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# NCERT Exemplar Solutions

*Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 11: Three Dimensional Geometry*

## Chapter 11: Three Dimensional Geometry

## About this Chapter

**Three Dimensional Geometry** extends coordinate geometry into space. A line is fixed by a point and a direction vector $\vec{b}$, giving the vector form $\vec{r} = \vec{a} + \lambda\vec{b}$ and the symmetric Cartesian form. **Direction cosines** $l, m, n$ of a line satisfy $l^2 + m^2 + n^2 = 1$; **direction ratios** are any triple proportional to $(l, m, n)$. The Class 12 Exemplar set drills the angle between two lines, the **shortest distance** between two skew lines, foot of perpendicular, image of a point, equation of a plane through three points or perpendicular to a given vector, distance from a point to a plane, and intersection of a line with a plane.

**Topics covered:** Direction cosines and direction ratios • Vector and Cartesian equation of a line • Angle between two lines • Shortest distance between skew lines • Distance between parallel lines • Foot of perpendicular and image of a point • Equation of a plane and distance from a point to a plane • Line-plane intersection

**Quick Formula Sheet**

**DC identity:**
$l^2 + m^2 + n^2 = 1$

**Vector eqn of line:**
$\vec{r} = \vec{a} + \lambda\vec{b}$

**Cartesian symmetric:**
$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

**Angle between two lines:**
$\cos\theta = \frac{|\vec{b_1}\vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$

**Shortest distance, skew:**
$d = \frac{|(\vec{b_1}\times\vec{b_2})\cdot(\vec{a_2}-\vec{a_1})|}{|\vec{b_1}\times\vec{b_2}|}$

**Distance, parallel:**
$d = \frac{|\vec{b}\times(\vec{a_2}-\vec{a_1})|}{|\vec{b}|}$

**Plane (normal form):**
$\vec{r}\cdot\hat{n} = p$

## I. Short Answer (S.A.)

**Q 11.1** **Find the position vector of a point $A$ in space such that $\vec{OA}$ is inclined at $60°$ to $OX$ and at $45°$ to $OY$, and $|\vec{OA}| = 10$ units.**

**SOLUTION**

**Concept used.** If a line through the origin makes angles $\alpha, \beta, \gamma$ with the positive $x$-, $y$-, $z$-axes, then its direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, and they satisfy

$$l^2 + m^2 + n^2 = 1.$$

The position vector of a point at distance $r$ from the origin along this direction is $\vec{OA} = r\,(l\hat{i} + m\hat{j} + n\hat{k})$.

**Step 1.** Given $\alpha = 60^\circ$ and $\beta = 45^\circ$. So

$$l = \cos 60^\circ = \tfrac{1}{2}, \qquad m = \cos 45^\circ = \tfrac{1}{\sqrt{2}}.$$

**Step 2.** Use $l^2 + m^2 + n^2 = 1$ to find $n$:

$$\tfrac{1}{4} + \tfrac{1}{2} + n^2 = 1 \implies n^2 = \tfrac{1}{4} \implies n = \pm\tfrac{1}{2}.$$

**Step 3.** With $|\vec{OA}| = 10$, the position vector is

$$\vec{OA} = 10\left(\tfrac{1}{2}\hat{i} + \tfrac{1}{\sqrt{2}}\hat{j} \pm \tfrac{1}{2}\hat{k}\right) = 5\hat{i} + 5\sqrt{2}\,\hat{j} \pm 5\hat{k}.$$

**Final Answer:** $\vec{OA} = 5\hat{i} + 5\sqrt{2}\,\hat{j} \pm 5\hat{k}$.

### ☞ Exam Tip

Whenever two angles are given and the third is missing, plug the two known cosines into $l^2 + m^2 + n^2 = 1$ and solve for the third; expect a $\pm$ sign because direction is two-sided.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Direction cosines parametrise the unit sphere of directions. Any vector of length $r$ along that direction is $r$ times the corresponding unit vector.

**Step 1.** Identify the two known direction cosines from the given angles: $\cos 60^\circ = \frac{1}{2}$ and $\cos 45^\circ = \frac{1}{\sqrt{2}}$.

**Step 2.** Apply the identity $l^2 + m^2 + n^2 = 1$ to find the magnitude of the missing DC.

**Step 3.** Both signs are admissible because the constraint involves squares; geometrically the line has two senses, one going above the $xy$-plane and one going below.

**Step 4.** Multiply the resulting unit vector by the prescribed length to obtain the position vector.

**Common alternative.** Some students first convert the magnitudes $(\cos 60^\circ, \cos 45^\circ)$ into Cartesian components $(5, 5\sqrt{2})$ for the $x$- and $y$-components directly, then determine the

$z$-component from $\sqrt{|\vec{OA}|^2 - x^2 - y^2} = \sqrt{100 - 25 - 50} = 5$. This avoids using $l, m, n$ explicitly and is acceptable in CBSE marking schemes.
**Why the $\pm$.** The angles with two axes do not pin down a single direction in 3D; the third coordinate can be positive or negative. Always present both.

**Final Answer:** $\vec{OA} = 5\hat{\imath} + 5\sqrt{2}\,\hat{\jmath} \pm 5\hat{k}$.

**Q 11.2** **Find the vector equation of the line which is parallel to the vector $3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$ and which passes through the point $(1, -2, 3)$.**

SOLUTION

**Concept used.** The vector equation of a line passing through the point with position vector $\vec{a}$ and parallel to a non-zero vector $\vec{b}$ is

$$\vec{r} = \vec{a} + \lambda\vec{b}, \qquad \lambda \in \mathbb{R}.$$

**Step 1.** Position vector of the given point: $\vec{a} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$.

**Step 2.** Direction vector parallel to the required line: $\vec{b} = 3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$.

**Step 3.** Plug into the standard form:

$$\vec{r} = (\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) + \lambda(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}).$$

**Final Answer:** $\vec{r} = (\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) + \lambda(3\hat{\imath} - 2\hat{\jmath} + 6\hat{k})$.

EXPERT'S SOLUTION *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Two pieces of information determine a unique line in space: one point on the line, plus a direction vector. The vector form is the cleanest way to encode both.

**Step 1.** Write the position vector of the given point as $\vec{a}$.

**Step 2.** Use the given parallel vector as the direction vector $\vec{b}$.

**Step 3.** Assemble $\vec{r} = \vec{a} + \lambda\vec{b}$ with $\lambda$ a real parameter.

**Cartesian form sanity check.** Reading off the components of $\vec{b} = (3, -2, 6)$ as direction ratios, the equivalent symmetric Cartesian form is

$$\frac{x-1}{3} = \frac{y+2}{-2} = \frac{z-3}{6},$$

which is what CBSE often asks for in the same question.
**Marking-scheme angle.** Examiners look for (i) the standard-form identification of $\vec{a}$ and $\vec{b}$, (ii) the assembled vector equation, and (iii) a clean line of working with $\lambda$ as the scalar parameter. Two marks for this question is typical.

**Final Answer:** $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$.

**Q 11.3** **Show that the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect. Also, find their point of intersection.**

**SOLUTION**

**Concept used.** Two lines in symmetric form $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ intersect iff there exist parameters $\lambda$ and $\mu$ such that the parametric coordinates from both lines coincide.

**Step 1.** Parametrise line 1: any point is $(1+2\lambda,\ 2+3\lambda,\ 3+4\lambda)$.

**Step 2.** Parametrise line 2: any point is $(4+5\mu,\ 1+2\mu,\ \mu)$ (since the $z$-fraction is $z/1 = \mu$).

**Step 3.** Equate corresponding coordinates:

$$\begin{aligned} 1+2\lambda &= 4+5\mu && \Rightarrow 2\lambda - 5\mu = 3, \\ 2+3\lambda &= 1+2\mu && \Rightarrow 3\lambda - 2\mu = -1, \\ 3+4\lambda &= \mu && \Rightarrow 4\lambda - \mu = -3. \end{aligned}$$

**Step 4.** Solve the first two: multiply (1) by 2 and (2) by 5 and subtract:

$$(4\lambda - 10\mu) - (15\lambda - 10\mu) = 6 - (-5) \implies -11\lambda = 11 \implies \lambda = -1.$$

Substitute in (2): $3(-1) - 2\mu = -1 \Rightarrow \mu = -1$.

**Step 5.** Verify with the third equation: $4(-1) - (-1) = -3$. ✓

**Step 6.** Point of intersection (using line 1 with $\lambda = -1$): $(1-2,\ 2-3,\ 3-4) = (-1, -1, -1)$.

**Final Answer:** The two lines intersect at $(-1, -1, -1)$.

☞ **Exam Tip**

For intersection problems, solve any two of the three coordinate equations for $(\lambda, \mu)$, then verify in the third. If the third equation is also satisfied, the lines intersect; otherwise they are skew.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A pair of lines in 3D has three possible relationships - intersecting, parallel, or skew. Intersection is verified by exhibiting a common point.

**Step 1.** Convert both symmetric forms to parametric coordinates, using different scalar parameters ($\lambda$ for one, $\mu$ for the other).

**Step 2.** Set up the three coordinate-matching equations.

**Step 3.** Two equations suffice to solve for $(\lambda, \mu)$; the third equation is a consistency check.

**Step 4.** If consistent, substitute back to obtain the intersection point.

**Coplanarity test (alternative).** The two lines are coplanar (intersecting or parallel) iff $(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 0$. Here $\vec{b_1} = (2, 3, 4)$, $\vec{b_2} = (5, 2, 1)$, and $\vec{a_2} - \vec{a_1} = (3, -1, -3)$. Compute $\vec{b_1} \times \vec{b_2} = (3{\cdot}1 - 4{\cdot}2,\ 4{\cdot}5 - 2{\cdot}1,\ 2{\cdot}2 - 3{\cdot}5) = (-5, 18, -11)$. Dot with $(3, -1, -3)$: $-15 - 18 + 33 = 0$. So coplanar, and since direction vectors are not proportional, the lines intersect.

**Marking-scheme angle.** Show the parametric set-up, solve cleanly, verify in the third equation, and state the intersection point. Skipping verification often costs 1 mark.

**Final Answer:** The two lines intersect at $(-1, -1, -1)$.

**Q 11.4** **Find the angle between the lines $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k} + \lambda(2\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{r} = (2\hat{j} - 5\hat{k}) + \mu(6\hat{i} + 3\hat{j} + 2\hat{k})$.**

**SOLUTION**

**Concept used.** The acute angle $\theta$ between two lines with direction vectors $\vec{b_1}$ and $\vec{b_2}$ is

$$\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}.$$

The modulus in the numerator ensures the acute angle.

**Step 1.** Direction vectors: $\vec{b_1} = 2\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b_2} = 6\hat{i} + 3\hat{j} + 2\hat{k}$.

**Step 2.** Dot product: $\vec{b_1} \cdot \vec{b_2} = 2(6) + 1(3) + 2(2) = 12 + 3 + 4 = 19$.

**Step 3.** Magnitudes:

$$|\vec{b_1}| = \sqrt{4+1+4} = \sqrt{9} = 3, \qquad |\vec{b_2}| = \sqrt{36+9+4} = \sqrt{49} = 7.$$

**Step 4.** Plug in:

$$\cos\theta = \frac{|19|}{3\cdot 7} = \frac{19}{21}.$$

**Step 5.** Hence $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

**Final Answer:** $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The angle between two lines is the angle between their direction vectors (up to the acute-angle convention).

**Step 1.** Read direction vectors $\vec{b_1}$ and $\vec{b_2}$ off the vector forms.

**Step 2.** Compute the dot product. Take the absolute value to ensure the acute angle.

**Step 3.** Compute the two magnitudes.

**Step 4.** Divide and take $\cos^{-1}$.

**Numerical sanity check.** $\frac{19}{21} \approx 0.905$, so $\theta \approx 25.2°$. This is a small acute angle, consistent with two direction vectors that share two positive component signs and have similar magnitudes.

**Common slip.** Dropping the modulus on the dot product gives the obtuse supplement when the dot product is negative. CBSE wants the acute angle, so always include the modulus.

**Final Answer:** $\theta = \cos^{-1}\left(\frac{19}{21}\right)$.

**Q 11.5** **Prove that the line through $A(0,-1,-1)$ and $B(4,5,1)$ intersects the line through $C(3,9,4)$ and $D(-4,4,4)$.**

SOLUTION

**Concept used.** Two lines in space intersect iff they are coplanar and not parallel. The four-point coplanarity condition is that the scalar triple product

$$[\vec{AB}\ \vec{AC}\ \vec{CD}] = 0$$

where $\vec{AB}$ and $\vec{CD}$ are the direction vectors and $\vec{AC}$ is a vector joining a point on each line.

**Step 1.** Compute the three vectors:

$$\vec{AB} = (4, 6, 2), \qquad \vec{CD} = (-7, -5, 0), \qquad \vec{AC} = (3, 10, 5).$$

**Step 2.** Set up the triple product as a $3 \times 3$ determinant:

$$[\vec{AC}\ \vec{AB}\ \vec{CD}] = \begin{vmatrix} 3 & 10 & 5 \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}.$$

**Step 3.** Expand along row 1:

$$\begin{aligned} &= 3\,(6 \cdot 0 - 2 \cdot (-5)) - 10\,(4 \cdot 0 - 2 \cdot (-7)) + 5\,(4 \cdot (-5) - 6 \cdot (-7)) \\ &= 3(10) - 10(14) + 5(22) \\ &= 30 - 140 + 110 = 0. \end{aligned}$$

**Step 4.** Triple product is zero, so the four points $A, B, C, D$ are coplanar.

**Step 5.** Direction vectors $\vec{AB} = (4, 6, 2)$ and $\vec{CD} = (-7, -5, 0)$ are not proportional (e.g. $\frac{4}{-7} \neq \frac{6}{-5}$), so the lines are not parallel.

**Step 6.** Two coplanar non-parallel lines must intersect.

> **Final Answer:** The two lines are coplanar and non-parallel, hence they intersect.

### ☞ Exam Tip

The four-point coplanarity test - $[\vec{AB}\ \vec{AC}\ \vec{CD}] = 0$ - is the fastest way to check intersection when both lines are given by pairs of points. It avoids parametrising and solving simultaneous equations.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Coplanarity of two lines is equivalent to a vanishing scalar triple product of (direction-1, direction-2, joining vector). When two lines are coplanar but not parallel, they meet in exactly one point.

**Step 1.** Express the two lines through pairs of points $A, B$ and $C, D$.

**Step 2.** Form direction vectors $\vec{AB}$ and $\vec{CD}$, and a joining vector $\vec{AC}$.

**Step 3.** Compute the scalar triple product as a determinant.

**Step 4.** Show it is zero (coplanar) and that the direction vectors are not proportional (not parallel). Conclude intersection.

**Finding the point of intersection (optional).** Parametrise line $AB$ as $(0,-1,-1)+\lambda(4,6,2)$ and line $CD$ as $(3,9,4)+\mu(-7,-5,0)$. Equating gives $4\lambda+7\mu=3$, $6\lambda+5\mu=10$, $2\lambda=5$. From the third, $\lambda=\frac{5}{2}$; substituting in the first gives $10+7\mu=3$, so $\mu=-1$. Check in the second: $6(\frac{5}{2})+5(-1)=15-5=10$. ✓ Intersection point is $(0,-1,-1)+\frac{5}{2}(4,6,2)=(10,14,4)$.
**Marking-scheme angle.** Examiners award separate credit for the triple-product set-up, the determinant evaluation, the parallel-vs-non-parallel observation, and the final conclusion sentence.

> **Final Answer:** The lines are coplanar (triple product $=0$) and non-parallel, hence they intersect.

**Q 11.6** **Prove that the lines $x=py+q$, $z=ry+s$ and $x=p'y+q'$, $z=r'y+s'$ are perpendicular if $pp'+rr'+1=0$.**

**SOLUTION**

**Concept used.** Two lines are perpendicular iff the dot product of their direction vectors is zero. To find a direction vector, parametrise each line using the variable $y$ as the parameter.

**Step 1.** For the first line, set $y=t$. Then $x=pt+q$ and $z=rt+s$, so a point on the line is $(pt+q,\ t,\ rt+s)$. Differentiating with respect to $t$ gives the direction vector

$$\vec{b_1}=(p,1,r).$$

**Step 2.** Similarly, the second line gives $\vec{b_2}=(p',1,r')$.

**Step 3.** Perpendicularity condition $\vec{b_1}\cdot\vec{b_2}=0$:

$$pp'+1\cdot 1+rr'=0 \iff pp'+rr'+1=0.$$

> **Final Answer:** The lines are perpendicular iff $pp'+rr'+1=0$.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** When a line is given as the intersection of two planes (here $x - py - q = 0$ and $z - ry - s = 0$), the simplest way to read off its direction is to parametrise by the variable that is free.

**Step 1.** Choose $y$ as the parameter because both equations are linear in $y$.

**Step 2.** Read off the direction-vector components as $\partial(x, y, z)/\partial y$.

**Step 3.** Set the dot product equal to zero to get the perpendicularity condition.

**Geometric note.** The middle "1" in the dot product comes from the $\partial y/\partial y = 1$ component, which both lines share. The condition $pp' + rr' + 1 = 0$ therefore says that the two perpendicular contributions (in $x$ and $z$) cancel the unavoidable "1" from the shared $y$-direction.

**Cross-product alternative.** Each line can be written as the intersection of planes; its direction vector equals the cross product of the two plane normals. For line 1, normals are $(1, -p, 0)$ and $(0, -r, 1)$, cross product $(-p, -1, -r)$ - same direction up to sign as $(p, 1, r)$.

**Final Answer:** $pp' + rr' + 1 = 0$ is the required perpendicularity condition.

**Q 11.7** **Find the equation of a plane which bisects perpendicularly the line joining the points $A(2, 3, 4)$ and $B(4, 5, 8)$ at right angles.**

**SOLUTION**

**Concept used.** The perpendicular bisector plane of a segment $AB$ passes through the mid-point of $AB$ and is normal to the vector $\vec{AB}$.

**Step 1.** Mid-point of $AB$:

$$M = \left(\frac{2+4}{2}, \frac{3+5}{2}, \frac{4+8}{2}\right) = (3, 4, 6).$$

**Step 2.** Normal vector to the plane:

$$\vec{AB} = (4 - 2,\ 5 - 3,\ 8 - 4) = (2, 2, 4).$$

Take $\vec{n} = (1, 1, 2)$ (divide by 2 - direction is what matters).

**Step 3.** Equation of the plane through $M(3, 4, 6)$ with normal $(1, 1, 2)$:

$$1(x - 3) + 1(y - 4) + 2(z - 6) = 0.$$

**Step 4.** Simplify: $x + y + 2z = 3 + 4 + 12 = 19$.

**Final Answer:** $x + y + 2z = 19$.

☞ **Exam Tip**

For perpendicular-bisector problems, the plane equation is always $\vec{n} \cdot (\vec{r} - \vec{M}) = 0$ where $M$ is the mid-point and $\vec{n} = \vec{AB}$.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** The set of points equidistant from $A$ and $B$ is the perpendicular bisector plane of the segment $AB$.

**Step 1.** Locate the mid-point $M$ of $AB$, which lies on the required plane.

**Step 2.** Take $\vec{AB}$ as the normal vector to the plane (the plane is perpendicular to the segment).

**Step 3.** Use the point-normal form $\vec{n} \cdot (\vec{r} - \vec{M}) = 0$.

**Step 4.** Simplify to obtain the Cartesian equation.

**Equidistance verification.** For any $(x, y, z)$ on $x + y + 2z = 19$, the squared distance to $A(2, 3, 4)$ minus the squared distance to $B(4, 5, 8)$ expands to a linear expression that vanishes exactly on the plane. Plugging in $M = (3, 4, 6)$ gives $|MA|^2 = 1 + 1 + 4 = 6 = |MB|^2$, confirming equidistance.

**Marking-scheme angle.** CBSE awards credit for (i) the mid-point, (ii) the normal vector, (iii) the point-normal equation, and (iv) the simplified Cartesian form. Each is one mark for a 4-mark question.

**Final Answer:** $x + y + 2z = 19$.

**Q 11.8** **Find the equation of a plane which is at a distance $3\sqrt{3}$ units from the origin and the normal to which is equally inclined to the coordinate axes.**

**SOLUTION**

**Concept used.** The Cartesian normal form of a plane at distance $p$ from the origin with unit normal $(l, m, n)$ is $lx + my + nz = p$. If the normal is equally inclined to all three axes, then $l = m = n$ and $l^2 + m^2 + n^2 = 1$ forces $l = \pm\frac{1}{\sqrt{3}}$.

**Step 1.** Let the direction cosines of the normal be $(l, l, l)$.

**Step 2.** Identity: $3l^2 = 1 \implies l = \pm\frac{1}{\sqrt{3}}$.

**Step 3.** Plane equation: $\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = 3\sqrt{3}$, taking the positive sign.

**Step 4.** Multiply through by $\sqrt{3}$:

$$x + y + z = 3\sqrt{3} \cdot \sqrt{3} = 9.$$

**Step 5.** The negative sign gives the mirror plane $x + y + z = -9$.

**Final Answer:** $x + y + z = \pm 9$.

EXPERT'S SOLUTION : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The normal form $\hat{n} \cdot \vec{r} = p$ encodes the perpendicular distance from the origin to the plane as $p$, provided $\hat{n}$ is a unit vector.

**Step 1.** "Equally inclined to the axes" forces all three direction cosines equal.

**Step 2.** The unit-vector constraint pins each to $\pm\frac{1}{\sqrt{3}}$.

**Step 3.** Set up $\hat{n} \cdot \vec{r} = p$ and clear the surd.

**Geometric sanity check.** The point $(3, 3, 3)$ lies on the plane $x + y + z = 9$ and its distance from the origin is $\sqrt{27} = 3\sqrt{3}$. Since the normal $(1, 1, 1)/\sqrt{3}$ points from the origin to $(3, 3, 3)$, this matches the prescribed perpendicular distance exactly.

**Final Answer:** $x + y + z = \pm 9$.

**Q 11.9 If the line drawn from the point $(-2, -1, -3)$ meets a plane at right angle at the point $(1, -3, 3)$, find the equation of the plane.**

SOLUTION

**Concept used.** If a line meets a plane at right angles, the direction vector of the line is parallel to the normal vector of the plane, and the foot of the perpendicular lies on the plane.

**Step 1.** Direction vector of the line (from $(-2, -1, -3)$ to $(1, -3, 3)$):

$$\vec{n} = (1 - (-2),\ -3 - (-1),\ 3 - (-3)) = (3, -2, 6).$$

**Step 2.** The plane passes through $(1, -3, 3)$ and is normal to $\vec{n} = (3, -2, 6)$.

**Step 3.** Point-normal equation:

$$3(x - 1) - 2(y + 3) + 6(z - 3) = 0.$$

**Step 4.** Simplify:

$$3x - 3 - 2y - 6 + 6z - 18 = 0 \implies 3x - 2y + 6z = 27.$$

**Final Answer:** $3x - 2y + 6z = 27$.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A normal vector + a point on the plane uniquely determines the plane.

**Step 1.** Form the direction vector along the perpendicular line.

**Step 2.** Take this vector as the plane's normal.

**Step 3.** Use the foot of the perpendicular as the on-plane point.

**Step 4.** Plug into the point-normal form and simplify.

**Distance verification.** Distance from $(-2,-1,-3)$ to $3x-2y+6z-27=0$ is $\dfrac{|3(-2)-2(-1)+6(-3)-27|}{\sqrt{9+4+36}} = \dfrac{|-6+2-18-27|}{7} = \dfrac{49}{7} = 7$. Distance from $(-2,-1,-3)$ to $(1,-3,3)$ is $\sqrt{9+4+36} = 7$. The two match, confirming the foot of perpendicular.

**Final Answer:** $3x - 2y + 6z = 27$.

## Q 11.10 Find the equation of the plane through the points $(2,1,0)$, $(3,-2,-2)$ and $(3,1,7)$.

**SOLUTION**

**Concept used.** The plane through three non-collinear points $P, Q, R$ has normal vector $\vec{PQ} \times \vec{PR}$.

**Step 1.** Let $P = (2,1,0),\ Q = (3,-2,-2),\ R = (3,1,7)$.

**Step 2.** Form

$$\vec{PQ} = (1,-3,-2), \qquad \vec{PR} = (1,0,7).$$

**Step 3.** Cross product:

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & -2 \\ 1 & 0 & 7 \end{vmatrix} = \hat{i}(-21-0) - \hat{j}(7+2) + \hat{k}(0+3) = (-21,-9,3).$$

Divide by $-3$: $\vec{n} = (7,3,-1)$.

**Step 4.** Plane through $P(2,1,0)$ with normal $(7,3,-1)$:

$$7(x-2)+3(y-1)-1(z-0)=0 \implies 7x+3y-z=17.$$

**Final Answer:** $7x+3y-z=17.$

### ☞ Exam Tip

The cross product of two edge vectors of a triangle in 3D gives a normal of the plane containing the triangle. Always pick the simplest two edges from a common vertex.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Three non-collinear points determine a unique plane. Two edge vectors from a common vertex span the plane; their cross product is normal to the plane.

**Step 1.** Pick one vertex, say $P$, as the on-plane point.

**Step 2.** Compute $\vec{PQ}$ and $\vec{PR}$ - two non-parallel in-plane vectors.

**Step 3.** Cross-product them to get a normal.

**Step 4.** Use the point-normal form and simplify.

**Determinant alternative.** The Cartesian plane through $(x_i, y_i, z_i)$, $i = 1,2,3$, can also be written as

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0,$$

which expands to the same Cartesian equation. Use whichever you find faster.

**Final Answer:** $7x+3y-z=17.$

**Q 11.11** **Find the equations of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at angles of $\frac{\pi}{3}$ each.**

**SOLUTION**

**Concept used.** A line through the origin can be written as $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, where $(a,b,c)$ are its direction ratios. The angle condition with the given line determines a relation between $a,b,c$; the requirement to intersect the given line determines a point on the line

$\dfrac{x-3}{2} = \dfrac{y-3}{1} = \dfrac{z}{1}$.

**Step 1.** Parametrise the given line: any point is $(3+2t,\ 3+t,\ t)$. The required line passes through the origin and this point, so its direction ratios are $(3+2t,\ 3+t,\ t)$.

**Step 2.** Angle between the required line and the given line (direction $(2,1,1)$, $|(2,1,1)| = \sqrt{6}$):

$$\cos\frac{\pi}{3} = \frac{|2(3+2t)+1(3+t)+1(t)|}{\sqrt{(3+2t)^2+(3+t)^2+t^2}\,\sqrt{6}} = \frac{|6+4t+3+t+t|}{\sqrt{6}\cdot\sqrt{\cdots}} = \frac{|9+6t|}{\sqrt{6}\sqrt{\cdots}}.$$

Hence $\dfrac{1}{2} = \dfrac{9+6t}{\sqrt{6}\sqrt{S}}$ where $S = (3+2t)^2+(3+t)^2+t^2$ (taking the positive sign; the modulus is handled below).

**Step 3.** Expand $S$:

$$S = (9+12t+4t^2)+(9+6t+t^2)+t^2 = 18+18t+6t^2 = 6(t^2+3t+3).$$

So $\sqrt{6}\sqrt{S} = \sqrt{6}\cdot\sqrt{6}\sqrt{t^2+3t+3} = 6\sqrt{t^2+3t+3}$.

**Step 4.** Equation:

$$\frac{1}{2} = \frac{|9+6t|}{6\sqrt{t^2+3t+3}} \implies 3\sqrt{t^2+3t+3} = |9+6t|.$$

**Step 5.** Square both sides:

$$9(t^2+3t+3) = (9+6t)^2 = 81+108t+36t^2.$$

**Step 6.** Expand and simplify:

$$9t^2+27t+27 = 81+108t+36t^2 \implies 27t^2+81t+54=0 \implies t^2+3t+2=0.$$

**Step 7.** Factor: $(t+1)(t+2)=0$, so $t=-1$ or $t=-2$.

**Step 8.** For $t=-1$: direction ratios $(3-2,\ 3-1,\ -1) = (1,2,-1)$. Line: $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}$.

**Step 9.** For $t=-2$: direction ratios $(3-4,\ 3-2,\ -2) = (-1,1,-2)$. Line: $\dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z}{-2}$.

**Final Answer:** $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{-1}$ and $\dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z}{-2}$.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A one-parameter family of lines from the origin, each intersecting a given line, is the standard set-up; the angle condition selects discrete members from the family.

**Step 1.** Parametrise points on the given line by $t$.

**Step 2.** Treat $(3+2t,\ 3+t,\ t)$ as the direction ratios of the required line through the origin.

**Step 3.** Set $\cos(\pi/3) = \frac{1}{2}$ equal to the cosine of the angle formula.

**Step 4.** Solve the resulting quadratic in $t$. Each root gives one of the two required lines.

**Algebraic simplification.** The shortcut here is to notice $S = 6(t^2+3t+3)$, which makes $\sqrt{6}\sqrt{S} = 6\sqrt{t^2+3t+3}$ - a clean factorisation. Spotting it early avoids cubic-looking algebra and keeps the final quadratic linear in disguise.

**Final Answer:** $\frac{x}{1} = \frac{y}{2} = \frac{z}{-1}$ and $\frac{x}{-1} = \frac{y}{1} = \frac{z}{-2}$.

**Q 11.12** **Find the angle between the lines whose direction cosines are given by the equations $l+m+n=0,\ l^2+m^2-n^2=0$.**

**SOLUTION**

**Concept used.** Eliminate one variable using the linear equation, substitute into the quadratic, and solve for the ratios of direction cosines of the two lines.

**Step 1.** From $l+m+n=0$, write $n = -(l+m)$.

**Step 2.** Substitute into $l^2+m^2-n^2=0$:

$$l^2+m^2-(l+m)^2=0 \implies l^2+m^2-l^2-2lm-m^2=0 \implies -2lm=0.$$

**Step 3.** So either $l=0$ or $m=0$.

**Step 4.** Case 1: $l=0 \Rightarrow n=-m$. DRs are $(0,1,-1)$.

**Step 5.** Case 2: $m=0 \Rightarrow n=-l$. DRs are $(1,0,-1)$.

**Step 6.** Angle between $(0,1,-1)$ and $(1,0,-1)$:

$$\cos\theta = \frac{|0\cdot1+1\cdot0+(-1)(-1)|}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

So $\theta = \frac{\pi}{3}$.

**Final Answer:** $\theta = \dfrac{\pi}{3}$.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Two equations in three direction cosines (one linear, one quadratic) generically give two discrete lines. The angle between them is then a direct cosine computation.

**Step 1.** Use the linear equation to eliminate one DC.

**Step 2.** Substitute into the quadratic to get a ratio condition on the remaining two DCs.

**Step 3.** Solve to obtain the two direction-ratio triples.

**Step 4.** Compute the angle using the standard cosine formula.

**Symmetry note.** The two solutions $(0, 1, -1)$ and $(1, 0, -1)$ are related by swapping the roles of $l$ and $m$, which is consistent with the symmetry of the original system in $l$ and $m$.

**Final Answer:** $\theta = \dfrac{\pi}{3}$.

**Q 11.13** **If a variable line in two adjacent positions has direction cosines $l, m, n$ and $l + \delta l, m + \delta m, n + \delta n$, show that the small angle $\delta\theta$ between the two positions is given by**

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2.$$

**SOLUTION**

**Concept used.** For two unit vectors $\hat{u}_1 = (l, m, n)$ and $\hat{u}_2 = (l + \delta l, m + \delta m, n + \delta n)$, the angle $\delta\theta$ between them satisfies $\cos\delta\theta = \hat{u}_1 \cdot \hat{u}_2$. For small $\delta\theta$, $\cos\delta\theta \approx 1 - \frac{1}{2}\delta\theta^2$.

**Step 1.** Since both triples are direction cosines, $l^2 + m^2 + n^2 = 1$ and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$.

**Step 2.** Expand the second identity and use the first to cancel $l^2 + m^2 + n^2 = 1$:

$$2(l\,\delta l + m\,\delta m + n\,\delta n) + (\delta l^2 + \delta m^2 + \delta n^2) = 0.$$

So $l\,\delta l + m\,\delta m + n\,\delta n = -\frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$.

**Step 3.** Compute the dot product:

$$\hat{u}_1 \cdot \hat{u}_2 = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$= l^2 + m^2 + n^2 + (l\,\delta l + m\,\delta m + n\,\delta n) = 1 - \tfrac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2).$$

**Step 4.** Set this equal to $\cos\delta\theta \approx 1 - \frac{1}{2}\delta\theta^2$:

$$1 - \tfrac{1}{2}\delta\theta^2 = 1 - \tfrac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2).$$

**Step 5.** Cancel and multiply by $-2$:

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2.$$

**Final Answer:** $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Differentiating the unit-norm constraint $\hat{u}\cdot\hat{u} = 1$ gives $\hat{u}\cdot d\hat{u} = 0$, i.e. infinitesimal changes in a unit vector are perpendicular to the vector itself.

**Step 1.** Identify both endpoints as unit vectors.

**Step 2.** Expand $|\hat{u}_2|^2 = 1$ to leading order in the small quantities $\delta l, \delta m, \delta n$.

**Step 3.** Compute the dot product to second order.

**Step 4.** Compare with the small-angle expansion of $\cos\delta\theta$.

**Geometric picture.** The two adjacent directions $\hat{u}_1$ and $\hat{u}_2$ are nearby points on the unit sphere; the small displacement vector $\delta\vec{u} = (\delta l, \delta m, \delta n)$ is approximately tangent to the sphere, and its squared length $|\delta\vec{u}|^2$ equals the squared arc length $\delta\theta^2$ to leading order.

**Final Answer:** $\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$.

**Q 11.14** **$O$ is the origin and $A$ is the point $(a, b, c)$. Find the direction cosines of the line $OA$ and the equation of the plane through $A$ at right angles to $OA$.**

**SOLUTION**

**Concept used.** A line from the origin to $(a, b, c)$ has direction ratios $(a, b, c)$ and length $\sqrt{a^2+b^2+c^2}$; its direction cosines are these ratios divided by the length. A plane perpendicular to a line and passing through a given point uses the line's direction as its normal.

**Step 1.** Direction ratios of $OA$: $(a, b, c)$. Length $|OA| = \sqrt{a^2+b^2+c^2}$.

**Step 2.** Direction cosines:

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right).$$

**Step 3.** Plane through $A(a,b,c)$ with normal $(a,b,c)$:

$$a(x-a)+b(y-b)+c(z-c)=0 \implies ax+by+cz=a^2+b^2+c^2.$$

**Final Answer:** DCs $= \dfrac{1}{\sqrt{a^2+b^2+c^2}}(a,b,c)$. Plane: $ax+by+cz=a^2+b^2+c^2$.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The DCs of any line through the origin are obtained by normalising its endpoint vector. A plane normal to that line through any specified point is given by the point-normal form.

**Step 1.** Compute $|OA|$.

**Step 2.** Divide $(a,b,c)$ component-wise by $|OA|$ to get the DCs.

**Step 3.** Use $(a,b,c)$ as the plane's normal and $(a,b,c)$ as the on-plane point in the point-normal form.

**Step 4.** Simplify to obtain the Cartesian equation.

**Distance from origin to plane.** The plane $ax+by+cz=a^2+b^2+c^2$ has perpendicular distance from the origin equal to $\dfrac{|a^2+b^2+c^2|}{\sqrt{a^2+b^2+c^2}}=\sqrt{a^2+b^2+c^2}=|OA|$. Consistent with the plane being normal to $OA$ at the point $A$.

**Final Answer:** DCs $= \dfrac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$, plane: $ax+by+cz=a^2+b^2+c^2$.

**Q 11.15** **Two systems of rectangular axes have the same origin. If a plane cuts them at distances $a,b,c$ and $a',b',c'$, respectively, from the origin, prove that**

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

**SOLUTION**

**Concept used.** The perpendicular distance $p$ from the origin to a plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (intercept form) is given by

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

This distance is a property of the plane itself - it does not depend on which rectangular coordinate system we use.

**Step 1.** In the first coordinate system, the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, so its perpendicular distance from the origin is $p$ with $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

**Step 2.** In the second coordinate system (different rotation, same origin), the plane is $\frac{x'}{a'}+\frac{y'}{b'}+\frac{z'}{c'}=1$, with perpendicular distance $p'$ satisfying $\frac{1}{p'^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$.

**Step 3.** The perpendicular distance from the (common) origin to the same plane is invariant under rotation of axes, so $p=p'$.

**Step 4.** Therefore $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$.

**Final Answer:** The identity is proved by invariance of the perpendicular distance from the origin to the plane under rotation of axes.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A rotation of axes preserves Euclidean distance, so any rotation-invariant quantity expressed in coordinates of one system equals the same expression in coordinates of the rotated system.

**Step 1.** Use the intercept-form perpendicular-distance formula for both coordinate systems.

**Step 2.** Note that the geometric distance from the origin to the plane is the same in both systems.

**Step 3.** Equate the two expressions for $1/p^2$.

**Why the intercept-distance formula works.** The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ has normal $(1/a, 1/b, 1/c)$ and constant term $1$, so the distance from the origin is $\frac{|1|}{\sqrt{1/a^2+1/b^2+1/c^2}}$. Inverting and squaring gives $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

**Final Answer:** $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}$ (invariance of origin-to-plane distance).

## II. Long Answer (L.A.)

**Q 11.16** **Find the foot of perpendicular from the point $(2, 3, -8)$ to the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.**

**SOLUTION**

**Concept used.** Rewrite the line in standard symmetric form $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ to read off a point on the line and the direction ratios. Parametrise; the foot of perpendicular is determined by the condition that the vector from the external point to the foot is perpendicular to the line's direction vector.

**Step 1.** Rewrite the line: $\frac{4-x}{2}=\frac{x-4}{-2}$, $\frac{1-z}{3}=\frac{z-1}{-3}$. So the standard form is

$$\frac{x-4}{-2}=\frac{y-0}{6}=\frac{z-1}{-3}.$$

Point on line: $(4,0,1)$. Direction ratios: $(-2,6,-3)$.

**Step 2.** Parametrise: any point $Q$ on the line is $(4-2t,\ 6t,\ 1-3t)$.

**Step 3.** Given external point $P=(2,3,-8)$. Vector $\vec{PQ}=(4-2t-2,\ 6t-3,\ 1-3t+8)=(2-2t,\ 6t-3,\ 9-3t)$.

**Step 4.** Perpendicularity: $\vec{PQ}\cdot(-2,6,-3)=0$.

$$-2(2-2t)+6(6t-3)+(-3)(9-3t)=0.$$

**Step 5.** Expand:

$$-4+4t+36t-18-27+9t=0 \implies 49t-49=0 \implies t=1.$$

**Step 6.** Foot of perpendicular $Q=(4-2,\ 6,\ 1-3)=(2,6,-2)$.

**Step 7.** Distance $|PQ|=\sqrt{(2-2)^2+(6-3)^2+(-2+8)^2}=\sqrt{0+9+36}=\sqrt{45}=3\sqrt{5}$.

**Final Answer:** Foot of perpendicular: $(2,6,-2)$. Distance: $3\sqrt{5}$ units.

### ☞ Exam Tip

When a line is written with a negative coefficient in front of $x$ or $z$ (e.g. $\frac{4-x}{2}$), always rewrite to bring it to standard form $\frac{x-x_0}{a}$. Mis-reading the direction ratios is the most common slip.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The foot of perpendicular from an external point to a line is the point on the line closest to the external point. The connecting vector is perpendicular to the line's direction vector.

**Step 1.** Bring the line equation to standard symmetric form.

**Step 2.** Parametrise points on the line.

**Step 3.** Impose perpendicularity of the connecting vector to the direction vector.

**Step 4.** Solve the resulting linear equation for $t$.

**Step 5.** Substitute back to get the foot and compute the distance.

**Cross-product distance check.** The distance from $P$ to the line through $A = (4, 0, 1)$ with direction $\vec{b} = (-2, 6, -3)$ can also be computed as $\frac{|\vec{AP} \times \vec{b}|}{|\vec{b}|}$. Here $\vec{AP} = (-2, 3, -9)$ and $\vec{AP} \times \vec{b} = (-9 \cdot 6 - (-3) \cdot 3,\ (-9)(-2) - (-2)(-3),\ (-2)(6) - 3(-2)) = (-45, 12, -6)$. Wait, let me recompute: $\vec{AP} = (2 - 4,\ 3 - 0,\ -8 - 1) = (-2, 3, -9)$. $\vec{AP} \times \vec{b} = (3(-3) - (-9)(6),\ (-9)(-2) - (-2)(-3),\ (-2)(6) - 3(-2)) =$ $(-9 + 54,\ 18 - 6,\ -12 + 6) = (45, 12, -6)$. Magnitude $= \sqrt{2025 + 144 + 36} = \sqrt{2205} = 21\sqrt{5}$. Divide by $|\vec{b}| = \sqrt{4 + 36 + 9} = 7$. Distance $= \frac{21\sqrt{5}}{7} = 3\sqrt{5}$. ✓

**Final Answer:** Foot: $(2, 6, -2)$, distance $= 3\sqrt{5}$.

**Q 11.17** **Find the distance of the point $(2, 4, -1)$ from the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$.**

**SOLUTION**

**Concept used.** Distance from an external point $P$ to a line through point $A$ with direction vector $\vec{b}$:

$$d = \frac{|\vec{AP} \times \vec{b}|}{|\vec{b}|}.$$

**Step 1.** Point on line: $A = (-5, -3, 6)$. Direction vector: $\vec{b} = (1, 4, -9)$.

**Step 2.** $\vec{AP} = (2 - (-5),\ 4 - (-3),\ -1 - 6) = (7, 7, -7)$.

**Step 3.** Cross product $\vec{AP} \times \vec{b}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 7 & -7 \\ 1 & 4 & -9 \end{vmatrix} = \hat{i}(7(-9) - (-7)(4)) - \hat{j}(7(-9) - (-7)(1)) + \hat{k}(7 \cdot 4 - 7 \cdot 1)$$

$$= \hat{i}(-63 + 28) - \hat{j}(-63 + 7) + \hat{k}(28 - 7) = (-35, 56, 21).$$

**Step 4.** Magnitude: $\sqrt{35^2 + 56^2 + 21^2} = \sqrt{1225 + 3136 + 441} = \sqrt{4802} = 49\sqrt{2}$. $(35 = 7 \cdot 5, 56 = 7 \cdot 8, 21 = 7 \cdot 3$; common factor 7 gives $7\sqrt{25 + 64 + 9} = 7\sqrt{98} = 49\sqrt{2}$.)

**Step 5.** $|\vec{b}| = \sqrt{1+16+81} = \sqrt{98} = 7\sqrt{2}$.

**Step 6.** Distance $d = \dfrac{49\sqrt{2}}{7\sqrt{2}} = 7$ units.

**Final Answer:** $d = 7$ units.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** The cross-product distance formula is the cleanest way to compute the perpendicular distance from a point to a line. No parametrisation is needed.

**Step 1.** Read off a point on the line ($A$) and the direction vector ($\vec{b}$) from the symmetric form.

**Step 2.** Form the vector $\vec{AP}$ from $A$ to the external point.

**Step 3.** Compute $\vec{AP} \times \vec{b}$ via the $3 \times 3$ determinant.

**Step 4.** Divide the magnitude by $|\vec{b}|$.

**Arithmetic tip.** When the cross product comes out as $(-35, 56, 21)$, notice the common factor 7 before squaring. Factoring early keeps the numbers small.

**Final Answer:** $d = 7$.

**Q 11.18 Find the length and the foot of perpendicular from the point $\left(1, \frac{3}{2}, 2\right)$ to the plane $2x - 2y + 4z + 5 = 0$.**

**SOLUTION**

**Concept used.** For a plane $Ax + By + Cz + D = 0$ and external point $P(x_0, y_0, z_0)$:

- Perpendicular distance: $d = \dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.
- Foot of perpendicular: $F = P - \dfrac{Ax_0 + By_0 + Cz_0 + D}{A^2 + B^2 + C^2}(A, B, C)$.

**Step 1.** Coefficients: $A = 2,\ B = -2,\ C = 4,\ D = 5$. Point: $(x_0, y_0, z_0) = (1, 3/2, 2)$.

**Step 2.** Numerator: $2(1) - 2(3/2) + 4(2) + 5 = 2 - 3 + 8 + 5 = 12$.

**Step 3.** Denominator: $\sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$.

**Step 4.** Distance: $d = \dfrac{|12|}{2\sqrt{6}} = \dfrac{12}{2\sqrt{6}} = \dfrac{6}{\sqrt{6}} = \sqrt{6}$.

**Step 5.** Foot of perpendicular: scale factor $\dfrac{12}{24} = \dfrac{1}{2}$.

$$F = \left(1 - \tfrac{1}{2}(2),\ \tfrac{3}{2} - \tfrac{1}{2}(-2),\ 2 - \tfrac{1}{2}(4)\right) = (0,\ \tfrac{5}{2},\ 0).$$

**Step 6.** Verify $F$ lies on the plane: $2(0) - 2(5/2) + 4(0) + 5 = 0 - 5 + 0 + 5 = 0$. ✓

> **Final Answer:** Length of perpendicular: $\sqrt{6}$ units. Foot: $\left(0, \dfrac{5}{2}, 0\right)$.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Standard plane-distance and foot-of-perpendicular formulas are direct substitutions; the main task is bookkeeping.

**Step 1.** Identify $A, B, C, D$ in the plane equation.

**Step 2.** Plug the external point into $Ax_0 + By_0 + Cz_0 + D$.

**Step 3.** Divide by $\sqrt{A^2 + B^2 + C^2}$ for the distance.

**Step 4.** Use $F = P - (\text{scale})\,(A, B, C)$ for the foot, where scale = (numerator)/(denominator-squared).

**Verification.** Always plug the computed foot back into the plane equation. Any non-zero residual signals an arithmetic slip earlier in the calculation.

> **Final Answer:** Distance $\sqrt{6}$, foot $(0, 5/2, 0)$.

**Q 11.19** **Find the equations of the line passing through the point $(3, 0, 1)$ and parallel to the planes $x + 2y = 0$ and $3y - z = 0$.**

**SOLUTION**

**Concept used.** A line parallel to two planes is perpendicular to both their normals. Its direction vector is therefore parallel to the cross product of the two normal vectors.

**Step 1.** Normal vectors: $\vec{n_1} = (1, 2, 0)$, $\vec{n_2} = (0, 3, -1)$.

**Step 2.** Direction of the required line: $\vec{b} = \vec{n_1} \times \vec{n_2}$:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 0 \\ 0 & 3 & -1 \end{vmatrix} = \hat{i}(-2 - 0) - \hat{j}(-1 - 0) + \hat{k}(3 - 0) = (-2, 1, 3).$$

**Step 3.** Line through $(3,0,1)$ with direction $(-2,1,3)$:

$$\vec{r} = 3\hat{i} + \hat{k} + \lambda(-2\hat{i} + \hat{j} + 3\hat{k}).$$

Cartesian form: $\dfrac{x-3}{-2} = \dfrac{y}{1} = \dfrac{z-1}{3}$.

**Final Answer:** $\dfrac{x-3}{-2} = \dfrac{y}{1} = \dfrac{z-1}{3}$.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A line parallel to a plane has direction perpendicular to that plane's normal. Parallel to two planes therefore means direction perpendicular to both normals - exactly the cross product.

**Step 1.** Read off the normal of each plane.

**Step 2.** Cross-product them to obtain the line's direction.

**Step 3.** Use the given point and the new direction in the line equation.

**Sanity check.** Direction $(-2,1,3)$ dotted with $\vec{n_1} = (1,2,0)$ gives $-2+2+0=0$. ✓ Dotted with $\vec{n_2} = (0,3,-1)$ gives $0+3-3=0$. ✓ Confirms perpendicularity to both normals.

**Final Answer:** $\vec{r} = 3\hat{i} + \hat{k} + \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$.

**Q 11.20** **Find the equation of the plane through the points $(2,1,-1)$ and $(-1,3,4)$, and perpendicular to the plane $x - 2y + 4z = 10$.**

**SOLUTION**

**Concept used.** A plane is determined by two in-plane vectors and an on-plane point. Two such in-plane vectors here are: (i) the vector joining the two given points, and (ii) the normal of the perpendicular plane (any vector in the perpendicular plane's plane of normality lies in the required plane). The required plane's normal is the cross product of these two in-plane vectors.

**Step 1.** In-plane vector from $(2,1,-1)$ to $(-1,3,4)$: $\vec{v_1} = (-3,2,5)$.

**Step 2.** Normal of the given perpendicular plane: $\vec{n_0} = (1,-2,4)$. This vector lies in (is parallel to) the required plane because the two planes are perpendicular.

**Step 3.** Normal of the required plane:

$$\vec{n} = \vec{v_1} \times \vec{n_0} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 2 & 5 \\ 1 & -2 & 4 \end{vmatrix}.$$

Expand: $\hat{i}(2 \cdot 4 - 5 \cdot (-2)) - \hat{j}((-3) \cdot 4 - 5 \cdot 1) + \hat{k}((-3) \cdot (-2) - 2 \cdot 1)$
$= \hat{i}(8 + 10) - \hat{j}(-12 - 5) + \hat{k}(6 - 2) = (18, 17, 4)$.

**Step 4.** Plane through $(2, 1, -1)$ with normal $(18, 17, 4)$:

$$18(x - 2) + 17(y - 1) + 4(z + 1) = 0 \implies 18x + 17y + 4z = 49.$$

**Final Answer:** $18x + 17y + 4z = 49$.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Two planes are perpendicular iff one's normal lies in the other plane. So the normal of the perpendicular plane is an in-plane direction of the required plane.

**Step 1.** Form the in-plane vector from the two given points.

**Step 2.** Use the normal of the perpendicular plane as a second in-plane vector.

**Step 3.** Cross-product them to get the required plane's normal.

**Step 4.** Plug into the point-normal form.

**Verification.** The required plane's normal $(18, 17, 4)$ dotted with the perpendicular plane's normal $(1, -2, 4)$: $18 - 34 + 16 = 0$. ✓ The two normals are perpendicular, confirming the two planes are perpendicular.

**Final Answer:** $18x + 17y + 4z = 49$.

**Q 11.21** **Find the shortest distance between the lines** $\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$ **and** $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

**SOLUTION**

**Concept used.** For two skew lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$,

$$d = \frac{\left|(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})\right|}{|\vec{b_1} \times \vec{b_2}|}.$$

**Step 1.** Read the lines:

$$\vec{a_1} = 8\hat{i} - 9\hat{j} + 10\hat{k}, \quad \vec{b_1} = 3\hat{i} - 16\hat{j} + 7\hat{k}.$$

$$\vec{a_2} = 15\hat{i} + 29\hat{j} + 5\hat{k}, \quad \vec{b_2} = 3\hat{i} + 8\hat{j} - 5\hat{k}.$$

**Step 2.** $\vec{a_2} - \vec{a_1} = 7\hat{i} + 38\hat{j} - 5\hat{k}$.

**Step 3.** $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$

$$\begin{aligned} &= \hat{i}((-16)(-5) - 7 \cdot 8) - \hat{j}(3 \cdot (-5) - 7 \cdot 3) + \hat{k}(3 \cdot 8 - (-16) \cdot 3) \\ &= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48) \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}. \end{aligned}$$

Factor out 12: $\vec{b_1} \times \vec{b_2} = 12(2\hat{i} + 3\hat{j} + 6\hat{k})$.

**Step 4.** $|\vec{b_1} \times \vec{b_2}| = 12\sqrt{4 + 9 + 36} = 12 \cdot 7 = 84$.

**Step 5.** Dot product:

$$(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1}) = 12\big(2 \cdot 7 + 3 \cdot 38 + 6 \cdot (-5)\big) = 12(14 + 114 - 30) = 12 \cdot 98 = 1176.$$

**Step 6.** Shortest distance:

$$d = \frac{|1176|}{84} = 14.$$

**Final Answer:** $d = 14$ units.

### ☞ Exam Tip

Factor any common numerical factor out of the cross product before squaring magnitudes. It reduces the arithmetic considerably and helps you catch errors visually.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** The cross-product distance formula for skew lines is the high-yield 5-mark template in Chapter 11. Drilling it on a numerically clean example like this is the best preparation for the board exam.

**Step 1.** Identify the two position vectors and the two direction vectors.

**Step 2.** Compute the difference of position vectors.

**Step 3.** Compute the cross product of direction vectors.

**Step 4.** Take the absolute value of the dot product of (cross product) with (difference of positions).

**Step 5.** Divide by the magnitude of the cross product.

**Skew vs parallel check.** $\vec{b_1} \times \vec{b_2} = 12(2,3,6) \neq \vec{0}$, so the two lines are not parallel. The non-zero numerator confirms they do not intersect either. The lines are skew, and the shortest distance is finite and positive.

**Final Answer:** $d = 14$.

**Q 11.22** **Find the equation of the plane which is perpendicular to the plane** $5x + 3y + 6z + 8 = 0$ **and which contains the line of intersection of the planes** $x + 2y + 3z - 4 = 0$ **and** $2x + y - z + 5 = 0$**.**

SOLUTION

**Concept used.** Any plane through the line of intersection of $P_1 : x + 2y + 3z - 4 = 0$ and $P_2 : 2x + y - z + 5 = 0$ has the form $P_1 + \lambda P_2 = 0$. The parameter $\lambda$ is fixed by the perpendicularity condition with $5x + 3y + 6z + 8 = 0$.

**Step 1.** Family of planes: $(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$, i.e.

$$(1 + 2\lambda)x + (2 + \lambda)y + (3 - \lambda)z + (-4 + 5\lambda) = 0.$$

**Step 2.** Normal of this plane: $(1 + 2\lambda,\ 2 + \lambda,\ 3 - \lambda)$. Normal of the perpendicular plane: $(5, 3, 6)$.

**Step 3.** Perpendicularity: dot product of the two normals $= 0$:

$$5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0.$$

**Step 4.** Expand: $5 + 10\lambda + 6 + 3\lambda + 18 - 6\lambda = 0 \Rightarrow 7\lambda + 29 = 0 \Rightarrow \lambda = -\dfrac{29}{7}$.

**Step 5.** Substitute back:

$$\left(1 - \tfrac{58}{7}\right)x + \left(2 - \tfrac{29}{7}\right)y + \left(3 + \tfrac{29}{7}\right)z + \left(-4 - \tfrac{145}{7}\right) = 0.$$

**Step 6.** Multiply through by 7:

$$(7 - 58)x + (14 - 29)y + (21 + 29)z + (-28 - 145) = 0,$$

$$-51x - 15y + 50z - 173 = 0 \implies 51x + 15y - 50z + 173 = 0.$$

**Final Answer:** $51x + 15y - 50z + 173 = 0$.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The family $P_1 + \lambda P_2 = 0$ generates every plane through the line of intersection of $P_1 = 0$ and $P_2 = 0$. One scalar condition (here perpendicularity) pins down a unique member.

**Step 1.** Write the parameter family.

**Step 2.** Read off the normal of the parametric plane.

**Step 3.** Impose the perpendicularity condition with the given plane.

**Step 4.** Solve for $\lambda$ and substitute back.

**Verification.** The normal $(51, 15, -50)$ dotted with $(5, 3, 6)$: $255 + 45 - 300 = 0$. ✓ Confirms perpendicularity to the prescribed plane.

> **Final Answer:** $51x + 15y - 50z + 173 = 0$.

**Q 11.23** **The plane $ax + by = 0$ is rotated about its line of intersection with the plane $z = 0$ through an angle $\alpha$. Prove that the equation of the plane in its new position is $ax + by \pm (\sqrt{a^2 + b^2}\ \tan\alpha)\, z = 0$.**

**SOLUTION**

**Concept used.** A plane through the intersection of $P_1 = ax + by = 0$ and $P_2 = z = 0$ has the form $P_1 + \lambda P_2 = ax + by + \lambda z = 0$. The rotation angle determines $|\lambda|$ via the cosine-of-angle formula between two planes.

**Step 1.** Family of planes: $ax + by + \lambda z = 0$, with normal $(a, b, \lambda)$.

**Step 2.** Cosine of angle between this plane and the original plane $ax + by = 0$ (normal $(a, b, 0)$):

$$\cos\alpha = \frac{|a \cdot a + b \cdot b + \lambda \cdot 0|}{\sqrt{a^2 + b^2 + \lambda^2}\,\sqrt{a^2 + b^2}} = \frac{a^2 + b^2}{\sqrt{a^2 + b^2}\sqrt{a^2 + b^2 + \lambda^2}} = \frac{\sqrt{a^2 + b^2}}{\sqrt{a^2 + b^2 + \lambda^2}}.$$

**Step 3.** Square: $\cos^2\alpha = \dfrac{a^2 + b^2}{a^2 + b^2 + \lambda^2}$. Cross-multiply:

$$(a^2 + b^2 + \lambda^2)\cos^2\alpha = a^2 + b^2.$$

**Step 4.** Rearrange:

$$\lambda^2\cos^2\alpha = (a^2 + b^2)(1 - \cos^2\alpha) = (a^2 + b^2)\sin^2\alpha.$$

**Step 5.** Solve for $\lambda$:

$$\lambda = \pm\frac{\sqrt{a^2 + b^2}\sin\alpha}{\cos\alpha} = \pm\sqrt{a^2 + b^2}\ \tan\alpha.$$

**Step 6.** Substitute back:

$$ax + by \pm \left(\sqrt{a^2 + b^2}\ \tan\alpha\right)z = 0.$$

**Final Answer:** $ax + by \pm \sqrt{a^2 + b^2}\ \tan\alpha \cdot z = 0.$

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A rotation about the line of intersection is equivalent to varying the parameter $\lambda$ in the family $P_1 + \lambda P_2 = 0$. The angle between the rotated and the original plane is a function of $\lambda$.

**Step 1.** Write the rotated plane as $ax + by + \lambda z = 0$.

**Step 2.** Compute the angle between the rotated and original plane via the cosine formula.

**Step 3.** Set this cosine equal to $\cos\alpha$ and solve for $\lambda$.

**Step 4.** Both signs of $\lambda$ are valid because rotation can be clockwise or counter-clockwise.

**Sanity check at $\alpha = 0$.** $\tan 0 = 0$, so $\lambda = 0$ and the equation reduces to $ax + by = 0$ - the original plane, as expected.

**Sanity check at $\alpha = \pi/2$.** $\tan(\pi/2)$ diverges, meaning the plane perpendicular to the original (through the same line of intersection) is $z = 0$ - which is exactly $P_2$. Consistent.

**Final Answer:** $ax + by \pm \sqrt{a^2 + b^2}\ \tan\alpha\ \cdot z = 0.$

**Q 11.24** **Find the equation of the plane through the intersection of the planes $\vec{r}\cdot(\hat{i} + 3\hat{j}) - 6 = 0$ and $\vec{r}\cdot(3\hat{i} - \hat{j} - 4\hat{k}) = 0$, whose perpendicular distance from the origin is unity.**

**SOLUTION**

**Concept used.** Family of planes: $P_1 + \lambda P_2 = 0$. Use the perpendicular-distance formula to pin down $\lambda$ from the unit-distance condition.

**Step 1.** Family in Cartesian form:

$$(x + 3y - 6) + \lambda(3x - y - 4z) = 0,$$

$$(1 + 3\lambda)x + (3 - \lambda)y + (-4\lambda)z - 6 = 0.$$

**Step 2.** Distance from origin:

$$d = \frac{|-6|}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+16\lambda^2}} = 1.$$

**Step 3.** So $\sqrt{(1+3\lambda)^2+(3-\lambda)^2+16\lambda^2} = 6$. Square:

$$(1+3\lambda)^2+(3-\lambda)^2+16\lambda^2 = 36.$$

**Step 4.** Expand:

$$1+6\lambda+9\lambda^2+9-6\lambda+\lambda^2+16\lambda^2 = 36,$$

$$26\lambda^2+10 = 36 \implies 26\lambda^2 = 26 \implies \lambda = \pm 1.$$

**Step 5.** Case $\lambda = 1$:
$(1+3)x+(3-1)y+(-4)z-6=0 \Rightarrow 4x+2y-4z-6=0 \Rightarrow 2x+y-2z-3=0.$

**Step 6.** Case $\lambda = -1$:
$(1-3)x+(3+1)y+(4)z-6=0 \Rightarrow -2x+4y+4z-6=0 \Rightarrow -x+2y+2z-3=0,$
i.e. $x-2y-2z+3=0.$

**Final Answer:** $2x+y-2z-3=0$ or $x-2y-2z+3=0.$

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The unit-distance condition pins $\lambda$ as a real number through a quadratic, generically giving two valid planes.

**Step 1.** Write the family $P_1+\lambda P_2=0$ in Cartesian form.

**Step 2.** Distance from the origin is $|D|/\sqrt{A^2+B^2+C^2}$ where $A, B, C, D$ are the coefficients.

**Step 3.** Set equal to 1, square, and solve for $\lambda$.

**Step 4.** Substitute each $\lambda$ back to get the two planes.

**Verification.** For $2x+y-2z=3$, distance from origin $= \frac{|3|}{\sqrt{4+1+4}} = \frac{3}{3} = 1.$ ✓ For $x-2y-2z=-3$, distance $= \frac{|-3|}{\sqrt{1+4+4}} = \frac{3}{3} = 1.$ ✓

**Final Answer:** Two planes: $2x+y-2z=3$ or $x-2y-2z=-3$.

**Q 11.25** **Show that the points $(\hat{i}-\hat{j}+3\hat{k})$ and $3(\hat{i}+\hat{j}+\hat{k})$ are equidistant from the plane $\vec{r}\cdot(5\hat{i}+2\hat{j}-7\hat{k})+9=0$ and lie on opposite sides of it.**

**SOLUTION**

**Concept used.** Signed distance from a point $P(x_0, y_0, z_0)$ to the plane $Ax + By + Cz + D = 0$ is $\dfrac{Ax_0 + By_0 + Cz_0 + D}{\sqrt{A^2 + B^2 + C^2}}$. Opposite signs $\Rightarrow$ opposite sides.

**Step 1.** Points: $P_1 = (1, -1, 3)$, $P_2 = (3, 3, 3)$.

**Step 2.** Plane: $5x + 2y - 7z + 9 = 0$, so $A = 5, B = 2, C = -7, D = 9$ and $\sqrt{A^2 + B^2 + C^2} = \sqrt{25 + 4 + 49} = \sqrt{78}$.

**Step 3.** Signed numerator at $P_1$: $5(1) + 2(-1) - 7(3) + 9 = 5 - 2 - 21 + 9 = -9$.

**Step 4.** Signed numerator at $P_2$: $5(3) + 2(3) - 7(3) + 9 = 15 + 6 - 21 + 9 = 9$.

**Step 5.** Magnitudes equal ($|9| = |-9|$), so distances are equal: $\dfrac{9}{\sqrt{78}}$ for both points.

**Step 6.** Signs are opposite ($-9$ vs $9$), so the points lie on opposite sides of the plane.

> **Final Answer:** Both points are at distance $\dfrac{9}{\sqrt{78}}$ from the plane and lie on opposite sides.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** The signed distance encodes both magnitude and side. Opposite signs of the signed distance correspond to opposite sides of the plane.

**Step 1.** Plug each point into the LHS of the plane equation (no modulus yet).

**Step 2.** Compare absolute values for "equidistant".

**Step 3.** Compare signs for "opposite sides".

**Mid-point lies on the plane.** The mid-point of $P_1P_2$ is $(2, 1, 3)$. Plug in: $5(2) + 2(1) - 7(3) + 9 = 10 + 2 - 21 + 9 = 0$. So the mid-point lies on the plane - exactly what equidistant points on opposite sides give.

> **Final Answer:** Equidistant: $\dfrac{9}{\sqrt{78}}$; opposite sides confirmed by sign reversal of the signed distance.

**Q 11.26** $\vec{AB} = 3\hat{i} - \hat{j} + \hat{k}$ **and** $\vec{CD} = -3\hat{i} + 2\hat{j} + 4\hat{k}$ **are two vectors. The position vectors of the points** $A$ **and** $C$ **are** $6\hat{i} + 7\hat{j} + 4\hat{k}$ **and** $-9\hat{j} + 2\hat{k}$**, respectively. Find the position vector of a point** $P$ **on the line** $AB$ **and a point** $Q$ **on the line** $CD$ **such that** $\vec{PQ}$ **is perpendicular to** $\vec{AB}$ **and** $\vec{CD}$ **both.**

## SOLUTION

**Concept used.** Parametrise points on each line, write $\vec{PQ}$ in terms of the two parameters, and impose the two perpendicularity conditions $\vec{PQ}\cdot\vec{AB}=0$ and $\vec{PQ}\cdot\vec{CD}=0$.

**Step 1.** Parametrise: $P = A+\lambda\vec{AB} = (6+3\lambda,\ 7-\lambda,\ 4+\lambda)$.

**Step 2.** Parametrise: $Q = C+\mu\vec{CD} = (0-3\mu,\ -9+2\mu,\ 2+4\mu)$.

**Step 3.** $\vec{PQ} = Q-P = (-3\mu-6-3\lambda,\ -9+2\mu-7+\lambda,\ 2+4\mu-4-\lambda)$
$= (-3\mu-3\lambda-6,\ 2\mu+\lambda-16,\ 4\mu-\lambda-2)$.

**Step 4.** Perpendicular to $\vec{AB}=(3,-1,1)$:

$$3(-3\mu-3\lambda-6)-1(2\mu+\lambda-16)+1(4\mu-\lambda-2)=0.$$

Expand: $-9\mu-9\lambda-18-2\mu-\lambda+16+4\mu-\lambda-2=0$
$\Rightarrow -7\mu-11\lambda-4=0 \Rightarrow 7\mu+11\lambda=-4.$ (i)

**Step 5.** Perpendicular to $\vec{CD}=(-3,2,4)$:

$$-3(-3\mu-3\lambda-6)+2(2\mu+\lambda-16)+4(4\mu-\lambda-2)=0.$$

Expand: $9\mu+9\lambda+18+4\mu+2\lambda-32+16\mu-4\lambda-8=0$
$\Rightarrow 29\mu+7\lambda-22=0 \Rightarrow 29\mu+7\lambda=22.$ (ii)

**Step 6.** Solve (i) and (ii): multiply (i) by 7 and (ii) by 11:

$$49\mu+77\lambda=-28, \qquad 319\mu+77\lambda=242.$$

Subtract: $270\mu=270 \Rightarrow \mu=1$. From (i): $7+11\lambda=-4 \Rightarrow \lambda=-1$.

**Step 7.** $P = A-\vec{AB} = (6-3, 7+1, 4-1) = (3,8,3)$.

**Step 8.** $Q = C+\vec{CD} = (0-3, -9+2, 2+4) = (-3,-7,6)$.

> **Final Answer:** $P = 3\hat{i}+8\hat{j}+3\hat{k}$ and $Q=-3\hat{i}-7\hat{j}+6\hat{k}$.

## EXPERT'S SOLUTION : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The common perpendicular of two skew (or coplanar) lines is the segment connecting the closest pair of points on the two lines. Its direction is parallel to $\vec{b_1}\times\vec{b_2}$.

**Step 1.** Parametrise $P$ on line 1 and $Q$ on line 2.

**Step 2.** Write $\vec{PQ}$ in terms of the two parameters.

**Step 3.** Set $\vec{PQ}\cdot\vec{b_1}=0$ and $\vec{PQ}\cdot\vec{b_2}=0$.

**Step 4.** Solve the resulting $2 \times 2$ linear system for $(\lambda, \mu)$.

**Step 5.** Substitute back to find $P$ and $Q$.

**Cross-product cross-check.** $\vec{AB} \times \vec{CD} = (3, -1, 1) \times (-3, 2, 4) = ((-1)(4) - 1(2),\ 1(-3) - 3(4),\ 3(2) - (-1)(-3)) = (-6, -15, 3) = 3(-2, -5, 1)$. The vector $\vec{PQ} = Q - P = (-3 - 3,\ -7 - 8,\ 6 - 3) = (-6, -15, 3) = 3(-2, -5, 1)$. ✓ $\vec{PQ}$ is indeed parallel to $\vec{AB} \times \vec{CD}$, confirming it is the common perpendicular.

**Final Answer:** $P = (3, 8, 3)$, $Q = (-3, -7, 6)$.

**Q 11.27** **Show that the straight lines whose direction cosines are given by $2l+2m-n = 0$ and $mn + nl + lm = 0$ are at right angles.**

SOLUTION

**Concept used.** Eliminate one DC using the linear equation, substitute into the quadratic, and find the two solution lines. Two lines are perpendicular iff the sum of products of corresponding direction ratios is zero.

**Step 1.** From $2l + 2m - n = 0$: $n = 2l + 2m$.

**Step 2.** Substitute into $mn + nl + lm = 0$:

$$m(2l + 2m) + l(2l + 2m) + lm = 0,$$

$$2lm + 2m^2 + 2l^2 + 2lm + lm = 0,$$

$$2l^2 + 5lm + 2m^2 = 0.$$

**Step 3.** Factor: $(2l + m)(l + 2m) = 0$. So $m = -2l$ or $l = -2m$.

**Step 4.** Case 1: $m = -2l \Rightarrow n = 2l + 2(-2l) = -2l$. DRs $\propto (1, -2, -2)$.

**Step 5.** Case 2: $l = -2m \Rightarrow n = 2(-2m) + 2m = -2m$. DRs $\propto (-2, 1, -2)$.

**Step 6.** Dot product of the two DR triples:
$(1)(-2) + (-2)(1) + (-2)(-2) = -2 - 2 + 4 = 0.$

**Step 7.** Perpendicular. ✓

**Final Answer:** The two lines are at right angles.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A linear + quadratic system in three DCs generically gives two discrete lines. Their relative angle (perpendicularity in particular) is read off from the dot product of their direction ratios.

**Step 1.** Use the linear equation to eliminate one DC.

**Step 2.** Substitute and factor the quadratic in the remaining two.

**Step 3.** Read off the two solution direction-ratio triples.

**Step 4.** Check perpendicularity by summing products of DRs.

**Symmetry insight.** The two solutions $(1, -2, -2)$ and $(-2, 1, -2)$ are related by swapping the roles of $l$ and $m$ - consistent with the symmetry of $2l + 2m - n = 0$ and $mn + nl + lm = 0$ under $l \leftrightarrow m$.

**Final Answer:** The lines are perpendicular (dot product of DRs $= 0$).

**Q 11.28** **If $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$ are the direction cosines of three mutually perpendicular lines, prove that the line whose direction cosines are proportional to $l_1 + l_2 + l_3,\ m_1 + m_2 + m_3,\ n_1 + n_2 + n_3$ makes equal angles with them.**

**SOLUTION**

**Concept used.** Three mutually perpendicular unit vectors form an orthonormal basis. Their pairwise dot products are zero and individual squared norms are 1.

**Step 1.** Orthonormality: $l_i^2 + m_i^2 + n_i^2 = 1$ for each $i$; and $l_i l_j + m_i m_j + n_i n_j = 0$ for $i \neq j$.

**Step 2.** Let $\vec{u} = (l_1 + l_2 + l_3,\ m_1 + m_2 + m_3,\ n_1 + n_2 + n_3)$.

**Step 3.** Compute $|\vec{u}|^2 = \sum_i (l_i^2 + m_i^2 + n_i^2) + 2\sum_{i<j}(l_i l_j + m_i m_j + n_i n_j) = 3(1) + 2(0) = 3.$

**Step 4.** Hence $|\vec{u}| = \sqrt{3}$, and the DCs of the line along $\vec{u}$ are obtained by dividing $\vec{u}$ by $\sqrt{3}$.

**Step 5.** Cosine of angle between $\vec{u}$ and line $i$ (with DCs $(l_i, m_i, n_i)$):

$$\cos\theta_i = \frac{l_i(l_1 + l_2 + l_3) + m_i(m_1 + m_2 + m_3) + n_i(n_1 + n_2 + n_3)}{\sqrt{3}}.$$

**Step 6.** Using orthonormality, only the $i = i$ terms survive:

$$\cos\theta_i = \frac{l_i^2 + m_i^2 + n_i^2}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

**Step 7.** The same value $\dfrac{1}{\sqrt{3}}$ for every $i$, so the angles are equal.

**Final Answer:** All three angles equal $\cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** An orthonormal frame in 3D is a rotation of the standard $\hat{i}, \hat{j}, \hat{k}$ frame, and the sum $\hat{e_1} + \hat{e_2} + \hat{e_3}$ in any orthonormal frame is a diagonal-type vector that makes equal angles with each basis vector.

**Step 1.** Use the orthonormal-frame property: pairwise dot products vanish, individual squared norms are 1.

**Step 2.** Compute $|\vec{u}|^2$ by expanding the sum.

**Step 3.** Compute $\vec{u} \cdot \hat{e_i}$ - only one term survives orthogonality.

**Step 4.** Conclude the cosine of the angle is $1/\sqrt{3}$ for every $i$.

**Geometric picture.** The vector $\hat{i} + \hat{j} + \hat{k}$ is the body diagonal of a unit cube and makes equal angles $\cos^{-1}(1/\sqrt{3}) \approx 54.7°$ with all three axes. The result here is the same statement in an arbitrary orthonormal frame, by rotational invariance.

**Final Answer:** Equal angles $\cos^{-1}(1/\sqrt{3})$ with each of the three perpendicular lines.

## III. Objective Type Questions (MCQ)

**Q 11.29** **Distance of the point $(\alpha, \beta, \gamma)$ from $y$-axis is**

**(A)** $\beta$ **(B)** $|\beta|$ **(C)** $|\beta| + |\gamma|$ **(D)** $\sqrt{\alpha^2 + \gamma^2}$.

**SOLUTION**

**Concept used.** The foot of perpendicular from $(\alpha, \beta, \gamma)$ to the $y$-axis is $(0, \beta, 0)$. The perpendicular distance is the distance from $(\alpha, \beta, \gamma)$ to $(0, \beta, 0)$.

**Step 1.** Compute $\sqrt{(\alpha - 0)^2 + (\beta - \beta)^2 + (\gamma - 0)^2} = \sqrt{\alpha^2 + \gamma^2}$.

**Step 2.** Matches option (D).

**Final Answer:** (D) $\sqrt{\alpha^2 + \gamma^2}$.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Distance to a coordinate axis is found by dropping the coordinate along that axis and taking the magnitude of what remains.

**Step 1.** Drop $\beta$ (the $y$-coordinate).

**Step 2.** Magnitude of $(\alpha, 0, \gamma)$ is $\sqrt{\alpha^2 + \gamma^2}$.

**General rule.** Distance from $(\alpha, \beta, \gamma)$ to $x$-axis is $\sqrt{\beta^2 + \gamma^2}$; to $y$-axis is $\sqrt{\alpha^2 + \gamma^2}$; to $z$-axis is $\sqrt{\alpha^2 + \beta^2}$.

**Final Answer:** (D).

**Q 11.30** **If the direction cosines of a line are $k, k, k$, then**

**(A)** $k > 0$ **(B)** $0 < k < 1$ **(C)** $k = 1$ **(D)** $k = \frac{1}{\sqrt{3}}$ **or** $-\frac{1}{\sqrt{3}}$.

**SOLUTION**

**Concept used.** Direction cosines satisfy $l^2 + m^2 + n^2 = 1$.

**Step 1.** Plug in $l = m = n = k$: $3k^2 = 1$.

**Step 2.** Solve: $k = \pm\frac{1}{\sqrt{3}}$.

**Final Answer:** (D) $k = \pm\frac{1}{\sqrt{3}}$.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The DC identity $l^2 + m^2 + n^2 = 1$ is the only constraint - it allows $k$ to be either positive or negative.

**Step 1.** Apply the identity with three equal DCs.

**Step 2.** Solve the simple square-root equation.

**Step 3.** Include both signs because a line has two senses.

**Final Answer:** (D).

**Q 11.31** **The distance of the plane $\vec{r}\cdot\left(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}\right)=1$ from the origin is**

**(A)** $1$ **(B)** $7$ **(C)** $\frac{1}{7}$ **(D) None of these.**

**SOLUTION**

**Concept used.** For a plane $\vec{r}\cdot\vec{n}=d$, distance from the origin is $\frac{|d|}{|\vec{n}|}$.

**Step 1.** $|\vec{n}|=\sqrt{(2/7)^2+(3/7)^2+(6/7)^2}=\sqrt{(4+9+36)/49}=\sqrt{49/49}=1.$

**Step 2.** Distance $=|1|/1=1.$

**Final Answer:** (A) 1.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** When the plane is given in the normal form $\vec{r}\cdot\hat{n}=p$ with $\hat{n}$ a unit vector, $p$ is directly the distance from the origin.

**Step 1.** Check whether $\vec{n}$ is already a unit vector.

**Step 2.** If yes, distance is just $|d|$.

**Step 3.** Here $|\vec{n}|=1$ exactly, so distance is $1$.

**Final Answer:** (A).

**Q 11.32** **The sine of the angle between the straight line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ and the plane $2x-2y+z=5$ is**

**(A)** $\frac{10}{6\sqrt{5}}$ **(B)** $\frac{4}{5\sqrt{2}}$ **(C)** $\frac{2\sqrt{3}}{5}$ **(D)** $\frac{\sqrt{2}}{10}$.

**SOLUTION**

**Concept used.** Angle between a line (direction $\vec{b}$) and a plane (normal $\vec{n}$) satisfies

$\sin\theta = \dfrac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}$ (because the angle between the line and the plane is the complement of the angle between the line and the normal).

**Step 1.** Direction vector: $\vec{b} = (3,4,5)$, $|\vec{b}| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$.

**Step 2.** Plane normal: $\vec{n} = (2,-2,1)$, $|\vec{n}| = \sqrt{4+4+1} = 3$.

**Step 3.** Dot product: $\vec{b}\cdot\vec{n} = 6-8+5 = 3$.

**Step 4.** $\sin\theta = \dfrac{|3|}{5\sqrt{2}\cdot 3} = \dfrac{1}{5\sqrt{2}} = \dfrac{\sqrt{2}}{10}$.

**Final Answer:** (D) $\dfrac{\sqrt{2}}{10}$.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Mnemonic: line-vs-plane angle uses $\sin$; line-vs-line angle uses $\cos$. The reason is that the line-plane angle is complementary to the line-normal angle.

**Step 1.** Identify $\vec{b}$ and $\vec{n}$.

**Step 2.** Compute the dot product and magnitudes.

**Step 3.** Use $\sin\theta = |\vec{b}\cdot\vec{n}|/(|\vec{b}||\vec{n}|)$.

**Rationalisation.** $\dfrac{1}{5\sqrt{2}} = \dfrac{1}{5\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}} = \dfrac{\sqrt{2}}{10}$ - which is option (D).

**Final Answer:** (D).

**Q 11.33** **The reflection of the point $(\alpha,\beta,\gamma)$ in the $xy$-plane is**

**(A)** $(\alpha,\beta,0)$ **(B)** $(0,0,\gamma)$ **(C)** $(-\alpha,-\beta,\gamma)$ **(D)** $(\alpha,\beta,-\gamma)$.

**SOLUTION**

**Concept used.** Reflection in the $xy$-plane (the plane $z=0$) preserves $x$ and $y$ coordinates and negates the $z$-coordinate.

**Step 1.** Apply the rule.

**Final Answer:** (D) $(\alpha,\beta,-\gamma)$.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Reflection in a coordinate plane negates only the coordinate perpendicular to that plane.

**Step 1.** $xy$-plane is $z = 0$, so reflection flips the $z$-coordinate.

**Final Answer:** (D).

**Q 11.34** **The area of the quadrilateral $ABCD$, where the vertices are $A(0, 4, 1)$, $B(2, 3, -1)$, $C(4, 5, 0)$, and $D(2, 6, 2)$, is**

**(A)** $9$ **sq. units** **(B)** $18$ **sq. units** **(C)** $27$ **sq. units** **(D)** $81$ **sq. units.**

**SOLUTION**

**Concept used.** If a quadrilateral $ABCD$ is a parallelogram, its area equals $|\vec{AB} \times \vec{AD}|$.

**Step 1.** $\vec{AB} = (2, -1, -2)$, $\vec{DC} = (4-2,\ 5-6,\ 0-2) = (2, -1, -2)$. So $\vec{AB} = \vec{DC}$: $ABCD$ is a parallelogram.

**Step 2.** $\vec{AD} = (2, 2, 1)$.

**Step 3.** $\vec{AB} \times \vec{AD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \hat{i}(-1+4) - \hat{j}(2+4) + \hat{k}(4+2) = (3, -6, 6)$.

**Step 4.** $|\vec{AB} \times \vec{AD}| = \sqrt{9+36+36} = \sqrt{81} = 9$.

**Final Answer:** (A) $9$ sq. units.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Cross-product magnitude gives the area of a parallelogram spanned by two adjacent edge vectors.

**Step 1.** Verify $ABCD$ is a parallelogram by checking $\vec{AB} = \vec{DC}$.

**Step 2.** Compute two adjacent edge vectors from a common vertex.

**Step 3.** Cross-product and take magnitude.

**Final Answer:** (A).

**Q 11.35** **The locus represented by $xy + yz = 0$ is**
**(A) A pair of perpendicular lines (B) A pair of parallel lines (C) A pair of parallel planes (D) A pair of perpendicular planes.**

**SOLUTION**

**Concept used.** Factor the equation: $xy + yz = y(x + z) = 0$. This gives $y = 0$ or $x + z = 0$ - two distinct planes.

**Step 1.** Factor: $y(x + z) = 0$.

**Step 2.** Planes: $y = 0$ (the $xz$-plane) and $x + z = 0$.

**Step 3.** Normals: $(0, 1, 0)$ and $(1, 0, 1)$.

**Step 4.** Dot product: $0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 = 0$. Perpendicular.

**Final Answer:** (D) A pair of perpendicular planes.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** A homogeneous equation that factors into two linear factors represents a union of two planes through the origin.

**Step 1.** Factor the second-degree expression.

**Step 2.** Read off the two planes from the factors.

**Step 3.** Check perpendicularity by dotting the two normals.

**Final Answer:** (D).

**Q 11.36** **The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with $x$-axis. The value of $\alpha$ is equal to**
**(A)** $\frac{\sqrt{3}}{2}$ **(B)** $\frac{\sqrt{2}}{3}$ **(C)** $\frac{2}{7}$ **(D)** $\frac{3}{7}$.

**SOLUTION**

**Concept used.** Angle between a line (direction $\vec{b}$) and a plane (normal $\vec{n}$) satisfies $\sin\theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}$. The $x$-axis has direction $(1, 0, 0)$.

**Step 1.** $\vec{b} = (1, 0, 0)$, $|\vec{b}| = 1$.

**Step 2.** $\vec{n} = (2, -3, 6)$, $|\vec{n}| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7$.

**Step 3.** Dot product: 2.

**Step 4.** $\sin\theta = \frac{|2|}{1 \cdot 7} = \frac{2}{7}$.

**Final Answer:** (C) $\frac{2}{7}$.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The $x$-axis is the line through the origin with direction $\hat{i} = (1, 0, 0)$.

**Step 1.** Plug $\vec{b} = (1, 0, 0)$ into the line-plane angle formula.

**Step 2.** Compute and simplify.

**Final Answer:** (C).

## IV. Fill in the Blanks

**Q 11.37** **A plane passes through the points** $(2, 0, 0)$, $(0, 3, 0)$ **and** $(0, 0, 4)$**. The equation of the plane is** __________**.**

**SOLUTION**

**Concept used.** A plane cutting the axes at $(a, 0, 0), (0, b, 0), (0, 0, c)$ has the intercept-form equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

**Step 1.** $a = 2, b = 3, c = 4$.

**Step 2.** Plane: $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

**Step 3.** Multiply by 12: $6x + 4y + 3z = 12$.

**Final Answer:** $6x + 4y + 3z = 12$ or equivalently $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Intercept form is the fastest plane equation when all three axis-intercepts are given.

**Step 1.** Read off the three intercepts.

**Step 2.** Plug into the intercept formula.

**Step 3.** Multiply through by the LCM of denominators for a clean integer form.

**Final Answer:** $6x + 4y + 3z = 12$.

**Q 11.38** **The direction cosines of the vector $2\hat{i} + 2\hat{j} - \hat{k}$ are ________.**

SOLUTION

**Concept used.** For a vector $(a, b, c)$, the direction cosines are $\frac{a}{|\vec{v}|}, \frac{b}{|\vec{v}|}, \frac{c}{|\vec{v}|}$.

**Step 1.** $|\vec{v}| = \sqrt{4 + 4 + 1} = 3$.

**Step 2.** DCs: $\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$.

**Final Answer:** $\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right)$.

EXPERT'S SOLUTION : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Direction cosines = direction ratios divided by the vector's magnitude.

**Step 1.** Compute the magnitude.

**Step 2.** Divide each component by it.

**Final Answer:** $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$.

**Q 11.39** **The vector equation of the line $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ is ________.**

SOLUTION

**Concept used.** The Cartesian symmetric form $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$ corresponds to the vector form $\vec{r} = \vec{a} + \lambda\vec{b}$ with $\vec{a} = (x_1, y_1, z_1)$ and $\vec{b} = (a, b, c)$.

**Step 1.** Point on line: $(5, -4, 6)$.

**Step 2.** Direction vector: $(3, 7, 2)$.

**Final Answer:** $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Symmetric Cartesian and vector forms are equivalent; converting is mechanical.

**Step 1.** Read the point from the numerators (sign-flipped).

**Step 2.** Read the direction vector from the denominators.

**Step 3.** Assemble.

**Final Answer:** $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

**Q 11.40** **The vector equation of the line through the points** $(3, 4, -7)$ **and** $(1, -1, 6)$ **is** __________.

**SOLUTION**

**Concept used.** Line through $\vec{a}$ and $\vec{b}$: $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

**Step 1.** $\vec{a} = (3, 4, -7)$, $\vec{b} = (1, -1, 6)$, $\vec{b} - \vec{a} = (-2, -5, 13)$.

**Step 2.** Equation: $\vec{r} = (3, 4, -7) + \lambda(-2, -5, 13)$.

**Final Answer:** $\vec{r} = 3\hat{i} + 4\hat{j} - 7\hat{k} + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$.

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Two-point vector form. Either point may serve as $\vec{a}$; the direction $\vec{b} - \vec{a}$ comes out with the sign convention chosen.

**Step 1.** Pick one point as the base $\vec{a}$.

**Step 2.** Use the difference vector as the direction.

**Final Answer:** $\vec{r} = (3\hat{i} + 4\hat{j} - 7\hat{k}) + \lambda(-2\hat{i} - 5\hat{j} + 13\hat{k})$.

**Q 11.41** **The Cartesian equation of the plane $\vec{r}\cdot(\hat{i}+\hat{j}-\hat{k})=2$ is ________.**

**SOLUTION**

**Concept used.** $\vec{r}\cdot\vec{n}=d$ converts to Cartesian by writing $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$ and reading off the dot product.

**Step 1.** Substitute: $(x)(1)+(y)(1)+(z)(-1)=2$.

**Step 2.** Simplify: $x+y-z=2$.

> **Final Answer:** $x+y-z=2$.

**EXPERT'S SOLUTION** *: Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Vector-form to Cartesian conversion is a direct dot product.

**Step 1.** Expand the dot product component-wise.

**Step 2.** Equate to the RHS scalar.

> **Final Answer:** $x+y-z=2$.

## V. True / False

**Q 11.42** **State True or False:** ***The unit vector normal to the plane $x+2y+3z-6=0$ is $\frac{1}{\sqrt{14}}\hat{i}+\frac{2}{\sqrt{14}}\hat{j}+\frac{3}{\sqrt{14}}\hat{k}$.***

**SOLUTION**

**True.**

**Concept used.** The plane $Ax+By+Cz+D=0$ has normal vector $(A,B,C)$; the unit normal is $(A,B,C)/\sqrt{A^2+B^2+C^2}$.

**Step 1.** $A,B,C=(1,2,3)$.

**Step 2.** $\sqrt{1+4+9}=\sqrt{14}$.

**Step 3.** Unit normal: $\frac{1}{\sqrt{14}}(1,2,3)$.

**Step 4.** Matches the given vector.

> **Final Answer:** **True.**

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** The plane's normal vector is read off directly from the coefficients of $x, y, z$ in the plane equation. Unit normal = normal / its magnitude.

**Step 1.** Read coefficients.

**Step 2.** Normalise.

**Final Answer:** True.

**Q 11.43** **State True or False:** ***The intercepts made by the plane*** $2x - 3y + 5z + 4 = 0$ ***on the coordinate axes are*** $-2, \frac{4}{3}, -\frac{4}{5}$.

**SOLUTION**

**True.**

**Concept used.** Rewrite the plane in intercept form by moving the constant and dividing.

**Step 1.** Rewrite: $2x - 3y + 5z = -4$.

**Step 2.** Divide by $-4$: $\frac{x}{-2} + \frac{y}{4/3} + \frac{z}{-4/5} = 1$.

**Step 3.** Intercepts: $-2, \frac{4}{3}, -\frac{4}{5}$.

**Final Answer:** **True**.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ exposes the intercepts directly.

**Step 1.** Move the constant to the RHS.

**Step 2.** Divide every term by the new RHS.

**Step 3.** Read off $a, b, c$ as the reciprocals of the coefficients.

**Final Answer:** True.

**Q 11.44** **State True or False:** ***The angle between the line*** $\vec{r} = (5\hat{i} - \hat{j} - 4\hat{k}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$ ***and the plane*** $\vec{r} \cdot (3\hat{i} - 4\hat{j} - \hat{k}) + 5 = 0$ ***is*** $\sin^{-1}\left(\frac{5}{2\sqrt{91}}\right)$.

**SOLUTION**

**True.**

**Concept used.** $\sin\theta = \dfrac{|\vec{b}\cdot\vec{n}|}{|\vec{b}||\vec{n}|}$.

**Step 1.** $\vec{b} = (2, -1, 1)$, $|\vec{b}| = \sqrt{4+1+1} = \sqrt{6}$.

**Step 2.** $\vec{n} = (3, -4, -1)$, $|\vec{n}| = \sqrt{9+16+1} = \sqrt{26}$.

**Step 3.** Dot product: $2(3) + (-1)(-4) + 1(-1) = 6 + 4 - 1 = 9$. Hmm let me re-check the claim: the question states $5/(2\sqrt{91})$. Let me recompute carefully.

**Step 4.** Dot product: $6 + 4 - 1 = 9$. So $\sin\theta = \dfrac{9}{\sqrt{6}\sqrt{26}} = \dfrac{9}{\sqrt{156}} = \dfrac{9}{2\sqrt{39}}$, which simplifies to $\dfrac{9}{2\sqrt{39}}$.

**Step 5.** This does not match $\dfrac{5}{2\sqrt{91}}$. Therefore the statement is **False**.

*Note: the question as printed has* $\sin^{-1}(5/(2\sqrt{91}))$ *which is not the correct value; recomputed answer is* $\sin^{-1}(9/(2\sqrt{39}))$*, so the statement is False.*

**Final Answer: False.**

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Line-plane angle uses $\sin$ on the dot product of (line direction) and (plane normal), normalised by the two magnitudes.

**Step 1.** Recompute carefully.

**Step 2.** Compare to the stated answer.

**Numerical recheck.** $\vec{b}\cdot\vec{n} = 2(3) + (-1)(-4) + (1)(-1) = 6 + 4 - 1 = 9$. $|\vec{b}||\vec{n}| = \sqrt{6\cdot 26} = \sqrt{156} = 2\sqrt{39}$. $\sin\theta = \dfrac{9}{2\sqrt{39}}$. The stated value $\dfrac{5}{2\sqrt{91}}$ differs - so the claim is False.

**Final Answer:** False.

**Q 11.45** **State True or False:** ***The angle between the planes*** $\vec{r}\cdot(2\hat{i} - 3\hat{j} + \hat{k}) = 1$ ***and*** $\vec{r}\cdot(\hat{i} - \hat{j}) = 4$ ***is*** $\cos^{-1}\left(\dfrac{5}{2\sqrt{7}}\right)$.

SOLUTION

**True.**

**Concept used.** Angle between two planes equals the angle between their normals: $\cos\theta = \dfrac{|\vec{n_1}\cdot\vec{n_2}|}{|\vec{n_1}||\vec{n_2}|}$.

**Step 1.** $\vec{n_1} = (2, -3, 1)$, $|\vec{n_1}| = \sqrt{4+9+1} = \sqrt{14}$.

**Step 2.** $\vec{n_2} = (1, -1, 0)$, $|\vec{n_2}| = \sqrt{2}$.

**Step 3.** Dot product: $2 + 3 + 0 = 5$.

**Step 4.** $\cos\theta = \dfrac{5}{\sqrt{14}\sqrt{2}} = \dfrac{5}{\sqrt{28}} = \dfrac{5}{2\sqrt{7}}$. Matches.

> **Final Answer:** **True**.

EXPERT'S SOLUTION : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Plane-plane angle uses $\cos$ on the dot product of normals.

**Step 1.** Read the two normals.

**Step 2.** Apply the cosine-of-angle formula.

> **Final Answer:** True.

**Q 11.46** **State True or False:** ***The line*** $\vec{r} = 2\hat{i} - 3\hat{j} - \hat{k} + \lambda(\hat{i} - \hat{j} + 2\hat{k})$ ***lies in the plane*** $\vec{r}\cdot(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0$**.**

SOLUTION

**True.**

**Concept used.** A line lies in a plane iff (i) a point on the line lies on the plane and (ii) the direction vector of the line is perpendicular to the plane's normal.

**Step 1.** Point on the line: $(2, -3, -1)$.

**Step 2.** Plug into plane $3x + y - z + 2 = 0$: $6 - 3 + 1 + 2 = 6$. Not zero. Hmm.

**Step 3.** Let me re-read the plane: $\vec{r}\cdot(3\hat{i} + \hat{j} - \hat{k}) + 2 = 0 \Rightarrow 3x + y - z = -2$. Plug in point: $6 - 3 + 1 = 4$, while RHS is $-2$. So the point does not lie on the plane.

**Step 4.** Therefore the statement is False.

> **Final Answer:** **False**.

**EXPERT'S SOLUTION** : *Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** Two conditions are needed for a line to lie in a plane.

**Step 1.** Check whether a point of the line lies on the plane.

**Step 2.** If not, the line cannot lie in the plane regardless of direction.

**Common slip.** Students sometimes only check perpendicularity of direction vs normal and conclude "the line lies in the plane". That condition only makes the line parallel to the plane (or in it); the on-plane point check is what distinguishes the two cases.

> **Final Answer:** False.

**Q 11.47** **State True or False:** *The vector equation of the line* $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ *is* $\vec{r} = 5\hat{i} - 4\hat{j} + 6\hat{k} + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

**SOLUTION**

**True.**

**Concept used.** Cartesian symmetric form converts directly to the vector form by reading off the point (numerators) and the direction vector (denominators).

**Step 1.** Point: $(5, -4, 6)$.

**Step 2.** Direction: $(3, 7, 2)$.

**Step 3.** Matches the stated vector equation exactly.

> **Final Answer:** **True**.

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** Mechanical conversion between Cartesian symmetric and vector parametric forms.

**Step 1.** Read the point.

**Step 2.** Read the direction.

**Step 3.** Compare.

> **Final Answer:** True.

**Q 11.48** **State True or False:** ***The equation of a line which is parallel to $2\hat{i} + \hat{j} + 3\hat{k}$ and which passes through the point $(5, -2, 4)$ is $\frac{x-5}{2} = \frac{y+2}{-1} = \frac{z-4}{3}$.***

**SOLUTION**

**False.**

**Concept used.** The direction ratios of a line equal the components of any vector parallel to it.

**Step 1.** Parallel vector: $(2, 1, 3)$, so DRs are $(2, 1, 3)$.

**Step 2.** Cartesian symmetric form through $(5, -2, 4)$: $\frac{x-5}{2} = \frac{y+2}{1} = \frac{z-4}{3}$.

**Step 3.** Stated equation has $-1$ in the $y$-denominator, which is incorrect.

> **Final Answer: False.**

**EXPERT'S SOLUTION** *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

**Concept used.** DR sign mistakes are the most common error in writing line equations. Always copy the parallel-vector components verbatim.

**Step 1.** Re-derive the equation.

**Step 2.** Spot the discrepancy in the $y$-denominator sign.

> **Final Answer:** False.

**Q 11.49** **State True or False:** ***If the foot of perpendicular drawn from the origin to a plane is $(5, -3, -2)$, then the equation of plane is $\vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.***

**SOLUTION**

**True.**

**Concept used.** If $F$ is the foot of perpendicular from the origin to a plane, then $\vec{OF}$ is normal to the plane, and the plane equation is $\vec{r} \cdot \vec{OF} = |\vec{OF}|^2$.

**Step 1.** $\vec{OF} = (5, -3, -2)$. $|\vec{OF}|^2 = 25 + 9 + 4 = 38$.

**Step 2.** Plane: $\vec{r} \cdot (5\hat{i} - 3\hat{j} - 2\hat{k}) = 38$.

**Step 3.** Matches the stated equation.

> **Final Answer: True.**

**EXPERT'S SOLUTION** : *Karan Patel, Ph.D Mathematics, IIT Delhi*

**Concept used.** The vector from origin to the foot of perpendicular is normal to the plane and its length equals the perpendicular distance from the origin to the plane.

**Step 1.** Identify $\vec{OF}$ as the plane's normal.

**Step 2.** Use $\vec{r} \cdot \vec{OF} = |\vec{OF}|^2$ (because $F$ itself lies on the plane and $\vec{F} \cdot \vec{F} = |\vec{F}|^2$).

**Final Answer:** True.

### Key Takeaways

- A line in space is uniquely determined by a point and a direction vector; the vector form is $\vec{r} = \vec{a} + \lambda\vec{b}$ and the Cartesian symmetric form is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.
- **Direction cosines** of a line always satisfy $l^2 + m^2 + n^2 = 1$. They are unique up to sign; direction ratios are unique only up to a non-zero scalar.
- The acute angle between two lines is given by $\cos\theta = \frac{|\vec{b_1} \cdot \vec{b_2}|}{|\vec{b_1}||\vec{b_2}|}$. The modulus is mandatory in CBSE answers.
- The shortest distance between two skew lines uses the scalar triple product: $d = \frac{|(\vec{b_1} \times \vec{b_2}) \cdot (\vec{a_2} - \vec{a_1})|}{|\vec{b_1} \times \vec{b_2}|}$.
- Distance between parallel lines reduces to $d = \frac{|\vec{b} \times (\vec{a_2} - \vec{a_1})|}{|\vec{b}|}$ since the two lines share a direction vector.
- A plane is determined by a point and a normal vector; the point-normal form is $\vec{n} \cdot (\vec{r} - \vec{a}) = 0$. The intercept form $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ is fastest when intercepts are given.
- Line-vs-plane angle uses $\sin$; line-vs-line and plane-vs-plane angles use $\cos$.
- The family $P_1 + \lambda P_2 = 0$ generates every plane through the intersection line of $P_1 = 0$ and $P_2 = 0$; one extra condition (distance, perpendicularity, etc.) fixes $\lambda$.

End of NCERT Exemplar Problems