# Relations and Functions

A relation R from set A to set B is a subset of the cartesian product A x B.
If (a, b) ~~belong~~ in R, we write a R b.

## Types of Relations (in a set A)

1. Empty Relation : no element of A is related to any element of A.

$\boxed{R = \text{phi (subset of) } A \times A}$ <- no pair

2. Universal Relation : every element of A is related to every element of A.

$\boxed{R = A \times A}$ <- all pairs

Both empty + universal relations are called trivial relations.

## Three Key Properties

(i) Reflexive : (a, a) in R, for every a in A

(ii) Symmetric : $(a_1, a_2)$ in R => $(a_2, a)$ in R

(iii) Transitive : $(a_1, a_2), (a_2, a_3)$ in R => $(a_1, a_3)$ in R for all $a_i$ in A.

# Equivalence Relation

A relation R on set A is called an equivalence relation if it is :

| Reflexive + Symmetric + Transitive | <-all 3 |
|---|---|

## Example (mod 2 on Z)

R = { (a, b) : 2 divides a - b }

Check :

R : 2 (a - a) = 0 -> reflexive

S : 2 (a - b) => 2 (b - a)

T : 2 (a-b), 2 (b-c)

=> 2 (a-b) + (b-c) = (a-c)

So R is an equivalence relation on Z.

## Equivalence Classes

An equivalence relation R on X partitions X into disjoint subsets $A_i$

(i) all elems of $A_i$ related to each other

(ii) no elem of $A_i$ related to $A_j$, i != j

(iii) union $A_i$ = X, $A_i$ $A_j$ = phi

e.g. mod 3 on Z gives [0], [1], [2] :

[0]={3r}, [1]={3r+1}, [2]={3r+2}, r in Z.

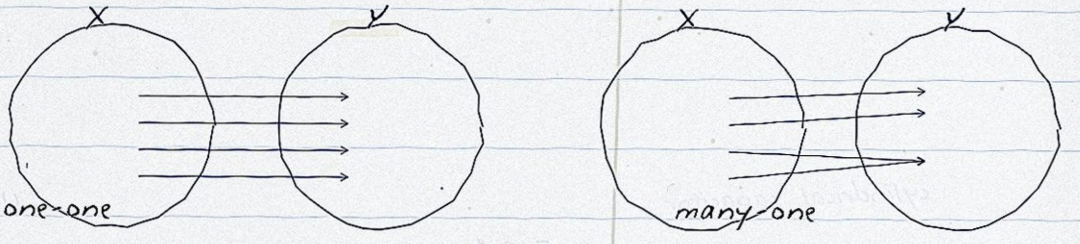
# Types of Functions

## One-One Function (Injective)

$f : X \to Y$ is one-one if distinct elements have distinct images.

$$\boxed{f(x) = f(x_2) \Rightarrow x_1 = x_2}$$ <- injectivity <- test

Otherwise $f$ is called ~~multi~~ many-one.




## Onto Function (Surjective)

$f : X \to Y$ is onto if every $y$ in $Y$ is the image of some $x$ in $X$.

$$\boxed{y \text{ in } Y, \text{ exists } x \text{ in } X : f(x) = y}$$ <- Range <- = Y

## Bijective Function

If $f$ is both one-one and onto, $f$ is called bijective.

Note : finite set $X$, $f : X \to X$ one-one $\Leftrightarrow$ onto.

# Function Examples

**Ex.** $f : R \to R$, $f(x) = 2x$

One-one ?

$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$ one-one

Onto ?

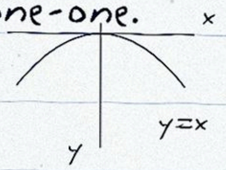
for $y$ in $R$, take $x = y/2$ in $R$

then $f(y/2) = 2 \cdot y/2 = y$ onto

$\boxed{\text{So } f \text{ is bijective.}}$ <- both

**Ex.** $f : R \to R$, $f(x) = x$

$f(-1) = 1 = f(1) \Rightarrow$ ~~bijective~~ not one-one.




Also $-2$ in co-dom has no pre-image

$(x \geq 0) \Rightarrow$ not onto.

**Ex.** $f : N \to N$, $f(x) = 2x$

one-one : $2x_1 = 2x_2 \Rightarrow x_1 = x_2$

not onto : $1$ in $N$ has no pre-image

(needs $x = 1/2$, not in $N$).

**Ex.** Modulus $f(x) = x$ , $R \to R$

$f(-2) = f(2) = 2 \Rightarrow$ not one-one;

Range $= [0, \infty) \Rightarrow$ not onto.

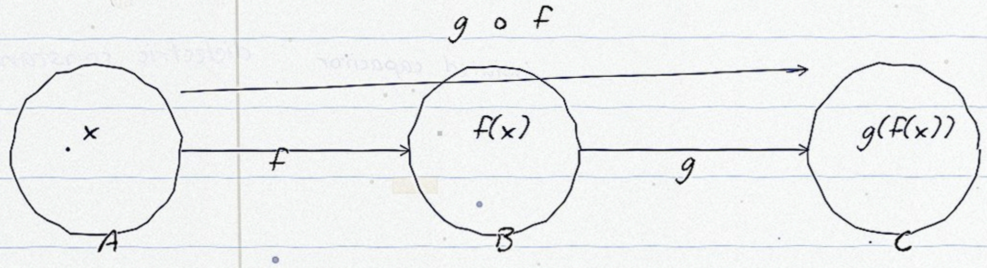
# Composition of Functions

Let $f : A \to B$ and $g : B \to C$.

Then the composition $g \circ f : A \to C$ is :

$$\boxed{(g \circ f)(x) = g(\, f(x)\, ), \quad \text{for all } x \text{ in } A}$$

apply f
<- then g




## Properties

(i) In general $g \circ f$ != $f \circ g$ (not commutative)

(ii) Associative : $h \circ (g \circ f) = (h \circ g) \circ f$

(iii) If $f, g$ both one-one $\Rightarrow$ $g \circ f$ one-one

(iv) If $f, g$ both onto $\Rightarrow$ $g \circ f$ onto

**Ex.** $f(x) = \cos x, \quad g(x) = 3\, x$

$g \circ f\,(x) = g(\cos x) = 3\,(\cos x)^2 = 3 \cos x$

$f \circ g\,(x) = f(3\, x^2) = \cos(3\, x^2)$

At $x = 0$ : $3 \cos 0 = 3, \quad \cos 0 = 1$

Therefore $g \circ f$ != $f \circ g$.

# Invertible Functions

A function $f : X \to Y$ is invertible if there exists $g : Y \to X$ such that :

$$\boxed{g \circ f = I_X, \quad f \circ g = I_Y}$$ <- identity <- on $X, Y$

Then $g$ is the inverse of $f$, written $g = f^{-1}$

## Key Theorem

$$\boxed{f \text{ invertible} \iff f \text{ is bijective}}$$ <- one-one <- + onto

So showing $f$ is one-one + onto is enough to prove invertibility, even without finding the actual ~~formula~~ inverse formula.

**Ex.** $f : N \to Y$, $f(x) = 4x + 3$

where $Y = \{ y \text{ in } N : y = 4x+3, \ x \text{ in } N \}$.

Step 1 : take $y = 4x + 3$

$\Rightarrow x = (y - 3) / 4$

Step 2 : define $g(y) = (y - 3) / 4$

Step 3 : verify

$g(f(x)) = ((4x+3) - 3)/4 = x$

$f(g(y)) = 4((y-3)/4) + 3 = y$

Hence $f$ is invertible, $f(y) = (y-3)/4$.

# Quick Recap & Tips

## Summary Table

| Relation R on A | Condition |
|---|---|
| Reflexive | (a, a) in R |
| Symmetric | (a,b) => (b,a) |
| Transitive | (a,b)+(b,c)=>(a,c) |
| Equivalence | all three above |
| Empty | R = phi |
| Universal | R = A x A |

## Function Quick Checks

- One-one : $f(x_1)=f(x_2) \Rightarrow x_1=x_2$
- Onto : Range f = co-domain.
- Bijective : one-one AND onto
- Invertible <=> bijective

## Counting Facts (small sets)

Number of one-one maps {1,2,3} -> itself

= 3! = 6 (permutations)

Number of onto maps {1,2,...,n} -> itself

= n! (one-one on finite set <=> onto).

Bonus : |A| = n => $2^{n^2-n}$ reflexive relations.