Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 11: Three Dimensional Geometry

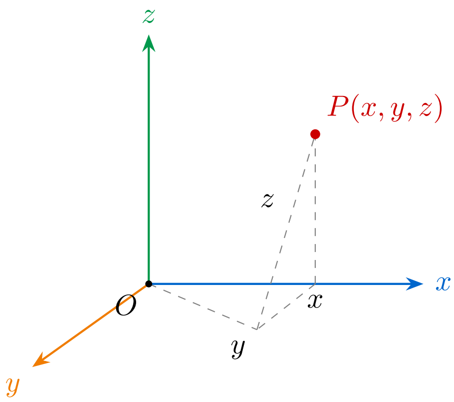
What this chapter covers: the directional language of 3D space — direction cosines and direction ratios, the vector and Cartesian equations of a line, the angle between two lines, and the shortest distance between two skew (or parallel) lines. The rationalised NCERT 2026–27 syllabus keeps only these; we include *equations of a plane*, *angle between two planes*, *angle between a line and a plane*, and *distance of a point from a plane* as JEE/NEET extensions, clearly tagged, because they are heavily tested in entrance exams.

1 The 3D Coordinate System and Direction Cosines

In Class XI we set up a right-handed system of three mutually perpendicular axes (x, y, z) and learnt how to locate a point $P(x, y, z)$ in space. Class 12 turns that static picture into a working language: every line carries an *intrinsic direction* which we encode using cosines of the angles it makes with the three axes. From this single idea we can write the equation of any line, measure the angle between any two lines, and compute the shortest distance between non-intersecting lines — all without drawing a single picture.

1.1 The 3D coordinate frame

Place three mutually perpendicular axes through a chosen origin O. By convention, x runs to the front-right, y to the back-right, and z vertically upward, so that the right-hand rule $\hat{i} \times \hat{j} = \hat{k}$ holds. Any point P in space has unique coordinates (x, y, z) where x, y, z are the signed perpendicular distances from the yz-, zx- and xy-planes respectively.



1.2 Direction angles and direction cosines

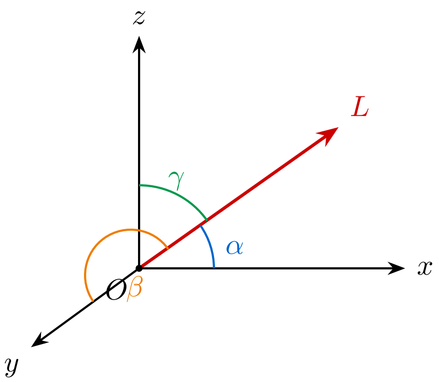
Let L be a directed line through the origin. Let α, β, γ be the angles L makes with the positive x-, y- and z-axes respectively. These are the **direction angles**. Their cosines

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

are the **direction cosines** (d.c.'s) of L.

A line in space can be traversed in two opposite directions, so it actually carries two sets of direction cosines: (l, m, n) and $(-l, -m, -n)$. We get a *unique* set only after fixing a direction of traversal.

If the line does not pass through the origin, draw a parallel through O and use its direction cosines — parallel lines share the same d.c.'s.



Fundamental Identity of Direction Cosines

For any line in space with direction cosines l, m, n:

$$l^2 + m^2 + n^2 = 1, \qquad \text{equivalently,} \qquad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

Equivalently, (l, m, n) is a unit vector along the line.

Why $l^2 + m^2 + n^2 = 1$?

Take a unit-length segment along L starting from O. Its projections on the three axes are exactly l, m, n (by definition of cosine). Since these three projections are the components of a unit vector, their squares sum to 1 by the 3D Pythagoras theorem.

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1.3 Direction ratios

Any triple of numbers (a, b, c) proportional to the direction cosines is called a set of **direction ratios** (d.r.'s) of the line. If (l, m, n) are the d.c.'s and (a, b, c) are d.r.'s, then for some non-zero scalar λ,

$$a = \lambda l, \quad b = \lambda m, \quad c = \lambda n.$$

Direction ratios are not unique — any non-zero scalar multiple of a d.r. triple gives the same line's direction. Every line therefore has *infinitely many* d.r. triples but only *two* d.c. triples.

Converting Direction Ratios to Direction Cosines

If (a, b, c) are direction ratios of a line, the direction cosines are

$$l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

The $\pm$ sign is fixed by the chosen direction of traversal (the same sign for all three).

Direction Cosines vs Direction Ratios at a Glance

Direction Cosines (l, m, n)	**Direction Ratios** (a, b, c)
Cosines of angles with the x, y, z-axes	Any numbers proportional to (l, m, n)
Unique up to sign (two sets per line)	Infinitely many sets per line
Always satisfy $l^2 + m^2 + n^2 = 1$	No size constraint
(l, m, n) is a unit vector along L	(a, b, c) is any vector along L

d.r. to d.c. — "Divide by the length"

To convert direction ratios (a, b, c) to direction cosines, divide each by the magnitude $\sqrt{a^2 + b^2 + c^2}$. It is exactly the unit-vector recipe from vector algebra: *vector divided by its length is a unit vector*.

1.4 Direction cosines of a line joining two points

Given $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the displacement vector $\vec{PQ}$ has components $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. Dividing by the distance $|PQ|$ gives the direction cosines.

Direction Cosines of PQ

$$l = \frac{x_2 - x_1}{|PQ|}, \quad m = \frac{y_2 - y_1}{|PQ|}, \quad n = \frac{z_2 - z_1}{|PQ|}$$

where

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Direction ratios may simply be taken as $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ — no division needed.

Worked Example. Find the direction cosines of the line joining $(-2, 4, -5)$ and $(1, 2, 3)$.

Direction ratios: $(1 - (-2),\ 2 - 4,\ 3 - (-5)) = (3, -2, 8)$.

Magnitude: $\sqrt{9 + 4 + 64} = \sqrt{77}$.

Direction cosines: $\left(\frac{3}{\sqrt{77}}, \frac{-2}{\sqrt{77}}, \frac{8}{\sqrt{77}}\right)$.

Quick Tip

For collinearity of three points A, B, C: compute d.r.'s of $\vec{AB}$ and $\vec{BC}$ and check proportionality. If $\vec{AB} : \vec{BC} =$ constant triple, then A, B, C are collinear. Don't bother computing all three distances and checking $AB + BC = AC$ — it works but is slower.

Common Mistake

A common slip: writing $(l, m, n) = (a, b, c)$ directly when given direction ratios. Direction ratios are *proportional to* d.c.'s, not equal. Always divide by $\sqrt{a^2 + b^2 + c^2}$.

1.5 Direction cosines of the coordinate axes

The x-axis makes angles $0°$, $90°$, $90°$ with x, y, z-axes, so its d.c.'s are $(1, 0, 0)$. Similarly the y- and z-axes have d.c.'s $(0, 1, 0)$ and $(0, 0, 1)$. These three triples are exactly the unit basis vectors $\hat{i}$, $\hat{j}$, $\hat{k}$.

Real-World Application

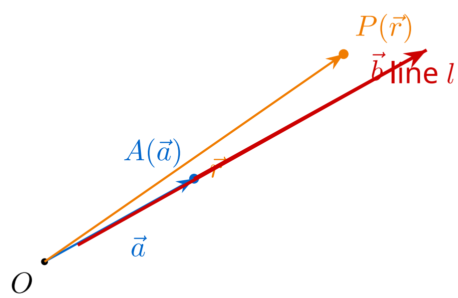
Direction cosines are how a flight controller specifies the heading of an aircraft relative to a fixed runway frame: the unit vector along the aircraft's nose, broken into $(\cos\alpha, \cos\beta, \cos\gamma)$ along the three runway axes. Pilots actually call them "direction cosines" — the maths textbook name and the cockpit name are identical.

2 Equation of a Line in Space

A line in 3D is determined by either (i) a point on it together with a direction, or (ii) two points on it. We write the line's equation in two equivalent forms: the *vector form* (compact, coordinate-free) and the *Cartesian form* (component-by-component, ready for substitution).

2.1 Line through a given point with given direction — Vector form

Let A be a fixed point with position vector $\vec{a}$, and let $\vec{b}$ be a vector parallel to the desired line. For any point P on the line, $\vec{AP}$ is a scalar multiple of $\vec{b}$, say $\vec{AP} = \lambda\vec{b}$ where $\lambda \in \mathbb{R}$. Writing $\vec{AP} = \vec{r} - \vec{a}$ (with $\vec{r}$ the position vector of P) gives the standard vector equation.



Vector Equation of a Line

$$\vec{r} = \vec{a} + \lambda\vec{b}, \qquad \lambda \in \mathbb{R}$$

where $\vec{a}$ = position vector of a point on the line, $\vec{b}$ = any vector parallel to the line, λ = real parameter that sweeps out the whole line.

Every value of λ produces one point of the line; conversely every point of the line corresponds to a unique λ.

2.2 Cartesian form

Write $\vec{a} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $\vec{b} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Substituting into the vector equation and matching coefficients of $\hat{i}, \hat{j}, \hat{k}$:

$$x = x_1 + \lambda a, \quad y = y_1 + \lambda b, \quad z = z_1 + \lambda c.$$

These are the *parametric equations*. Eliminating λ gives the symmetric Cartesian form.

Cartesian Equation of a Line (Symmetric Form)

A line through (x_1, y_1, z_1) with direction ratios (a, b, c) is

$$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}}$$

If we use direction cosines (l, m, n) instead, replace a, b, c by l, m, n.

Vector Form vs Cartesian Form — Same Line, Two Languages

Vector form	Cartesian form
$\vec{r} = \vec{a} + \lambda\vec{b}$	$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
Compact, coordinate-free	Three explicit equations in x, y, z
Best for dot/cross products, angles, distances	Best for substitution, finding intersection points
$\vec{a}$ identifies a point, $\vec{b}$ gives direction	(x_1, y_1, z_1) identifies a point, (a, b, c) gives direction
Parameter $\lambda \in \mathbb{R}$ generates all points	Parameter eliminated; relation between x, y, z

Worked Example. Vector and Cartesian equation of the line through $(5, 2, -4)$ parallel to $3\hat{i} + 2\hat{j} - 8\hat{k}$.

Vector form: $\vec{r} = (5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$.

Cartesian form: $\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z + 4}{-8}$.

2.3 Line through two given points

If the line passes through $A(\vec{a})$ and $B(\vec{b})$, take the direction vector as $\vec{b} - \vec{a}$. Then:

Line Through Two Points

Vector form: $\vec{r} = \vec{a} + \lambda\,(\vec{b} - \vec{a})$.
Cartesian form:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

where (x_1, y_1, z_1), (x_2, y_2, z_2) are the two given points.

Quick Tip

When converting symmetric Cartesian form to vector form, read off (x_1, y_1, z_1) as the position vector and the denominators as the direction vector. Example: $\dfrac{x-5}{3} = \dfrac{y+4}{7} = \dfrac{z-6}{2}$ becomes $\vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$.

Common Mistake

If a denominator in symmetric form is zero, say $\dfrac{x-1}{2} = \dfrac{y-3}{0} = \dfrac{z+1}{5}$, do **not** divide by zero. Interpret it as $y - 3 = 0$ (i.e., $y = 3$) combined with $\frac{x-1}{2} = \frac{z+1}{5}$. The line lies in the plane $y = 3$.

3 Angle Between Two Lines

The angle between two lines is, by convention, the *acute* angle between them. Because direction is determined only up to sign, we always wrap the dot product in an absolute value so that the cosine is non-negative.

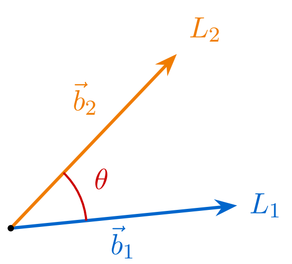
3.1 Vector form

Given lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, the angle θ between them depends only on $\vec{b}_1, \vec{b}_2$:

Angle Between Two Lines (Vector Form)

$$\boxed{\cos\theta = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1|\,|\vec{b}_2|}}, \qquad 0 \le \theta \le \tfrac{\pi}{2}.$$

The absolute value forces θ to be acute regardless of the sign of $\vec{b}_1 \cdot \vec{b}_2$.



3.2 Cartesian form

Given two lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2):

Angle Between Two Lines (Cartesian Form)

$$\cos\theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) are used, the denominator collapses to 1 and

$$\cos\theta = |\,l_1l_2 + m_1m_2 + n_1n_2\,|.$$

3.3 Conditions for parallel and perpendicular lines

Special Cases

Perpendicular: $\vec{b}_1 \cdot \vec{b}_2 = 0$, i.e., $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Parallel: $\vec{b}_1 = k\,\vec{b}_2$ for some k, i.e., $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

Worked Example. Find the angle between $\vec{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\vec{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$.

$\vec{b}_1 \cdot \vec{b}_2 = 3 + 4 + 12 = 19$; $|\vec{b}_1| = 3$, $|\vec{b}_2| = 7$. So $\cos\theta = \frac{19}{21}$ and $\theta = \cos^{-1}(19/21)$.

Quick Tip

Spotting perpendicular lines: just compute the dot product of the direction vectors. If it's zero, $\theta = 90^\circ$ — no need to compute magnitudes. For parallel lines, check if d.r.'s are proportional.

Common Mistake

Forgetting the absolute value in the angle formula can give an obtuse θ. Angles between *lines* (not rays) are always taken in $[0, \pi/2]$. If you compute $\cos\theta = -0.5$, take the magnitude and report $\theta = 60^\circ$, not 120°.

Why we use absolute value

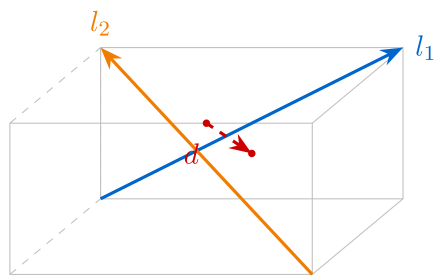
A line has no preferred direction — reversing $\vec{b}$ to $-\vec{b}$ flips the sign of the dot product. To get a well-defined acute angle between the two *lines* (not the two directed segments), we take $|\vec{b}_1 \cdot \vec{b}_2|$.

4 Shortest Distance Between Two Lines

Two lines in 3D can be *intersecting* (shortest distance = 0), *parallel* (shortest distance = constant perpendicular distance), or *skew* (neither intersecting nor parallel — they live in different planes and never meet). For skew lines the segment of shortest distance is uniquely defined: it is perpendicular to both lines.

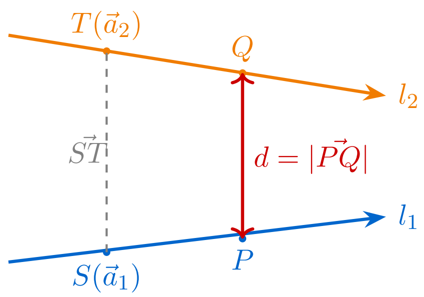
4.1 Skew lines

Skew lines are pairs of lines in 3D that do not lie in any common plane. The classic example: in a rectangular room, the diagonal of the ceiling and the diagonal of an adjacent wall (chosen carefully) do not meet and are not parallel.



4.2 Shortest distance between two skew lines — Vector form

Take points S on l_1 and T on l_2 with position vectors $\vec{a}_1$ and $\vec{a}_2$. The shortest-distance segment $\vec{PQ}$ is perpendicular to both direction vectors, so it lies along $\vec{b}_1 \times \vec{b}_2$. The magnitude of $\vec{PQ}$ is the projection of $\vec{ST} = \vec{a}_2 - \vec{a}_1$ onto the unit vector $\hat{n} = (\vec{b}_1 \times \vec{b}_2)/|\vec{b}_1 \times \vec{b}_2|$.



Shortest Distance: Skew Lines (Vector Form)

For $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ (skew):

$$d = \frac{\left|\,(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)\,\right|}{|\vec{b}_1 \times \vec{b}_2|}$$

If $\vec{b}_1 \times \vec{b}_2 = \vec{0}$ the lines are parallel; use the parallel formula in §4.4.

Geometric reading

The numerator is the absolute value of the *scalar triple product* $[\vec{b}_1\ \vec{b}_2\ (\vec{a}_2 - \vec{a}_1)]$ — six times the volume of the parallelepiped with $\vec{b}_1$, $\vec{b}_2$, $\vec{a}_2 - \vec{a}_1$ as edges. The denominator is the area of the base parallelogram spanned by $\vec{b}_1, \vec{b}_2$. Volume $\div$ base area = perpendicular height = shortest distance.

4.3 Shortest distance — Cartesian form

Given two skew lines

$$l_1 : \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}, \qquad l_2 : \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2},$$

the formula uses a determinant in the numerator (a scalar triple product) and the magnitude of the cross product in the denominator.

Shortest Distance: Skew Lines (Cartesian Form)

$$d = \frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2-b_2c_1)^2+(c_1a_2-c_2a_1)^2+(a_1b_2-a_2b_1)^2}}$$

The denominator is just $|\vec{b}_1 \times \vec{b}_2|$ written out.

Condition for two lines to be coplanar (JEE/NEET Extension)

The two lines intersect (i.e., are coplanar and not parallel) precisely when the scalar triple product

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = 0.$$

In Cartesian form, the determinant in the numerator above must vanish. This is the standard “coplanarity of two lines” test — frequently asked in JEE.

4.4 Distance between two parallel lines

If $\vec{b}_1 \parallel \vec{b}_2$, say $\vec{b}_1 = \vec{b}_2 = \vec{b}$, the cross product $\vec{b}_1 \times \vec{b}_2 = \vec{0}$ and the skew-line formula collapses. Instead, drop a perpendicular from any point on l_2 onto l_1.

Distance Between Parallel Lines

For $\vec{r} = \vec{a}_1 + \lambda\vec{b}$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}$:

$$d = \frac{|\,\vec{b} \times (\vec{a}_2 - \vec{a}_1)\,|}{|\vec{b}|}$$

Worked Example. Shortest distance between $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

$\vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$; $\vec{b}_1 \times \vec{b}_2 = 3\hat{i} - \hat{j} - 7\hat{k}$; $|\vec{b}_1 \times \vec{b}_2| = \sqrt{59}$.

Numerator: $|(3)(1) + (-1)(0) + (-7)(-1)| = |3+7| = 10$. So $d = 10/\sqrt{59}$.

Quick Tip

Always check whether two lines are parallel *before* using the skew formula. If $\vec{b}_1 \times \vec{b}_2 = \vec{0}$, the skew formula divides by zero. Test proportionality of d.r.'s; if proportional, switch to the parallel-lines formula.

Common Mistake

Students often write $(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_1 - \vec{a}_2)$ with the subtraction reversed. It only affects the sign — and we take an absolute value anyway — so don't fret over which order. But *do* fret over absolute value: forgetting it can give a negative "distance", which is meaningless.

Real-World Application

Skew-line distance is exactly how GPS triangulation works in 3D: the receiver's clock-bias problem reduces to finding the closest approach between several lines emanating from satellites. The same maths underpins air-traffic conflict detection: software predicts the minimum approach (skew distance) between two aircraft trajectories and triggers alerts when d falls below a safety threshold.

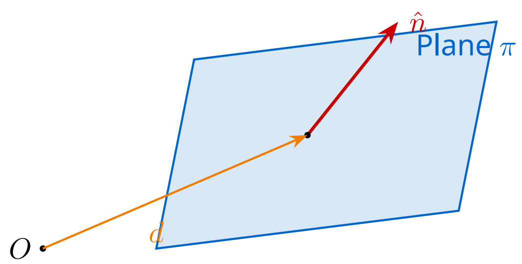
5 Equation of a Plane [JEE/NEET Extension]

The rationalised NCERT 2026–27 syllabus removed planes from Chapter 11, but JEE Main and NEET continue to test all four standard plane topics: equation of a plane, angle between planes, angle between a line and a plane, and distance of a point from a plane. We cover them here at the same level of rigour as the rest of the chapter — mark these sections with the [JEE/NEET extension] tag so you know they are competitive-exam material.

5.1 Plane in normal form

A plane is uniquely determined by a point on it and a vector *normal* (perpendicular) to it. Let $\hat{n}$ be a unit normal to the plane and $d \geq 0$ the perpendicular distance of the plane from the origin. Then for any point P on the plane with position vector $\vec{r}$:

$$\vec{r} \cdot \hat{n} = d.$$



Plane in Normal Form

Vector form: $\vec{r} \cdot \hat{n} = d$ (where $\hat{n}$ is a unit normal).
Cartesian form: $lx + my + nz = d$, where (l, m, n) are the direction cosines of $\hat{n}$.

5.2 Plane through a point with given normal

Let $A(\vec{a})$ lie on the plane and let $\vec{N}$ be any (not necessarily unit) normal vector. For any point $P(\vec{r})$ on the plane, $\vec{AP} \perp \vec{N}$, so:

$$(\vec{r} - \vec{a}) \cdot \vec{N} = 0.$$

Plane Through a Point, Given Normal

Vector form: $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$.
Cartesian form: If $A = (x_1, y_1, z_1)$ and $\vec{N} = A\hat{i} + B\hat{j} + C\hat{k}$, then

$$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0.$$

Expanding: $Ax + By + Cz = D$ where $D = Ax_1 + By_1 + Cz_1$.
The coefficients (A, B, C) *are* the direction ratios of the normal — read them straight off the equation.

5.3 General equation of a plane

Every plane in 3D can be written as

$$Ax + By + Cz + D = 0, \quad (A, B, C) \neq (0, 0, 0).$$

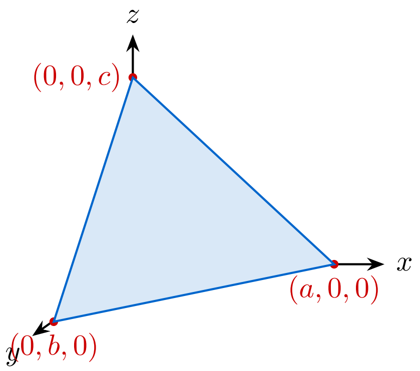
(A, B, C) is a normal vector to the plane. To convert to normal form, divide through by $\pm\sqrt{A^2 + B^2 + C^2}$.

5.4 Intercept form

A plane that cuts the x-, y- and z-axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively has the elegant equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

This is the 3D analogue of the 2D intercept form $x/a + y/b = 1$.



5.5 Plane through three non-collinear points

If A, B, C have position vectors $\vec{a}, \vec{b}, \vec{c}$, the vectors $\vec{b} - \vec{a}$ and $\vec{c} - \vec{a}$ lie in the plane, so a normal is $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$.

Plane Through Three Points

Vector form: $(\vec{r} - \vec{a}) \cdot \left[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right] = 0.$
Cartesian form:

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0.$$

Quick Tip

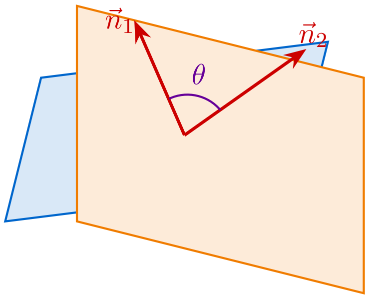
The fastest way to find the equation of a plane through three points by hand: compute one cross product $(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})$ to get the normal $\vec{N}$, then write $\vec{N} \cdot (\vec{r} - \vec{a}) = 0$ and expand. Avoid the determinant unless asked — the cross product is shorter.

6 Angles and Distances Involving Planes [JEE/NEET Extension]

With a plane represented by its normal vector, all questions about angles and distances reduce to dot products between normals and direction vectors. This is why representing a plane by its normal is so powerful.

6.1 Angle between two planes

Two planes intersect in a line; the *dihedral angle* between them equals the angle between their normals (or its supplement). To get the acute angle, take the absolute value of the dot product.



Angle Between Two Planes

Vector form: If $\vec{r}\cdot\vec{n}_1 = d_1$ and $\vec{r}\cdot\vec{n}_2 = d_2$, then

$$\cos\theta = \frac{|\vec{n}_1\cdot\vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}.$$

Cartesian form: Planes $A_1x+B_1y+C_1z+D_1=0$ and $A_2x+B_2y+C_2z+D_2=0$ make an angle

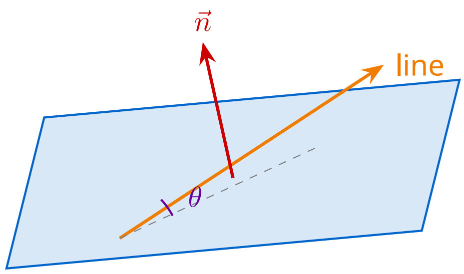
$$\cos\theta = \frac{|A_1A_2+B_1B_2+C_1C_2|}{\sqrt{(A_1^2+B_1^2+C_1^2)\,(A_2^2+B_2^2+C_2^2)}}.$$

Planes are perpendicular iff $\vec{n}_1\cdot\vec{n}_2=0$; parallel iff $\vec{n}_1 \parallel \vec{n}_2$.

6.2 Angle between a line and a plane

If $\vec{b}$ is the direction of the line and $\vec{n}$ the normal of the plane, the angle between the line and the plane is the *complement* of the angle between the line and the normal. So if θ is the angle between the line and the plane:

$$\sin\theta = \cos(\tfrac{\pi}{2}-\theta) = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\,|\vec{n}|}.$$



Angle Between a Line and a Plane

Vector form: For line $\vec{r}=\vec{a}+\lambda\vec{b}$ and plane $\vec{r}\cdot\vec{n}=d$:

$$\boxed{\sin\theta = \frac{|\vec{b}\cdot\vec{n}|}{|\vec{b}|\,|\vec{n}|}}$$

Cartesian form: For a line with d.r.'s (a, b, c) and a plane $Ax+By+Cz+D=0$:

$$\sin\theta = \frac{|Aa+Bb+Cc|}{\sqrt{A^2+B^2+C^2}\,\sqrt{a^2+b^2+c^2}}.$$

Common Mistake

A persistent error: writing $\cos\theta$ instead of $\sin\theta$ for the angle between a line and a plane. The dot product $\vec{b}\cdot\vec{n}$ measures the angle between the line and the *normal* — the angle between the line and the plane is the complement, so sin replaces cos.

Line parallel or perpendicular to a plane

Line parallel to plane: $\vec{b}\cdot\vec{n}=0$ (line is perpendicular to the normal).
Line perpendicular to plane: $\vec{b}\parallel\vec{n}$, i.e., $\vec{b}=k\vec{n}$ for some scalar k.
These conditions feed many JEE problems: "find λ such that the line is parallel to the plane."

6.3 Distance of a point from a plane

The signed distance is the projection of $(\vec{p}-\vec{a})$ onto the unit normal, where $\vec{a}$ is any point on the plane. In Cartesian form, plug the point into the plane equation and normalise.

Distance of a Point from a Plane

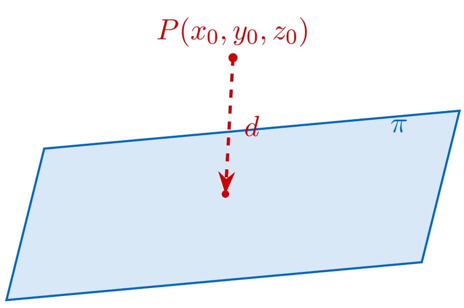
Vector form: Distance from $P(\vec{p})$ to the plane $\vec{r}\cdot\hat{n}=d$ is

$$\text{distance} = |\vec{p}\cdot\hat{n}-d|.$$

Cartesian form: Distance from $P(x_0, y_0, z_0)$ to $Ax+By+Cz+D=0$:

$$\boxed{\text{distance} = \frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}}$$

The sign of $Ax_0+By_0+Cz_0+D$ tells you which side of the plane P lies on.



Worked Example. Distance from $(1, 2, -1)$ to the plane $2x - 3y + 6z + 5 = 0$:

$$d = \frac{|2(1) - 3(2) + 6(-1) + 5|}{\sqrt{4 + 9 + 36}} = \frac{|2 - 6 - 6 + 5|}{\sqrt{49}} = \frac{5}{7}.$$

Quick Tip

For the distance from the *origin* to a plane $Ax + By + Cz + D = 0$, just compute $|D|/\sqrt{A^2 + B^2 + C^2}$. No point substitution needed — $x_0 = y_0 = z_0 = 0$ removes the first three terms.

Real-World Application

Computer graphics engines clip 3D scenes against the six planes of the camera's view frustum. For each vertex, they compute $Ax + By + Cz + D$ to see which side of each plane it lies on; if negative on all six, the vertex is inside and gets drawn. Speed of rendering depends literally on how fast this dot product runs.

7 Quick Reference Summary

A single-page summary of every formula in this chapter, organised for last-minute revision. Read it once aloud before walking into the exam.

Direction Cosines & Direction Ratios

- $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$ with $l^2 + m^2 + n^2 = 1$.
- d.r.'s (a, b, c) are any non-zero scalar multiple of (l, m, n).
- d.c.'s from d.r.'s: $l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}$, similarly m, n.
- d.r.'s of line PQ: $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Line in Space

- Vector form: $\vec{r} = \vec{a} + \lambda\vec{b}$.
- Cartesian form: $\dfrac{x - x_1}{a} = \dfrac{y - y_1}{b} = \dfrac{z - z_1}{c}$.
- Through two points: $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$.

Angles & Distances Between Lines

- Angle: $\cos\theta = \dfrac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1|\,|\vec{b}_2|}$.
- Perpendicular: $a_1a_2 + b_1b_2 + c_1c_2 = 0$. Parallel: $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.
- Skew distance: $d = \dfrac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|}$.

- Parallel distance: $d = \dfrac{|\vec{b} \times (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}|}$.

Plane (JEE/NEET Extension)

- Normal form: $\vec{r} \cdot \hat{n} = d$, or $lx + my + nz = d$.
- Point + normal: $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$, i.e., $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$.
- General: $Ax + By + Cz + D = 0$; normal d.r.'s are (A, B, C).
- Intercept: $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$.

Angles & Distances with Planes (JEE/NEET Extension)

- Angle between two planes: $\cos\theta = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1|\,|\vec{n}_2|}$.
- Angle between line and plane: $\sin\theta = \dfrac{|\vec{b} \cdot \vec{n}|}{|\vec{b}|\,|\vec{n}|}$.
- Distance from (x_0, y_0, z_0) to $Ax + By + Cz + D = 0$: $\dfrac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

7.1 Master Comparison Table

Quantity	Vector form	Cartesian form
Line through $\vec{a}$ along $\vec{b}$	$\vec{r} = \vec{a} + \lambda\vec{b}$	$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
Angle between two lines	$\cos\theta = \frac{\vert\vec{b}_1 \cdot \vec{b}_2\vert}{\vert\vec{b}_1\vert\,\vert\vec{b}_2\vert}$	$\cos\theta = \frac{\vert\sum a_i a_i'\vert}{\sqrt{(\sum a_i^2)(\sum a_i'^2)}}$
Shortest skew distance	$\frac{\vert(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)\vert}{\vert\vec{b}_1 \times \vec{b}_2\vert}$	Determinant $\div \vert\vec{b}_1 \times \vec{b}_2\vert$
Parallel-line distance	$\frac{\vert\vec{b} \times (\vec{a}_2 - \vec{a}_1)\vert}{\vert\vec{b}\vert}$	$\frac{\vert\vec{b} \times (\vec{a}_2 - \vec{a}_1)\vert}{\vert\vec{b}\vert}$
Plane (normal form)	$\vec{r} \cdot \hat{n} = d$	$lx + my + nz = d$
Angle between planes	$\cos\theta = \frac{\vert\vec{n}_1 \cdot \vec{n}_2\vert}{\vert\vec{n}_1\vert\,\vert\vec{n}_2\vert}$	$\cos\theta = \frac{\vert\sum A_i A_i'\vert}{\sqrt{(\sum A_i^2)(\sum A_i'^2)}}$
Line $\angle$ plane	$\sin\theta = \frac{\vert\vec{b} \cdot \vec{n}\vert}{\vert\vec{b}\vert\,\vert\vec{n}\vert}$	$\sin\theta = \frac{\vert Aa + Bb + Cc\vert}{\sqrt{(A^2+B^2+C^2)(a^2+b^2+c^2)}}$
Point–plane distance	$\vert\vec{p} \cdot \hat{n} - d\vert$	$\frac{\vert Ax_0 + By_0 + Cz_0 + D\vert}{\sqrt{A^2+B^2+C^2}}$

7.2 Strategy for solving 3D-geometry problems

1. **Read the data; pick a representation.** Coordinates of points $\rightarrow$ Cartesian first; position vectors with $\hat{i}, \hat{j}, \hat{k} \rightarrow$ vector first. You can switch any time.
2. **Identify what is asked.** Angle? Distance? Equation of a line/plane? Each has a single formula — pick it.
3. **Set up the building blocks.** Direction vector for a line, normal vector for a plane, position vector of a point. Almost every formula uses these three objects.

4. **Compute dot or cross products.** Most quantities reduce to one of: $\vec{u}\cdot\vec{v}$ (angle, distance), $\vec{u}\times\vec{v}$ (normal, area, perpendicular direction), $[\vec{u}\ \vec{v}\ \vec{w}]$ (volume, skew distance, coplanarity).
5. **Sanity-check.** Lengths positive, $\cos\theta \in [0,1]$ (acute), units consistent. A negative "distance" is a sign error.

7.3 Final word

Three dimensional geometry rewards a tidy mental model. Keep three pictures in mind:

- A **line** = (point) + (direction vector $\vec{b}$).
- A **plane** = (point) + (normal vector $\vec{n}$).
- The **angle/distance formulas** are nothing but dot/cross products of these $\vec{b}$'s and $\vec{n}$'s.

Once you can read off the direction vector of a line and the normal vector of a plane on autopilot, every numerical question in the chapter is a one- or two-step vector calculation. The eight formulas in the Quick Reference Summary, used in combination, cover every problem you will see in the boards and in JEE/NEET.

Best of luck with your preparation!

Visualise the geometry, trust the vector algebra.