{2}\log(x^2+y^2) + C_1.$$

Final Answer: $\tan^{-1}(y/x) = \dfrac{1}{2}\log(x^2+y^2) + C$.

EXPERT'S SOLUTION : *Priya Gupta, Ph.D Mathematics, IIT Delhi*

Strategic angle. Splitting $\dfrac{1-v}{1+v^2} = \dfrac{1}{1+v^2} - \dfrac{v}{1+v^2}$ produces two textbook integrals: $\tan^{-1} v$ and $\frac{1}{2}\log(1+v^2)$.

Step 1. Substitute and separate: $\dfrac{(1-v)\,dv}{1+v^2} = \dfrac{dx}{x}$.

Step 2. Integrate: $\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log|x| + C_1$.

Step 3. Replace $v = y/x$; combine logs into $\frac{1}{2}\log(x^2+y^2)$.

> **Final Answer:** $\tan^{-1}(y/x) = \dfrac{1}{2}\log(x^2+y^2) + C$.

Q 9.4 $(x^2 - y^2)\,dx + 2xy\,dy = 0$.

SOLUTION

Concept used. Solve for $\dfrac{dy}{dx}$ and confirm homogeneity.

Step 1. Rearrange: $\dfrac{dy}{dx} = -\dfrac{x^2-y^2}{2xy} = \dfrac{y^2-x^2}{2xy}$. Divide top and bottom by x^2:

$$\frac{dy}{dx} = \frac{v^2-1}{2v}, \quad v = y/x.$$

Step 2. Substitute $y = vx$, $y' = v + xv'$:

$$v + xv' = \frac{v^2-1}{2v} \implies xv' = \frac{v^2-1}{2v} - v = \frac{v^2-1-2v^2}{2v} = -\frac{1+v^2}{2v}.$$

Step 3. Separate:

$$\frac{2v}{1+v^2}\,dv = -\frac{dx}{x}.$$

Step 4. Integrate: $\log(1+v^2) = -\log|x| + C_1$. Combine:

$$\log|x(1+v^2)| = C_1.$$

Step 5. Back-substitute: $|x| \cdot \left(1 + \dfrac{y^2}{x^2}\right) = e^{C_1}$, i.e.

$$\frac{x^2+y^2}{|x|} = e^{C_1} \implies x^2 + y^2 = C|x|,$$

or simply $x^2 + y^2 = Cx$ (signs absorbed).

> **Final Answer:** $x^2 + y^2 = Cx$.

EXPERT'S SOLUTION : *Aanya Bhat, M.Sc Mathematics, IIT Madras*

Picture-first. The family $x^2 + y^2 = Cx$ is a family of circles passing through the origin with centres on the x-axis at $(C/2, 0)$ and radius $|C|/2$.

Step 1. Substitute $y = vx$, leading to $xv' = -\dfrac{1+v^2}{2v}$.

Step 2. Integrate $\dfrac{2v\,dv}{1+v^2} = -\dfrac{dx}{x}$ to $\log(1+v^2) + \log|x| = C_1$.

Step 3. Hence $x(1+v^2) = C$, i.e. $x^2 + y^2 = Cx$.

Final Answer: $x^2 + y^2 = Cx$.

♥ Geometry behind the DE

$(x^2 - y^2)\,dx + 2xy\,dy = 0$ is exactly the equation satisfied by the family of circles through the origin with centres on the x-axis. Homogeneous DEs often encode geometric invariance under scaling about the origin.

Q 9.5 $x^2 \dfrac{dy}{dx} = x^2 - 2y^2 + xy$.

SOLUTION

Concept used. Divide through by x^2 to expose y/x.

Step 1. Divide:

$$\frac{dy}{dx} = 1 - 2\left(\frac{y}{x}\right)^2 + \frac{y}{x}.$$

Step 2. Substitute $y = vx$:

$$v + xv' = 1 - 2v^2 + v \implies xv' = 1 - 2v^2.$$

Step 3. Separate:

$$\frac{dv}{1-2v^2} = \frac{dx}{x}.$$

Step 4. Rewrite the LHS using partial fractions with $a^2 = 1/2$:

$$\frac{1}{1-2v^2} = \frac{1}{(1-\sqrt{2}v)(1+\sqrt{2}v)}.$$

Use the standard result

$$\int \frac{dv}{a^2 - v^2} = \frac{1}{2a}\log\left|\frac{a+v}{a-v}\right|$$

with $1-2v^2 = 2(\frac{1}{2}-v^2)$ and $a = 1/\sqrt{2}$:

$$\int \frac{dv}{1-2v^2} = \frac{1}{2}\int \frac{dv}{\frac{1}{2}-v^2} = \frac{1}{2}\cdot\frac{1}{2\cdot 1/\sqrt{2}}\log\left|\frac{1/\sqrt{2}+v}{1/\sqrt{2}-v}\right| = \frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right|.$$

Step 5. Therefore

$$\frac{1}{2\sqrt{2}}\log\left|\frac{1+\sqrt{2}v}{1-\sqrt{2}v}\right| = \log|x| + C_1.$$

Step 6. Back-substitute $v = y/x$:

$$\frac{1}{2\sqrt{2}}\log\left|\frac{x+\sqrt{2}\,y}{x-\sqrt{2}\,y}\right| = \log|x| + C_1,$$

which we write as $\log\left|\frac{x+\sqrt{2}\,y}{x-\sqrt{2}\,y}\right| = 2\sqrt{2}\log|x| + C.$

Final Answer: $\log\left|\frac{x+\sqrt{2}\,y}{x-\sqrt{2}\,y}\right| = 2\sqrt{2}\,\log|x| + C.$

EXPERT'S SOLUTION : *Rohit Singh, B.Tech CSE, IIT Roorkee*

Strategic angle. The integrand $1/(1-2v^2)$ is a textbook $\frac{1}{a^2-v^2}$ form with $a = 1/\sqrt{2}$. The standard log formula closes it.

Step 1. After substitution: $xv' = 1-2v^2$, separable.

Step 2. Use $\int \frac{dv}{a^2-v^2} = \frac{1}{2a}\log\left|\frac{a+v}{a-v}\right|$ with $a = 1/\sqrt{2}$.

Step 3. Final answer involves $\log\left|\frac{x+\sqrt{2}y}{x-\sqrt{2}y}\right|$.

Final Answer: $\log\left|\frac{x+\sqrt{2}\,y}{x-\sqrt{2}\,y}\right| = 2\sqrt{2}\log|x| + C.$

Q 9.6 $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx.$

SOLUTION

Concept used. Solve for $\frac{dy}{dx}$. The RHS divided by x depends on y/x alone; homogeneous.

Step 1. $x\,dy = (y + \sqrt{x^2+y^2})\,dx$, so

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} = \frac{y}{x} + \sqrt{1+(y/x)^2}.$$

(Here we take $x > 0$; for $x < 0$ a sign appears but is absorbed into the constant.)

Step 2. Substitute $y = vx$:

$$v + xv' = v + \sqrt{1+v^2} \implies xv' = \sqrt{1+v^2}.$$

Step 3. Separate:

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}.$$

Step 4. Integrate. Standard result: $\int dv/\sqrt{1+v^2} = \log\left|v+\sqrt{1+v^2}\right|$.

$$\log\left|v+\sqrt{1+v^2}\right| = \log|x| + C_1.$$

Step 5. Back-substitute $v = y/x$:

$$\log\left|\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{|x|}\right| = \log|x| + C_1,$$

i.e. $\frac{y+\sqrt{x^2+y^2}}{x} = Cx$ (signs absorbed), so $y+\sqrt{x^2+y^2} = Cx^2$.

Final Answer: $y+\sqrt{x^2+y^2} = Cx^2$.

EXPERT'S SOLUTION : *Sanya Verma, M.Sc Mathematics, IIT Kanpur*

Strategic angle. The standard integral $\int dv/\sqrt{1+v^2} = \sinh^{-1} v = \log(v+\sqrt{1+v^2})$ closes the problem after a single substitution.

Step 1. $y = vx$ gives $xv' = \sqrt{1+v^2}$.

Step 2. Integrate: $\log(v+\sqrt{1+v^2}) = \log|x| + C_1$.

Step 3. Hence $y+\sqrt{x^2+y^2} = Cx^2$.

Final Answer: $y+\sqrt{x^2+y^2} = Cx^2$.

Q 9.7 $\left\{x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right\}y\,dx=\left\{y\sin\left(\frac{y}{x}\right)-x\cos\left(\frac{y}{x}\right)\right\}x\,dy.$

SOLUTION

Concept used. Both sides become y/x-dependent after dividing by x^2; substitute $y=vx$.

Step 1. Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{y\{x\cos(y/x)+y\sin(y/x)\}}{x\{y\sin(y/x)-x\cos(y/x)\}}.$$

Step 2. Divide top and bottom by x^2, using $y/x=v$:

$$\frac{dy}{dx}=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v}.$$

Step 3. Substitute $y=vx$, $y'=v+xv'$:

$$v+xv'=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v}.$$

Step 4. Isolate:

$$xv'=\frac{v(\cos v+v\sin v)}{v\sin v-\cos v}-v=\frac{v(\cos v+v\sin v)-v(v\sin v-\cos v)}{v\sin v-\cos v}=\frac{2v\cos v}{v\sin v-\cos v}.$$

Step 5. Separate:

$$\frac{v\sin v-\cos v}{v\cos v}\,dv=\frac{2\,dx}{x}\implies\left(\tan v-\frac{1}{v}\right)dv=\frac{2\,dx}{x}.$$

Step 6. Integrate:

$$-\log|\cos v|-\log|v|=2\log|x|+C_1,$$

i.e. $\log|v\cos v|=-2\log|x|+C_2$, hence $|v\cos v|\cdot x^2=C$.

Step 7. Back-substitute $v=y/x$: $\frac{y}{x}\cos\frac{y}{x}\cdot x^2=C$, i.e.

$$xy\cos\left(\frac{y}{x}\right)=C.$$

> **Final Answer:** $xy\cos(y/x)=C$.

EXPERT'S SOLUTION : *Ananya Kapoor, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. The numerator simplifies cleanly to $2v\cos v$ after subtracting v, and the resulting $(tanv-1/v)$ integrates to $-\log|v\cos v|$.

Step 1. $y=vx$ produces $xv'=\dfrac{2v\cos v}{v\sin v-\cos v}$.

Step 2. Separate and split into $\tan v - 1/v$, which integrates to $-\log|\cos v| - \log|v|$.

Step 3. Combine and exponentiate: $xy\cos(y/x) = C$.

Final Answer: $xy\cos(y/x) = C$.

Q 9.8 $x\dfrac{dy}{dx} - y + x\sin\left(\dfrac{y}{x}\right) = 0$.

SOLUTION

Concept used. Rearrange to $\dfrac{dy}{dx} = \dfrac{y}{x} - \sin(y/x)$, a function of y/x only.

Step 1. Substitute $y = vx$:

$$v + xv' = v - \sin v \implies xv' = -\sin v.$$

Step 2. Separate:

$$-\frac{dv}{\sin v} = \frac{dx}{x} \implies -\csc v\,dv = \frac{dx}{x}.$$

Step 3. Integrate. Standard result: $\int \csc v\,dv = \log|\tan(v/2)|$. So

$$-\log|\tan(v/2)| = \log|x| + C_1 \implies \log|\tan(v/2)| = -\log|x| - C_1.$$

Step 4. Exponentiate: $\tan(v/2) = \dfrac{C}{x}$ where $C = \pm e^{-C_1}$.

Step 5. Back-substitute $v = y/x$:

$$\tan\left(\frac{y}{2x}\right) = \frac{C}{x} \implies x\tan\left(\frac{y}{2x}\right) = C.$$

Final Answer: $x\tan\left(\dfrac{y}{2x}\right) = C$.

EXPERT'S SOLUTION : *Dev Nair, M.Sc Mathematics, IIT Bombay*

Quick reading. After substitution the equation collapses to $xv' = -\sin v$. Use the standard $\int \csc v\,dv = \log|\tan(v/2)|$.

Step 1. $v = y/x$ gives $xv' = -\sin v$.

Step 2. Integrate: $\log|\tan(v/2)| = -\log|x| + C_1$.

Step 3. Hence $x\tan(y/2x) = C$.

Final Answer: $x\tan(y/2x) = C$.

Q 9.9 $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2x\,dy = 0$.

SOLUTION

Concept used. Solve for $\frac{dy}{dx}$ and confirm homogeneity.

Step 1. Rearrange: $y\,dx = \big(2x - x\log(y/x)\big)dy$, so

$$\frac{dy}{dx} = \frac{y}{2x - x\log(y/x)} = \frac{y/x}{2 - \log(y/x)}.$$

Step 2. Substitute $y = vx$:

$$v + xv' = \frac{v}{2-\log v} \implies xv' = \frac{v}{2-\log v} - v = \frac{v - v(2-\log v)}{2-\log v} = \frac{v(\log v - 1)}{2-\log v}.$$

Step 3. Separate:

$$\frac{2-\log v}{v(\log v - 1)}\,dv = \frac{dx}{x}.$$

Step 4. Substitute $u = \log v - 1$, $du = dv/v$. Then $\log v = u+1$ and $2 - \log v = 1 - u$:

$$\frac{1-u}{u}\,du = \frac{dx}{x} \implies \left(\frac{1}{u} - 1\right)du = \frac{dx}{x}.$$

Step 5. Integrate: $\log|u| - u = \log|x| + C_1$. Substitute $u = \log v - 1$:

$$\log|\log v - 1| - (\log v - 1) = \log|x| + C_1,$$

i.e. $\log|\log v - 1| - \log v + 1 = \log|x| + C_1$, hence (combining constants)

$$\log\left|\frac{\log v - 1}{x}\right| = \log v + C.$$

Step 6. Back-substitute $v = y/x$:

$$\log\left|\frac{\log(y/x) - 1}{x}\right| = \log(y/x) + C.$$

Equivalently $\frac{\log(y/x)-1}{x} = K\cdot\frac{y}{x}$, i.e. $\log(y/x) - 1 = Ky$ (renaming constant).

Final Answer: $\log(y/x) - 1 = Cy$.

EXPERT'S SOLUTION : *Pranav Banerjee, M.Tech Applied Mathematics, IIT Delhi*

Strategic angle. The substitution $u = \log v - 1$ collapses the seemingly messy fraction into the simple form $(1/u - 1)\,du = dx/x$.

Step 1. Substitute $y = vx$; get $xv' = \dfrac{v(\log v - 1)}{2 - \log v}$.

Step 2. Substitute $u = \log v - 1$; reduce to $(1/u - 1)du = dx/x$.

Step 3. Integrate, back-substitute, and simplify to $\log(y/x) - 1 = Cy$.

> **Final Answer:** $\log(y/x) - 1 = Cy$.

Q 9.10 $\left(1 + e^{x/y}\right) dx + e^{x/y}\left(1 - \dfrac{x}{y}\right) dy = 0.$

SOLUTION

Concept used. Here it is convenient to view x as a function of y. Solve for $\dfrac{dx}{dy}$ and use the substitution $x = vy$.

Step 1. Solve for $\dfrac{dx}{dy}$:

$$\frac{dx}{dy} = -\frac{e^{x/y}(1 - x/y)}{1 + e^{x/y}}.$$

Step 2. Substitute $x = vy$, so $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$:

$$v + y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v}.$$

Step 3. Isolate $y\,dv/dy$:

$$y\frac{dv}{dy} = -\frac{e^v(1 - v)}{1 + e^v} - v = \frac{-e^v(1 - v) - v(1 + e^v)}{1 + e^v} = \frac{-e^v + ve^v - v - ve^v}{1 + e^v} = -\frac{v + e^v}{1 + e^v}.$$

Step 4. Separate:

$$\frac{1 + e^v}{v + e^v}\,dv = -\frac{dy}{y}.$$

Step 5. The LHS is exactly $\dfrac{d(v + e^v)}{v + e^v}$:

$$\log|v + e^v| = -\log|y| + C_1.$$

Step 6. Exponentiate and absorb: $(v + e^v)y = C$. Substitute $v = x/y$:

$$\left(\frac{x}{y} + e^{x/y}\right)y = C \implies x + y\,e^{x/y} = C.$$

Final Answer: $x + y\,e^{x/y} = C$.

EXPERT'S SOLUTION : *Tara Krishna, Ph.D Mathematics, IIT Bombay*

Strategic angle. The numerator $1 + e^v$ is exactly $\frac{d}{dv}(v + e^v)$, so the LHS is $d(v + e^v)/(v + e^v)$, which integrates to a single log.

Step 1. Use $x = vy$; get $y\frac{dv}{dy} = -\frac{v + e^v}{1 + e^v}$.

Step 2. Separate and notice $(1 + e^v)\,dv = d(v + e^v)$.

Step 3. Integrate: $\log|v + e^v| = -\log|y| + C_1$, so $y(v + e^v) = C$, i.e. $x + ye^{x/y} = C$.

Final Answer: $x + y\,e^{x/y} = C$.

☞ Exam Tip

When the equation contains $e^{x/y}$ (note the order x over y), substitute $x = vy$, not $y = vx$. The general rule: substitute so that the inner argument of the awkward function becomes the new variable.

Q 9.11 $(x + y)\,dy + (x - y)\,dx = 0$; $y = 1$ **when** $x = 1$.

SOLUTION

Concept used. Solve for $\frac{dy}{dx}$; both numerator and denominator are degree-1, so the DE is homogeneous.

Step 1. $\frac{dy}{dx} = -\frac{x - y}{x + y} = \frac{y - x}{x + y}$. Divide by x:

$$\frac{dy}{dx} = \frac{v - 1}{1 + v}, \quad v = y/x.$$

Step 2. Substitute $y = vx$:

$$v + xv' = \frac{v - 1}{1 + v} \implies xv' = \frac{v - 1 - v(1 + v)}{1 + v} = -\frac{1 + v^2}{1 + v}.$$

Step 3. Separate:

$$-\frac{1 + v}{1 + v^2}\,dv = \frac{dx}{x}.$$

Step 4. Split: $\dfrac{1+v}{1+v^2} = \dfrac{1}{1+v^2} + \dfrac{v}{1+v^2}$. Integrate:

$$-\left[\tan^{-1} v + \frac{1}{2}\log(1+v^2)\right] = \log|x| + C_1.$$

Step 5. Back-substitute $v = y/x$:

$$-\tan^{-1}\frac{y}{x} - \frac{1}{2}\log\frac{x^2+y^2}{x^2} = \log|x| + C_1,$$

i.e. $\tan^{-1}(y/x) + \dfrac{1}{2}\log(x^2+y^2) = -C_1$. Let $C = -C_1$:

$$\tan^{-1}\frac{y}{x} + \frac{1}{2}\log(x^2+y^2) = C.$$

Step 6. Apply $(1,1)$: $\tan^{-1} 1 + \frac{1}{2}\log 2 = C \implies C = \dfrac{\pi}{4} + \dfrac{1}{2}\log 2.$

> **Final Answer:** $\log(x^2+y^2) + 2\tan^{-1}(y/x) = \dfrac{\pi}{2} + \log 2.$

EXPERT'S SOLUTION : *Ishita Banerjee, M.Sc Mathematics, ISI Kolkata*

Quick reading. A cousin of Q3: same integrand pattern, opposite sign. Same trick: split $\dfrac{1+v}{1+v^2}$ into $\dfrac{1}{1+v^2} + \dfrac{v}{1+v^2}$.

Step 1. Substitute, separate, integrate: $\tan^{-1} v + \frac{1}{2}\log(1+v^2) + \log|x| = C_1$.

Step 2. Back-substitute and combine: $\tan^{-1}(y/x) + \frac{1}{2}\log(x^2+y^2) = C$.

Step 3. $(1,1) \Rightarrow C = \pi/4 + (1/2)\log 2$.

> **Final Answer:** $\log(x^2+y^2) + 2\tan^{-1}(y/x) = \pi/2 + \log 2$.

Q 9.12 $x^2\,dy + (xy+y^2)\,dx = 0$; $y = 1$ **when** $x = 1$.

SOLUTION

Concept used. Solve for $\dfrac{dy}{dx}$; numerator and denominator are both degree-2.

Step 1. $\dfrac{dy}{dx} = -\dfrac{xy+y^2}{x^2} = -\dfrac{y}{x} - \left(\dfrac{y}{x}\right)^2.$

Step 2. Substitute $y = vx$:

$$v + xv' = -v - v^2 \implies xv' = -2v - v^2 = -v(2+v).$$

Step 3. Separate:

$$-\frac{dv}{v(2+v)} = \frac{dx}{x}.$$

Step 4. Partial fractions: $\dfrac{1}{v(2+v)} = \dfrac{1}{2}\left(\dfrac{1}{v} - \dfrac{1}{2+v}\right)$. So

$$-\frac{1}{2}\left(\frac{1}{v} - \frac{1}{2+v}\right) dv = \frac{dx}{x}.$$

Step 5. Integrate: $-\dfrac{1}{2}(\log|v| - \log|2+v|) = \log|x| + C_1$, i.e.

$$\frac{1}{2}\log\left|\frac{2+v}{v}\right| = \log|x| + C_1.$$

Multiply by 2 and exponentiate:

$$\frac{2+v}{v} = Cx^2.$$

Step 6. Back-substitute $v = y/x$: $\dfrac{2+y/x}{y/x} = \dfrac{2x+y}{y} = Cx^2$, so $2x + y = Cx^2 y$.

Step 7. Apply $(1,1)$: $2 + 1 = C \cdot 1 \implies C = 3$.

Final Answer: $2x + y = 3x^2 y$.

EXPERT'S SOLUTION : *Karan Verma, Ph.D Mathematics, IIT Delhi*

Strategic angle. The right-side reduces to $-v(2+v)$; partial fractions handle $1/[v(2+v)]$ cleanly.

Step 1. $xv' = -v(2+v)$.

Step 2. $\dfrac{1}{v(2+v)} = \frac{1}{2}(1/v - 1/(2+v))$, so $-\frac{1}{2}\log|v/(2+v)| = \log|x| + C_1$.

Step 3. Exponentiate: $(2+v)/v = Cx^2$, giving $2x + y = Cx^2 y$. Use $(1,1)$ to find $C = 3$.

Final Answer: $2x + y = 3x^2 y$.

Q 9.13 $\left[x\sin^2(y/x) - y\right] dx + x\,dy = 0$; $y = \pi/4$ **when** $x = 1$.

SOLUTION

Concept used. Substitute $y = vx$ to reduce to a separable equation in v and x.

Step 1. Solve for $\frac{dy}{dx}$:

$$x\frac{dy}{dx} = y - x\sin^2(y/x) \implies \frac{dy}{dx} = \frac{y}{x} - \sin^2(y/x).$$

Step 2. Substitute $y = vx$:

$$v + xv' = v - \sin^2 v \implies xv' = -\sin^2 v.$$

Step 3. Separate:

$$-\frac{dv}{\sin^2 v} = \frac{dx}{x} \implies -\csc^2 v\, dv = \frac{dx}{x}.$$

Step 4. Integrate: $\cot v = \log|x| + C_1$ (since $\int -\csc^2 v\, dv = \cot v$).

Step 5. Back-substitute $v = y/x$:

$$\cot(y/x) = \log|x| + C_1.$$

Step 6. Apply $(1, \pi/4)$: $\cot(\pi/4) = \log 1 + C_1 \implies 1 = 0 + C_1 \implies C_1 = 1.$

> **Final Answer:** $\cot(y/x) = \log|x| + 1,$ i.e. $\log|ex| = \cot(y/x).$

EXPERT'S SOLUTION : *Aditya Chatterjee, M.Sc Mathematics, IIT Madras*

Quick reading. The equation telescopes after substitution: $xv' = -\sin^2 v$, integrand $-\csc^2 v$, antiderivative $\cot v$.

Step 1. $y = vx \Rightarrow xv' = -\sin^2 v.$

Step 2. $\cot v = \log|x| + C_1.$

Step 3. $(1, \pi/4)$ gives $C_1 = 1$, so $\cot(y/x) = \log|x| + 1 = \log|ex|.$

> **Final Answer:** $\log|ex| = \cot(y/x).$

Q 9.14 $\frac{dy}{dx} - \frac{y}{x} + \csc\left(\frac{y}{x}\right) = 0;\ \ y = 0$ **when** $x = 1.$

SOLUTION

Concept used. Rearrange to $\frac{dy}{dx} = \frac{y}{x} - \csc(y/x)$, a function of y/x alone.

Step 1. Substitute $y = vx$:

$$v + xv' = v - \csc v \implies xv' = -\csc v.$$

Step 2. Separate:

$$-\sin v\, dv = \frac{dx}{x}.$$

Step 3. Integrate: $\cos v = \log|x| + C_1$.

Step 4. Back-substitute: $\cos(y/x) = \log|x| + C_1$.

Step 5. Apply $(1, 0)$: $\cos 0 = \log 1 + C_1 \implies 1 = 0 + C_1 \implies C_1 = 1$.

> **Final Answer:** $\cos(y/x) = \log|x| + 1$, i.e. $\cos(y/x) = \log|ex|$.

EXPERT'S SOLUTION *: Pooja Pillai, M.Sc Mathematics, IIT Kanpur*

Quick reading. After $y = vx$, equation is $xv' = -\csc v$, separable with $-\sin v\, dv = dx/x$.

Step 1. $xv' = -\csc v$.

Step 2. Integrate: $\cos v = \log|x| + C_1$.

Step 3. $(1, 0) \Rightarrow C_1 = 1$; final $\cos(y/x) = \log|ex|$.

> **Final Answer:** $\cos(y/x) = \log|ex|$.

Q 9.15 $2xy + y^2 - 2x^2\frac{dy}{dx} = 0$; $y = 2$ **when** $x = 1$.

SOLUTION

Concept used. Solve for $\frac{dy}{dx}$, confirm homogeneity, substitute $y = vx$.

Step 1. $\frac{dy}{dx} = \frac{2xy + y^2}{2x^2} = \frac{y}{x} + \frac{y^2}{2x^2}$.

Step 2. Substitute $y = vx$:

$$v + xv' = v + \frac{v^2}{2} \implies xv' = \frac{v^2}{2}.$$

Step 3. Separate:

$$\frac{2\,dv}{v^2} = \frac{dx}{x}.$$

Step 4. Integrate: $-\frac{2}{v} = \log|x| + C_1$.

Step 5. Back-substitute $v = y/x$: $-\frac{2x}{y} = \log|x| + C_1$.

Step 6. Apply $(1, 2)$: $-\frac{2}{2} = 0 + C_1 \implies C_1 = -1$. So $\log|x| - \frac{2x}{y} = -1$, i.e. $\frac{2x}{y} = \log|x| + 1 = \log|ex|$.

Final Answer: $\frac{2x}{y} = \log|ex|$, i.e. $y = \frac{2x}{\log|ex|}$.

EXPERT'S SOLUTION *: Diya Kumar, M.Sc Mathematics, IIT Bombay*

Quick reading. After $y = vx$, equation collapses to $xv' = v^2/2$, a simple separable form.

Step 1. $xv' = v^2/2$.

Step 2. $-2/v = \log|x| + C_1$.

Step 3. Back-substitute and apply $(1, 2)$: $C_1 = -1$. So $y = 2x/\log|ex|$.

Final Answer: $y = \frac{2x}{\log|ex|}$.

Q 9.16 **A homogeneous DE of the form $\frac{dx}{dy} = h\left(\frac{x}{y}\right)$ can be solved by making the substitution: (A) $y = vx$ (B) $v = yx$ (C) $x = vy$ (D) $x = v$.**

SOLUTION

Concept used. When the RHS depends on x/y, treat y as the independent variable. The natural substitution is $x = vy$, which gives $\frac{dx}{dy} = v + y\frac{dv}{dy}$, and reduces the DE to a separable equation in v and y.

Step 1. The form is $\frac{dx}{dy} = h(x/y)$; we want the new variable to equal x/y.

Step 2. Let $v = x/y$, i.e. $x = vy$.

Step 3. Then $\frac{dx}{dy} = v + y\frac{dv}{dy}$, and the DE becomes $y\frac{dv}{dy} = h(v) - v$, which is separable.

Final Answer: Correct option: **(C)** $x = vy$.

EXPERT'S SOLUTION *: Yash Rao, M.Sc Mathematics, IIT Bombay*

Quick reading. Match the substitution to the form of the equation. "$dx/dy = h(x/y)$" calls for $x = vy$.

Step 1. RHS is a function of x/y, so let $v = x/y$, giving $x = vy$.

Step 2. This reduces the DE to a separable form in (v, y).

Step 3. Option (C).

Final Answer: (C).

Q 9.17 **Which of the following is a homogeneous differential equation?**

(A) $(4x + 6y + 5)\,dy - (3y + 2x + 4)\,dx = 0$

(B) $(xy)\,dx - (x^3 + y^3)\,dy = 0$

(C) $(x^3 + 2y^2)\,dx + 2xy\,dy = 0$

(D) $y^2\,dx + (x^2 - xy - y^2)\,dy = 0$.

SOLUTION

Concept used. Solve each for $\frac{dy}{dx}$ (or $\frac{dx}{dy}$) and test homogeneity of degree zero in the RHS, i.e. whether $F(\lambda x, \lambda y) = F(x, y)$.

Step 1. **(A)** $\frac{dy}{dx} = \frac{3y + 2x + 4}{4x + 6y + 5}$. The constants 4 and 5 in numerator and denominator are degree-0 terms, while the rest are degree-1. Under $(x, y) \to (\lambda x, \lambda y)$ the constants do not scale, so the expression is *not* homogeneous of degree zero. **Not homogeneous.**

Step 2. **(B)** $\frac{dy}{dx} = \frac{xy}{x^3 + y^3}$. Numerator is degree 2, denominator is degree 3. Under scaling: $\frac{\lambda^2 xy}{\lambda^3(x^3 + y^3)} = \frac{1}{\lambda}F(x, y) \neq F(x, y)$. Degree of F is -1. **Not homogeneous (degree zero).**

Step 3. **(C)** $\frac{dy}{dx} = -\frac{x^3 + 2y^2}{2xy}$. Numerator: x^3 is degree 3, $2y^2$ is degree 2 – different degrees, so the numerator is *not* homogeneous. **Not homogeneous.**

Step 4. **(D)** $\frac{dy}{dx} = -\frac{y^2}{x^2 - xy - y^2}$. Numerator and denominator are both degree 2.

Under scaling: $\frac{\lambda^2 y^2}{\lambda^2(x^2 - xy - y^2)} = \frac{y^2}{x^2 - xy - y^2}$, unchanged. **Homogeneous of degree zero.**

Final Answer: Correct option: **(D)**.

EXPERT'S SOLUTION *: Aditi Joshi, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. For an MCQ, scan each option's RHS for uniform degree of numerator and denominator and absence of stand-alone constants.

Step 1. (A) has stand-alone constants $4, 5$. Out.

Step 2. (B): degrees 2 and 3, unequal. Out.

Step 3. (C): numerator has mixed degrees 3 and 2. Out.

Step 4. (D): all terms in both numerator and denominator are degree 2. Homogeneous of degree zero. Correct.

Final Answer: (D).

♥ Quick homogeneity check

A polynomial in x and y is homogeneous of degree n iff every monomial has total degree exactly n. A polynomial DE is homogeneous iff its numerator and denominator (as polynomials in x, y) are individually homogeneous, with the *same* degree.

Key Takeaways

- A DE is homogeneous iff $\frac{dy}{dx}$ is a function of y/x alone (equivalently $F(\lambda x, \lambda y) = F(x, y)$).
- Standard substitution: $y = vx$ (or $x = vy$ if $dx/dy = h(x/y)$). It always reduces the equation to separable form.
- Apply the initial condition only after writing the general solution.
- Stand-alone constants and unequal degrees in numerator/denominator break homogeneity.