Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 13: Probability

What this chapter covers: conditional probability, the multiplication theorem, independent events, the theorem of total probability, and Bayes' theorem. The rationalised NCERT keeps only these. We include *random variables, probability distributions, mean and variance, Bernoulli trials, and the binomial distribution* as JEE/NEET extensions — they are heavily tested in entrance exams.

1 Conditional Probability

In Class 11 we computed $P(E)$ for events in a uniform sample space. Class 12 asks a sharper question: *if we already know that event F has occurred, what is the probability of event E?* The information that F has happened shrinks the relevant sample space from S down to F, and probabilities have to be re-scaled accordingly.

This single idea — updating probability in the light of new information — powers the entire chapter and most of modern statistics.

1.1 Motivating example: three coin tosses

Toss three fair coins. The sample space is

$$S = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}, \quad |S| = 8,$$

with each outcome equally likely (probability $1/8$).

Let E: "at least two heads appear" and F: "the first coin shows tail".

$$\begin{aligned} E &= \{\text{HHH, HHT, HTH, THH}\}, \quad P(E) = 4/8 = 1/2, \\ F &= \{\text{THH, THT, TTH, TTT}\}, \quad P(F) = 4/8 = 1/2, \\ E \cap F &= \{\text{THH}\}, \quad P(E \cap F) = 1/8. \end{aligned}$$

Given that F has occurred, our universe of outcomes shrinks to the four equally likely outcomes of F. Of those, only THH lies in E. So

$$P(E \mid F) = \frac{1}{4} = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{1/2}.$$

1.2 Formal definition

Conditional Probability

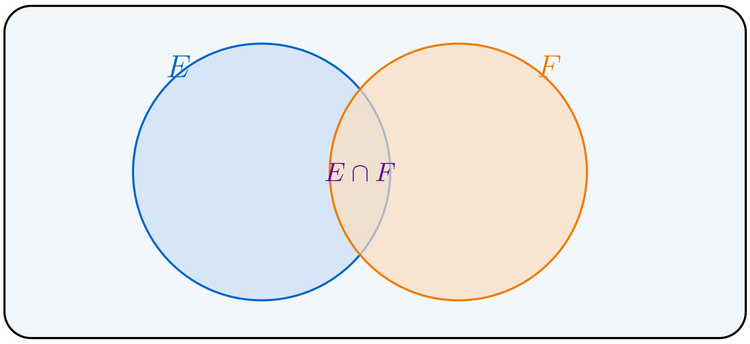
For two events E, F of the same sample space with $P(F) \neq 0$,

$$\boxed{P(E \mid F) = \frac{P(E \cap F)}{P(F)}}$$

Read "probability of E given F". The condition $P(F) > 0$ is essential — conditioning on an impossible event is undefined.

Conditioning on F is exactly equivalent to redefining the sample space as F and computing fresh probabilities inside it. The formula is just the rescaling factor: divide by $P(F)$ to renormalise.

Sample space S



Conditioning on F keeps only the orange region as the new universe.

The intuition in one line

$P(E \mid F)$ asks: *within the F-world, how big a slice does E take up?* The numerator $P(E \cap F)$ is that slice; the denominator $P(F)$ is the size of the F-world.

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We can also write conditional probability using counting (when outcomes are equally likely):

$$P(E \mid F) = \frac{n(E \cap F)}{n(F)}.$$

$P(E \mid F)$ vs $P(F \mid E)$

These are different numbers in general. The conditioning event always goes *after* the vertical bar and *always* divides:

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)} \neq P(F \mid E) = \frac{P(E \cap F)}{P(E)}.$$

A common slip: "probability of disease given positive test" (which we want) is *not* "probability of positive test given disease" (which is easier to find). Bayes' theorem, later in this chapter, exists precisely to convert one into the other.

1.3 Properties of conditional probability

Let E, F, A, B be events in S with $P(F) \neq 0$.

Property 1. $P(S \mid F) = P(F \mid F) = 1$.

Proof. $P(S \mid F) = P(S \cap F)/P(F) = P(F)/P(F) = 1$, and similarly $P(F \mid F) = P(F)/P(F) = 1$.

Property 2. For any events A, B,

$$P\big((A \cup B) \mid F\big) = P(A \mid F) + P(B \mid F) - P\big((A \cap B) \mid F\big).$$

In particular, if A and B are disjoint, $P((A \cup B) \mid F) = P(A \mid F) + P(B \mid F)$.

Proof sketch. Apply distributivity inside F and the inclusion-exclusion identity, then divide each piece by $P(F)$.

Property 3. $P(E' \mid F) = 1 - P(E \mid F)$.

Proof. From Property 1 with $S = E \cup E'$ (disjoint): $1 = P(E \mid F) + P(E' \mid F)$.

Conditional probability obeys all the usual rules

Once F is fixed, $P(\cdot \mid F)$ behaves like an ordinary probability function on S: addition, complement, and inclusion-exclusion all work as before. Only the *number* on the right of the bar is special.

1.4 Worked examples

Example 1. Family with two children. Sample space $\{(b,b),(b,g),(g,b),(g,g)\}$, equally likely. Let E: both are boys; F: at least one is a boy. Then $F =$

$\{(b,b),(b,g),(g,b)\}$, $P(F) = 3/4$, and $E \cap F = \{(b,b)\}$, $P(E \cap F) = 1/4$. So

$$P(E \mid F) = \frac{1/4}{3/4} = \frac{1}{3}.$$

Notice this is *not* $1/2$, which is a common wrong answer. The information "at least one boy" eliminates only (g,g), leaving three equally likely cases of which one is (b,b).

Example 2. A card is drawn from 10 cards numbered 1–10. Given that the number is greater than 3, find the probability that it is even.

Let A: number is even $= \{2,4,6,8,10\}$, and B: number $> 3 = \{4,5,6,7,8,9,10\}$. Then $A \cap B = \{4,6,8,10\}$ and

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{4/10}{7/10} = \frac{4}{7}.$$

Example 3 (school). 1000 students, 430 girls. 10% of the girls study in Class XII. Find the probability that a randomly chosen student studies in Class XII given that the student is a girl.

Let E: studies Class XII; F: is a girl. Then $P(F) = 430/1000 = 0.43$ and $P(E \cap F) = 43/1000 = 0.043$. So

$$P(E \mid F) = \frac{0.043}{0.43} = 0.1,$$

which is just the 10% rate, as expected.

2 Multiplication Theorem on Probability

Re-arranging the definition of conditional probability gives the **multiplication theorem**, the workhorse for computing the probability of two events both happening.

2.1 The theorem

Multiplication Theorem (Two Events)

For any events E, F with $P(E), P(F) > 0$,

$$\boxed{P(E \cap F) = P(E)\,P(F \mid E) = P(F)\,P(E \mid F).}$$

The two forms are interchangeable; pick whichever conditional you can compute more easily.

For three events the formula extends:

$$P(E \cap F \cap G) = P(E)\,P(F \mid E)\,P\big(G \mid E \cap F\big).$$

And in general, for n events $A_1, A_2, \ldots, A_n$,

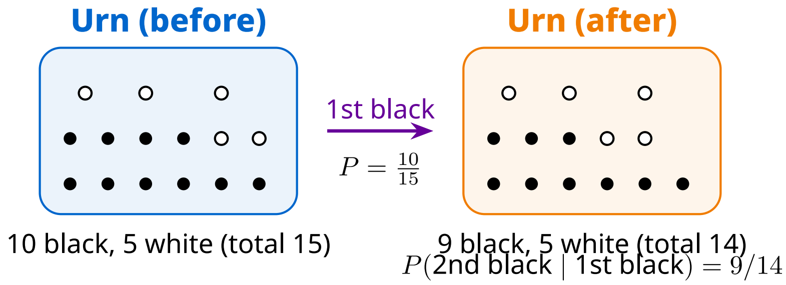
$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)\,P(A_2 \mid A_1)\,P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap \cdots \cap A_{n-1}).$$

2.2 Classic urn problem

Example. An urn contains 10 black and 5 white balls. Two balls are drawn one after the other *without replacement*. What is the probability that both are black?

Let E: first ball black; F: second ball black. Then $P(E) = 10/15$. After one black ball is removed, 9 black and 5 white remain (14 total), so $P(F \mid E) = 9/14$. By the multiplication theorem,

$$P(E \cap F) = \frac{10}{15} \times \frac{9}{14} = \frac{90}{210} = \frac{3}{7}.$$



Example (three cards). Three cards are drawn successively without replacement from a pack of 52. Find $P(\text{first two kings and third an ace})$. Letting K = king, A = ace,

$$P(KKA) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{4}{50} = \frac{2}{5525}.$$

"Without replacement" = update after each draw

Every time a ball/card is removed, the next conditional probability uses the *reduced* pool. Always update both numerator and denominator. With *replacement*, the pool is unchanged and the draws become independent (next section).

Where multiplication theorem appears

Quality-control sampling without replacement, drug-trial dropout chains, and reliability of a series system ("the whole works only if every component works") all reduce to repeated applications of the multiplication theorem.

3 Independent Events

A subtle but crucial distinction: "F has occurred" may or may not change the probability of E. When it does *not*, the events are **independent**.

3.1 Definition

Independent Events

Two events E and F (with $P(E), P(F) > 0$) are **independent** if any one of the following equivalent conditions holds:

$$P(E \mid F) = P(E),$$
$$P(F \mid E) = P(F),$$
$$P(E \cap F) = P(E) \cdot P(F).$$

The third form (the *product rule*) is the cleanest test and works even when $P(E)$ or $P(F)$ is zero.

Intuition: knowing one event has occurred gives no information about the other.

Canonical example. Draw a single card from a 52-card deck. Let E: "a spade" and F: "an ace". Then $P(E) = 13/52 = 1/4$, $P(F) = 4/52 = 1/13$, and $E \cap F$ is just the ace of spades, $P(E \cap F) = 1/52$. Check: $1/4 \times 1/13 = 1/52$. ✓ So E and F are independent.

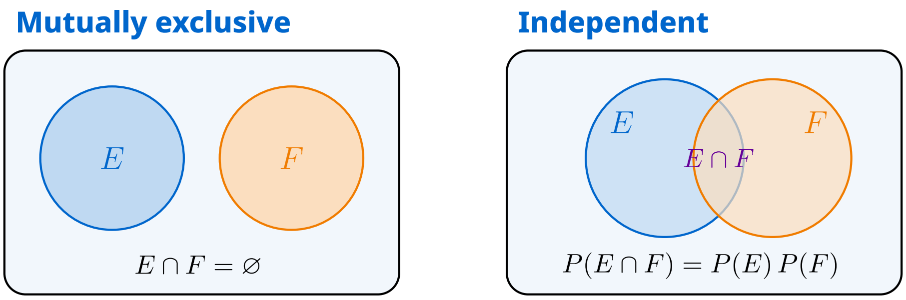
3.2 Independent vs Mutually Exclusive

These two terms are routinely confused but mean very different things.

Mutually Exclusive (Disjoint)	Independent
Defined in terms of *outcomes*: $E \cap F = \varnothing$.	Defined in terms of *probabilities*: $P(E \cap F) = P(E)\,P(F)$.
The two events cannot both occur.	Occurrence of one tells you nothing about the other.
$P(E \cup F) = P(E) + P(F)$.	$P(E \cap F) = P(E)\,P(F)$.
If $P(E), P(F) > 0$, they are *never* independent.	If $P(E), P(F) > 0$ and independent, they are *never* mutually exclusive.
Example: rolling a die, E = "≤ 2" and F = "≥ 5".	Example: tossing two coins, E = "H on first" and F = "H on second".

Disjoint $\neq$ Independent

A frequent exam trap. If $P(E), P(F) > 0$ and $E \cap F = \varnothing$, then $P(E \cap F) = 0 \neq P(E)P(F)$, so the events are **dependent**. Disjoint events are about as far from independent as you can get — knowing one occurred tells you the other definitely did not.



3.3 Three or more independent events

Three events A, B, C are **mutually independent** if *all four* of the following hold:

$$\begin{aligned} P(A \cap B) &= P(A)P(B), \\ P(B \cap C) &= P(B)P(C), \\ P(A \cap C) &= P(A)P(C), \\ P(A \cap B \cap C) &= P(A)P(B)P(C). \end{aligned}$$

Pairwise independence alone (the first three) does *not* imply mutual independence; the joint product rule is an extra requirement.

Useful consequence. If E and F are independent, then so are $\{E, F'\}$, $\{E', F\}$, and $\{E', F'\}$.

Proof for E and F'. Using $E = (E \cap F) \cup (E \cap F')$ (disjoint),

$$P(E \cap F') = P(E) - P(E \cap F) = P(E) - P(E)P(F) = P(E)(1 - P(F)) = P(E)P(F').$$

Independence under complementation

If $E \perp F$, then any one of them (or both) can be replaced by its complement and independence is preserved. This is enormously useful when computing "at least one" probabilities: $P(\text{at least one of } A, B) = 1 - P(A')P(B')$.

Example. A problem is given to A and B independently, with $P(A \text{ solves}) = 1/2$ and $P(B \text{ solves}) = 1/3$. Find $P(\text{problem solved})$.

$$P(\text{solved}) = 1 - P(A' \cap B') = 1 - \tfrac{1}{2} \cdot \tfrac{2}{3} = 1 - \tfrac{1}{3} = \tfrac{2}{3}.$$

"Independent multiplies", "Disjoint adds"

Disjoint $\to$ add (probabilities of unions). Independent $\to$ multiply (probabilities of intersections). The two cases use opposite operations and apply in opposite circumstances.

4 Total Probability Theorem

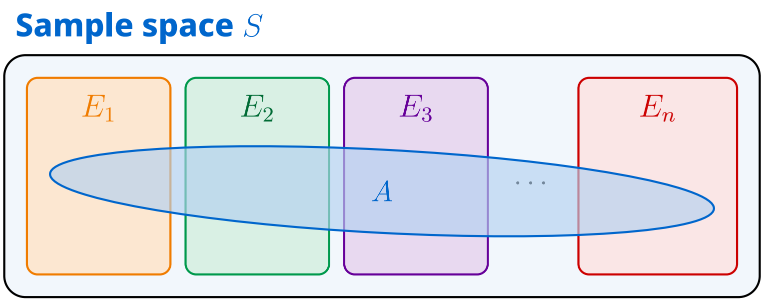
We often know an event A can occur via several mutually exclusive "routes" — the bag from which a ball came, the machine that produced an item, the source of a customer. The **total probability theorem** stitches these route-wise probabilities into the overall $P(A)$.

4.1 Partition of a sample space

A collection of events $E_1, E_2, \dots, E_n$ is a **partition** of S if:

1. $E_i \cap E_j = \varnothing$ for $i \neq j$ (pairwise disjoint),
2. $E_1 \cup E_2 \cup \cdots \cup E_n = S$ (exhaustive),
3. $P(E_i) > 0$ for every i.

A simple two-event partition is $\{E, E'\}$. More generally we may have $\{E_1, \dots, E_n\}$ representing n mutually exclusive "causes" or "cases".



The event A intersects each block E_i; the pieces $A \cap E_i$ partition A.

4.2 Statement and proof

Theorem of Total Probability

Let $\{E_1, E_2, \dots, E_n\}$ be a partition of S with $P(E_i) > 0$. For any event A,

$$\boxed{P(A) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i).}$$

Proof. Since $S = E_1 \cup \cdots \cup E_n$,

$$A = A \cap S = A \cap (E_1 \cup \cdots \cup E_n) = (A \cap E_1) \cup \cdots \cup (A \cap E_n).$$

The pieces $A \cap E_i$ are pairwise disjoint (because the E_i are), so

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i),$$

using the multiplication theorem on each term. ■

4.3 Worked example: construction job

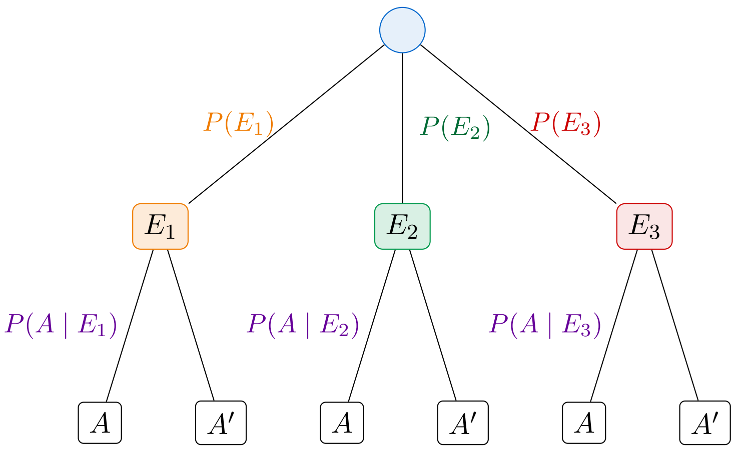
A construction project: $P(\text{strike}) = 0.65$. Given a strike, $P(\text{on time}) = 0.32$. Given no strike, $P(\text{on time}) = 0.80$. Find the (unconditional) probability of completion on time.

Let A: project completes on time; B: strike occurs. Then $\{B, B'\}$ partitions S, with $P(B) = 0.65, P(B') = 0.35$. By total probability,

$$P(A) = P(B)P(A \mid B)+P(B')P(A \mid B') = 0.65\times0.32+0.35\times0.80 = 0.208+0.28 = 0.488.$$

4.4 Probability tree for total probability

The cleanest way to set up these problems is a **probability tree**: one branch per partition cell, then a sub-branch per outcome.



$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)$$

To find $P(A)$: multiply along each branch that ends in A, then add. To find $P(A')$: multiply along each branch that ends in A', then add.

Tree-diagram bookkeeping

Label every branch with its (conditional) probability. The probabilities on branches *coming out of the same node* must sum to 1. The probability of a leaf is the product of branch probabilities from the root.

5 Bayes' Theorem

Total probability goes *forward* (causes $\to$ effect): given the cause-probabilities $P(E_i)$ and the cause-given-effect probabilities $P(A \mid E_i)$, it computes $P(A)$.

Bayes' theorem goes *backward* (effect $\to$ cause): given that A has occurred, what is the probability it came from a particular cause E_i? This "reverse probability" question is everywhere in modern inference — medical diagnosis, spam filters, signal detection, machine learning.

5.1 Statement and proof

Bayes' Theorem

Let $\{E_1, E_2, \dots, E_n\}$ be a partition of S with each $P(E_i) > 0$, and let A be any event with $P(A) > 0$. Then for each i,

$$\boxed{P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{\sum_{j=1}^{n} P(E_j)\,P(A \mid E_j)}.}$$

Proof. By definition,

$$P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\,P(A \mid E_i)}{P(A)},$$

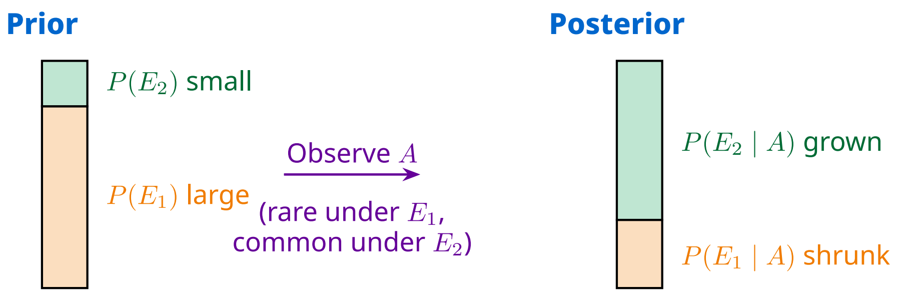
where the numerator uses the multiplication theorem and the denominator is total probability. ■

Terminology.

- $P(E_i)$ — **prior probability** (before A is observed).
- $P(A \mid E_i)$ — **likelihood** of the data A under cause E_i.
- $P(E_i \mid A)$ — **posterior probability** (updated after observing A).

The events E_i are called *hypotheses* or *causes*; Bayes' theorem updates our belief in each cause in light of the observed effect.

5.2 Visualising Bayes' theorem



Bayes' theorem is a *rule for updating belief* after evidence arrives. Causes whose likelihood of producing the observed evidence is high get *boosted*; causes whose likelihood is low get *discounted*.

5.3 Worked example 1: two bags

Bag I: 3 red, 4 black. Bag II: 5 red, 6 black. A bag is chosen at random and a ball drawn; it is red. Find the probability the bag was Bag II.

Let E_1, E_2 = bags chosen, A = red ball drawn. Priors: $P(E_1) = P(E_2) = 1/2$. Likelihoods: $P(A \mid E_1) = 3/7$ and $P(A \mid E_2) = 5/11$.

$$P(E_2 \mid A) = \frac{P(E_2)P(A \mid E_2)}{P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)} = \frac{\frac{1}{2} \cdot \frac{5}{11}}{\frac{1}{2} \cdot \frac{3}{7} + \frac{1}{2} \cdot \frac{5}{11}} = \frac{35}{68}.$$

5.4 Worked example 2: HIV testing (Bayes in action)

An HIV test detects the disease 90% of the time when present, but also gives 1% false-positive results in healthy people. In a population where 0.1% actually have HIV, what is the probability that a randomly chosen person who tests positive truly has HIV?

Let E: has HIV; A: tests positive. Priors: $P(E) = 0.001$, $P(E') = 0.999$. Likelihoods: $P(A \mid E) = 0.9$, $P(A \mid E') = 0.01$.

$$P(E \mid A) = \frac{0.001 \times 0.9}{0.001 \times 0.9 + 0.999 \times 0.01} = \frac{0.0009}{0.01089} \approx 0.083.$$

Only about 8.3%. This counter-intuitive result — that most positive tests on a low-prevalence disease are false alarms — is a classic illustration of why Bayesian reasoning matters in medicine.

Bayes in medical diagnosis and spam filters

A modern spam filter computes $P(\text{spam} \mid \text{words observed})$ by combining the prior fraction of spam in your inbox with the likelihood of each word under spam vs ham. Self-driving car perception, COVID test interpretation, and search-engine query suggestions all run the same calculation under the hood.

5.5 Worked example 3: machines and defective bolts

Factory has machines A, B, C producing 25%, 35%, 40% of the bolts; defect rates are 5%, 4%, 2% respectively. A bolt is found defective. Find the probability it came from machine B.

Let B_i denote "produced by machine i"; E: bolt is defective. Then

$$P(B_2 \mid E) = \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{0.0140}{0.0345} = \frac{28}{69}.$$

The Bayes shortcut: "prior $\times$ likelihood, normalise".

Posterior $\propto$ Prior $\times$ Likelihood. Compute prior$\times$likelihood for every hypothesis, then divide each one by the sum (the normalisation). The hardest part is identifying the partition; the algebra is mechanical.

Don't confuse prior, likelihood, and posterior

$P(E_i)$ = how often hypothesis i is true *in general*. $P(A \mid E_i)$ = how often A occurs *when E_i is true*. $P(E_i \mid A)$ = how often E_i is true *when we have seen A*. Mixing these up is the single biggest source of errors in Bayes-theorem problems.

5.6 Worked example 4: truthful witness

A man speaks truth 3 times out of 4. He throws a die and reports a six. Find the actual probability of a six.

Let S_1: a six actually occurs; S_2: a six does not occur; E: the man reports a six. Priors $P(S_1) = 1/6$, $P(S_2) = 5/6$. Likelihoods $P(E \mid S_1) = 3/4$ (he tells the truth) and $P(E \mid S_2) = 1/4$ (he lies). Then

$$P(S_1 \mid E) = \frac{(1/6)(3/4)}{(1/6)(3/4) + (5/6)(1/4)} = \frac{3/24}{3/24 + 5/24} = \frac{3}{8}.$$

Even with a fairly reliable witness, the prior rarity of sixes drags the posterior probability of an actual six below 50%.

5.7 Worked example 5: three coin boxes (the classic)

Three identical boxes: Box I has two gold coins, Box II has two silver coins, Box III has one gold and one silver. A box is chosen at random; a coin is drawn; it is gold. What is the probability that the *other* coin in that box is also gold?

The "other coin is gold" is precisely the event "Box I was chosen". Let E_1, E_2, E_3 denote "Box i chosen" and A: gold coin drawn. Priors $P(E_i) = 1/3$. Likelihoods:

$$P(A \mid E_1) = 1, \quad P(A \mid E_2) = 0, \quad P(A \mid E_3) = \tfrac{1}{2}.$$

By Bayes',

$$P(E_1 \mid A) = \frac{(1/3)(1)}{(1/3)(1) + (1/3)(0) + (1/3)(1/2)} = \frac{1/3}{1/2} = \frac{2}{3}.$$

The answer $2/3$ surprises many students; the intuition is that Box I has *two* ways to produce a gold draw while Box III has only one, so observing gold makes Box I twice as likely.

5.8 Worked example 6: doctor's transport

A doctor visits a patient by train, bus, scooter, or other means with probabilities $3/10, 1/5, 1/10, 2/5$ respectively. The probabilities of arriving late by these means are $1/4, 1/3, 1/12, 0$. Given that the doctor arrived late, find the probability he came by train.

Let T_i: came by mode i; E: arrived late.

$$P(T_1 \mid E) = \frac{(3/10)(1/4)}{(3/10)(1/4) + (1/5)(1/3) + (1/10)(1/12) + (2/5)(0)} = \frac{3/40}{9/60} = \frac{1}{2}.$$

6 Random Variables and Distributions [JEE/NEET Extension]

The remaining topics — random variables, probability distributions, mean and variance, Bernoulli trials, and the binomial distribution — were removed in the NCERT 2026–27 rationalisation but remain heavily tested in JEE Main, JEE Advanced, and NEET. We include them here for completeness.

6.1 Random variable

A **random variable** (rv) is a real-valued function $X : S \to \mathbb{R}$. It assigns a number to each outcome of a random experiment.

Example. Toss a coin twice. $S = \{HH, HT, TH, TT\}$. Let X = number of heads. Then

$$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0.$$

A *discrete* random variable takes finitely many (or countably many) values. We restrict attention to the discrete case — this is what NCERT and entrance exams test.

6.2 Probability distribution

The **probability distribution** of a discrete rv X is the list of values X can take, each with its probability.

Probability Distribution

X takes values $x_1, x_2, \dots, x_n$ with probabilities $p_1, p_2, \dots, p_n$ where

$$p_i = P(X = x_i), \quad p_i \geq 0, \quad \sum_{i=1}^{n} p_i = 1.$$

The pair (x_i, p_i) is usually presented as a table.

Example. Two coins tossed; X = number of heads.

x	0	1	2
$P(X = x)$	$1/4$	$1/2$	$1/4$

The probabilities sum to 1, as they must.

6.3 Mean (Expected Value) and Variance

Mean and Variance of a Discrete rv

$$\mu = E(X) = \sum_{i=1}^{n} x_i\, p_i \quad \text{(expectation / mean)},$$

$$\sigma^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \sum_{i=1}^{n} x_i^2\, p_i - \mu^2,$$

$$\sigma = \sqrt{\mathrm{Var}(X)} \quad \text{(standard deviation)}.$$

The mean is the long-run average value of X; the variance measures spread around the mean.

Worked example. For the two-coin example above,

$$\mu = 0 \cdot \tfrac{1}{4} + 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} = 1, \quad E(X^2) = 0 + \tfrac{1}{2} + 1 = \tfrac{3}{2}, \quad \sigma^2 = \tfrac{3}{2} - 1 = \tfrac{1}{2}.$$

6.4 Bernoulli trials

A **Bernoulli trial** is a random experiment satisfying:

1. There are exactly two outcomes: *success* (S) and *failure* (F).
2. Trials are repeated a fixed number of times n.
3. Trials are *independent*.
4. The probability p of success is the same on every trial; let $q = 1 - p$ be the probability of failure.

Tossing a coin n times, drawing n balls with replacement, or asking n true/false questions are all classic Bernoulli setups.

6.5 Binomial distribution

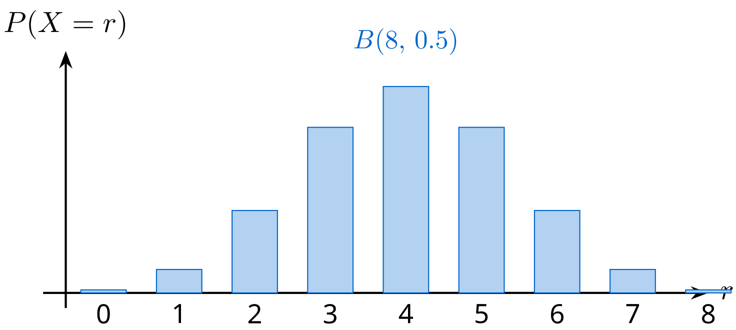
Let X = number of successes in n Bernoulli trials. Then X follows the **binomial distribution** $B(n, p)$:

Binomial Distribution

$$\boxed{P(X = r) = \binom{n}{r} p^r q^{n-r}, \quad r = 0, 1, 2, \ldots, n,\ q = 1 - p.}$$

Mean: $\mu = np$. Variance: $\sigma^2 = npq$. Standard deviation: $\sigma = \sqrt{npq}$.

Why $\binom{n}{r} p^r q^{n-r}$*?* A specific arrangement with r successes and $n-r$ failures has probability $p^r q^{n-r}$ (by independence). The number of such arrangements is $\binom{n}{r}$. Multiply.



Symmetric around the mean $np = 4$ when $p = 1/2$.

Worked example. A fair coin is tossed 6 times. X = number of heads. Find $P(X \geq 4)$.

$$P(X = r) = \binom{6}{r}\left(\tfrac{1}{2}\right)^6, \quad P(X \geq 4) = \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = \frac{15+6+1}{64} = \frac{22}{64} = \frac{11}{32}.$$

Worked example. The probability of a defective item is $p = 0.1$. From a lot, ten items are picked. Find the probability that (a) none is defective, (b) at most 2 are defective.

Here $n = 10, p = 0.1, q = 0.9$.

$$P(X = 0) = \binom{10}{0}(0.1)^0(0.9)^{10} = (0.9)^{10} \approx 0.3487.$$
$$P(X \leq 2) = \binom{10}{0}(0.9)^{10} + \binom{10}{1}(0.1)(0.9)^9 + \binom{10}{2}(0.1)^2(0.9)^8 \approx 0.9298.$$

So about 35% of lots are fault-free, and 93% have at most two defective items. The mean is $\mu = np = 1$ defective per ten items, with $\sigma = \sqrt{npq} = \sqrt{0.9} \approx 0.95$.

Recognising a binomial problem

Three checks: (i) a fixed number of trials, (ii) two outcomes per trial, (iii) constant success probability with independence. If all three hold, the binomial formula applies. "Without replacement" violates (iii) — use multiplication theorem or hypergeometric ideas instead.

Mean of binomial is np, not $n/2$

A common slip is using $\mu = n/2$ regardless of p. The correct mean is $\mu = np$. For $n = 100, p = 0.3$, $\mu = 30$, not 50.

Where binomial shows up

Defective-item counts in factory batches, number of customer conversions out of n ad impressions, exam scores on multiple-choice tests under random guessing, and proportions of votes in opinion polls are all modelled (at first approximation) by binomials.

7 Quick Reference Summary

A one-stop lookup card for the entire chapter.

7.1 Key formulae

Master Formula List

Conditional probability: $P(E \mid F) = \dfrac{P(E \cap F)}{P(F)}, \quad P(F) > 0.$

Multiplication theorem (2 events): $P(E \cap F) = P(E)P(F \mid E) = P(F)P(E \mid F).$

Multiplication (3 events): $P(E \cap F \cap G) = P(E)P(F \mid E)P(G \mid E \cap F).$

Independence: $P(E \cap F) = P(E)P(F).$

Total probability: $P(A) = \sum_i P(E_i)P(A \mid E_i).$

Bayes' theorem: $P(E_i \mid A) = \dfrac{P(E_i)P(A \mid E_i)}{\sum_j P(E_j)P(A \mid E_j)}.$

JEE/NEET extras — Random Variables

Probability distribution: $\sum_i p_i = 1, \; p_i \geq 0.$

Mean: $E(X) = \sum_i x_i p_i.$

Variance: $\text{Var}(X) = E(X^2) - [E(X)]^2.$

Binomial pmf: $P(X = r) = \binom{n}{r} p^r q^{n-r}.$

Binomial mean / variance: $\mu = np, \; \sigma^2 = npq.$

7.2 Concept comparison

Concept	**Conditional** $P(E \mid F)$	**Unconditional** $P(E)$
Sample space used	Restricted to F	Whole of S
Information assumed	F has occurred	Nothing has occurred
Formula	$P(E \cap F)/P(F)$	$\lvert E\rvert/\lvert S\rvert$ (for equally likely)
Sums to 1 over	$P(E \mid F) + P(E' \mid F) = 1$	$P(E) + P(E') = 1$

Property	Independent	Mutually Exclusive
Defining relation	$P(E \cap F) = P(E)P(F)$	$E \cap F = \varnothing$
Can both occur?	Yes	No
$P(E \cup F)$	$P(E) + P(F) - P(E)P(F)$	$P(E) + P(F)$
$P(E \mid F)$	$P(E)$	0 (if $P(F) > 0$)

7.3 Problem-solving checklist

When a problem says "find the probability that...given that...":

1. Identify the events. Label them E, F (or $E_1, \ldots, E_n$ and A).
2. Decide which formula applies: direct definition, multiplication, total probability, or Bayes'.
3. Identify the partition (for total probability / Bayes'). Verify it is exhaustive and disjoint.
4. Compute priors $P(E_i)$ and likelihoods $P(A \mid E_i)$.
5. For total probability: add over branches. For Bayes': single branch on top, sum of branches on bottom.
6. Sanity-check: probabilities lie in $[0, 1]$, posteriors sum to 1 across all hypotheses.

7.4 Choosing the right tool

Question shape	Tool
"$P(E \mid F)$ directly?"	Definition: $P(E \cap F)/P(F)$
"Probability both occur?"	Multiplication theorem
"Probability "at least one"?"	$1 - P(A')P(B')$ if independent; inclusion–exclusion otherwise
"Overall probability of A, given cause-probabilities?"	Total probability theorem
"Probability the cause was E_i, given A occurred?"	Bayes' theorem
"n independent trials, count successes?"	Binomial distribution $B(n, p)$
"Average value of X?"	$E(X) = \sum x_i p_i$

7.5 Final word

Probability in Class 12 is really one idea repeated: *condition on what you know*. Conditional probability gives the definition; the multiplication theorem turns the definition around; independence is the special case where conditioning changes nothing; total probability stitches conditional pieces together; and Bayes' theorem reverses the direction of the conditioning.

Once you can draw a clean probability tree, label every branch, and identify whether the question asks “forward” (total probability) or “backward” (Bayes’), the rest is arithmetic. Practise enough urn, card, die, and machine problems to feel comfortable setting up the partition, and the chapter is yours.

Best of luck with your preparation!

Update your priors, trust the evidence, and verify the partition.