



Collegedunia NCERT Formula Sheet

Class 12 Mathematics — Chapter 11

Chapter 11: Three Dimensional Geometry

Quantity / Symbol	Meaning	Key relation
α, β, γ	Direction angles with x, y, z axes	$l = \cos \alpha, m = \cos \beta, n = \cos \gamma$
l, m, n	Direction cosines (DCs) of a line	$l^2 + m^2 + n^2 = 1$
a, b, c	Direction ratios (DRs) of a line	DRs \propto DCs; any nonzero multiple works
\vec{a}	Position vector of a point on the line	Anchors the line in space
\vec{b}	Direction vector ($a\hat{i} + b\hat{j} + c\hat{k}$)	Components are the DRs of the line
λ, μ	Scalar parameters	Slide a point along the line
θ	Acute angle between two lines	$0 \leq \theta \leq \pi/2$, hence absolute value used

1 Direction Cosines & Direction Ratios

This section sets up the algebra of a line's orientation in space — the direction angles, their cosines (DCs), the proportional direction ratios (DRs), and the DCs of a line through two given points.

Direction angles and direction cosines

A directed line through the origin makes angles α, β, γ with the positive x, y, z axes. The cosines $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$ are the line's **direction cosines**. Reversing the line's direction replaces each angle by its supplement, so the signs of l, m, n flip.

Fundamental relation between DCs

$$l^2 + m^2 + n^2 = 1$$

where l, m, n are the direction cosines of a line.

Every set of DCs lies on the **unit sphere**. Use this to test whether three numbers can be DCs, or to find one DC from the other two.

Direction ratios from direction cosines

Any three numbers a, b, c proportional to l, m, n are **direction ratios**:

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$$

DRs are **not unique** — ka, kb, kc ($k \neq 0$) work too. DCs are unique up to overall sign (line direction).

DCs from DRs

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \quad n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where a, b, c are DRs of the line.

Normalise the DR triple by its magnitude. Same sign throughout fixes one of the two opposite directions of the line.

DRs and DCs of a line through two points

For $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$:

DRs: $x_2 - x_1, y_2 - y_1, z_2 - z_1$

DCs: $\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$

where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

The DRs are simply the **coordinate differences**. Reversing the order $Q \rightarrow P$ flips all signs — still valid DRs.

DCs of the coordinate axes

x -axis: $(1, 0, 0)$ y -axis: $(0, 1, 0)$ z -axis: $(0, 0, 1)$. A line equally inclined to all three axes has DCs $(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}})$.

DRs are not DCs

Direction ratios $(2, -1, -2)$ do **not** satisfy $a^2 + b^2 + c^2 = 1$ in general. Divide by $\sqrt{a^2 + b^2 + c^2}$ before using them as DCs (e.g. $\cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$).

Collinearity test via DRs

Three points A, B, C are collinear iff the DRs of \overrightarrow{AB} and \overrightarrow{BC} are **proportional**. Equivalently, all three coordinate-difference ratios match.

2 Equation of a Line in Space

A line in space is fixed by either (i) one point plus a direction, or (ii) two points. This section lists both, in vector and Cartesian forms.

Line through a point, parallel to \vec{b} — vector form

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

where \vec{a} = position vector of a fixed point on the line; \vec{b} = direction vector; $\lambda \in \mathbb{R}$ is the parameter.

As λ varies over \mathbb{R} , \vec{r} traces **every point** of the line. \vec{b} may be replaced by any nonzero scalar multiple — the line is unchanged.

Line through a point, given DRs — Cartesian form

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

where (x_1, y_1, z_1) = fixed point on the line; a, b, c = direction ratios.

Each fraction equals the parameter λ . If a DR is **zero** (say $a = 0$), write that coordinate fixed: $x = x_1$, and use the remaining two-fraction equality.

Cartesian form using DCs

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

where l, m, n are the direction cosines of the line.

Same structure as the DR form, but the common value of each ratio now equals the **signed distance** from (x_1, y_1, z_1) along the line.

Parametric equations of a line

$$x = x_1 + \lambda a, \quad y = y_1 + \lambda b, \quad z = z_1 + \lambda c$$

Useful when you need to **substitute** a general point of the line into another equation (e.g. to find an intersection or foot of perpendicular).

Line through two points — vector form

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

where \vec{a}, \vec{b} are position vectors of the two given points.

Direction vector is $\vec{b} - \vec{a}$. $\lambda = 0$ gives

the first point, $\lambda = 1$ gives the second, $0 < \lambda < 1$ traces the **segment** between them.

Line through two points — Cartesian form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ are the two given points.

Built directly from two-point DRs $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

Converting between vector and Cartesian forms

From $\vec{r} = \vec{a} + \lambda\vec{b}$ with $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$: set $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, equate components, and **eliminate** λ to land at the Cartesian form.

3 Angle Between Two Lines

The acute angle between two lines is the angle between any pair of direction vectors, with the absolute value taken on the cosine so that $0 \leq \theta \leq \pi/2$.

Angle between two lines — vector form

$$\cos \theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

where \vec{b}_1, \vec{b}_2 are direction vectors of the two lines.

The **absolute value** picks the acute angle. Skew or intersecting lines: the formula uses only the directions, not the anchor points.

Angle between two lines — DR form

$$\cos \theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

where $(a_1, b_1, c_1), (a_2, b_2, c_2)$ are DRs of the two lines.

Just the Cartesian expansion of $\vec{b}_1 \cdot \vec{b}_2 / (|\vec{b}_1| |\vec{b}_2|)$.

Angle between two lines — DC form

$$\cos \theta = |l_1l_2 + m_1m_2 + n_1n_2|$$

$$\sin \theta = \sqrt{(l_1m_2 - l_2m_1)^2 + (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2}$$

Denominators vanish because $l^2 + m^2 + n^2 = 1$. The **sin θ** form uses the cross-product components of the two unit direction vectors.

Perpendicular and parallel lines

Perpendicular ($\theta = 90^\circ$):

$$a_1a_2 + b_1b_2 + c_1c_2 = 0 \quad \text{or} \quad l_1l_2 + m_1m_2 + n_1n_2 = 0$$

Parallel ($\theta = 0$):

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Perpendicularity uses the **dot product**; parallelism uses **proportionality** of DRs.

Don't forget the absolute value

$\cos \theta = \vec{b}_1 \cdot \vec{b}_2 / (|\vec{b}_1| |\vec{b}_2|)$ (without bars) gives the angle between the *directed* vectors, which can be obtuse. The angle **between two lines** is always the acute one — always wrap the cosine in $|\cdot|$.

4 Shortest Distance Between Two Lines

Two lines in space may intersect, be parallel, or be skew. The shortest-distance formula handles each case; skew lines give the only nontrivial geometry.

Skew lines

Lines that are **neither parallel nor intersecting** are called skew. They lie in different planes. The shortest segment joining them is perpendicular to both lines, and its length is the shortest distance.

Shortest distance between skew lines — vector

For $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$:

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right|$$

Projection of the connector $\vec{a}_2 - \vec{a}_1$ onto the unit vector $\hat{n} = (\vec{b}_1 \times \vec{b}_2)/|\vec{b}_1 \times \vec{b}_2|$ perpendicular to both lines.

Shortest distance — Cartesian form

For $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$:

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

Numerator: scalar triple product of the connector and the two direction vectors.
Denominator: $|\vec{b}_1 \times \vec{b}_2|$ in Cartesian form.

Condition for two lines to intersect (coplanar)

$$d = 0 \iff (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0$$

$$\text{Cartesian: } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Vanishing scalar triple product means the three vectors are **coplanar** — so the lines lie in one plane and (being non-parallel) intersect.

Distance between two parallel lines

For $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$ (same \vec{b}):

$$d = \left| \frac{\vec{b} \times (\vec{a}_2 - \vec{a}_1)}{|\vec{b}|} \right|$$

$\vec{b} \times \vec{b}_2$ is zero for parallel lines, so the skew formula breaks. Use the foot-of-perpendicular construction instead: project the connector across the common direction.

How to tell the case before computing

- DRs proportional \Rightarrow **parallel** (use parallel-distance formula; zero iff lines coincide).
- DRs not proportional + scalar triple product = 0 \Rightarrow **intersecting** (shortest distance = 0).
- DRs not proportional + scalar triple product $\neq 0 \Rightarrow$ **skew** (use shortest-distance formula).

Sign and absolute-value handling

The numerator of the skew-distance formula is a scalar (could be negative); always wrap in $|\cdot|$. Don't drop the modulus in the Cartesian determinant version either — distance is non-negative.

JEE/NEET Extension: Foot of perpendicular

To find the foot F of the perpendicular from a point P to the line $\vec{r} = \vec{a} + \lambda\vec{b}$: write $F = \vec{a} + \lambda\vec{b}$, impose $(\vec{F} - \vec{P}) \cdot \vec{b} = 0$, solve for λ . The perpendicular distance is $|\vec{F} - \vec{P}|$, and the **image** of P in the line is $2\vec{F} - \vec{P}$.

JEE/NEET Extension: Distance via cross product

Perpendicular distance from point P (position \vec{p}) to line $\vec{r} = \vec{a} + \lambda\vec{b}$:

$$d = \frac{|\vec{b} \times (\vec{p} - \vec{a})|}{|\vec{b}|}$$

— same structure as the parallel-lines formula, with $\vec{a}_2 - \vec{a}_1$ replaced by $\vec{p} - \vec{a}$.

Quick Reference — Chapter 11

Topic	Key formula
DCs identity	$l^2 + m^2 + n^2 = 1$
DCs from DRs	$l, m, n = \pm \frac{a, b, c}{\sqrt{a^2 + b^2 + c^2}}$
DCs from two points	$\frac{x_2 - x_1}{PQ}, \frac{y_2 - y_1}{PQ}, \frac{z_2 - z_1}{PQ}$ with $PQ = \sqrt{\sum (x_2 - x_1)^2}$
Line through point, $\parallel \vec{b}$ (vector)	$\vec{r} = \vec{a} + \lambda \vec{b}$
Line through point, DRs (Cartesian)	$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
Line through two points (vector)	$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$
Line through two points (Cartesian)	$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$
Angle (vector)	$\cos \theta = \frac{ \vec{b}_1 \cdot \vec{b}_2 }{ \vec{b}_1 \vec{b}_2 }$
Angle (DRs)	$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{\sum a_1^2} \sqrt{\sum a_2^2}}$
Angle (DCs)	$\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2 $
Perpendicular condition	$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$
Parallel condition	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
Skew SD (vector)	$d = \frac{ \vec{b}_1 \times \vec{b}_2 \cdot (\vec{a}_2 - \vec{a}_1) }{ \vec{b}_1 \times \vec{b}_2 }$
Parallel-lines distance	$d = \frac{ \vec{b} \times (\vec{a}_2 - \vec{a}_1) }{ \vec{b} }$
Coplanarity (lines intersect)	$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0$

Note: The 2024–26 NCERT rationalisation removed the Plane sections (normal form, intercept form, point-normal, three-point, line of intersection, angle between line and plane, distance from point to plane) from this chapter. They are not in the current syllabus and so are not covered here.