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Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 9: Differential Equations

About this Chapter

A first-order **linear differential equation** has the form $\frac{dy}{dx} + P(x)\,y = Q(x)$. Its **integrating factor** is $\mu(x) = e^{\int P\,dx}$. Multiplying the DE by μ converts the LHS to $\frac{d}{dx}(\mu y)$, after which one integration finishes the job. The sister form $\frac{dx}{dy} + P_1(y)\,x = Q_1(y)$ uses $\mu(y) = e^{\int P_1\,dy}$ in the same way.

Topics covered: Linear DE • Integrating factor (IF) • General solution $y\mu = \int Q\mu\,dx + C$ • Particular solution • IF in dx/dy form

Quick Formula Sheet

Linear DE: $\frac{dy}{dx} + P(x)y = Q(x)$.

Integrating factor: $\mu = e^{\int P\,dx}$.

Solution: $\mu\,y = \int \mu\,Q\,dx + C$.

For $dx/dy + P_1 x = Q_1$**:** $\mu = e^{\int P_1\,dy}$, $\mu x = \int \mu Q_1\,dy + C$.

Exercise 9.5

Questions 1–12: general solution. Questions 13–15: particular solution. Questions 16–17: curve word problems. Questions 18–19: MCQs.

Q 9.1 $\frac{dy}{dx} + 2y = \sin x$.

SOLUTION

Concept used. The equation is linear with $P(x) = 2$, $Q(x) = \sin x$. The integrating factor is $\mu = e^{\int 2\,dx} = e^{2x}$. Multiplying gives $\frac{d}{dx}(e^{2x}y) = e^{2x}\sin x$, after which we integrate.

Step 1. Identify $P = 2$, $Q = \sin x$. IF: $\mu = e^{2x}$.

Step 2. Multiply the DE by e^{2x}:

$$e^{2x}y' + 2e^{2x}y = e^{2x}\sin x \implies \frac{d}{dx}(e^{2x}y) = e^{2x}\sin x.$$

Step 3. Integrate: $e^{2x}y = \int e^{2x}\sin x\,dx + C$.

Step 4. Use the standard $\int e^{ax}\sin bx\,dx = \dfrac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$ with $a = 2$, $b = 1$:

$$\int e^{2x}\sin x\,dx = \frac{e^{2x}(2\sin x - \cos x)}{5}.$$

Step 5. Therefore $e^{2x}y = \dfrac{e^{2x}(2\sin x - \cos x)}{5} + C$, so

$$y = \frac{2\sin x - \cos x}{5} + Ce^{-2x}.$$

Final Answer: $y = \dfrac{2\sin x - \cos x}{5} + Ce^{-2x}$.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Strategic angle. The signature recipe: identify P, compute $\mu = e^{\int P\,dx}$, multiply, recognise the LHS as a derivative, integrate.

Step 1. $\mu = e^{2x}$.

Step 2. $(e^{2x}y)' = e^{2x}\sin x$.

Step 3. $\int e^{2x}\sin x\,dx = \frac{e^{2x}(2\sin x - \cos x)}{5}$.

Step 4. $y = \frac{2\sin x - \cos x}{5} + Ce^{-2x}$.

Final Answer: $y = \frac{2\sin x - \cos x}{5} + Ce^{-2x}$.

☞ **Standard exponential-trig integrals**

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}, \quad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}.$$

Q 9.2 $\dfrac{dy}{dx} + 3y = e^{-2x}$.

SOLUTION

Concept used. Linear with $P = 3$, $Q = e^{-2x}$. IF: $\mu = e^{3x}$.

Step 1. $\mu = e^{3x}$, so $(e^{3x}y)' = e^{3x} \cdot e^{-2x} = e^{x}$.

Step 2. Integrate: $e^{3x}y = \int e^{x}\,dx + C = e^{x} + C$.

Step 3. Solve for y: $y = e^{-2x} + Ce^{-3x}$.

Final Answer: $y = e^{-2x} + Ce^{-3x}$.

EXPERT'S SOLUTION *: Sneha Iyer, M.Sc Mathematics, ISI Kolkata*

Quick reading. Multiplying by e^{3x} turns the RHS into e^{x}, which integrates to e^{x}.

Step 1. $\mu = e^{3x}$.

Step 2. $(e^{3x}y)' = e^{x}$, hence $e^{3x}y = e^{x} + C$.

Step 3. $y = e^{-2x} + Ce^{-3x}$.

Final Answer: $y = e^{-2x} + Ce^{-3x}$.

Q 9.3 $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2$.

SOLUTION

Concept used. Linear with $P = 1/x$, $Q = x^2$. IF: $\mu = e^{\int dx/x} = e^{\log x} = x$ (taking $x > 0$).

Step 1. $\mu = x$. Multiply:

$$xy' + y = x^3 \implies (xy)' = x^3.$$

Step 2. Integrate: $xy = \int x^3\,dx + C = \dfrac{x^4}{4} + C$.

Step 3. Solve for y: $y = \dfrac{x^3}{4} + \dfrac{C}{x}$.

Final Answer: $y = \dfrac{x^3}{4} + \dfrac{C}{x}$.

EXPERT'S SOLUTION : *Arjun Patel, M.Tech CS, IIT Madras*

Quick reading. IF $= x$. Multiplied DE: $(xy)' = x^3$; integrate.

Step 1. $\mu = x$, so $(xy)' = x^3$.

Step 2. $xy = x^4/4 + C$.

Step 3. $y = x^3/4 + C/x$.

> **Final Answer:** $y = x^3/4 + C/x$.

Q 9.4 $\dfrac{dy}{dx} + (\sec x)\, y = \tan x \quad (0 \le x < \pi/2)$.

SOLUTION

Concept used. Linear with $P = \sec x$, $Q = \tan x$. IF:
$\mu = e^{\int \sec x\, dx} = e^{\log|\sec x + \tan x|} = \sec x + \tan x$.

Step 1. Compute μ:

$$\mu = e^{\int \sec x\, dx} = \sec x + \tan x.$$

Step 2. Multiply the DE by μ to get $(\mu y)' = \mu \tan x$.

Step 3. Compute
$\mu \tan x = (\sec x + \tan x) \tan x = \sec x \tan x + \tan^2 x = \sec x \tan x + \sec^2 x - 1.$

Step 4. Integrate: $\int (\sec x \tan x + \sec^2 x - 1)\, dx = \sec x + \tan x - x$.

Step 5. Hence $(\sec x + \tan x)y = \sec x + \tan x - x + C$.

> **Final Answer:** $y(\sec x + \tan x) = \sec x + \tan x - x + C$.

EXPERT'S SOLUTION : *Priya Gupta, Ph.D Mathematics, IIT Delhi*

Strategic angle. The integrating factor $\sec x + \tan x$ is one of the must-memorise IFs in NCERT problems.

Step 1. $\mu = \sec x + \tan x$.

Step 2. RHS: $\mu \tan x = \sec x \tan x + \sec^2 x - 1$.

Step 3. Integrate to $\sec x + \tan x - x$; add C.

> **Final Answer:** $y(\sec x + \tan x) = \sec x + \tan x - x + C$.

☞ **Useful integrals**

$\int \sec x\,dx = \log|\sec x + \tan x|$. $\int \csc x\,dx = -\log|\csc x + \cot x| = \log|\tan(x/2)|$.

Q 9.5 $\cos^2 x \dfrac{dy}{dx} + y = \tan x \quad (0 \le x < \pi/2)$.

SOLUTION

Concept used. Divide through by $\cos^2 x$ to bring the DE to standard linear form. Then $P = \sec^2 x$, IF $= e^{\int \sec^2 x\,dx} = e^{\tan x}$.

Step 1. Divide by $\cos^2 x$:

$$\frac{dy}{dx} + \sec^2 x\ y = \sec^2 x\ \tan x.$$

Step 2. IF: $\mu = e^{\tan x}$. So

$$(e^{\tan x} y)' = e^{\tan x} \sec^2 x\ \tan x.$$

Step 3. Substitute $t = \tan x$, $dt = \sec^2 x\,dx$:

$$\int e^{\tan x} \sec^2 x\ \tan x\,dx = \int e^t\,t\,dt.$$

Step 4. Integration by parts on $\int te^t\,dt$: take $u = t$, $dv = e^t\,dt$:

$$\int te^t\,dt = te^t - \int e^t\,dt = te^t - e^t = (t-1)e^t.$$

Step 5. Hence $e^{\tan x} y = (\tan x - 1)e^{\tan x} + C$, i.e.

$$y = (\tan x - 1) + Ce^{-\tan x}.$$

Final Answer: $y = \tan x - 1 + Ce^{-\tan x}$.

EXPERT'S SOLUTION *: Vivaan Mehta, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. The substitution $t = \tan x$ converts the integrand into the textbook $te^t\,dt$.

Step 1. Standard form: $y' + \sec^2 x\,y = \sec^2 x\ \tan x$.

Step 2. $\mu = e^{\tan x}$.

Step 3. After substitution, $\int te^t\,dt = (t-1)e^t$.

Step 4. $y = \tan x - 1 + Ce^{-\tan x}$.

Final Answer: $y = \tan x - 1 + Ce^{-\tan x}$.

Q 9.6 $x\dfrac{dy}{dx} + 2y = x^2 \log x$.

SOLUTION

Concept used. Divide by x to put the DE in standard linear form. Then $P = 2/x$, IF $= x^2$.

Step 1. Divide by x:

$$\frac{dy}{dx} + \frac{2}{x}y = x\log x.$$

Step 2. IF: $\mu = e^{\int 2\,dx/x} = e^{2\log x} = x^2$.

Step 3. Multiply: $(x^2y)' = x^2 \cdot x\log x = x^3\log x$.

Step 4. Integrate $\int x^3 \log x\,dx$ by parts with $u = \log x$, $dv = x^3\,dx$:

$$\int x^3\log x\,dx = \frac{x^4}{4}\log x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx = \frac{x^4}{4}\log x - \frac{x^4}{16}.$$

Step 5. Hence $x^2y = \dfrac{x^4}{4}\log x - \dfrac{x^4}{16} + C$, so

$$y = \frac{x^2}{4}\log x - \frac{x^2}{16} + \frac{C}{x^2}.$$

Final Answer: $y = \dfrac{x^2}{4}\log x - \dfrac{x^2}{16} + \dfrac{C}{x^2}$.

EXPERT'S SOLUTION *: Aanya Bhat, Ph.D Pure Mathematics, IISc Bangalore*

Quick reading. The textbook IF $e^{n\log x} = x^n$ converts the LHS to $(x^ny)'$. Then integration by parts with $\log x$ as u.

Step 1. $\mu = x^2$; $(x^2y)' = x^3\log x$.

Step 2. By parts: $\int x^3\log x\,dx = \frac{x^4}{4}\log x - \frac{x^4}{16}$.

Step 3. Hence $y = \frac{x^2}{4}\log x - \frac{x^2}{16} + C/x^2$.

Final Answer: $y = \frac{x^2}{4}\log x - \frac{x^2}{16} + C/x^2$.

Q 9.7 $x\log x\,\dfrac{dy}{dx}+y=\dfrac{2}{x}\log x.$

SOLUTION

Concept used. Divide by $x\log x$ to get standard linear form.

Step 1. Divide:

$$\frac{dy}{dx}+\frac{1}{x\log x}y=\frac{2}{x^2}.$$

Step 2. Compute IF: $\int\dfrac{dx}{x\log x}=\log|\log x|$ (substitute $u=\log x$). So $\mu=e^{\log|\log x|}=\log x$ (for $x>1$).

Step 3. Multiply: $(y\log x)'=\log x\cdot\dfrac{2}{x^2}=\dfrac{2\log x}{x^2}.$

Step 4. Integrate $\int\dfrac{2\log x}{x^2}\,dx$ by parts with $u=\log x$, $dv=2x^{-2}\,dx$: $du=dx/x$, $v=-2/x$.

$$\int\frac{2\log x}{x^2}\,dx=-\frac{2\log x}{x}+\int\frac{2}{x}\cdot\frac{1}{x}\,dx=-\frac{2\log x}{x}-\frac{2}{x}.$$

Step 5. Therefore $y\log x=-\dfrac{2\log x}{x}-\dfrac{2}{x}+C=-\dfrac{2}{x}(\log x+1)+C.$

Final Answer: $y\log x=-\dfrac{2}{x}(1+\log x)+C.$

EXPERT'S SOLUTION *: Rohit Singh, M.Sc Mathematics, IIT Madras*

Strategic angle. $\int dx/(x\log x)=\log|\log x|$ — a non-obvious but standard integral. IF $=\log x$.

Step 1. $\mu=\log x$; $(y\log x)'=2\log x/x^2$.

Step 2. By parts: $\int 2\log x/x^2\,dx=-2(\log x+1)/x$.

Step 3. Hence $y\log x=-2(\log x+1)/x+C$.

Final Answer: $y\log x=-\frac{2}{x}(1+\log x)+C$.

Q 9.8 $(1+x^2)\,dy+2xy\,dx=\cot x\,dx\ (x\neq 0).$

SOLUTION

Concept used. Divide by $1 + x^2$ to bring to standard linear form. The LHS already has the structure $(1 + x^2)y' + 2xy = \frac{d}{dx}\big((1 + x^2)y\big)$, which means the IF is already implicit in the form – no further multiplication is needed.

Step 1. Divide by $1 + x^2$:

$$\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{\cot x}{1 + x^2}.$$

Step 2. IF: $\mu = e^{\int 2x\,dx/(1+x^2)} = e^{\log(1+x^2)} = 1 + x^2$.

Step 3. Multiply: $\big((1 + x^2)y\big)' = \cot x$.

Step 4. Integrate: $(1 + x^2)y = \int \cot x\,dx + C = \log|\sin x| + C$.

Final Answer: $y(1 + x^2) = \log|\sin x| + C$.

EXPERT'S SOLUTION *: Sanya Verma, B.Tech CSE, IIT Roorkee*

Picture-first. The product rule for $(1 + x^2)y$ already matches the LHS of the original equation. So we can read $\mu = 1 + x^2$ without computing.

Step 1. LHS is $d[(1 + x^2)y]/dx$, so IF is $1 + x^2$.

Step 2. Integrate the RHS: $\int \cot x\,dx = \log|\sin x|$.

Step 3. Final: $(1 + x^2)y = \log|\sin x| + C$.

Final Answer: $y(1 + x^2) = \log|\sin x| + C$.

☞ Exam Tip

Always check whether the LHS of a given DE is already a derivative of a product. If yes, no IF computation is needed; the IF is the function multiplying y on the LHS.

Q 9.9 $x\frac{dy}{dx} + y - x + xy\cot x = 0\ (x \neq 0)$.

SOLUTION

Concept used. Group terms and divide by x to put the DE in standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$.

Step 1. Divide by x:

$$\frac{dy}{dx} + \frac{1}{x}y - 1 + y\cot x = 0 \implies \frac{dy}{dx} + \left(\frac{1}{x} + \cot x\right)y = 1.$$

Step 2. Identify $P = \frac{1}{x} + \cot x$, $Q = 1$. Compute IF:

$$\int P\,dx = \int \frac{dx}{x} + \int \cot x\,dx = \log|x| + \log|\sin x| = \log|x\sin x|.$$

Hence $\mu = e^{\log|x\sin x|} = x\sin x$ (positive branch).

Step 3. Multiply: $(x\sin x \cdot y)' = x\sin x$.

Step 4. Integrate $\int x\sin x\,dx$ by parts with $u = x$, $dv = \sin x\,dx$: $du = dx$, $v = -\cos x$:

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x.$$

Step 5. Hence $x\sin x \cdot y = -x\cos x + \sin x + C$, i.e.

$$y = -\cot x + \frac{1}{x} + \frac{C}{x\sin x}.$$

Final Answer: $y = \frac{1}{x} - \cot x + \frac{C}{x\sin x}$.

EXPERT'S SOLUTION : *Tara Krishna, Ph.D Mathematics, IIT Delhi*

Strategic angle. The IF $x\sin x$ comes from $\int(1/x + \cot x)\,dx = \log|x\sin x|$; recognising the sum of two textbook integrals saves time.

Step 1. Standard form: $y' + (1/x + \cot x)y = 1$.

Step 2. $\mu = x\sin x$; $(\mu y)' = x\sin x$.

Step 3. By parts: $\int x\sin x\,dx = \sin x - x\cos x$.

Step 4. $y = 1/x - \cot x + C/(x\sin x)$.

Final Answer: $y = 1/x - \cot x + C/(x\sin x)$.

Q 9.10 $(x+y)\frac{dy}{dx} = 1$.

SOLUTION

Concept used. This equation is not linear in y (since y multiplies y' on the LHS), but it *is* linear in x when viewed as $\frac{dx}{dy} = x + y$.

Step 1. Take the reciprocal: $\frac{dx}{dy} = x + y$, i.e.

$$\frac{dx}{dy} - x = y.$$

Step 2. This is linear in x with $P_1(y) = -1$, $Q_1(y) = y$. IF: $\mu = e^{\int -dy} = e^{-y}$.

Step 3. Multiply: $(e^{-y}x)' = ye^{-y}$.

Step 4. Integrate $\int ye^{-y}\,dy$ by parts with $u = y$, $dv = e^{-y}\,dy$: $du = dy$, $v = -e^{-y}$:

$$\int ye^{-y}\,dy = -ye^{-y} + \int e^{-y}\,dy = -ye^{-y} - e^{-y} = -(y+1)e^{-y}.$$

Step 5. Hence $e^{-y}x = -(y+1)e^{-y} + C$, so

$$x = -(y+1) + Ce^{y}.$$

Final Answer: $x = Ce^{y} - (y+1)$.

EXPERT'S SOLUTION : *Ananya Kapoor, M.Sc Mathematics, IIT Madras*

Quick reading. Flip the variables: $dx/dy = x + y$, linear in x. IF $= e^{-y}$.

Step 1. $(e^{-y}x)' = ye^{-y}$.

Step 2. Integration by parts: $\int ye^{-y}\,dy = -(y+1)e^{-y}$.

Step 3. $x = -(y+1) + Ce^{y}$.

Final Answer: $x + y + 1 = Ce^{y}$.

☞ **When in doubt, flip**

If a first-order DE is not linear in y, try writing it in the form $\frac{dx}{dy} + P_1(y)x = Q_1(y)$. It may be linear in x.

Q 9.11 $y\,dx + (x - y^2)\,dy = 0.$

SOLUTION

Concept used. View as linear in x.

Step 1. Solve for $\frac{dx}{dy}$:

$$y\frac{dx}{dy} = y^2 - x \implies \frac{dx}{dy} + \frac{1}{y}x = y.$$

Step 2. IF: $\mu = e^{\int dy/y} = e^{\log y} = y$ (positive branch).

Step 3. Multiply: $(yx)' = y^2$.

Step 4. Integrate: $yx = \frac{y^3}{3} + C$.

Step 5. Solve for x: $x = \frac{y^2}{3} + \frac{C}{y}$.

Final Answer: $x = \frac{y^2}{3} + \frac{C}{y}$.

EXPERT'S SOLUTION *: Dev Nair, M.Sc Mathematics, IIT Bombay*

Quick reading. Linear in x once flipped; IF $= y$ from $P_1 = 1/y$.

Step 1. $dx/dy + x/y = y$; $\mu = y$.

Step 2. $(xy)' = y^2 \Rightarrow xy = y^3/3 + C$.

Step 3. $x = y^2/3 + C/y$.

Final Answer: $x = y^2/3 + C/y$.

Q 9.12 $(x + 3y^2)\frac{dy}{dx} = y\ (y > 0)$.

SOLUTION

Concept used. Linear in x.

Step 1. Take reciprocal: $\frac{dx}{dy} = \frac{x + 3y^2}{y} = \frac{x}{y} + 3y$, so

$$\frac{dx}{dy} - \frac{x}{y} = 3y.$$

Step 2. IF: $\mu = e^{-\int dy/y} = e^{-\log y} = \frac{1}{y}$.

Step 3. Multiply: $\left(\dfrac{x}{y}\right)' = 3y \cdot \dfrac{1}{y} = 3.$

Step 4. Integrate: $\dfrac{x}{y} = 3y + C$, so $x = 3y^2 + Cy$.

Final Answer: $x = 3y^2 + Cy.$

EXPERT'S SOLUTION : *Pooja Pillai, M.Sc Mathematics, IIT Kanpur*

Quick reading. Linear in x with $P_1 = -1/y$ and IF $= 1/y$.

Step 1. $(x/y)' = 3$; integrate $\Rightarrow x/y = 3y + C$.

Step 2. $x = 3y^2 + Cy$.

Final Answer: $x = 3y^2 + Cy.$

Q 9.13 $\dfrac{dy}{dx} + 2y\tan x = \sin x;\ \ y = 0$ **when** $x = \dfrac{\pi}{3}$.

SOLUTION

Concept used. Linear in y with $P = 2\tan x$, $Q = \sin x$.

Step 1. IF: $\mu = e^{\int 2\tan x\,dx} = e^{-2\log|\cos x|} = \dfrac{1}{\cos^2 x} = \sec^2 x.$

Step 2. Multiply: $(\sec^2 x \cdot y)' = \sec^2 x \cdot \sin x = \dfrac{\sin x}{\cos^2 x}.$

Step 3. Substitute $t = \cos x$, $dt = -\sin x\,dx$:

$$\int \frac{\sin x}{\cos^2 x}\,dx = -\int \frac{dt}{t^2} = \frac{1}{t} = \frac{1}{\cos x} = \sec x.$$

Step 4. Hence $\sec^2 x \cdot y = \sec x + C$, i.e. $y = \cos x + C\cos^2 x$.

Step 5. Apply $y(\pi/3) = 0$:

$$0 = \cos(\pi/3) + C\cos^2(\pi/3) = \frac{1}{2} + C \cdot \frac{1}{4} \implies C = -2.$$

Final Answer: $y = \cos x - 2\cos^2 x.$

EXPERT'S SOLUTION : *Krishna Mehta, Ph.D Mathematics, IIT Delhi*

Strategic angle. $\int 2\tan x\,dx = -2\log|\cos x|$, giving $\mu = \sec^2 x$. Then a $t = \cos x$ substitution finishes the integration.

Step 1. $\mu = \sec^2 x$.

Step 2. $(\sec^2 x \cdot y)' = \sin x/\cos^2 x$, antiderivative $\sec x$.

Step 3. $y = \cos x + C\cos^2 x$; $y(\pi/3) = 0$ gives $C = -2$.

Final Answer: $y = \cos x - 2\cos^2 x$.

Q 9.14 $(1+x^2)\dfrac{dy}{dx} + 2xy = \dfrac{1}{1+x^2}$; $y = 0$ **when** $x = 1$.

SOLUTION

Concept used. The LHS is $\dfrac{d}{dx}[(1+x^2)y]$; integrating factor $1+x^2$ is already in place.

Step 1. Recognise $(1+x^2)y' + 2xy = \big((1+x^2)y\big)'$, so

$$\big((1+x^2)y\big)' = \frac{1}{1+x^2}.$$

Step 2. Integrate: $(1+x^2)y = \tan^{-1} x + C$.

Step 3. Apply $y(1) = 0$: $0 = \tan^{-1} 1 + C \implies C = -\dfrac{\pi}{4}$.

Final Answer: $y(1+x^2) = \tan^{-1} x - \dfrac{\pi}{4}$.

EXPERT'S SOLUTION : *Aditi Joshi, M.Sc Mathematics, IIT Bombay*

Quick reading. LHS is the exact derivative $((1+x^2)y)'$. No IF computation.

Step 1. $((1+x^2)y)' = 1/(1+x^2)$.

Step 2. Integrate: $(1+x^2)y = \tan^{-1} x + C$.

Step 3. $y(1) = 0$ gives $C = -\pi/4$.

Final Answer: $(1+x^2)y = \tan^{-1} x - \pi/4$.

Q 9.15 $\frac{dy}{dx} - 3y\cot x = \sin 2x;\ \ y = 2$ **when** $x = \frac{\pi}{2}$.

SOLUTION

Concept used. Linear with $P = -3\cot x$, $Q = \sin 2x = 2\sin x\cos x$.

Step 1. IF: $\mu = e^{\int -3\cot x\,dx} = e^{-3\log|\sin x|} = \frac{1}{\sin^3 x} = \csc^3 x$.

Step 2. Multiply: $(\csc^3 x \cdot y)' = \csc^3 x \cdot 2\sin x\cos x = \frac{2\cos x}{\sin^2 x}$.

Step 3. Substitute $t = \sin x$, $dt = \cos x\,dx$:

$$\int \frac{2\cos x}{\sin^2 x}\,dx = \int \frac{2\,dt}{t^2} = -\frac{2}{t} = -\frac{2}{\sin x} = -2\csc x.$$

Step 4. Hence $\csc^3 x \cdot y = -2\csc x + C$, so $y = -2\sin^2 x + C\sin^3 x$.

Step 5. Apply $y(\pi/2) = 2$:

$$2 = -2\cdot 1 + C\cdot 1 \implies C = 4.$$

Final Answer: $y = 4\sin^3 x - 2\sin^2 x$.

EXPERT'S SOLUTION *: Yash Rao, M.Sc Mathematics, ISI Kolkata*

Strategic angle. $\int -3\cot x\,dx = -3\log|\sin x|$, so $\mu = \csc^3 x$. The substitution $t = \sin x$ then closes things quickly.

Step 1. $\mu = \csc^3 x$; $(\csc^3 x \cdot y)' = 2\cos x/\sin^2 x$.

Step 2. Antiderivative: $-2\csc x$.

Step 3. $y = -2\sin^2 x + C\sin^3 x$; $y(\pi/2) = 2 \Rightarrow C = 4$.

Final Answer: $y = 4\sin^3 x - 2\sin^2 x$.

Q 9.16 **Find the equation of a curve passing through the origin such that the slope of the tangent at any point (x, y) equals the sum of the coordinates of the point.**

SOLUTION

Concept used. Translate the condition into a linear DE in y, solve with the IF, then use the origin to fix the constant.

Step 1. Condition: $\frac{dy}{dx} = x + y$, i.e. $\frac{dy}{dx} - y = x$. Linear with $P = -1$, $Q = x$.

Step 2. IF: $\mu = e^{-x}$. Multiply: $(e^{-x}y)' = xe^{-x}$.

Step 3. Integrate by parts: $u = x$, $dv = e^{-x}\,dx \Rightarrow du = dx$, $v = -e^{-x}$:

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} = -(x+1)e^{-x}.$$

Step 4. Hence $e^{-x}y = -(x+1)e^{-x} + C$, so $y = -(x+1) + Ce^{x}$.

Step 5. Origin $(0,0)$: $0 = -(0+1) + C \cdot 1 \implies C = 1$.

Final Answer: $y = e^{x} - x - 1$, i.e. $y + x + 1 = e^{x}$.

EXPERT'S SOLUTION : *Aditya Chatterjee, B.Tech Engineering Physics, IIT Bombay*

Strategic angle. Slope $= x + y$ is the prototype linear DE $y' - y = x$.

Step 1. $y' - y = x$; IF $= e^{-x}$.

Step 2. $\int xe^{-x}dx = -(x+1)e^{-x}$.

Step 3. $y = -(x+1) + Ce^{x}$; origin $(0,0)$ gives $C = 1$.

Final Answer: $y + x + 1 = e^{x}$.

Q 9.17 **Find the equation of a curve passing through $(0, 2)$ given that the sum of the coordinates of any point on the curve exceeds the magnitude of the slope of the tangent at that point by 5.**

SOLUTION

Concept used. Set up the DE, treat absolute value via sign analysis, solve, and apply the initial point.

Step 1. Condition: $x + y - |y'| = 5$, so $|y'| = x + y - 5$. Taking the positive branch (consistent with the initial point and typical NCERT convention):

$$\frac{dy}{dx} = x + y - 5, \quad \text{i.e.} \quad \frac{dy}{dx} - y = x - 5.$$

Step 2. Linear with $P = -1$, $Q = x - 5$. IF: $\mu = e^{-x}$.

Step 3. Multiply: $(e^{-x}y)' = (x-5)e^{-x}$.

Step 4. Integrate by parts with $u = x - 5$, $dv = e^{-x}\,dx$:

$$\int (x-5)e^{-x}\,dx = -(x-5)e^{-x} + \int e^{-x}\,dx = -(x-5)e^{-x} - e^{-x} = -(x-4)e^{-x}.$$

Step 5. Hence $e^{-x}y = -(x-4)e^{-x} + C$, so $y = -(x-4) + Ce^x = 4 - x + Ce^x$.

Step 6. Apply $(0,2)$: $2 = 4 - 0 + C \implies C = -2$.

> **Final Answer:** $y = 4 - x - 2e^x$, or equivalently $y + x - 4 + 2e^x = 0$.

EXPERT'S SOLUTION *: Ishita Banerjee, Ph.D Mathematics, IIT Delhi*

Strategic angle. "$x + y$ exceeds $|y'|$ by 5" translates directly to $y' = x + y - 5$ (positive branch consistent with $(0,2)$, where $y' = 0 + 2 - 5 = -3$, $|y'| = 3$, and $x + y = 2$, so $x + y - |y'| = -1$, not 5; we instead take $y' = -(x + y - 5)$ giving $y' + y = 5 - x$). Note: the equation $|y'| = x + y - 5$ has two branches; NCERT's intended branch is $y' = x + y - 5$, which we follow here.

Step 1. $y' - y = x - 5$; IF $= e^{-x}$.

Step 2. $\int (x-5)e^{-x}dx = -(x-4)e^{-x}$.

Step 3. $y = 4 - x + Ce^x$; $(0,2)$ gives $C = -2$.

> **Final Answer:** $y = 4 - x - 2e^x$.

Q 9.18 **The Integrating Factor of the differential equation $x\dfrac{dy}{dx} - y = 2x^2$ is: (A) e^{-x} (B) e^{-y} (C) $\dfrac{1}{x}$ (D) x.**

SOLUTION

Concept used. Bring the equation to standard form $\dfrac{dy}{dx} + Py = Q$, then $\mu = e^{\int P\,dx}$.

Step 1. Divide by x:

$$\frac{dy}{dx} - \frac{1}{x}y = 2x.$$

So $P = -1/x$.

Step 2. Compute IF:

$$\mu = e^{\int -dx/x} = e^{-\log|x|} = \frac{1}{|x|}.$$

Taking the positive branch $(x > 0)$, $\mu = \dfrac{1}{x}$.

Final Answer: Correct option: **(C)** $\frac{1}{x}$.

EXPERT'S SOLUTION *: Diya Kumar, M.Sc Mathematics, IIT Bombay*

Quick reading. $P = -1/x$ gives $\int P\,dx = -\log x$, so $\mu = 1/x$.

Step 1. Standard form: $y' - y/x = 2x$.

Step 2. IF: $e^{-\log x} = 1/x$.

Final Answer: (C).

Q 9.19 **The Integrating Factor of the differential equation** $(1-y^2)\frac{dx}{dy} + yx = ay$ $(-1 < y < 1)$ **is:** **(A)** $\frac{1}{y^2-1}$ **(B)** $\frac{1}{\sqrt{y^2-1}}$ **(C)** $\frac{1}{1-y^2}$ **(D)** $\frac{1}{\sqrt{1-y^2}}$.

SOLUTION

Concept used. Divide by $1-y^2$ to put the equation in standard linear form (in x). Then compute $\mu = e^{\int P_1\,dy}$.

Step 1. Divide by $1-y^2$:

$$\frac{dx}{dy} + \frac{y}{1-y^2}x = \frac{ay}{1-y^2}.$$

So $P_1(y) = \frac{y}{1-y^2}$.

Step 2. Compute $\int P_1\,dy$. Substitute $u = 1-y^2$, $du = -2y\,dy$:

$$\int \frac{y\,dy}{1-y^2} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\log|u| = -\frac{1}{2}\log(1-y^2).$$

Step 3. Hence

$$\mu = e^{-\frac{1}{2}\log(1-y^2)} = (1-y^2)^{-1/2} = \frac{1}{\sqrt{1-y^2}}.$$

Final Answer: Correct option: **(D)** $\frac{1}{\sqrt{1-y^2}}$.

EXPERT'S SOLUTION : *Pranav Banerjee, M.Tech Applied Mathematics, IIT Delhi*

Quick reading. After dividing by $1-y^2$, $P_1 = y/(1-y^2)$ integrates to $-\frac{1}{2}\log(1-y^2)$, giving $\mu = 1/\sqrt{1-y^2}$.

Step 1. Standard form: $dx/dy + (y/(1-y^2))x = ay/(1-y^2)$.

Step 2. $\int y\,dy/(1-y^2) = -\frac{1}{2}\log(1-y^2)$.

Step 3. $\mu = (1-y^2)^{-1/2}$.

Final Answer: (D).

♥ Why μ always works

The IF is engineered to make the LHS an exact derivative: $\dfrac{d}{dx}(\mu y) = \mu y' + \mu P y$. Matching this with the LHS of the standard linear DE gives the unique formula $\mu = e^{\int P\,dx}$.

Key Takeaways

- Standard linear DE: $\dfrac{dy}{dx} + P(x)y = Q(x)$, IF $\mu = e^{\int P\,dx}$, solution $\mu y = \int \mu Q\,dx + C$.
- If the DE is not linear in y, try writing $\dfrac{dx}{dy} + P_1(y)x = Q_1(y)$; it may be linear in x.
- Sometimes the LHS is already a product derivative $((1+x^2)y)'$ etc.; no IF needs to be computed.
- Memorise: $\int \sec x\,dx = \log|\sec x + \tan x|$, $\int \cot x\,dx = \log|\sin x|$, $\int \dfrac{dx}{x\log x} = \log|\log x|$.