Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 10: Vector Algebra

What this chapter covers: the algebra and geometry of vectors — magnitude and direction, the eight standard vector types, addition by the triangle and parallelogram laws, components along $\hat{i}, \hat{j}, \hat{k}$, the section formula, the scalar (dot) product with its projection meaning, and the vector (cross) product with the right-hand rule and parallelogram area. We treat the rationalised NCERT in full and add *scalar triple product* as a JEE/NEET extension, since it remains a high-frequency entrance-exam tool.

1 Scalars, Vectors and Basic Concepts

A length of 1.6 m, a mass of 2 kg, a temperature of $30°$C — each is captured by a single number with a unit. Quantities of this kind are called **scalars**. But a force of 5 N pushing east is fundamentally different from 5 N pushing west; here the number is not enough, the *direction* matters too. Quantities that carry both magnitude and direction — and combine according to the parallelogram law — are called **vectors**.

This chapter develops the algebraic and geometric machinery for handling vectors: how to represent them, add them, scale them, decompose them along axes, and multiply two of them in the two distinct ways (dot and cross) that physics and geometry demand.

1.1 Scalar vs vector at a glance

Scalar	Vector
Magnitude only	Magnitude *and* direction
Specified by a single real number with unit	Specified by a directed line segment or by components
Added by ordinary arithmetic	Added by the triangle / parallelogram law
Examples: mass, length, time, speed, work, energy, density, temperature	Examples: displacement, velocity, acceleration, force, momentum, weight

Speed vs velocity, distance vs displacement

Speed (60 km/h) is a scalar; velocity (60 km/h *towards north*) is a vector. Distance travelled along a curved path is a scalar; displacement (the straight-line vector from start to finish) is a vector. In NCERT-style classification questions, the keyword to spot is direction.

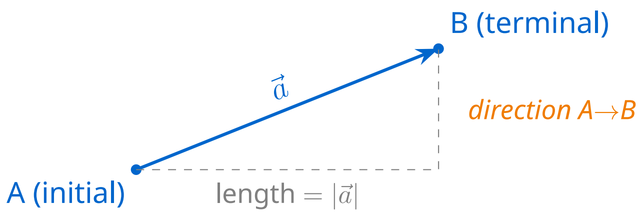
1.2 Directed line segment, magnitude, direction

Take any straight line ℓ in space. It admits two opposite directions; pick one (mark with an arrowhead) and you obtain a **directed line**. Restrict to a segment AB on ℓ with A as the starting point and B as the end point — this gives a **directed line segment** $\overrightarrow{AB}$, which has both a length and a direction. Such an object is a vector.

Vector — the basic object

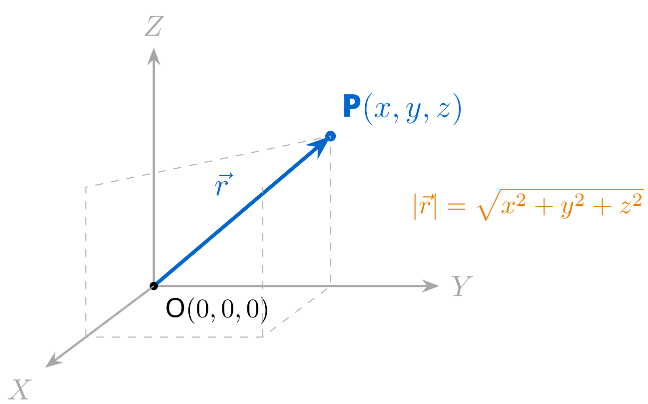
A vector $\vec{a} = \overrightarrow{AB}$ has

- **Initial point** A, **terminal point** B.
- **Magnitude** (length) $|\vec{a}| = |\overrightarrow{AB}|$, always non-negative. The notation $|\vec{a}| < 0$ is meaningless.
- **Direction** indicated by the arrowhead from A to B.



Position vector. Fix an origin O$(0, 0, 0)$ in a right-handed Cartesian frame. For any point P(x, y, z) in space, the vector $\overrightarrow{OP}$ from the origin to P is the **position vector** of P, usually denoted $\vec{r}$. Its magnitude is given by the distance formula:

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}.$$



1.3 Direction cosines and direction ratios

Let $\vec{r} = \overrightarrow{OP}$ make angles α, β, γ with the positive x-, y-, z-axes. The cosines of these angles,

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma,$$

are called the **direction cosines** of $\vec{r}$.

From the right-angled triangles formed by P and its foot of perpendicular on each axis,

$$\cos\alpha = \tfrac{x}{r}, \quad \cos\beta = \tfrac{y}{r}, \quad \cos\gamma = \tfrac{z}{r}, \quad r = |\vec{r}|.$$

Direction cosines identity

For any non-zero vector $\vec{r}$ with direction cosines l, m, n:

$$\boxed{l^2 + m^2 + n^2 = 1}$$

Any three numbers a, b, c proportional to l, m, n (so $a = lr,\ b = mr,\ c = nr$) are called **direction ratios**. Direction ratios are not unique; direction cosines are (up to sign for the reverse direction).

Recovering DCs from DRs

Given direction ratios a, b, c, the direction cosines are

$$l = \tfrac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \tfrac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \tfrac{c}{\sqrt{a^2+b^2+c^2}}.$$

Just normalise the triple (a, b, c) to a unit triple.

Why directions matter in engineering

A crane cable that supports 5000 N along the line $(2, 3, 6)$ does not exert the same force on the boom as a cable supporting 5000 N along $(1, 0, 0)$. Direction cosines convert a line in space into the fraction of the load each axis bears — exactly the quantity a structural engineer needs.

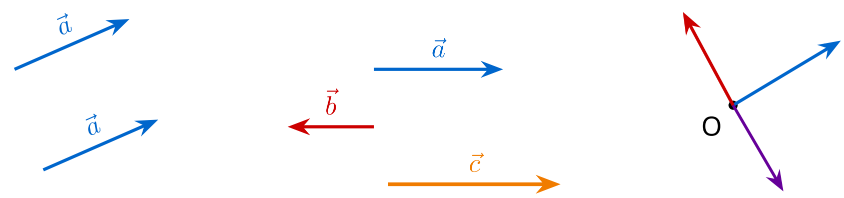
2 Types of Vectors

NCERT names eight standard types of vectors. They are not exclusive categories — a single vector can be unit *and* position *and* free at once. The names just pick out properties we frequently refer to.

2.1 The eight named types

- **Zero (null) vector** $\vec{0}$ — initial and terminal points coincide, magnitude 0, direction indeterminate (treated as "any" direction). Notation: $\overrightarrow{AA} = \vec{0}$.

- **Unit vector** $\hat{a}$ — a vector of magnitude 1. Direction is what carries information; magnitude is fixed.
- **Coinitial vectors** — two or more vectors sharing the same initial point.
- **Collinear vectors** — vectors parallel to the same straight line, regardless of magnitude or direction.
- **Equal vectors** $\vec{a} = \vec{b}$ — same magnitude *and* same direction (positions of initial points don't matter).
- **Negative of a vector** $-\vec{a}$ — same magnitude, opposite direction. So $\overrightarrow{BA} = -\overrightarrow{AB}$.
- **Parallel vectors** — collinear vectors with the same direction (some authors merge this with collinear).
- **Free vector** — a vector regarded as unchanged under parallel translation; magnitude and direction matter, location does not. Throughout this chapter, all vectors are free vectors.



Equal (same length, same direction) **Collinear** (parallel to one line) **Coinitial** (common starting point)

2.2 Equal vs collinear vs coplanar — a comparison

Property	Definition	Test
Equal	Same magnitude *and* same direction	$\vec{a} = \vec{b} \iff$ all components equal.
Collinear	Parallel to the same straight line	$\vec{b} = \lambda\vec{a}$ for some scalar λ (positive or negative).
Parallel (same sense)	Collinear with the same direction	$\vec{b} = \lambda\vec{a}$ with $\lambda > 0$.
Coplanar	All lie in (or parallel to) a common plane	Three vectors $\vec{a}, \vec{b}, \vec{c}$ coplanar $\iff$ scalar triple product $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.

Equal $\Rightarrow$ Collinear, but not the reverse

Two equal vectors are automatically collinear (they sit on parallel lines and even point the same way with the same length). But two collinear vectors with different magnitudes — or opposite directions — are *not* equal. "Equal" is the strictest of the three; "coplanar" is the loosest.

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Three checks for "same vector"?

M-D-S: **M**agnitude same, **D**irection same, location **S**hift allowed. If both M and D match, the vectors are equal — you can slide one onto the other.

3 Addition of Vectors and Scalar Multiplication

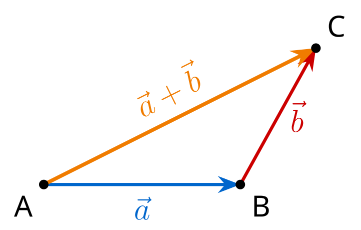
The whole point of representing displacements, forces, and velocities by arrows is that we can combine them *geometrically*. Two laws govern that combination — the triangle law and the parallelogram law — and a third operation, scalar multiplication, lets us stretch or flip a vector without rotating it.

3.1 Triangle law

If a particle moves from A to B (vector $\overrightarrow{AB}$), and then from B to C (vector $\overrightarrow{BC}$), its net displacement is from A to C. We define

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$

To add two free vectors $\vec{a}$ and $\vec{b}$: translate $\vec{b}$ so its tail sits at the tip of $\vec{a}$. The sum $\vec{a} + \vec{b}$ is the arrow from the tail of $\vec{a}$ to the tip of (translated) $\vec{b}$.



A direct consequence: if A, B, C form a triangle, then traversing it once gives the zero vector,

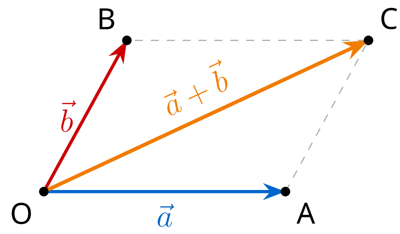
$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}.$$

3.2 Parallelogram law

Place $\vec{a}$ and $\vec{b}$ so they share an initial point O. Complete the parallelogram OACB with $\overrightarrow{OA} = \vec{a}$ and $\overrightarrow{OB} = \vec{b}$. Then

$$\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC},$$

where $\overrightarrow{OC}$ is the diagonal of the parallelogram from the common point.



Triangle law $\equiv$ Parallelogram law

The two laws are not competing rules — they are equivalent. Translating $\vec{b}$ so its tail sits at the tip of $\vec{a}$ (triangle setup) recreates exactly the closing diagonal of the parallelogram. Use whichever picture suits the problem.

Properties of vector addition

For all vectors $\vec{a}, \vec{b}, \vec{c}$:

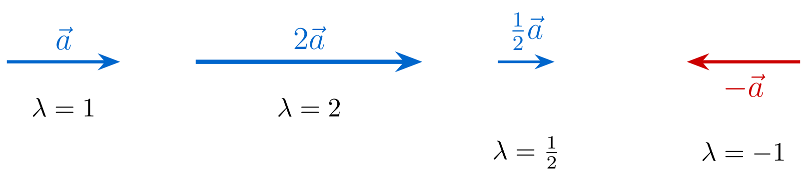
- Commutative: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
- Associative: $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$
- Additive identity: $\vec{a}+\vec{0}=\vec{a}$
- Additive inverse: $\vec{a}+(-\vec{a})=\vec{0}$

Subtraction. We define $\vec{a}-\vec{b}$ as $\vec{a}+(-\vec{b})$. Geometrically, reverse $\vec{b}$ first, then add by the triangle law.

3.3 Scalar multiplication

Multiplying a vector $\vec{a}$ by a real number (scalar) λ produces a new vector $\lambda\vec{a}$ that is collinear with $\vec{a}$:

- Magnitude: $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$.
- Direction: same as $\vec{a}$ if $\lambda > 0$, opposite if $\lambda < 0$, undefined if $\lambda = 0$ (gives $\vec{0}$).



Unit vector along $\vec{a}$

For any non-zero $\vec{a}$, the unit vector in its direction is

$$\hat{a} = \frac{1}{|\vec{a}|}\,\vec{a}.$$

Verification: $|\hat{a}| = \frac{1}{|\vec{a}|}\,|\vec{a}| = 1$, and direction is preserved (since $1/|\vec{a}| > 0$).

Vector of given magnitude along $\vec{a}$

To get a vector of magnitude m in the direction of $\vec{a}$, compute $m\,\hat{a} = \dfrac{m}{|\vec{a}|}\vec{a}$. Always normalise first, then scale — never multiply $\vec{a}$ blindly by m.

$|\vec{a}+\vec{b}| \neq |\vec{a}|+|\vec{b}|$ in general

Magnitudes add only when the two vectors point exactly the same way. Otherwise $|\vec{a}+\vec{b}| < |\vec{a}|+|\vec{b}|$ (this is the triangle inequality). For perpendicular vectors, $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2$ (Pythagoras), not the simple sum.

4 Components, Position Vectors and Section Formula

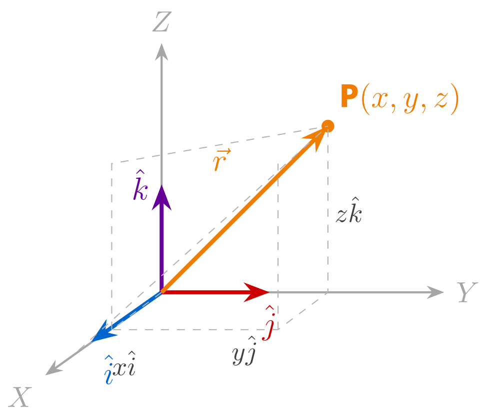
A vector in space is uniquely fixed by three numbers — its components along three mutually perpendicular axes. This converts every geometric question about vectors into an algebraic one, which is what makes vector algebra so powerful.

4.1 The standard basis $\hat{i}, \hat{j}, \hat{k}$

In a right-handed Cartesian frame, let A$(1,0,0)$, B$(0,1,0)$, C$(0,0,1)$. The position vectors of these points,

$$\hat{i} = \overrightarrow{OA}, \quad \hat{j} = \overrightarrow{OB}, \quad \hat{k} = \overrightarrow{OC},$$

are unit vectors along the x-, y-, z-axes. Any vector in space can be written uniquely as a linear combination of them.



Component form of a vector

The position vector of P(x, y, z) is

$$\vec{r} = \overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}, \quad |\vec{r}| = \sqrt{x^2 + y^2 + z^2}.$$

Here x, y, z are the **scalar components** of $\vec{r}$, while $x\hat{i}, y\hat{j}, z\hat{k}$ are its **vector components** along the three axes.

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and any scalar λ:

- $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$
- $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$
- $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$
- $\vec{a} = \vec{b} \iff a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$

Components turn geometry into algebra

Once a vector is written as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, addition becomes “add corresponding components”, scaling becomes “multiply each component”, and direction questions become questions about ratios of components. Almost every NCERT problem reduces to this.

4.2 Vector joining two points

If P$_1(x_1, y_1, z_1)$ and P$_2(x_2, y_2, z_2)$, then by the triangle law $\overrightarrow{OP_1} + \overrightarrow{P_1P_2} = \overrightarrow{OP_2}$, so

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}.$$

The magnitude is the familiar distance formula:

$$|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Direction “from P to Q” vs “from Q to P”

$\overrightarrow{PQ}$ always means “terminal minus initial” = (position vector of Q) − (position vector of P). A sign flip in the question phrasing flips every component. Read carefully.

4.3 Section formula

Let P and Q have position vectors $\vec{a}$ and $\vec{b}$. Suppose R divides the line segment PQ in the ratio $m : n$.

Internal division. R lies between P and Q with $\frac{PR}{RQ} = \frac{m}{n}$:

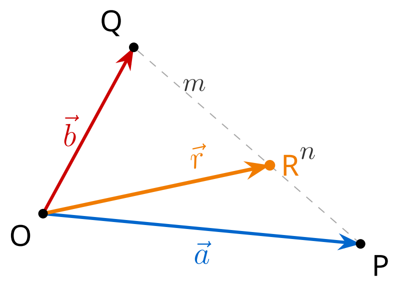
$$\boxed{\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m+n}}$$

External division. R lies on the extension of PQ with $\frac{PR}{RQ} = \frac{m}{n}$:

$$\overrightarrow{OR} = \frac{m\vec{b} - n\vec{a}}{m-n}.$$

Midpoint. Setting $m = n$ in the internal formula:

$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}.$$



Centre of mass

For a uniform rod with end-point position vectors $\vec{a}$ and $\vec{b}$, the centre of mass sits at the midpoint $\frac{\vec{a}+\vec{b}}{2}$. For weighted endpoints (masses m at Q, n at P), the centre of mass shifts to exactly the internal section formula $\frac{m\vec{b}+n\vec{a}}{m+n}$ — the same algebra physicists use for two-particle systems.

4.4 Worked example — components and section formula

Example. Find the unit vector in the direction of $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, and locate the point R that divides the segment from P$(1, 2, -1)$ to Q$(4, -1, 5)$ internally in the ratio $2 : 1$.

Solution.

- $|\vec{a}| = \sqrt{4+9+1} = \sqrt{14}$, so $\hat{a} = \frac{1}{\sqrt{14}}(2\hat{i} + 3\hat{j} + \hat{k}) = \frac{2}{\sqrt{14}}\hat{i} + \frac{3}{\sqrt{14}}\hat{j} + \frac{1}{\sqrt{14}}\hat{k}$.
- Position vectors: $\vec{p} = \hat{i} + 2\hat{j} - \hat{k}$, $\vec{q} = 4\hat{i} - \hat{j} + 5\hat{k}$. With $m = 2,\ n = 1$:

$$\overrightarrow{OR} = \frac{2\vec{q} + 1\cdot\vec{p}}{3} = \frac{(8\hat{i} - 2\hat{j} + 10\hat{k}) + (\hat{i} + 2\hat{j} - \hat{k})}{3} = \frac{9\hat{i} + 0\hat{j} + 9\hat{k}}{3} = 3\hat{i} + 3\hat{k}.$$

- So R $= (3, 0, 3)$.

Check by ratio

After computing R, verify $\overrightarrow{PR} : \overrightarrow{RQ} = m : n$. Here $\overrightarrow{PR} = 2\hat{i} - 2\hat{j} + 4\hat{k}$ and $\overrightarrow{RQ} = \hat{i} - \hat{j} + 2\hat{k}$ — the first is exactly twice the second, confirming $2 : 1$.

5 Scalar (Dot) Product

We can multiply two vectors in two structurally different ways. The first — the scalar product — takes two vectors and returns a number; geometrically, it captures *how much one vector points along the other*, which is exactly what we need for work, projection, and the angle between vectors.

5.1 Definition and geometric meaning

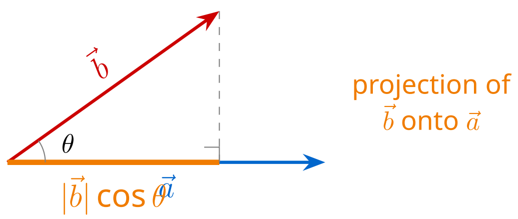
Scalar (Dot) Product

For non-zero vectors $\vec{a}, \vec{b}$ with angle $\theta \in [0, \pi]$ between them,

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta.$$

If either vector is $\vec{0}$, define $\vec{a} \cdot \vec{b} = 0$. The result is a *scalar*.

The factor $|\vec{b}|\cos\theta$ is the signed length of the projection of $\vec{b}$ onto $\vec{a}$. So the dot product is "magnitude of $\vec{a}$ times the projection of $\vec{b}$ onto $\vec{a}$" — a number that is positive if the vectors lean the same way, zero if they are perpendicular, negative if they oppose.



5.2 Algebraic properties

- **Commutative**: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- **Distributive over** $+$: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.
- **Scalars factor out**: $(\lambda\vec{a}) \cdot \vec{b} = \lambda(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (\lambda\vec{b})$.
- **Self-product**: $\vec{a} \cdot \vec{a} = |\vec{a}|^2 \geq 0$, with equality iff $\vec{a} = \vec{0}$.

Standard basis dots. The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular and unit length, so:

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, \qquad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0.$$

5.3 Component formula and consequences

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, expanding by distributivity and using the basis dots:

Dot Product in Components

$$\boxed{\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3}$$

Angle between vectors:

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}.$$

Perpendicularity test: $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$ (for non-zero vectors).

Projection formulas. The scalar projection of $\vec{b}$ on $\vec{a}$ (a number) and the vector projection (a vector along $\hat{a}$) are

$$\text{proj}_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}, \qquad \vec{\text{proj}}_{\vec{a}}\vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2}\right)\vec{a}.$$

Sign of $\vec{a} \cdot \vec{b}$ tells you the angle

- $\vec{a} \cdot \vec{b} > 0 \implies \theta$ acute (vectors lean the same way)
- $\vec{a} \cdot \vec{b} = 0 \implies \theta = 90^\circ$ (perpendicular)
- $\vec{a} \cdot \vec{b} < 0 \implies \theta$ obtuse (vectors lean against each other)

No need to compute θ explicitly when the question only asks for "acute / right / obtuse".

Work done by a force

When a constant force $\vec{F}$ moves an object through displacement $\vec{d}$, the work done is $W = \vec{F} \cdot \vec{d} = |\vec{F}||\vec{d}|\cos\theta$. A force perpendicular to motion does zero work — which is why a satellite in a perfectly circular orbit needs no power input: gravity is always radial, motion is always tangential, and $\vec{F} \cdot \vec{d} = 0$ at every instant.

Dot product is *not* associative

The expression $(\vec{a} \cdot \vec{b}) \cdot \vec{c}$ is meaningless — $\vec{a} \cdot \vec{b}$ is a scalar, and you can't dot a scalar with a vector. Always think before chaining dot products. The valid combination $(\vec{a} \cdot \vec{b})\vec{c}$ is just a scalar times a vector.

6 Vector (Cross) Product

The second way to multiply two vectors — the cross product — returns another *vector*, perpendicular to both, whose magnitude is the area of the parallelogram they span. This single object encodes turning, torque, area, and orientation all at once.

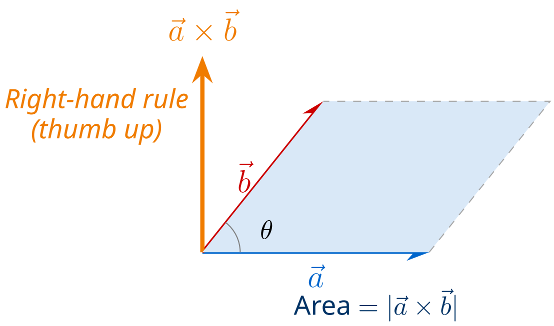
6.1 Definition and the right-hand rule

Vector (Cross) Product

For non-zero, non-parallel vectors $\vec{a}, \vec{b}$ with angle $\theta \in [0, \pi]$ between them,

$$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}|\,\sin\theta\,\hat{n},$$

where $\hat{n}$ is the unit vector perpendicular to both $\vec{a}$ and $\vec{b}$, oriented by the **right-hand rule**: curl the fingers of the right hand from $\vec{a}$ towards $\vec{b}$; the thumb points along $\hat{n}$.
If $\vec{a}$ or $\vec{b}$ is $\vec{0}$ or if they are parallel ($\sin\theta = 0$), then $\vec{a} \times \vec{b} = \vec{0}$.



6.2 Key properties

- **Anti-commutative**: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Reversing order flips the direction (the right hand now curls the other way).
- **Distributive over** $+$: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
- **Scalars factor out**: $(\lambda\vec{a}) \times \vec{b} = \lambda(\vec{a} \times \vec{b}) = \vec{a} \times (\lambda\vec{b})$.
- **Self-cross is zero**: $\vec{a} \times \vec{a} = \vec{0}$ (parallel to itself, $\sin 0 = 0$).
- **Parallel test**: $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = \vec{0}$ (for non-zero vectors).

Standard basis crosses. Using the right-hand rule on a right-handed frame:

$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j},$$

and reversing the order in any of these flips the sign.

Cyclic order of basis crosses

Arrange $\hat{i} \to \hat{j} \to \hat{k} \to \hat{i}$ around a circle. Going *clockwise* (in the direction of the arrows): each cross gives the *next* unit vector with a plus sign. Going *counter-clockwise*: the same crosses give a minus sign.

6.3 Determinant formula

Cross Product as a Determinant

For $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

6.4 Area formulas

The geometric meaning of $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin\theta$ is the area of the parallelogram with $\vec{a}, \vec{b}$ as adjacent sides — it is exactly the base $|\vec{a}|$ times the perpendicular height $|\vec{b}|\,\sin\theta$.

Area via Cross Product

Parallelogram with adjacent sides $\vec{a}, \vec{b}$:

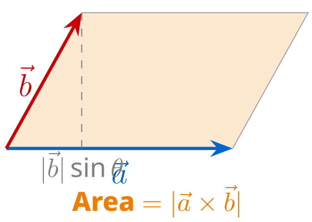
$$\text{Area} = |\vec{a} \times \vec{b}|.$$

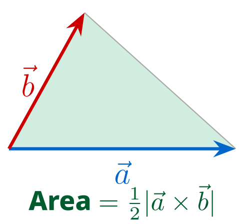
Triangle with two sides $\vec{a}, \vec{b}$ from a common vertex:

$$\text{Area} = \tfrac{1}{2}\,|\vec{a} \times \vec{b}|.$$

Triangle ABC with vertices A, B, C:

$$\text{Area} = \tfrac{1}{2}\,|\overrightarrow{AB} \times \overrightarrow{AC}|.$$





Torque and angular momentum

A spanner of length $\vec{r}$ pushed with force $\vec{F}$ exerts a turning effect (torque)

$\vec{\tau} = \vec{r} \times \vec{F}$. The direction of $\vec{\tau}$ is along the axis of rotation, by the right-hand rule. Pushing the spanner straight along its handle ($\vec{r} \parallel \vec{F}$) produces zero torque — the cross product vanishes, and the bolt does not turn.

$\vec{a} \times \vec{b}$ vs $\vec{a} \cdot \vec{b}$ — output type

The cross product returns a *vector*; the dot product returns a *scalar*. Writing "$\vec{a} \cdot \vec{b}\,\hat{k}$" for a cross product result, or "$|\vec{a} \times \vec{b}|$ is the angle" instead of the area, are classic slip-ups. Always identify the output type first.

6.5 Dot vs Cross product — a comparison

Aspect	**Dot product $\vec{a} \cdot \vec{b}$**	**Cross product $\vec{a} \times \vec{b}$**
Type of output	Scalar (number)	Vector
Formula	$\|\vec{a}\|\|\vec{b}\| \cos\theta$	$\|\vec{a}\|\|\vec{b}\| \sin\theta\, \hat{n}$
Commutative?	Yes: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$	No: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
Zero when...	$\theta = 90^\circ$ (perpendicular)	$\theta = 0$ or π (parallel)
Self-product	$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$	$\vec{a} \times \vec{a} = \vec{0}$
Geometric meaning	Projection of one onto the other	Area of parallelogram spanned
Physical use (typical)	Work, flux, projection	Torque, angular momentum, area, normal

$\sin^2\theta + \cos^2\theta = 1$ identity for vectors

$|\vec{a} \cdot \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$. Useful when the question gives you one of them and asks for the other.

6.6 Worked example — area and angle

Example. For $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$, find (i) $\vec{a} \cdot \vec{b}$, (ii) the angle θ between them, and (iii) the area of the triangle whose two sides from one vertex are $\vec{a}$ and $\vec{b}$.

Solution.

- $\vec{a} \cdot \vec{b} = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3 = 6$.
- $|\vec{a}| = \sqrt{3},\ |\vec{b}| = \sqrt{14}$, so $\cos\theta = \frac{6}{\sqrt{3}\sqrt{14}} = \frac{6}{\sqrt{42}}$, giving $\theta = \cos^{-1}\left(\frac{6}{\sqrt{42}}\right) \approx 22.2^\circ$.
- Cross product:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = (3-2)\hat{i} - (3-1)\hat{j} + (2-1)\hat{k} = \hat{i} - 2\hat{j} + \hat{k}.$$

- $|\vec{a}\times\vec{b}| = \sqrt{1+4+1} = \sqrt{6}$, so the triangle area is $\frac{1}{2}\sqrt{6}$ square units.

Sign error in the $\hat{j}$ row

When expanding the 3×3 determinant for a cross product, the middle row carries a minus sign. Writing $+(a_1b_3 - a_3b_1)\hat{j}$ instead of $-(a_1b_3 - a_3b_1)\hat{j}$ is the single most common arithmetic slip in this chapter. The signs are $+, -, +$ across the top row of cofactors.

7 Scalar Triple Product [JEE/NEET Extension]

The scalar triple product was removed from the rationalised NCERT but remains a staple of JEE Main, JEE Advanced and many engineering-entrance papers. It is the most efficient tool for testing coplanarity and computing volumes from coordinates, so a serious student should still know it.

7.1 Definition

Scalar Triple Product

For three vectors $\vec{a}, \vec{b}, \vec{c}$ the **scalar triple product** (or **box product**) is

$$[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}).$$

In components, with $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$, $\vec{c} = (c_1, c_2, c_3)$:

$$[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

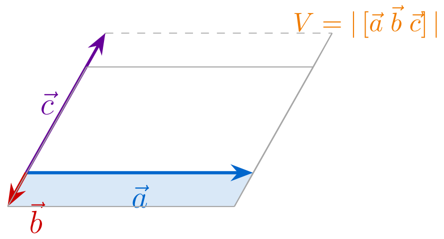
The result is a *scalar*.

7.2 Geometric meaning — volume of a parallelepiped

If $\vec{a}, \vec{b}, \vec{c}$ are the three edges of a parallelepiped meeting at one vertex, the magnitude of their scalar triple product is the volume:

$$V = \big|\,[\vec{a}\,\vec{b}\,\vec{c}]\,\big| = |\vec{a}\cdot(\vec{b}\times\vec{c})|.$$

The reasoning: $|\vec{b}\times\vec{c}|$ is the area of the base parallelogram, and $|\vec{a}|\cos\phi$ (where ϕ is the angle between $\vec{a}$ and the base normal $\vec{b}\times\vec{c}$) is the perpendicular height. Their product is base $\times$ height $=$ volume.



7.3 Properties and the coplanarity test

- **Cyclic invariance:** $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$. Swapping any two flips the sign: $[\vec{a}\,\vec{b}\,\vec{c}] = -[\vec{b}\,\vec{a}\,\vec{c}]$.
- **Dot and cross interchange:** $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$.
- **Linear in each slot:** $[\lambda\vec{a}\,\vec{b}\,\vec{c}] = \lambda[\vec{a}\,\vec{b}\,\vec{c}]$, and similarly for $\vec{b},\vec{c}$.
- **Two equal entries make it zero:** $[\vec{a}\,\vec{a}\,\vec{c}] = 0$ (the parallelepiped collapses).

Coplanarity Test (JEE-favourite)

Three vectors $\vec{a},\vec{b},\vec{c}$ are coplanar if and only if

$$[\vec{a}\,\vec{b}\,\vec{c}] = 0.$$

Four points A, B, C, D are coplanar iff $[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}] = 0$.

Vector triple product (further extension). For completeness:

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}.$$

This "BAC–CAB" identity expands a nested cross product into a linear combination of $\vec{b}$ and $\vec{c}$.

BAC–CAB rule

$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$. Read it as "**BAC minus CAB**" — the middle vector of the inner cross times the dot of the outer with the far one, minus the far vector times the dot of the outer with the middle.

When to reach for the box product

If a JEE question asks for the volume of a tetrahedron with vertices A, B, C, D, the fastest path is

$$V = \tfrac{1}{6}\left|\,[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]\,\right|.$$

The $\frac{1}{6}$ comes from a tetrahedron being one-sixth of the corresponding parallelepiped.

8 Quick Reference Summary

Magnitudes, components, distance

- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, \quad |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$
- DCs: $l = x/r,\ m = y/r,\ n = z/r$, with $l^2 + m^2 + n^2 = 1$
- Distance: $|\overrightarrow{P_1P_2}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Unit vector: $\hat{a} = \vec{a}/|\vec{a}|$

Section formula

Internal: $\overrightarrow{OR} = \dfrac{m\vec{b} + n\vec{a}}{m + n}$, External: $\overrightarrow{OR} = \dfrac{m\vec{b} - n\vec{a}}{m - n}$, Midpoint: $\dfrac{\vec{a} + \vec{b}}{2}$.

Dot product

- $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta = a_1b_1 + a_2b_2 + a_3b_3$
- Perpendicular: $\vec{a} \cdot \vec{b} = 0$
- $\cos\theta = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$
- Projection of $\vec{b}$ on $\vec{a}$ (scalar): $\dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}|}$

Cross product

- $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$ (right-hand rule)
- Determinant form: $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- Anti-commutative: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
- Parallel: $\vec{a} \times \vec{b} = \vec{0}$
- Area of parallelogram $= |\vec{a} \times \vec{b}|$; triangle $= \frac{1}{2}|\vec{a} \times \vec{b}|$
- Identity: $|\vec{a} \cdot \vec{b}|^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$

Standard basis products

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, \quad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0.$$
$$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = \hat{j}.$$

JEE/NEET extensions

- Scalar triple product: $[\vec{a}\,\vec{b}\,\vec{c}] = \vec{a}\cdot(\vec{b}\times\vec{c}) = \det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$
- Volume of parallelepiped $= |[\vec{a}\,\vec{b}\,\vec{c}]|$; tetrahedron $= \frac{1}{6}|[\vec{a}\,\vec{b}\,\vec{c}]|$
- Coplanarity: $[\vec{a}\,\vec{b}\,\vec{c}] = 0$
- Vector triple product (BAC–CAB): $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

One-line takeaways

- Vector = magnitude + direction; scalar = magnitude alone.
- Add vectors head-to-tail (triangle) or by parallelogram diagonal.
- Components turn every geometry problem into algebra on $\hat{i}, \hat{j}, \hat{k}$.
- Dot product = projection $\to$ angle test, work.
- Cross product = area + axis $\to$ torque, normal direction, parallelogram area.
- Box product = signed volume $\to$ coplanarity, tetrahedron volume.