Collegedunia NCERT Formula Sheet

Class 12 Mathematics — Chapter 10

Chapter 10: Vector Algebra

Quantity	Symbol / Form	Key relation
Magnitude of $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$	$\lvert\vec{r}\rvert$	$\lvert\vec{r}\rvert = \sqrt{x^2 + y^2 + z^2}$
Direction cosines	$l,\ m,\ n$	$l^2 + m^2 + n^2 = 1$
Unit vector along $\vec{a}$	$\hat{a}$	$\hat{a} = \frac{\vec{a}}{\lvert\vec{a}\rvert}$
Standard basis vectors	$\hat{i},\ \hat{j},\ \hat{k}$	mutually $\perp$, magnitude 1
Dot product (angle form)	$\vec{a} \cdot \vec{b}$	$\lvert\vec{a}\rvert\lvert\vec{b}\rvert \cos\theta$
Cross product (angle form)	$\vec{a} \times \vec{b}$	$\lvert\vec{a}\rvert\lvert\vec{b}\rvert \sin\theta\, \hat{n}$

1 Vectors & Scalars — Basics

This section sets up the core definitions: what makes a quantity a vector, how its magnitude is measured, and how its direction is encoded by direction cosines.

Scalar vs Vector

A **scalar** has only magnitude (e.g., mass, time, temperature, work, distance). A **vector** has both magnitude and direction (e.g., displacement, velocity, force, weight). A directed line segment $\overrightarrow{AB}$ with initial point A and terminal point B represents a vector.

Magnitude of a vector

$|\overrightarrow{AB}| =$ length of segment AB

For position vector $\vec{r} = \overrightarrow{OP}$ of point $P(x, y, z)$:

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Magnitude is the **length** of the vector (always ≥ 0). The notation $|\vec{a}| < 0$ has **no meaning**.

Position vector

The position vector of a point $P(x, y, z)$ with respect to origin O is $\overrightarrow{OP} = \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Position vectors of A, B, C are usually denoted $\vec{a}, \vec{b}, \vec{c}$.

Direction cosines

If $\vec{r}$ makes angles α, β, γ with the x, y, z axes:

$$l = \cos\alpha = \frac{x}{r}, \quad m = \cos\beta = \frac{y}{r}, \quad n = \cos\gamma = \frac{z}{r}$$

$\boxed{l^2 + m^2 + n^2 = 1}$

Direction cosines fix the **orientation** of $\vec{r}$ in 3D. The identity $l^2 + m^2 + n^2 = 1$ is a **must-know**; it is the constraint that any unit vector satisfies.

Direction ratios

Any numbers a, b, c proportional to l, m, n are direction ratios of $\vec{r}$.

If $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ then a, b, c are direction ratios, and

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

In general $a^2 + b^2 + c^2 \neq 1$; only the **normalised** ratios are the direction cosines.

2 Types of Vectors

NCERT classifies vectors by their geometry and how they relate to each other. Each type unlocks specific shortcuts in proofs and component manipulations.

Standard types

Zero / null vector $\vec{0}$: initial and terminal points coincide; magnitude 0, no definite direction. **Unit vector** $\hat{a}$: magnitude 1. **Coinitial vectors**: share the same initial point. **Collinear vectors**: parallel to the same line (irrespective of magnitude or sense). **Equal vectors**: same magnitude AND direction (positions of initial points irrelevant). **Negative of** $\vec{a}$: same magnitude, opposite direction, written $-\vec{a}$. **Free vectors**: can be parallel-translated without changing magnitude/direction; this chapter deals with free vectors only.

Unit vector in direction of $\vec{a}$

$$\hat{a} = \frac{1}{|\vec{a}|}\vec{a}, \qquad \vec{a} \neq \vec{0}$$

$$|\hat{a}| = 1$$

Divide a non-zero vector by its own magnitude to get the unit vector pointing the **same way**. Useful whenever you need just the **direction information** of $\vec{a}$.

Collinearity condition

$\vec{a} \parallel \vec{b} \iff \vec{b} = \lambda\vec{a}$ for some scalar $\lambda \neq 0$

In component form, with $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$:

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} = \frac{b_3}{a_3} = \lambda$$

Two vectors are collinear iff their corresponding components are **proportional**. Sign of λ tells you same / opposite sense.

3 Addition of Vectors

Adding vectors gives the resultant displacement. NCERT presents two equivalent geometric laws plus the algebraic component rule, all governed by the same four properties.

Triangle law

If $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$, then

$$\vec{a} + \vec{b} = \vec{AB} + \vec{BC} = \vec{AC}$$

For a closed triangle: $\vec{AB} + \vec{BC} + \vec{CA} = \vec{0}$

Place the tail of $\vec{b}$ at the head of $\vec{a}$; the resultant runs from tail of $\vec{a}$ to head of $\vec{b}$. **Closed polygon** of vectors $\Rightarrow$ sum is zero.

Parallelogram law

If $\vec{a}, \vec{b}$ are adjacent sides of a parallelogram from a common point O, then

$\vec{a} + \vec{b} =$ diagonal through O

$$\vec{OA} + \vec{OB} = \vec{OC}$$

Equivalent to the triangle law (because $\vec{AC} = \vec{OB}$). Use it for **forces or velocities** acting from a common point.

Properties of vector addition

$\vec{a}+\vec{b}=\vec{b}+\vec{a}$ **(commutative)**
$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ **(associative)**

$\vec{a}+\vec{0}=\vec{0}+\vec{a}=\vec{a}$ **(additive identity)**
$\vec{a}+(-\vec{a})=\vec{0}$ **(additive inverse)**
Associativity lets you drop brackets and write $\vec{a}+\vec{b}+\vec{c}$. $\vec{0}$ is the **neutral element** of addition.

Difference of vectors

$\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$
$\overrightarrow{AB}-\overrightarrow{CB}=\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$
Subtraction is just addition of the **reversed** vector — flip $\vec{b}$ and apply the triangle law.

4 Scalar Multiplication

Scaling a vector by a real number changes only its magnitude (and possibly flips its sense). This operation is what lets us build all vectors in a plane / space from a basis.

Multiplication by a scalar

$\lambda\vec{a}$ is a vector **collinear** with $\vec{a}$.
$|\lambda\vec{a}|=|\lambda||\vec{a}|$
Direction: same as $\vec{a}$ if $\lambda>0$; opposite if $\lambda<0$.
Stretches or shrinks $\vec{a}$ by factor $|\lambda|$ and reverses it when $\lambda<0$. $\lambda=0$ gives $\vec{0}$.

Distributive laws (scalars k, m; vectors $\vec{a}, \vec{b}$)

$k\vec{a}+m\vec{a}=(k+m)\vec{a}$
$k(m\vec{a})=(km)\vec{a}$
$k(\vec{a}+\vec{b})=k\vec{a}+k\vec{b}$
These distributive rules make vector algebra behave like ordinary algebra — **factor out scalars** freely.

5 Component Form

In 3D, any vector decomposes uniquely along the standard basis $\hat{i},\hat{j},\hat{k}$. Component form turns geometric questions into arithmetic ones.

Component (rectangular) form

$\vec{r}=\overrightarrow{OP}=x\hat{i}+y\hat{j}+z\hat{k}$
x,y,z: **scalar components**; $x\hat{i},\ y\hat{j},\ z\hat{k}$: **vector components**
$|\vec{r}|=\sqrt{x^2+y^2+z^2}$
Pythagoras in 3D: the magnitude is the diagonal of the rectangular box with sides $|x|,|y|,|z|$.

Algebra in components

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$.
$\vec{a}+\vec{b}=(a_1+b_1)\hat{i}+(a_2+b_2)\hat{j}+(a_3+b_3)\hat{k}$
$\vec{a}-\vec{b}=(a_1-b_1)\hat{i}+(a_2-b_2)\hat{j}+(a_3-b_3)\hat{k}$
$\lambda\vec{a}=(\lambda a_1)\hat{i}+(\lambda a_2)\hat{j}+(\lambda a_3)\hat{k}$
$\vec{a}=\vec{b}\iff a_1=b_1,\ a_2=b_2,\ a_3=b_3$
Operate **componentwise** — that is the entire algebra of vectors once a basis is fixed.

Vector joining two points

For $P_1(x_1,y_1,z_1)$, $P_2(x_2,y_2,z_2)$:
$\overrightarrow{P_1P_2}=(x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$
$|\overrightarrow{P_1P_2}|=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
"Head minus tail." The magnitude is just the 3D **distance formula** from Class XI.

Unit vector via direction cosines

If l,m,n are direction cosines of $\vec{r}$:
$\hat{r}=l\hat{i}+m\hat{j}+n\hat{k}=\cos\alpha\,\hat{i}+\cos\beta\,\hat{j}+\cos\gamma\,\hat{k}$
Every unit vector can be **written from its direction angles**, and conversely l,m,n are read off as its components.

6 Section Formula

Locates the point that divides a join PQ in a given ratio — internal and external — using position vectors.

Section formula — internal division

R divides PQ internally in ratio $m : n$ ($m, n > 0$):

$$\overrightarrow{OR} = \vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$$

where $\vec{a} = \overrightarrow{OP}$, $\vec{b} = \overrightarrow{OQ}$.

R **lies between** P and Q. Heavier weight goes to the **opposite** endpoint — the m-part of the ratio is closer to Q.

Section formula — external division

R divides PQ externally in ratio $m : n$ ($m \neq n$):

$$\overrightarrow{OR} = \frac{m\vec{b} - n\vec{a}}{m - n}$$

R **lies outside** segment PQ, on the line PQ produced. Same formula with a **sign flip** on n and on the denominator.

Midpoint formula

If R is the midpoint of PQ (i.e. $m = n$):

$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$$

Special case $m : n = 1 : 1$ of internal section formula — the **average** of the two position vectors.

Internal vs External

Internal: $\dfrac{m\vec{b} + n\vec{a}}{m + n}$. External: $\dfrac{m\vec{b} - n\vec{a}}{m - n}$.

Common slip — using $+$ in the denominator for external division. Use $m - n$ and flip the sign on $\vec{a}$ as well.

7 Scalar (Dot) Product

The dot product turns two vectors into a number; it measures how much they point in the same direction. Critical for angle, projection and perpendicularity questions.

Dot product — angle form

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta, \quad 0 \leq \theta \leq \pi$

If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, define $\vec{a} \cdot \vec{b} = 0$.

Result is a **real number** (scalar). Positive when θ acute, zero when perpendicular, negative when obtuse.

Dot product — component form

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$:

$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

In particular $\vec{a} \cdot \vec{a} = |\vec{a}|^2 = a_1^2 + a_2^2 + a_3^2$.

Just **multiply componentwise and add**. The fastest way to compute angles and projections from coordinates.

Standard basis dot products

$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$

$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

Basis vectors are **orthonormal** — pairwise perpendicular and individually unit length.

Properties of dot product

Commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Distributive: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

Scalar pull-out: $(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda\vec{b})$

Perpendicularity: $\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$ (for non-zero vectors)

The dot product is the standard test for **perpendicularity** once both vectors are non-zero.

Angle between two vectors

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}, \quad \theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$$

Combine the angle form and the component form to read off θ **directly from coordinates**.

Projection of a vector

Projection of $\vec{a}$ on $\vec{b}$ (scalar):

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \vec{a}\cdot\hat{b}$$

Projection **vector** of $\vec{a}$ on $\vec{b}$:

$$\vec{p} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Length of the shadow $\vec{a}$ casts on the line of $\vec{b}$. Sign: $+$ if θ acute, $-$ if obtuse, 0 if perpendicular.

Direction cosines via dot product

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$:

$$\cos\alpha = \frac{\vec{a}\cdot\hat{i}}{|\vec{a}|} = \frac{a_1}{|\vec{a}|}, \quad \cos\beta = \frac{a_2}{|\vec{a}|}, \quad \cos\gamma = \frac{a_3}{|\vec{a}|}$$

Scalar components a_1, a_2, a_3 are the **projections** of $\vec{a}$ on the coordinate axes.

JEE/NEET Extension — Cauchy-Schwarz & Triangle Inequality

For any vectors $\vec{a}, \vec{b}$:

$|\vec{a}\cdot\vec{b}| \le |\vec{a}||\vec{b}|$ (Cauchy-Schwarz)

$|\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$ (Triangle inequality)

Equality in Cauchy-Schwarz $\Leftrightarrow$ $\vec{a}, \vec{b}$ collinear; in triangle inequality $\Leftrightarrow$ they point the same way.

8 Vector (Cross) Product

The cross product produces a third vector perpendicular to both inputs; its magnitude equals the area swept. It is the key tool for area, torque, and 3D perpendicularity.

Cross product — angle form

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}, \quad 0 \le \theta \le \pi$$

where $\hat{n}$ is a unit vector $\perp$ both $\vec{a}, \vec{b}$ with $(\vec{a}, \vec{b}, \hat{n})$ a **right-handed** system.

If $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$: $\vec{a}\times\vec{b} = \vec{0}$.

Result is a **vector**. Curl right-hand fingers from $\vec{a}$ to $\vec{b}$; thumb gives $\hat{n}$.

Cross product — determinant form

For $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b} = b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$:

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$$

Expand by the **first row**. Negative sign on the $\hat{j}$ term is the cofactor pattern — easy to forget.

Standard basis cross products

$\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = \vec{0}$

$\hat{i}\times\hat{j} = \hat{k}, \quad \hat{j}\times\hat{k} = \hat{i}, \quad \hat{k}\times\hat{i} = \hat{j}$

$\hat{j}\times\hat{i} = -\hat{k}, \quad \hat{k}\times\hat{j} = -\hat{i}, \quad \hat{i}\times\hat{k} = -\hat{j}$

Cyclic order $\hat{i} \to \hat{j} \to \hat{k} \to \hat{i}$ gives $+$; reverse the order and flip the sign.

Cyclic i-j-k

Imagine $\hat{i}, \hat{j}, \hat{k}$ on a clock going **clockwise**. Moving **forward** ($\hat{i} \to \hat{j}, \hat{j} \to \hat{k}, \hat{k} \to \hat{i}$) gives the **next** unit vector with a $+$ sign. Moving **backward** flips the sign.

Properties of cross product

Anti-commutative: $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

Distributive: $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Scalar pull-out: $\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$

Parallelism: $\vec{a}\times\vec{b} = \vec{0} \iff \vec{a} \parallel \vec{b}$ (for non-zero $\vec{a}, \vec{b}$)

$\vec{a}\times\vec{a} = \vec{0}$ (since $\theta = 0$)

Swapping operands **flips the sign** — order matters for $\times$, unlike $\cdot$.

Angle between vectors via ×

sin $\theta = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$

$\hat{n} = \dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$ — unit vector $\perp$ both $\vec{a}, \vec{b}$

Cross-product gives both the **angle (via sin)** and the **perpendicular direction** in one shot.

Area: parallelogram & triangle

For $\vec{a}, \vec{b}$ as **adjacent sides of a parallelogram**:

Area $= |\vec{a} \times \vec{b}|$

For $\vec{a}, \vec{b}$ as **two sides of a triangle** from a common vertex:

Area $= \dfrac{1}{2}|\vec{a} \times \vec{b}|$

$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|$ **sin** θ is exactly the **geometric area** swept by the two vectors. Halve it for the triangle.

Area of triangle from three vertices

Triangle ABC with position vectors $\vec{a}, \vec{b}, \vec{c}$:

Area $= \dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \dfrac{1}{2}|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})|$

Pick any vertex, build two edge vectors from it, take half the magnitude of their cross product.

Dot = scalar, Cross = vector

$\vec{a} \cdot \vec{b}$ is a **number**; $\vec{a} \times \vec{b}$ is a **vector**.
Also: $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$, but $\vec{a} \times \vec{b} = \vec{0} \Rightarrow \vec{a} \parallel \vec{b}$ — exactly **opposite** geometric meanings.

Quick Reference — All Vector Identities at a Glance

Concept	Formula	Quick note
Magnitude of $\vec{r}$	$\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$	3D Pythagoras
Direction cosines identity	$l^2 + m^2 + n^2 = 1$	Always true
Unit vector	$\hat{a} = \vec{a}/\|\vec{a}\|$	Direction-only of $\vec{a}$
Collinearity	$\vec{b} = \lambda\vec{a}$	Components proportional
Triangle law	$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$	Closed polygon $\Rightarrow \vec{0}$
$\vec{a} + \vec{b}$ in components	$(a_1 + b_1)\hat{\imath} + (a_2 + b_2)\hat{\jmath} + (a_3 + b_3)\hat{k}$	Componentwise
Distance P_1P_2	$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$	3D distance
Section (internal)	$\vec{r} = \frac{m\vec{b} + n\vec{a}}{m + n}$	R inside PQ
Section (external)	$\vec{r} = \frac{m\vec{b} - n\vec{a}}{m - n}$	R outside PQ
Midpoint	$\vec{r} = \frac{\vec{a} + \vec{b}}{2}$	$m = n = 1$
Dot product (angle)	$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$	Scalar
Dot product (components)	$a_1b_1 + a_2b_2 + a_3b_3$	Multiply and add
Angle: $\cos\theta$	$\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|}$	From $\cdot$
Projection of $\vec{a}$ on $\vec{b}$	$\frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|}$	Signed length
Perpendicularity	$\vec{a} \cdot \vec{b} = 0$	non-zero $\vec{a}, \vec{b}$
Cross product (angle)	$\vec{a} \times \vec{b} = \|\vec{a}\|\|\vec{b}\| \sin\theta\, \hat{n}$	Vector, RH rule
Cross product (det.)	$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$	Expand row 1
Angle: $\sin\theta$	$\frac{\|\vec{a} \times \vec{b}\|}{\|\vec{a}\|\|\vec{b}\|}$	From $\times$
Area of parallelogram	$\|\vec{a} \times \vec{b}\|$	Adjacent sides
Area of triangle	$\frac{1}{2}\|\vec{a} \times \vec{b}\|$	Two sides from vertex
Parallelism	$\vec{a} \times \vec{b} = \vec{0}$	non-zero $\vec{a}, \vec{b}$
Anti-commutativity	$\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$	Sign flips on swap

End of Chapter 10 Formula Sheet — Vector Algebra