Collegedunia NCERT Solutions

Step-by-step solutions, alternate methods & exam tips for Class 12 Mathematics

Chapter 9: Differential Equations

About this Chapter

The miscellaneous exercise blends every technique of the chapter: identifying **order and degree**, **verifying solutions**, solving **variable-separable**, **homogeneous** and **linear** differential equations, and pinning down particular solutions from initial conditions. A correct strategy choice is half the work.

Topics covered: Order/degree • Verification • Separable • Homogeneous • Linear (IF) • Initial conditions

Quick Formula Sheet

Order/Degree: highest derivative; its power once polynomial in derivatives.

Linear DE: $y' + Py = Q \implies \mu = e^{\int P\,dx}$, $\mu y = \int \mu Q\,dx + C$.

Homogeneous DE: $\frac{dy}{dx} = F(y/x)$: use $y = vx$.
Form $dx/dy = h(x/y)$: use $x = vy$.

Miscellaneous Exercise

A mix of order/degree, verification, separable, homogeneous and linear questions, plus initial-value problems and MCQs.

Q 9.1 **Indicate the order and degree (if defined) of each of the following differential equations.**

(i) $\frac{d^2y}{dx^2} + 5x\left(\frac{dy}{dx}\right)^2 - 6y = \log x.$

(ii) $\left(\frac{dy}{dx}\right)^3 - 4\left(\frac{dy}{dx}\right)^2 + 7y = \sin x.$

(iii) $\frac{d^4y}{dx^4} - \sin\left(\frac{d^3y}{dx^3}\right) = 0.$

SOLUTION

Concept used. Order is the order of the highest derivative; degree is the power of that highest-order derivative once the equation is polynomial in all derivatives.

Step 1. **(i)** Derivatives: y'' (order 2), y' (order 1). Highest order $= 2$. Both derivatives appear with positive-integer exponents (1 and 2); no transcendental wrapper around them. Polynomial form holds. Power of y'' is 1, so degree $= 1$.

Step 2. **(ii)** The only derivative is y'. Order $= 1$. Powers 3 and 2 on y' are fine; no $\sin / \cos / \log$ around y'. Polynomial form holds. The highest-order derivative (y') is raised to power 3. So degree $= 3$.

Step 3. **(iii)** Derivatives: $y^{(4)}$ (order 4), y''' (order 3). Highest order $= 4$. However, y''' is wrapped inside $\sin$; polynomial form fails. Hence degree is not defined.

> **Final Answer:** (i) Order 2, Degree 1; (ii) Order 1, Degree 3; (iii) Order 4, Degree not defined.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Quick reading. For each item: max derivative superscript → order. Then scan for $\sin / \cos / \log / e^{(\cdot)}$ around a derivative; if found, degree undefined; if not, degree is the power of the highest-order derivative.

Step 1. (i) Max superscript 2; no transcendental wrapper. Power of y'' is 1. (2, 1).

Step 2. (ii) Max superscript 1; no wrapper. Power of y' is 3. (1, 3).

Step 3. (iii) Max superscript 4; $\sin(y''')$ kills polynomial form. (4, undefined).

> **Final Answer:** (2,1), (1,3), (4, not defined).

✗ Common Mistake

In (ii) the exponent 3 does *not* make the order 3. Order looks only at how many times y is differentiated – the highest derivative is y' regardless of its exponent.

Q 9.2 **For each verify that the given function (implicit or explicit) is a solution of the corresponding DE.**

(i) $xy = ae^x + be^{-x} + x^2 \;:\; x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} - xy + x^2 - 2 = 0.$

(ii) $y = e^x(a\cos x + b\sin x) \;:\; \dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0.$

(iii) $y = x\sin 3x \; : \; \dfrac{d^2y}{dx^2} + 9y - 6\cos 3x = 0.$

(iv) $x^2 = 2y^2 \log y \; : \; (x^2+y^2)\dfrac{dy}{dx} - xy = 0.$

SOLUTION

Concept used. For each part, differentiate the given function the required number of times and substitute into the DE; the result must be an identity in x.

(i) Differentiate $xy = ae^x + be^{-x} + x^2$ implicitly with respect to x:

$$y + xy' = ae^x - be^{-x} + 2x.$$

Differentiate again:

$$y' + y' + xy'' = ae^x + be^{-x} + 2 \implies 2y' + xy'' = ae^x + be^{-x} + 2.$$

But $ae^x + be^{-x} = xy - x^2$ (from the original equation). Hence

$$xy'' + 2y' = (xy - x^2) + 2 \implies xy'' + 2y' - xy + x^2 - 2 = 0.$$

This is exactly the given DE.

Final Answer: (i) Verified.

(ii) $y = e^x(a\cos x + b\sin x)$. Differentiate using the product rule:

$$y' = e^x(a\cos x + b\sin x) + e^x(-a\sin x + b\cos x) = e^x[(a+b)\cos x + (b-a)\sin x].$$

Differentiate again:

$$y'' = e^x[(a+b)\cos x + (b-a)\sin x] + e^x[-(a+b)\sin x + (b-a)\cos x] = e^x[2b\cos x - 2a\sin x].$$

Compute $y'' - 2y' + 2y$:

$$e^x[2b\cos x - 2a\sin x] - 2e^x[(a+b)\cos x + (b-a)\sin x] + 2e^x[a\cos x + b\sin x].$$

Coefficient of $e^x\cos x$: $2b - 2(a+b) + 2a = 0.$
Coefficient of $e^x\sin x$: $-2a - 2(b-a) + 2b = 0.$
Hence $y'' - 2y' + 2y = 0.$

Final Answer: (ii) Verified.

(iii) $y = x\sin 3x$. Product rule:

$$y' = \sin 3x + 3x\cos 3x.$$

Second derivative:

$$y'' = 3\cos 3x + 3\cos 3x - 9x\sin 3x = 6\cos 3x - 9y.$$

Substitute: $y'' + 9y = 6\cos 3x$, i.e. $y'' + 9y - 6\cos 3x = 0$.

Final Answer: (iii) Verified.

(iv) $x^2 = 2y^2 \log y$. Differentiate implicitly:

$$2x = 4y\log y \cdot y' + 2y^2 \cdot \frac{1}{y} \cdot y' = (4y\log y + 2y)\, y' = 2y(2\log y + 1)\, y'.$$

So

$$y' = \frac{x}{y(2\log y + 1)}.$$

From the original relation $2\log y = \dfrac{x^2}{y^2}$, so $2\log y + 1 = \dfrac{x^2+y^2}{y^2}$. Hence

$$y' = \frac{x}{y \cdot \dfrac{x^2+y^2}{y^2}} = \frac{xy}{x^2+y^2}.$$

Therefore $(x^2+y^2)y' = xy$, i.e. $(x^2+y^2)\dfrac{dy}{dx} - xy = 0$.

Final Answer: (iv) Verified.

EXPERT'S SOLUTION : *Sneha Iyer, Ph.D Mathematics, IIT Delhi*

Strategic angle. For implicit relations like (i) and (iv), substitute the original equation back into the derivative chain to eliminate a, b (in i) or $\log y$ (in iv).

Step 1. (i) Two differentiations of xy produce $2y' + xy'' = ae^x + be^{-x} + 2$. Use original to replace $ae^x + be^{-x}$.

Step 2. (ii) $y'' = e^x(2b\cos x - 2a\sin x)$; check that the linear combination vanishes.

Step 3. (iii) $y'' = 6\cos 3x - 9y$; rearrange.

Step 4. (iv) $y' = \dfrac{x}{y(2\log y + 1)}$; substitute $2\log y = x^2/y^2$ to get $y' = xy/(x^2+y^2)$.

Final Answer: All four verified.

Q 9.3 Prove that $x^2 - y^2 = c(x^2+y^2)^2$ is the general solution of $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter.

SOLUTION

Concept used. The DE is homogeneous (both sides degree 3). Substitute $y = vx$, reduce to separable, integrate, and compare with the proposed solution.

Step 1. Solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} = \frac{x(x^2 - 3y^2)}{y(y^2 - 3x^2)}.$$

Step 2. Divide top and bottom by x^3 and let $v = y/x$:

$$\frac{dy}{dx} = \frac{1 - 3v^2}{v(v^2 - 3)} = \frac{1 - 3v^2}{v^3 - 3v}.$$

Step 3. Substitute $y = vx$, so $y' = v + xv'$:

$$v + xv' = \frac{1 - 3v^2}{v^3 - 3v}.$$

Subtract v and simplify the numerator:

$$xv' = \frac{1 - 3v^2 - v(v^3 - 3v)}{v^3 - 3v} = \frac{1 - v^4}{v^3 - 3v}.$$

Step 4. Separate:

$$\frac{v^3 - 3v}{1 - v^4}\, dv = \frac{dx}{x}.$$

Factor the denominator $1 - v^4 = (1 - v^2)(1 + v^2)$. Rewrite the numerator: $v^3 - 3v = v(v^2 + 1) - 4v$. Hence

$$\frac{v^3 - 3v}{(1 - v^2)(1 + v^2)} = \frac{v}{1 - v^2} - \frac{4v}{1 - v^4}.$$

Decompose the remaining piece: $\frac{4v}{1 - v^4} = \frac{2v}{1 - v^2} + \frac{2v}{1 + v^2}$ (verified by common denominator $1 - v^4$: $2v(1 + v^2) + 2v(1 - v^2) = 4v$). Therefore

$$\frac{v^3 - 3v}{1 - v^4} = \frac{v}{1 - v^2} - \frac{2v}{1 - v^2} - \frac{2v}{1 + v^2} = -\frac{v}{1 - v^2} - \frac{2v}{1 + v^2}.$$

Step 5. Integrate:

$$\int \left(-\frac{v}{1 - v^2} - \frac{2v}{1 + v^2} \right) dv = \int \frac{dx}{x}.$$

Use $\int v\, dv/(1 - v^2) = -\frac{1}{2} \log|1 - v^2|$ and $\int 2v\, dv/(1 + v^2) = \log(1 + v^2)$:

$$\tfrac{1}{2} \log|1 - v^2| - \log(1 + v^2) = \log|x| + C_1.$$

Step 6. Multiply by 2 and combine: $\log|1 - v^2| - 2\log(1 + v^2) = 2\log|x| + 2C_1$, i.e.

$$\log\left|\frac{1 - v^2}{(1 + v^2)^2}\right| = 2\log|x| + 2C_1 \implies \frac{1 - v^2}{(1 + v^2)^2} = Kx^2 \quad (K = \pm e^{2C_1}).$$

Step 7. Replace $v = y/x$. Then $1 - v^2 = (x^2 - y^2)/x^2$ and $1 + v^2 = (x^2 + y^2)/x^2$:

$$\frac{(x^2 - y^2)/x^2}{(x^2 + y^2)^2/x^4} = Kx^2 \implies \frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2} = Kx^2 \implies x^2 - y^2 = K(x^2 + y^2)^2.$$

Renaming $K = c$ gives the proposed general solution.

> **Final Answer:** Proven: $x^2 - y^2 = c(x^2 + y^2)^2$.

EXPERT'S SOLUTION : *Arjun Patel, M.Sc Mathematics, IIT Madras*

Strategic angle. Homogeneous DE of degree 3; substitute $y = vx$, simplify the RHS to $(1 - v^4)/(v^3 - 3v)$, then separate. The trick is the algebraic identity $\dfrac{v^3 - 3v}{1 - v^4} = -\dfrac{v}{1 - v^2} - \dfrac{2v}{1 + v^2}$.

Step 1. After $y = vx$: $xv' = (1 - v^4)/(v^3 - 3v)$.

Step 2. Decompose; integrate to $\frac{1}{2}\log|1 - v^2| - \log(1 + v^2) = \log|x| + C_1$.

Step 3. Exponentiate, back-substitute, simplify: $x^2 - y^2 = c(x^2 + y^2)^2$.

> **Final Answer:** Verified.

Q 9.4 **Find the general solution of the differential equation** $\dfrac{dy}{dx} + \dfrac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} = 0$.

SOLUTION

Concept used. Variable-separable. Use $\int dy/\sqrt{1 - y^2} = \sin^{-1} y$.

Step 1. Rearrange:

$$\frac{dy}{\sqrt{1 - y^2}} = -\frac{dx}{\sqrt{1 - x^2}}.$$

Step 2. Integrate both sides:

$$\sin^{-1} y = -\sin^{-1} x + C.$$

Step 3. Rewrite: $\sin^{-1} x + \sin^{-1} y = C$.

> **Final Answer:** $\sin^{-1} x + \sin^{-1} y = C$.

EXPERT'S SOLUTION : *Priya Gupta, M.Sc Mathematics, IIT Bombay*

Quick reading. Both sides are textbook $\sin^{-1}$ integrands.

Step 1. $\dfrac{dy}{\sqrt{1-y^2}} = -\dfrac{dx}{\sqrt{1-x^2}}$.

Step 2. Integrate: $\sin^{-1} y + \sin^{-1} x = C$.

Final Answer: $\sin^{-1} x + \sin^{-1} y = C$.

Q 9.5 **Show that the general solution of** $\dfrac{dy}{dx} + \dfrac{y^2+y+1}{x^2+x+1} = 0$ **is** $(x+y+1) = A(1-x-y-2xy)$**, where** A **is a parameter.**

SOLUTION

Concept used. Separable; integrate using $\int \dfrac{dt}{t^2+t+1} = \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{2t+1}{\sqrt{3}}$ (complete the square).

Step 1. Separate:

$$\frac{dy}{y^2+y+1} = -\frac{dx}{x^2+x+1}.$$

Step 2. Both denominators complete the square as $\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}$. Hence

$$\int \frac{dt}{t^2+t+1} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2t+1}{\sqrt{3}}.$$

Step 3. Integrate:

$$\frac{2}{\sqrt{3}} \tan^{-1} \frac{2y+1}{\sqrt{3}} = -\frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C_1.$$

Step 4. Multiply through by $\sqrt{3}/2$ and combine the inverse-tangents using

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1} \frac{A+B}{1-AB}.$$

With $A = \dfrac{2x+1}{\sqrt{3}}$, $B = \dfrac{2y+1}{\sqrt{3}}$:

$$A+B = \frac{2(x+y)+2}{\sqrt{3}}, \qquad 1-AB = 1 - \frac{(2x+1)(2y+1)}{3} = \frac{3-(2x+1)(2y+1)}{3}.$$

Now $(2x+1)(2y+1) = 4xy + 2x + 2y + 1$, so

$$3 - (2x+1)(2y+1) = 2 - 4xy - 2x - 2y = 2(1-2xy-x-y).$$

Hence

$$\frac{A+B}{1-AB} = \frac{2(x+y+1)/\sqrt{3}}{2(1-x-y-2xy)/3} = \frac{3(x+y+1)}{\sqrt{3}(1-x-y-2xy)} = \frac{\sqrt{3}(x+y+1)}{1-x-y-2xy}.$$

Step 5. Therefore (taking $\tan$ of both sides of the combined log-of-tangent equation)

$$\tan^{-1}\frac{2x+1}{\sqrt{3}} + \tan^{-1}\frac{2y+1}{\sqrt{3}} = C',$$

and so

$$\frac{\sqrt{3}(x+y+1)}{1-x-y-2xy} = \tan C' = \sqrt{3}A \quad \text{(rename constant: } A = \tan C'/\sqrt{3}\text{)}.$$

This gives

$$x+y+1 = A(1-x-y-2xy),$$

as required.

Final Answer: $x+y+1 = A(1-x-y-2xy)$.

EXPERT'S SOLUTION : *Aanya Bhat, Ph.D Pure Mathematics, IISc Bangalore*

Strategic angle. Complete the square in each quadratic; use the $\tan^{-1}$ addition formula to combine the two inverse-tangents.

Step 1. Separate and integrate to $\tan^{-1}\frac{2y+1}{\sqrt{3}} + \tan^{-1}\frac{2x+1}{\sqrt{3}} = C'$.

Step 2. Apply $\tan^{-1} A + \tan^{-1} B = \tan^{-1}\frac{A+B}{1-AB}$ and simplify.

Step 3. Final: $x+y+1 = A(1-x-y-2xy)$.

Final Answer: Verified.

Q 9.6 Find the equation of the curve passing through $\left(0, \frac{\pi}{4}\right)$ whose differential equation is $\sin x \cos y \, dx + \cos x \sin y \, dy = 0$**.**

SOLUTION

Concept used. Variable-separable.

Step 1. Divide by $\cos x \cos y$:

$$\tan x \, dx + \tan y \, dy = 0.$$

Step 2. Integrate:

$$-\log|\cos x| - \log|\cos y| = C_1 \implies \log|\cos x \cos y| = -C_1.$$

Exponentiate: $\cos x \cos y = C$ (sign absorbed).

Step 3. Apply $(0, \pi/4)$: $\cos 0 \cdot \cos(\pi/4) = 1 \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$. So $C = \frac{1}{\sqrt{2}}$.

Final Answer: $\cos x \cos y = \frac{1}{\sqrt{2}}$, i.e. $\sqrt{2}\cos x \cos y = 1$.

EXPERT'S SOLUTION *: Rohit Singh, M.Sc Mathematics, ISI Kolkata*

Quick reading. Each term has $\tan u\, du$ after division; integrate to $\log|\cos x \cos y|$.

Step 1. $\tan x\, dx + \tan y\, dy = 0$.

Step 2. Integrate: $\cos x \cos y = C$.

Step 3. $(0, \pi/4)$ gives $C = 1/\sqrt{2}$.

Final Answer: $\sqrt{2}\cos x \cos y = 1$.

Q 9.7 **Find the particular solution of $(1 + e^{2x})\, dy + (1 + y^2)e^x\, dx = 0$, given $y = 1$ when $x = 0$.**

SOLUTION

Concept used. Separable; substitution $t = e^x$ on the x-side.

Step 1. Separate:

$$\frac{dy}{1 + y^2} = -\frac{e^x\, dx}{1 + e^{2x}}.$$

Step 2. For the RHS, let $t = e^x$, $dt = e^x\, dx$:

$$\int \frac{e^x\, dx}{1 + e^{2x}} = \int \frac{dt}{1 + t^2} = \tan^{-1} t = \tan^{-1}(e^x).$$

Step 3. Integrate LHS: $\int dy/(1 + y^2) = \tan^{-1} y$. So

$$\tan^{-1} y = -\tan^{-1}(e^x) + C.$$

Step 4. Apply $(0, 1)$: $\tan^{-1} 1 = -\tan^{-1} 1 + C \implies C = 2\tan^{-1} 1 = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$.

Final Answer: $\tan^{-1} y + \tan^{-1}(e^x) = \frac{\pi}{2}$.

EXPERT'S SOLUTION : *Sanya Verma, B.Tech Engineering Physics, IIT Bombay*

Quick reading. Both integrals are $\tan^{-1}$ forms after the substitution $t = e^x$.

Step 1. $\dfrac{dy}{1+y^2} = -\dfrac{e^x\,dx}{1+e^{2x}}$.

Step 2. $\tan^{-1} y + \tan^{-1}(e^x) = C$.

Step 3. $y(0) = 1$ gives $C = \pi/2$.

Final Answer: $\tan^{-1} y + \tan^{-1} e^x = \pi/2$.

Q 9.8 Solve $y\,e^{x/y}\,dx = \left(xe^{x/y} + y^2\right) dy\ (y \neq 0)$.

SOLUTION

Concept used. Solve for $\dfrac{dx}{dy}$, then substitute $x = vy$ (since $e^{x/y}$ argues for the x/y variable).

Step 1. Solve for $\dfrac{dx}{dy}$:

$$ye^{x/y}\frac{dx}{dy} = xe^{x/y} + y^2 \implies \frac{dx}{dy} = \frac{x}{y} + \frac{y}{e^{x/y}}.$$

Step 2. Substitute $x = vy$, $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$:

$$v + y\frac{dv}{dy} = v + ye^{-v} \implies y\frac{dv}{dy} = ye^{-v}.$$

Step 3. Cancel y (since $y \neq 0$): $\dfrac{dv}{dy} = e^{-v}$, i.e. $e^v\,dv = dy$.

Step 4. Integrate: $e^v = y + C$. Back-substitute $v = x/y$:

$$e^{x/y} = y + C.$$

Final Answer: $e^{x/y} = y + C$.

EXPERT'S SOLUTION : *Tara Krishna, M.Sc Mathematics, IIT Kanpur*

Strategic angle. $e^{x/y}$ argues for treating y as independent and using $x = vy$.

Step 1. Rearrange to $dx/dy = x/y + ye^{-x/y}$.

Step 2. $x = vy$ gives $y\,dv/dy = ye^{-v}$, so $e^v dv = dy$.

Step 3. Integrate: $e^{x/y} = y + C$.

Final Answer: $e^{x/y} = y + C$.

Q 9.9 **Find a particular solution of $(x-y)(dx+dy) = dx - dy$, given $y = -1$ when $x = 0$.**

Hint: put $x - y = t$.

SOLUTION

Concept used. Substitute $t = x - y$ as suggested. Then $dt = dx - dy$, i.e. $dy = dx - dt$.

Step 1. Replace $x - y \to t$ and $dx - dy \to dt$. Also $dx + dy = dx + (dx - dt) = 2\,dx - dt$.

Step 2. The equation becomes

$$t(2\,dx - dt) = dt \implies 2t\,dx = dt + t\,dt = (1+t)\,dt.$$

Step 3. Separate:

$$2\,dx = \frac{1+t}{t}\,dt = \left(\frac{1}{t} + 1\right)dt.$$

Step 4. Integrate:

$$2x = \log|t| + t + C_1.$$

Step 5. Back-substitute $t = x - y$:

$$2x = \log|x-y| + (x-y) + C_1 \implies x + y = \log|x-y| + C_1.$$

Step 6. Apply $(0,-1)$: $0 + (-1) = \log|0-(-1)| + C_1 = \log 1 + C_1 = C_1$. So $C_1 = -1$.

Final Answer: $\log|x-y| = x + y + 1$.

EXPERT'S SOLUTION *: Ananya Kapoor, Ph.D Mathematics, IIT Delhi*

Strategic angle. The suggested substitution makes the equation separable in t and x after a single rewrite.

Step 1. $t = x - y$, $dt = dx - dy$, $dx + dy = 2\,dx - dt$.

Step 2. $2t\,dx = (1+t)\,dt$, so $2\,dx = (1/t + 1)\,dt$.

Step 3. Integrate; back-substitute; apply $(0,-1)$: $C_1 = -1$.

Final Answer: $\log|x-y| = x+y+1$.

Q 9.10 **Solve** $\left(\dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} - \dfrac{y}{\sqrt{x}}\right)\dfrac{dx}{dy} = 1\ (x \neq 0)$.

SOLUTION

Concept used. Re-interpret as a linear DE in x vs y. Take reciprocals.

Step 1. Reciprocal:

$$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}, \quad \text{i.e.} \quad \frac{dy}{dx} + \frac{y}{\sqrt{x}} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}.$$

Step 2. Linear in y with $P = 1/\sqrt{x}$, $Q = e^{-2\sqrt{x}}/\sqrt{x}$. Compute IF: $\int \dfrac{dx}{\sqrt{x}} = 2\sqrt{x}$. So $\mu = e^{2\sqrt{x}}$.

Step 3. Multiply: $(e^{2\sqrt{x}}y)' = e^{2\sqrt{x}} \cdot \dfrac{e^{-2\sqrt{x}}}{\sqrt{x}} = \dfrac{1}{\sqrt{x}}$.

Step 4. Integrate: $e^{2\sqrt{x}}y = 2\sqrt{x} + C$, so

$$y = (2\sqrt{x} + C)e^{-2\sqrt{x}}.$$

Final Answer: $y\,e^{2\sqrt{x}} = 2\sqrt{x} + C$.

EXPERT'S SOLUTION *: Dev Nair, M.Sc Mathematics, IIT Madras*

Strategic angle. Take the reciprocal; the equation is linear in y with IF $e^{2\sqrt{x}}$.

Step 1. $y' + y/\sqrt{x} = e^{-2\sqrt{x}}/\sqrt{x}$.

Step 2. $\int dx/\sqrt{x} = 2\sqrt{x}$, so $\mu = e^{2\sqrt{x}}$.

Step 3. $(e^{2\sqrt{x}}y)' = 1/\sqrt{x}$; integrate to $2\sqrt{x} + C$.

Final Answer: $ye^{2\sqrt{x}} = 2\sqrt{x} + C$.

Q 9.11 **Find a particular solution of** $\dfrac{dy}{dx} + y\cot x = 4x\csc x\ (x \neq 0)$, **given** $y = 0$ **when** $x = \dfrac{\pi}{2}$.

SOLUTION

Concept used. Linear with $P = \cot x$, $Q = 4x \csc x$. IF: $\mu = e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x$.

Step 1. Multiply: $(y \sin x)' = \sin x \cdot 4x \csc x = 4x$.

Step 2. Integrate: $y \sin x = 2x^2 + C$.

Step 3. Apply $(\pi/2, 0)$: $0 \cdot 1 = 2(\pi/2)^2 + C \implies C = -\dfrac{\pi^2}{2}$.

> **Final Answer:** $y \sin x = 2x^2 - \dfrac{\pi^2}{2}$.

EXPERT'S SOLUTION *: Pooja Pillai, M.Sc Mathematics, IIT Bombay*

Quick reading. $\mu = \sin x$; multiply through; RHS becomes $4x$, integrating to $2x^2$.

Step 1. $(y \sin x)' = 4x$.

Step 2. $y \sin x = 2x^2 + C$; $(\pi/2, 0) \Rightarrow C = -\pi^2/2$.

> **Final Answer:** $y \sin x = 2x^2 - \pi^2/2$.

Q 9.12 **Find a particular solution of $(x+1)\dfrac{dy}{dx} = 2e^{-y} - 1$, given $y = 0$ when $x = 0$.**

SOLUTION

Concept used. Separable.

Step 1. Separate:

$$\frac{dy}{2e^{-y} - 1} = \frac{dx}{x+1}.$$

Step 2. Multiply numerator and denominator on the LHS by e^y:

$$\frac{e^y\, dy}{2 - e^y} = \frac{dx}{x+1}.$$

Step 3. For the LHS, let $u = 2 - e^y$, $du = -e^y\, dy$:

$$\int \frac{e^y\, dy}{2 - e^y} = -\int \frac{du}{u} = -\log|u| = -\log|2 - e^y|.$$

Step 4. Integrate RHS: $\log|x+1|$. So

$$-\log|2 - e^y| = \log|x+1| + C_1.$$

Multiply by -1:

$$\log|2 - e^y| = -\log|x+1| - C_1, \quad \text{i.e.} \quad (2 - e^y)(x+1) = C \quad (C = \pm e^{-C_1}).$$

Step 5. Apply $(0, 0)$: $(2-1)(0+1) = C \implies C = 1$.

Final Answer: $(2-e^y)(x+1) = 1$.

EXPERT'S SOLUTION : *Krishna Mehta, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Multiplying numerator and denominator by e^y turns the integrand into $\dfrac{d(2-e^y)}{2-e^y}$.

Step 1. Separate; multiply LHS by e^y/e^y.

Step 2. Integrate: $-\log|2-e^y| = \log|x+1| + C_1$.

Step 3. Exponentiate to $(2-e^y)(x+1) = C$; $(0, 0)$ gives $C = 1$.

Final Answer: $(2-e^y)(x+1) = 1$.

Q 9.13 **The general solution of** $\dfrac{y\,dx - x\,dy}{y} = 0$ **is:** **(A)** $xy = C$ **(B)** $x = Cy^2$ **(C)** $y = Cx$ **(D)** $y = Cx^2$.

SOLUTION

Concept used. Multiply through by y to get $y\,dx - x\,dy = 0$; recognise as $d(x/y) = 0$ or $d(y/x) = 0$ form.

Step 1. $y\,dx - x\,dy = 0$ is the same as $\dfrac{y\,dx - x\,dy}{x^2} = 0$, i.e. $d\left(\dfrac{y}{x}\right) = 0$ (since $d(y/x) = (x\,dy - y\,dx)/x^2$; this equals zero iff $y\,dx - x\,dy = 0$).

Step 2. Therefore $\dfrac{y}{x} = \text{const}$, i.e. $y = Cx$.

Final Answer: Correct option: **(C)** $y = Cx$.

EXPERT'S SOLUTION : *Diya Kumar, M.Sc Mathematics, IIT Bombay*

Quick reading. $y\,dx = x\,dy \Rightarrow dy/y = dx/x \Rightarrow \log y = \log x + C_1 \Rightarrow y = Cx$.

Step 1. Separate: $dy/y = dx/x$.

Step 2. Integrate: $\log|y| = \log|x| + C_1$, hence $y = Cx$.

Final Answer: (C).

Q 9.14 **The general solution of an equation of the form $\frac{dx}{dy} + P_1 x = Q_1$ is:**

(A) $y\, e^{\int P_1\, dy} = \int (Q_1 e^{\int P_1\, dy})\, dy + C$
(B) $y \cdot e^{\int P_1\, dx} = \int (Q_1 e^{\int P_1\, dx})\, dx + C$
(C) $x \cdot e^{\int P_1\, dy} = \int (Q_1 e^{\int P_1\, dy})\, dy + C$
(D) $x \cdot e^{\int P_1\, dx} = \int (Q_1 e^{\int P_1\, dx})\, dx + C$.

SOLUTION

Concept used. For a DE of the form $\frac{dx}{dy} + P_1(y)\, x = Q_1(y)$, the integrating factor is $\mu(y) = e^{\int P_1\, dy}$, and the general solution is $\mu\, x = \int \mu\, Q_1\, dy + C$.

Step 1. The dependent variable here is x, not y, so multiplication is by $e^{\int P_1\, dy}$, not by anything in dx.

Step 2. Multiplying gives $(x\, e^{\int P_1\, dy})' = Q_1 e^{\int P_1\, dy}$.

Step 3. Integrate with respect to y:

$$x\, e^{\int P_1\, dy} = \int Q_1\, e^{\int P_1\, dy}\, dy + C.$$

Step 4. This matches option (C).

Final Answer: Correct option: **(C)**.

EXPERT'S SOLUTION : *Yash Rao, Ph.D Mathematics, IIT Delhi*

Quick reading. Independent variable is y; dependent is x. Both "dy" and "x" should appear in the formula.

Step 1. Standard formula transports: $x\mu = \int Q_1 \mu\, dy + C$.

Step 2. $\mu = e^{\int P_1\, dy}$.

Step 3. Option (C) matches.

Final Answer: (C).

Q 9.15 **The general solution of $e^x\,dy + (ye^x + 2x)\,dx = 0$ is: (A) $xe^y + x^2 = C$ (B) $xe^y + y^2 = C$ (C) $ye^x + x^2 = C$ (D) $ye^y + x^2 = C$.**

SOLUTION

Concept used. The equation is linear in y once divided by e^x.

Step 1. Divide by e^x:

$$dy + (y + 2xe^{-x})\,dx = 0 \implies \frac{dy}{dx} + y = -2xe^{-x}.$$

Step 2. Linear with $P = 1$, IF $\mu = e^x$. Multiply:

$$(e^x y)' = e^x \cdot (-2xe^{-x}) = -2x.$$

Step 3. Integrate: $e^x y = -x^2 + C$, i.e. $ye^x + x^2 = C$.

Final Answer: Correct option: **(C)** $ye^x + x^2 = C$.

EXPERT'S SOLUTION *: Aditi Joshi, M.Sc Mathematics, IIT Bombay*

Quick reading. The LHS of the original DE $e^x\,dy + ye^x\,dx = d(ye^x)$ is already an exact derivative.

Step 1. $d(ye^x) + 2x\,dx = 0$.

Step 2. Integrate: $ye^x + x^2 = C$.

Final Answer: (C).

♥ Exact differentials

A common shortcut: if the LHS of a DE is the exact differential of some product $\mu \cdot y$, just integrate it directly. Spotting these patterns saves the integrating-factor computation altogether.

Key Takeaways

- Pick the method by inspecting the equation: order/degree (read off), separable (factor RHS), homogeneous (uniform-degree numerator and denominator), linear (write in $y' + Py = Q$ form).
- If the equation is not linear in y, try linearity in x via dx/dy.
- Always apply the initial condition AFTER writing the general solution.

- Common substitutions: $y = vx$, $x = vy$, $t = e^x$, $t = x - y$.
- Memorise the standard integrals: $\int \sec x\, dx$, $\int \cot x\, dx$, $\int e^{ax} \sin bx\, dx$, $\int e^{ax} \cos bx\, dx$, $\int dy/(y \log y)$, $\int dx/\sqrt{x}$.