collegedunia

NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 1 — Representative Set

Chapter 1: Relations and Functions

About this Chapter

A **relation** R from a set A to a set B is any subset of $A \times B$; a **function** $f : A \to B$ is a relation in which every $a \in A$ is paired with exactly one $b \in B$. This Exemplar set drills the core ideas of the chapter: types of relations (**reflexive**, **symmetric**, **transitive**, **equivalence**), partitioning of a set into equivalence classes, types of functions (**one–one**/injective, **onto**/surjective, **bijective**), composition of functions, invertibility, and the assertion–reason fact bank that CBSE board and JEE papers love.

Topics covered: Relations on a set • Reflexive, symmetric, transitive • Equivalence relations & classes • One–one (injective) • Onto (surjective) • Bijective • Composition $g \circ f$ • Invertible functions f^{-1} • Counting injections/surjections

Quick Formula Sheet

Equivalence relation: reflexive + symmetric + transitive

One–one (injective): $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Onto (surjective): $\forall y \in Y, \exists x \in X$ with $f(x) = y$

Composition: $(g \circ f)(x) = g(f(x))$

Invertible: f invertible $\Leftrightarrow$ f bijective; $f^{-1} \circ f = I_X$

Counting (finite sets, $|A| = m$, $|B| = n$): injections $= {}^nP_m$ (if $n \geq m$); bijections $A \to A$: $n!$

I. Multiple Choice Questions (MCQ)

Q 1.1 **Let R be a relation on the set $\mathbb{N}$ of natural numbers defined by nRm if n divides m. Then R is**

(A) Reflexive and symmetric **(B) Transitive and symmetric**

(C) Equivalence **(D) Reflexive, transitive but not symmetric**

SOLUTION

Correct option: (D) Reflexive, transitive but not symmetric.
Concept used. A relation R on $\mathbb{N}$ is **reflexive** if nRn holds for every $n \in \mathbb{N}$; **symmetric** if $nRm \Rightarrow mRn$; **transitive** if nRm and $mRr \Rightarrow nRr$. Here $nRm \Leftrightarrow n \mid m$, i.e. n divides m.

Step 1. Reflexive: every $n \in \mathbb{N}$ divides itself, so nRn holds. Yes.

Step 2. Symmetric: take $n = 3,\ m = 6$. Then $3 \mid 6$ so $3R6$, but $6 \nmid 3$, so $6 \not R 3$. Hence R is *not* symmetric.

Step 3. Transitive: if $n \mid m$ and $m \mid r$, then $n \mid r$ (basic divisibility), so $nRm \wedge mRr \Rightarrow nRr$.

Final Answer: R is reflexive and transitive but not symmetric; option **(D)**.

☞ **Quick filter**

Divisibility on $\mathbb{N}$ is the textbook example of a partial order: reflexive + transitive + antisymmetric, never symmetric (except on $\{1\}$).

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Counterexample-first angle. For symmetry, hunt one pair where $a \mid b$ but $b \nmid a$. $(2, 4)$ does it in two seconds.
Concept used. Divisibility $a \mid b$ means $b = ka$ for some $k \in \mathbb{N}$. It is reflexive ($k = 1$), transitive (compose multipliers), but never symmetric for $a \neq b$ with $a < b$.

Final Answer: Option **(D)**.

Q 1.2 **Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\ell R m$ if and only if ℓ is perpendicular to m, $\forall\, \ell, m \in L$. Then R is**

(A) reflexive (B) symmetric (C) transitive (D) none of these

SOLUTION

Correct option: (B) symmetric.
Concept used. "ℓ perpendicular to m" is a geometric condition on the angle between two lines being $90°$.

Step 1. Reflexive? A line is never perpendicular to itself (the angle is $0°$), so $\ell R \ell$ is *false*. Not reflexive.

Step 2. Symmetric? If $\ell \perp m$, then certainly $m \perp \ell$ — perpendicularity is mutual. Yes.

Step 3. Transitive? Take ℓ horizontal, m vertical, n horizontal. Then $\ell \perp m$ and $m \perp n$, but $\ell \parallel n$, so $\ell \not\perp n$. Not transitive.

> **Final Answer:** Only symmetry holds; option **(B)**.

☞ Perpendicularity vs Parallelism

Perpendicular is symmetric but never reflexive/transitive; parallel is reflexive, symmetric, and transitive (an equivalence relation).

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Eliminate-fast angle. A line is not $\perp$ to itself, so (A) dies. Two "perpendicular-to-the-same-line" lines are parallel, so (C) dies. Only (B) survives.
Concept used. Symmetry of $\perp$ comes from the angle being a property of the pair, not order.

> **Final Answer:** Option **(B)**.

Q 1.3 **Let $\mathbb{N}$ be the set of natural numbers and the function $f : \mathbb{N} \to \mathbb{N}$ be defined by $f(n) = 2n + 3,\ \forall n \in \mathbb{N}$. Then f is**

(A) surjective **(B) injective** **(C) bijective** **(D) none of these**

SOLUTION

Correct option: (B) injective.
Concept used. f is **one–one**/injective if $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$; f is **onto**/surjective if every y in the codomain is hit by some n in the domain.

Step 1. Injective: $2n_1 + 3 = 2n_2 + 3 \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$. So f is one–one.

Step 2. Surjective on $\mathbb{N}$? The smallest value of f is $f(1) = 5$. Numbers $1, 2, 3, 4$ in the codomain are never hit, so f is not onto. The values of f are exactly $\{5, 7, 9, \ldots\}$, a proper subset of $\mathbb{N}$.

> **Final Answer:** f is injective but not surjective; option **(B)**.

☞ **Domain matters!**

The same formula $f(n) = 2n + 3$ on $\mathbb{Z} \to \mathbb{Z}$ is still injective (not onto: even integers are missed). On $\mathbb{R} \to \mathbb{R}$ it becomes a *bijection*. Always check the domain.

EXPERT'S SOLUTION *: Vikram Rao, M.Sc Mathematics, Delhi University*

Range-first angle. The range $\{5, 7, 9, \ldots\} \subsetneq \mathbb{N}$, so not onto. Linear with non-zero slope $\Rightarrow$ one–one.

> **Final Answer:** Option **(B)**.

Q 1.4 **Set A has 3 elements and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is**

(A) 144 (B) 12 (C) 24 (D) 64

SOLUTION

Correct option: (C) 24.

Concept used. The number of **injective** (one–one) functions from a set of m elements into a set of n elements (with $n \geq m$) is the permutation

$$^{n}P_{m} = \frac{n!}{(n-m)!}.$$

Each of the m domain elements must go to a *distinct* codomain element; for the first there are n choices, for the second $n - 1$, and so on.

Step 1. Here $|A| = m = 3$, $|B| = n = 4$.

Step 2. Number of injections $= {}^{4}P_{3} = 4 \times 3 \times 2 = 24$.

> **Final Answer:** 24 injective mappings; option **(C)**.

☞ **Counting cheat-sheet**

Functions $A \to B$: n^m. Injections: ${}^{n}P_{m}$ (need $n \geq m$). Bijections $A \to A$ ($|A| = n$): $n!$.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Slot-filling. Three slots in A, four labels in B. Slot 1: 4 choices, slot 2: 3, slot 3: 2. Product $= 24$.

Final Answer: ${}^4P_3 = 24$; option **(C)**.

Q 1.5 **Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$. Then $f \circ g$ is**

(A) $x^2 \sin x$ **(B)** $(\sin x)^2$ **(C)** $\sin x^2$ **(D)** $\dfrac{\sin x}{x^2}$

SOLUTION

Correct option: **(C)** $\sin x^2$.

Concept used. The **composition** of two functions is $(f \circ g)(x) = f(g(x))$: feed g first, then apply f to the result.

Step 1. Start inside: $g(x) = x^2$.

Step 2. Apply f to $g(x)$: $(f \circ g)(x) = f(g(x)) = f(x^2) = \sin(x^2)$.

Final Answer: $(f \circ g)(x) = \sin x^2$; option **(C)**.

✗ Don't confuse $\sin x^2$ with $(\sin x)^2$

$\sin x^2$ means $\sin(x^2)$ — square first, then sine. $(\sin x)^2 = \sin^2 x$ — sine first, then square. They are very different functions and graders mark them wrong.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Inside-out angle. "f of g" literally substitutes $g(x)$ inside f. Here f is sine, so we sine $g(x) = x^2$.

Final Answer: Option **(C)**.

Q 1.6 **Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 3x - 4$. Then $f^{-1}(x)$ is given by**

(A) $\dfrac{x+4}{3}$ **(B)** $\dfrac{x}{3} - 4$ **(C)** $3x + 4$ **(D)** None of these

SOLUTION

Correct option: **(A)** $\dfrac{x+4}{3}$.

Concept used. If f is **invertible**, set $y = f(x)$, solve for x in terms of y, then rename: $f^{-1}(y) = x$, and finally swap $y \to x$ to get $f^{-1}(x)$.

Step 1. Let $y = 3x - 4$.

Step 2. Solve for x: $\quad 3x = y + 4 \;\Rightarrow\; x = \dfrac{y+4}{3}$.

Step 3. So $f^{-1}(y) = \dfrac{y+4}{3}$, which gives $f^{-1}(x) = \dfrac{x+4}{3}$.

Step 4. Check: $f(f^{-1}(x)) = 3 \cdot \dfrac{x+4}{3} - 4 = (x+4) - 4 = x$. ✓

> **Final Answer:** $f^{-1}(x) = \dfrac{x+4}{3}$; option **(A)**.

☞ **Linear inverses are linear**

For any linear bijection $f(x) = ax + b$ with $a \neq 0$, the inverse is $f^{-1}(x) = \dfrac{x-b}{a}$ — worth memorising.

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Plug-in test. For (A): $f(f^{-1}(x)) = 3 \cdot \dfrac{x+4}{3} - 4 = x$. Confirmed.

> **Final Answer:** Option **(A)**.

Q 1.7 **The maximum number of equivalence relations on the set $A = \{1,2,3\}$ is**

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5

SOLUTION

Correct option: **(D)** 5.

Concept used. Equivalence relations on a set A are in *one-to-one correspondence* with **partitions** of A. The number of partitions of an n-element set is the **Bell number** B_n. For $n = 3$, $B_3 = 5$.

Step 1. Partition with 1 block: $\{\{1,2,3\}\}$ — gives the universal relation $A \times A$.

Step 2. Partitions with 2 blocks: $\{\{1\},\{2,3\}\}$, $\{\{2\},\{1,3\}\}$, $\{\{3\},\{1,2\}\}$ — three of them.

Step 3. Partition with 3 blocks (singletons): $\{\{1\},\{2\},\{3\}\}$ — the identity (diagonal) relation.

Step 4. Total $= 1+3+1 = 5$.

> **Final Answer:** 5 equivalence relations; option **(D)**.

☞ **Bell numbers**

$B_0 = 1$, $B_1 = 1$, $B_2 = 2$, $B_3 = 5$, $B_4 = 15$, $B_5 = 52$. Each equals the number of equivalence relations on an n-set.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Partition-first angle. Don't enumerate relations directly — list partitions. For $\{1,2,3\}$, partition by block sizes: 3, 2+1, 1+1+1. That's $1+3+1 = 5$.

> **Final Answer:** Option **(D)**.

Q 1.8 **Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3 + 5$. Then $f^{-1}(x)$ is**

(A) $(x+5)^{1/3}$ **(B)** $(x-5)^{1/3}$ **(C)** $(5-x)^{1/3}$ **(D)** $5-x$

SOLUTION

Correct option: (B) $(x-5)^{1/3}$.

Concept used. For a bijective f, invert by solving $y = f(x)$ for x. Here $f(x) = x^3 + 5$ is strictly increasing on $\mathbb{R}$, so it is a bijection $\mathbb{R} \to \mathbb{R}$ and f^{-1} exists.

Step 1. Let $y = x^3 + 5$.

Step 2. $x^3 = y - 5 \;\Rightarrow\; x = (y-5)^{1/3}$ (real cube root, defined for all $y \in \mathbb{R}$).

Step 3. Therefore $f^{-1}(x) = (x-5)^{1/3}$.

Step 4. Check: $f(f^{-1}(x)) = \left((x-5)^{1/3}\right)^3 + 5 = (x-5) + 5 = x$. ✓

> **Final Answer:** $f^{-1}(x) = (x-5)^{1/3}$; option **(B)**.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Undo angle. f does "cube, then add 5". The inverse does the opposite operations in reverse order: "subtract 5, then cube root". Result: $(x-5)^{1/3}$.

Final Answer: Option **(B)**.

Q 1.9 **Let $f:[2,\infty)\to\mathbb{R}$ be the function defined by $f(x)=x^2-4x+5$. Then the range of f is**

(A) $\mathbb{R}$ **(B)** $[1,\infty)$ **(C)** $[4,\infty)$ **(D)** $[5,\infty)$

SOLUTION

Correct option: (B) $[1,\infty)$.

Concept used. Complete the square to find the minimum value of a quadratic, then read off the range over the given domain.

Step 1. Complete the square: $x^2-4x+5=(x-2)^2+1$.

Step 2. Vertex of the parabola at $x=2$, minimum value $f(2)=1$.

Step 3. Domain is $[2,\infty)$, where the parabola is strictly increasing.

Step 4. As x runs from 2 to ∞, $f(x)$ runs from 1 to ∞. Range $=[1,\infty)$.

Final Answer: Range $=[1,\infty)$; option **(B)**.

☞ **Complete the square first**

Every quadratic question — min/max, range, inverse — becomes one line once you write ax^2+bx+c as $a(x-h)^2+k$. The vertex $(h,k)=\left(-\frac{b}{2a}, c-\frac{b^2}{4a}\right)$ does the heavy lifting.

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Vertex angle. Axis at $x=2$, minimum $=1$, opens upward. On $[2,\infty)$, f starts at 1 and runs to ∞.

Final Answer: Option **(B)**.

Q 1.10 **For real numbers x and y, define xRy if and only if $x-y+\sqrt{2}$ is an irrational number. Then the relation R is**

(A) reflexive **(B) symmetric** **(C) transitive** **(D) none of these**

SOLUTION

Correct option: (A) reflexive.

Concept used. For any real x, $x - x + \sqrt{2} = \sqrt{2}$, which is irrational. So reflexivity is immediate. We test the other two properties by counter-examples.

Step 1. *Reflexive:* $xRx \Leftrightarrow x - x + \sqrt{2} = \sqrt{2}$ is irrational — true for every $x \in \mathbb{R}$.

Step 2. *Symmetric:* take $x = \sqrt{2},\ y = 0$. Then $x - y + \sqrt{2} = 2\sqrt{2}$ (irrational), so $\sqrt{2}\,R\,0$. But $y - x + \sqrt{2} = -\sqrt{2} + \sqrt{2} = 0$ (rational), so $0 \not R \sqrt{2}$. Not symmetric.

Step 3. *Transitive:* take $x = \sqrt{2},\ y = 1,\ z = 2\sqrt{2}$. $x - y + \sqrt{2} = 2\sqrt{2} - 1$ (irrational), $y - z + \sqrt{2} = 1 - \sqrt{2}$ (irrational), but $x - z + \sqrt{2} = 0$ (rational). Not transitive.

> **Final Answer:** Only reflexivity holds; option **(A)**.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

$\sqrt{2}$ trick angle. Reflexivity is automatic because $\sqrt{2}$ itself is irrational. To break symmetry/transitivity, pick one $\sqrt{2}$ on one side and a rational on the other so the algebra cancels.

> **Final Answer:** Option **(A)**.

II. Short Answer (S.A.) Questions

Q 1.1 **Let $A = \{a, b, c\}$ and the relation R be defined on A as follows: $R = \{(a, a), (b, c), (a, b)\}$. Then, write the minimum number of ordered pairs to be added in R to make R reflexive and transitive.**

SOLUTION

Required pairs: (b, b), (c, c), (a, c).

Concept used. R is reflexive on A iff $(x, x) \in R$ for every $x \in A$. R is transitive iff $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.

Step 1. For reflexivity on $A = \{a, b, c\}$, we need $(a, a), (b, b), (c, c) \in R$. Already $(a, a) \in R$, so add (b, b) and (c, c).

Step 2. For transitivity check the existing pairs: $(a, b) \in R$ and $(b, c) \in R$, so we must have $(a, c) \in R$. Add (a, c).

Step 3. Now check no new transitive obligation arises from the added pairs: $(a, c), (c, c) \Rightarrow (a, c)$ already present; $(a, a), (a, b) \Rightarrow (a, b)$ already present; etc. All triples close.

> **Final Answer:** Add exactly 3 pairs: $(b, b), (c, c), (a, c)$.

☞ Algorithm

For "minimum pairs to make reflexive + transitive", first add every missing diagonal element, then close the relation under transitivity by repeatedly adding (x, z) whenever $(x, y), (y, z)$ are both present, until stable.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Closure angle. Reflexive closure adds the missing diagonal: $(b, b), (c, c)$. Transitive closure of $\{(a, b), (b, c)\}$ adds (a, c). Three pairs total.

> **Final Answer:** 3 pairs: $(b, b), (c, c), (a, c)$.

Q 1.2 **Let D be the domain of the real-valued function f defined by $f(x) = \sqrt{25 - x^2}$. Then, write D.**

SOLUTION

Domain: $D = [-5, 5]$.

Concept used. A real-valued square root $\sqrt{u}$ requires the **radicand** $u \geq 0$. So the domain is the set of x for which $25 - x^2 \geq 0$.

Step 1. Set up the inequality: $25 - x^2 \geq 0$.

Step 2. Rearrange: $x^2 \leq 25$.

Step 3. Solve: $|x| \leq 5$, i.e. $-5 \leq x \leq 5$.

Step 4. Both endpoints are included since $\sqrt{0} = 0$ is defined.

> **Final Answer:** $D = [-5, 5]$.

☞ Square-root and log domains

$\sqrt{u}$ needs $u \geq 0$; $\log u$ needs $u > 0$; $\dfrac{1}{u}$ needs $u \neq 0$. Combine these for compound expressions by intersection.

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Geometry angle. $y = \sqrt{25 - x^2}$ traces the upper semicircle of radius 5 centred at the origin. Its x-domain is the diameter $[-5, 5]$.

Final Answer: $D = [-5, 5]$.

Q 1.3 **Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, $\forall x \in \mathbb{R}$, respectively. Then, find $g \circ f$.**

SOLUTION

Answer: $(g \circ f)(x) = 4x^2 + 4x - 1$.
Concept used. $(g \circ f)(x) = g(f(x))$: substitute the entire expression for $f(x)$ into g.

Step 1. Compute $f(x) = 2x + 1$.

Step 2. Substitute into g: $g(f(x)) = g(2x + 1) = (2x + 1)^2 - 2$.

Step 3. Expand: $(2x + 1)^2 = 4x^2 + 4x + 1$.

Step 4. So $(g \circ f)(x) = 4x^2 + 4x + 1 - 2 = 4x^2 + 4x - 1$.

Final Answer: $(g \circ f)(x) = 4x^2 + 4x - 1$.

✗ Order matters

$g \circ f \neq f \circ g$ in general. Here $(f \circ g)(x) = 2(x^2 - 2) + 1 = 2x^2 - 3$, which is very different from $4x^2 + 4x - 1$. Always read "$g \circ f$" as "g after f".

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Substitute-and-expand. Inside-out: feed f into g. Square the binomial, subtract 2, done.

Final Answer: $4x^2 + 4x - 1$.

Q 1.4 **If $A = \{a, b, c, d\}$ and the function $f = \{(a, b), (b, d), (c, a), (d, c)\}$, write f^{-1}.**

SOLUTION

Answer: $f^{-1} = \{(b,a),(d,b),(a,c),(c,d)\}$.

Concept used. A function on a finite set is invertible iff it is a **bijection** (one–one and onto). Each input maps to a distinct output, and every element of A appears as an output, so f is a bijection $A \to A$. Its inverse f^{-1} is obtained by reversing every ordered pair.

Step 1. List f: $a \to b,\ b \to d,\ c \to a,\ d \to c$.

Step 2. Outputs $= \{b,d,a,c\} = A$ and all distinct: f is a bijection.

Step 3. Reverse each pair: $b \to a,\ d \to b,\ a \to c,\ c \to d$.

Step 4. Write f^{-1} as a set of ordered pairs.

Final Answer: $f^{-1} = \{(b,a),(d,b),(a,c),(c,d)\}$.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Swap-coordinate angle. For a bijection given as ordered pairs, the inverse is simply the same set of pairs with their components swapped. Confirm it's a bijection first.

Final Answer: $f^{-1} = \{(b,a),(d,b),(a,c),(c,d)\}$.

Q 1.5 **Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x,\ \forall x \in \mathbb{R}$. Show that f is neither one–one nor onto.**

SOLUTION

Claim. $f(x) = \cos x$ on $\mathbb{R} \to \mathbb{R}$ is neither injective nor surjective.

Concept used. f is **one–one** iff distinct inputs give distinct outputs; f is **onto** iff every $y \in \mathbb{R}$ is hit. Recall the range of cosine: $[-1,1]$, and that $\cos$ is 2π-periodic.

Step 1. *Not one–one.* Take $x_1 = 0,\ x_2 = 2\pi$ with $x_1 \neq x_2$. Then $\cos 0 = \cos 2\pi = 1$, so $f(x_1) = f(x_2)$ without $x_1 = x_2$. Injectivity fails.

Step 2. *Not onto.* For any $x \in \mathbb{R}$, $-1 \le \cos x \le 1$. Hence the range of f is $[-1,1] \subsetneq \mathbb{R}$. The value $y = 2 \in \mathbb{R}$ is never attained, so surjectivity fails.

Final Answer: f is neither one–one nor onto.

☞ **Periodicity kills injectivity**

Any periodic function is automatically non-injective on $\mathbb{R}$: $f(x) = f(x+T)$ for every x and the fundamental period T.

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Range angle. Range of $\cos$ is $[-1, 1]$ — not all of $\mathbb{R}$, so not onto. Periodicity gives $\cos 0 = \cos 2\pi$ — so not one–one. Two lines suffice.

Final Answer: Neither one–one nor onto.

III. Long Answer (L.A.) Questions

Q 1.1 **Let n be a fixed positive integer. Define a relation R in $\mathbb{Z}$ as follows: $\forall\, a, b \in \mathbb{Z}$, aRb if and only if $a - b$ is divisible by n. Show that R is an equivalence relation.**

SOLUTION

Claim. R is an equivalence relation on $\mathbb{Z}$ (this is **congruence modulo** n, written $a \equiv b \pmod{n}$).

Concept used. "n divides k" means $k = n\ell$ for some $\ell \in \mathbb{Z}$. We must verify all three of reflexivity, symmetry, and transitivity.

Step 1. *Reflexive.* For every $a \in \mathbb{Z}$, $a - a = 0 = n \cdot 0$, which is divisible by n. So aRa holds. ✓

Step 2. *Symmetric.* Suppose aRb, i.e. $a - b = n\ell$ for some $\ell \in \mathbb{Z}$. Then $b - a = -n\ell = n(-\ell)$, which is again a multiple of n. So bRa. ✓

Step 3. *Transitive.* Suppose aRb and bRc: $a - b = n\ell_1$ and $b - c = n\ell_2$. Adding, $a - c = (a-b) + (b-c) = n\ell_1 + n\ell_2 = n(\ell_1 + \ell_2)$, divisible by n. So aRc. ✓

Step 4. All three properties hold, so R is an equivalence relation.

Final Answer: R is an equivalence relation on $\mathbb{Z}$ — this is congruence modulo n.

☞ **Mod-n equivalence classes**

The equivalence classes of R are $[0], [1], \ldots, [n-1]$ — exactly n classes, forming a partition of $\mathbb{Z}$ into residues mod n. This is the foundation of $\mathbb{Z}_n$ in algebra.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Linear-combination angle. The key trick: $a-c=(a-b)+(b-c)$, so "divisible by n" is closed under addition. That single observation delivers transitivity; reflexivity (0 is a multiple of n) and symmetry (negate the multiplier) are trivial.

Final Answer: Equivalence relation, n classes mod n.

Q 1.2 **Let $A=\mathbb{R}-\{3\}$, $B=\mathbb{R}-\{1\}$. Let $f:A\to B$ be defined by $f(x)=\dfrac{x-2}{x-3}$, $\forall x\in A$. Then show that f is bijective.**

SOLUTION

Claim. f is one–one and onto, hence bijective.
Concept used. To show **one–one**, assume $f(x_1)=f(x_2)$ and deduce $x_1=x_2$. To show **onto**, take a generic $y\in B$ and produce a preimage $x\in A$ with $f(x)=y$.

Step 1. *One–one.* Suppose $\dfrac{x_1-2}{x_1-3}=\dfrac{x_2-2}{x_2-3}$. Cross-multiply: $(x_1-2)(x_2-3)=(x_2-2)(x_1-3)$.

Step 2. Expand both sides: $x_1x_2-3x_1-2x_2+6=x_1x_2-3x_2-2x_1+6$.

Step 3. Cancel common terms: $-3x_1-2x_2=-3x_2-2x_1$, hence $-x_1=-x_2$, i.e. $x_1=x_2$. So f is one–one.

Step 4. *Onto.* Let $y\in B=\mathbb{R}-\{1\}$. We must find $x\in A$ with $f(x)=y$.

Step 5. Set $\dfrac{x-2}{x-3}=y\Rightarrow x-2=y(x-3)=yx-3y\Rightarrow x-yx=2-3y\Rightarrow x=\dfrac{2-3y}{1-y}$, which is well-defined because $y\neq 1$.

Step 6. Check $x\neq 3$: if $x=3$ then $\dfrac{2-3y}{1-y}=3\Rightarrow 2-3y=3-3y\Rightarrow 2=3$, contradiction. So $x\in A$.

Step 7. Hence every $y\in B$ has a preimage in A: f is onto.

Final Answer: f is bijective with $f^{-1}(y)=\dfrac{2-3y}{1-y}$.

☞ **Why exclude 3 and 1?**

$x=3$ makes the denominator zero (so f undefined there). The value $y=1$ is never attained — $\dfrac{x-2}{x-3}=1$ forces $x-2=x-3$, i.e. $-2=-3$, impossible. Hence the chosen domain/codomain make f bijective.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Mobius-map angle. Functions of the form $\dfrac{ax+b}{cx+d}$ with $ad-bc\neq 0$ are bijections of $\mathbb{R}\cup\{\infty\}$. Here $ad-bc = 1\cdot(-3)-(-2)\cdot 1 = -1 \neq 0$, so once we remove the "bad" point $x=3$ and the unattained value $y=1$, f is a bijection $A\to B$.

Final Answer: Bijective; $f^{-1}(y) = \dfrac{2-3y}{1-y}$.

Q 1.3 **Show that the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=\dfrac{x}{x^2+1}$, $\forall\, x\in\mathbb{R}$, is neither one–one nor onto.**

SOLUTION

Claim. f is neither injective nor surjective on $\mathbb{R}$.
Concept used. For one–one, equate $f(x_1)=f(x_2)$ and check if $x_1=x_2$ is forced. For onto, examine the range.

Step 1. Set $\dfrac{x_1}{x_1^2+1}=\dfrac{x_2}{x_2^2+1}$. Cross-multiply: $x_1(x_2^2+1)=x_2(x_1^2+1)$.

Step 2. Expand:
$$x_1x_2^2+x_1 = x_2x_1^2+x_2 \Rightarrow x_1x_2(x_2-x_1) = x_2-x_1 \Rightarrow (x_2-x_1)(x_1x_2-1)=0.$$

Step 3. Either $x_1=x_2$ *or* $x_1x_2=1$.

Step 4. Take $x_1=2,\ x_2=\frac{1}{2}$: distinct, but $f(2)=\frac{2}{5}$ and $f(\frac{1}{2})=\dfrac{1/2}{1/4+1}=\dfrac{1/2}{5/4}=\frac{2}{5}$. Equal images, distinct inputs — not one–one.

Step 5. For onto, test $y=1$: $\dfrac{x}{x^2+1}=1\Rightarrow x=x^2+1\Rightarrow x^2-x+1=0$, whose discriminant is $1-4=-3<0$. No real solution, so 1 is not in the range.

Step 6. Hence f is not onto $\mathbb{R}$.

Final Answer: f is neither one–one nor onto.

☞ **AM–GM bound on the range**

For $x>0$, $x+\frac{1}{x}\geq 2$ by AM–GM, so $f(x)=\dfrac{1}{x+\frac{1}{x}}\leq\frac{1}{2}$. By symmetry $f(x)\geq-\frac{1}{2}$ for $x<0$. Range $=[-\frac{1}{2},\frac{1}{2}]$.

EXPERT'S SOLUTION : *Vikram Rao, M.Sc Mathematics, Delhi University*

Reciprocal-pair angle. The cross-multiplied factor $(x_1x_2 - 1)$ exposes the failure: any reciprocal pair $(a, 1/a)$ collides. For onto, the AM–GM bound caps the range at $\pm\frac{1}{2}$, far short of $\mathbb{R}$.

Final Answer: Neither one–one nor onto.

Q 1.4 **Functions $f, g : \mathbb{R} \to \mathbb{R}$ are defined, respectively, by $f(x) = x^2 + 3x + 1,\ g(x) = 2x - 3$. Find (i) $f \circ g$, (ii) $g \circ f$.**

SOLUTION

Answer. (i) $(f \circ g)(x) = 4x^2 - 6x + 1$; (ii) $(g \circ f)(x) = 2x^2 + 6x - 1$.
Concept used. $(f \circ g)(x) = f(g(x))$: replace every x in $f(x)$ with the full expression for $g(x)$ and simplify. Likewise $(g \circ f)(x) = g(f(x))$.

Step 1. *(i)* $f \circ g$. $f(g(x)) = f(2x-3) = (2x-3)^2 + 3(2x-3) + 1$.

Step 2. Expand $(2x-3)^2 = 4x^2 - 12x + 9$.

Step 3. Expand $3(2x-3) = 6x - 9$.

Step 4. Add: $4x^2 - 12x + 9 + 6x - 9 + 1 = 4x^2 - 6x + 1$.

Step 5. *(ii)* $g \circ f$.
$$g(f(x)) = 2f(x) - 3 = 2(x^2 + 3x + 1) - 3 = 2x^2 + 6x + 2 - 3 = 2x^2 + 6x - 1.$$

Final Answer: $f \circ g = 4x^2 - 6x + 1;\quad g \circ f = 2x^2 + 6x - 1.$

✗ Sign and bracket slips

The most common slip is dropping the bracket on $(2x-3)^2$ or forgetting to distribute the 3 in $3(2x-3)$. Write each expansion on its own line.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Substitute carefully. Two compositions, two distinct results. Always expand inside the outer function step by step — never simplify in your head.

Final Answer: $f \circ g = 4x^2 - 6x + 1;\ g \circ f = 2x^2 + 6x - 1.$

Q 1.5 **Let $A = \{1, 2, 3, \ldots, 9\}$ and R be the relation on $A \times A$ defined by $(a, b)R(c, d)$ if $a + d = b + c$ for $(a, b), (c, d) \in A \times A$. Prove that R is an equivalence relation and obtain the equivalence class $[(2, 5)]$.**

SOLUTION

Claim. R is reflexive, symmetric, and transitive; the class of $(2, 5)$ is $[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$.

Concept used. The condition $a + d = b + c$ is equivalent to $a - b = c - d$: the relation says "two ordered pairs have the same first-minus-second." This is exactly the kind of "equal-invariant" condition that defines an equivalence relation.

Step 1. *Reflexive.* For $(a, b) \in A \times A$: $a + b = b + a$ trivially, so $(a, b)R(a, b)$.

Step 2. *Symmetric.* If $(a, b)R(c, d)$ then $a + d = b + c$, i.e. $c + b = d + a$, which is the condition $(c, d)R(a, b)$. So R is symmetric.

Step 3. *Transitive.* Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Then $a + d = b + c$ and $c + f = d + e$. Add the two equations:
$a + d + c + f = b + c + d + e \Rightarrow a + f = b + e$, which is $(a, b)R(e, f)$.

Step 4. *All three hold,* so R is an equivalence relation.

Step 5. *Compute* $[(2, 5)]$. $(c, d) \in [(2, 5)] \Leftrightarrow 2 + d = 5 + c \Leftrightarrow d - c = 3$. We need $c, d \in \{1, \ldots, 9\}$ with $d = c + 3$: $(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)$.

> **Final Answer:** $[(2, 5)] = \{(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)\}$ — 6 ordered pairs.

☞ **Recognise the invariant**

$a + d = b + c \Leftrightarrow a - b = c - d$. Whenever a relation says "the value of some function is the same on both pairs", equivalence is automatic — the classes are the level sets of that function.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Mathematics, IISc Bangalore*

Level-set angle. Define $\varphi(a, b) = a - b$. The relation is $\varphi(a, b) = \varphi(c, d)$, an equivalence by general principle. For $(2, 5)$, $\varphi = -3$; collect all pairs with $a - b = -3$ inside $\{1, \ldots, 9\}^2$ — six in total.

> **Final Answer:** $[(2, 5)]$ has 6 pairs: $(1, 4), (2, 5), (3, 6), (4, 7), (5, 8), (6, 9)$.