NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12th Mathematics, Chapter 2

Chapter 2: Inverse Trigonometric Functions

About this Chapter

An **inverse trigonometric function** returns the angle whose trigonometric ratio is a given number. Because the six trig functions are many-one over their full domains, we restrict the domain to make each one bijective; the inverse is then defined on the resulting **principal value branch**. This Exemplar set drills the core ideas of Chapter 2: principal values, the six branch tables, the identities $\sin^{-1}x + \cos^{-1}x = \pi/2$, $\tan^{-1}x + \cot^{-1}x = \pi/2$, the sum/difference formulas for $\tan^{-1}$, the double-angle $2\tan^{-1}$ identities, and the conversion shortcuts $\sin^{-1}(1/x) = \csc^{-1}x$. By the end of this chapter the student should be able to evaluate, simplify, and solve equations involving these functions with full confidence.

Topics covered: Principal value branches • Domain and range of inverse trig • $\sin^{-1}x + \cos^{-1}x = \pi/2$ • $\tan^{-1}x + \cot^{-1}x = \pi/2$ • Sum/difference of $\tan^{-1}$ • $2\tan^{-1}x$ identities • Graphs of inverse trig • Solving equations

Quick Formula Sheet

Principal value ranges:

$\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

$\cos^{-1} : [-1, 1] \to [0, \pi]$

$\tan^{-1} : \mathbb{R} \to (-\pi/2, \pi/2)$

Complementary identities:

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

Sum of arctangents:

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$, $xy < 1$

Double-angle:

$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$, $|x| < 1$

$2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$, $|x| \le 1$

Negative-argument:

$\sin^{-1}(-x) = -\sin^{-1}x$; $\cos^{-1}(-x) = \pi - \cos^{-1}x$

Short Answer Type Questions

Q 2.1 **Find the value of** $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right)$.

SOLUTION

Concept used. The composition identities are conditional: $\tan^{-1}(\tan x) = x$ holds only when $x \in (-\pi/2, \pi/2)$, and $\cos^{-1}(\cos x) = x$ only when $x \in [0, \pi]$. When the inner angle falls outside the principal range, we first add or subtract a multiple of the period so the equivalent argument lies inside the principal branch, then apply the identity.

Step 1. First term. The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$, but $\frac{5\pi}{6} \notin (-\pi/2, \pi/2)$. Use the periodicity $\tan(\theta - \pi) = \tan\theta$:

$$\tan\frac{5\pi}{6} = \tan\left(\frac{5\pi}{6} - \pi\right) = \tan\left(-\frac{\pi}{6}\right).$$

Since $-\frac{\pi}{6} \in (-\pi/2, \pi/2)$, we may apply the identity:

$$\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right) = -\frac{\pi}{6}.$$

Step 2. Second term. The principal range of $\cos^{-1}$ is $[0, \pi]$, but $\frac{13\pi}{6} \notin [0, \pi]$. Use the periodicity $\cos(\theta - 2\pi) = \cos\theta$:

$$\cos\frac{13\pi}{6} = \cos\left(\frac{13\pi}{6} - 2\pi\right) = \cos\frac{\pi}{6}.$$

Since $\frac{\pi}{6} \in [0, \pi]$, the identity applies:

$$\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \cos^{-1}\left(\cos\frac{\pi}{6}\right) = \frac{\pi}{6}.$$

Step 3. Add. $-\frac{\pi}{6} + \frac{\pi}{6} = 0.$

> **Final Answer:** $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) + \cos^{-1}\left(\cos\frac{13\pi}{6}\right) = 0.$

✗ Common Pitfall

A frequent error is to write $\tan^{-1}(\tan\frac{5\pi}{6}) = \frac{5\pi}{6}$ by "cancelling" the inverse with the function. The cancellation is valid only inside the principal range. Always first push the inner angle into $(-\pi/2, \pi/2)$ for $\tan^{-1} \circ \tan$, and into $[0, \pi]$ for $\cos^{-1} \circ \cos$.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Bombay*

Reduce-then-cancel angle. Every $\tan^{-1}(\tan\alpha)$ or $\cos^{-1}(\cos\alpha)$ problem becomes mechanical once you remember the trick: shift α by integer multiples of the period of the inner function until the result sits inside the principal range of the outer inverse, then cancel.

Concept used. $\tan$ has period π and $\cos$ has period 2π. So $\tan(\alpha + k\pi) = \tan\alpha$ and $\cos(\alpha + 2k\pi) = \cos\alpha$ for every integer k, which lets us replace the inner angle by an equivalent one in the principal branch.

Step 1. $\frac{5\pi}{6} = \pi - \frac{\pi}{6}$, so subtracting one period of $\tan$ gives $\frac{5\pi}{6} - \pi = -\frac{\pi}{6}$, which is inside $(-\pi/2, \pi/2)$. Therefore $\tan^{-1}\left(\tan\frac{5\pi}{6}\right) = -\frac{\pi}{6}$.

Step 2. $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, so subtracting one period of $\cos$ gives $\frac{\pi}{6} \in [0, \pi]$. Therefore $\cos^{-1}\left(\cos\frac{13\pi}{6}\right) = \frac{\pi}{6}$.

Step 3. Add: $-\frac{\pi}{6} + \frac{\pi}{6} = 0$.

Why this matters. CBSE and JEE problems routinely test this "which branch?" instinct. Always identify the principal range first, then reduce.

Final Answer: 0.

Q 2.2 **Evaluate** $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$.

SOLUTION

Concept used. For $x \in [-1, 1]$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$ (negative-argument rule). Also, $\cos^{-1}(\cos\theta) = \theta$ only for $\theta \in [0, \pi]$, so we must check the resulting angle lies in this range before cancelling.

Step 1. **Evaluate the inverse.** We need an angle in $[0, \pi]$ whose cosine is $-\sqrt{3}/2$. Since $\cos\frac{\pi}{6} = \sqrt{3}/2$ and $\cos$ is negative in the second quadrant, the required angle is $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. Thus $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

Step 2. **Add the angles.**

$$\frac{5\pi}{6} + \frac{\pi}{6} = \frac{6\pi}{6} = \pi.$$

Step 3. **Apply outer cosine.** $\cos\pi = -1$.

Final Answer: $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=-1.$

☞ **Quadrant facts**

cos is positive in Q1 and Q4, negative in Q2 and Q3. The principal range $[0,\pi]$ covers Q1 and Q2, so $\cos^{-1}$ of a negative number always lands in Q2.

EXPERT'S SOLUTION *: Sneha Patel, M.Sc Mathematics, ISI Kolkata*

Direct-substitution angle. Compute the inverse exactly, add, then take cosine. No identities needed beyond the negative-argument shift.
Concept used. $\cos^{-1}(-x)=\pi-\cos^{-1}x$ for $x\in[-1,1]$; $\cos^{-1}(\sqrt{3}/2)=\pi/6$ from the standard 30-60-90 reference triangle.

Step 1. Apply the negative-argument rule:

$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}.$$

Step 2. Sum with $\pi/6$: $\frac{5\pi}{6}+\frac{\pi}{6}=\pi.$

Step 3. Final cosine: $\cos\pi=-1.$

Why this matters. The negative-argument rule for $\cos^{-1}$ $(\pi-\cos^{-1}x)$ is different from the rule for $\sin^{-1}$ $(-\sin^{-1}x)$. Mixing them up is a top-three mistake on this chapter.

Final Answer: $-1.$

Q 2.3 **Prove that** $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=7.$

SOLUTION

Concept used. If $\cot^{-1}3=\theta$ then $\cot\theta=3$, so $\tan\theta=\frac{1}{3}$. The tangent double-angle formula is $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$, and the angle subtraction formula is $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$. Finally, $\cot\phi=\frac{1}{\tan\phi}$.

Step 1. Let $\theta=\cot^{-1}3$. Then $\cot\theta=3$ and $\tan\theta=\frac{1}{3}$.

Step 2. Compute $\tan 2\theta$:

$$\tan 2\theta = \frac{2 \cdot \frac{1}{3}}{1 - \left(\frac{1}{3}\right)^2} = \frac{2/3}{1 - 1/9} = \frac{2/3}{8/9} = \frac{2}{3} \cdot \frac{9}{8} = \frac{18}{24} = \frac{3}{4}.$$

Step 3. Compute $\tan\left(\frac{\pi}{4} - 2\theta\right)$ with $\tan\frac{\pi}{4} = 1$ and $\tan 2\theta = \frac{3}{4}$:

$$\tan\left(\frac{\pi}{4} - 2\theta\right) = \frac{1 - \frac{3}{4}}{1 + 1 \cdot \frac{3}{4}} = \frac{1/4}{7/4} = \frac{1}{7}.$$

Step 4. Take reciprocal to get cotangent:

$$\cot\left(\frac{\pi}{4} - 2\theta\right) = \frac{1}{1/7} = 7.$$

Final Answer: $\cot\left(\frac{\pi}{4} - 2\cot^{-1} 3\right) = 7.$

☞ Substitution trick

For any $\cot^{-1}$ or $\tan^{-1}$ expression, let the inverse equal a single angle θ at the very first step. Everything downstream is then just plain trig identities on θ.

EXPERT'S SOLUTION *: Vivaan Iyer, Ph.D Mathematics, IIT Delhi*

Identity-chaining angle. Rewrite $2\cot^{-1} 3$ as $2\tan^{-1}(1/3)$ and then apply the $2\tan^{-1}$ shortcut directly to get a single $\tan^{-1}$, which collapses cleanly with $\pi/4$.

Concept used. $\cot^{-1} x = \tan^{-1}(1/x)$ for $x > 0$, and $2\tan^{-1} t = \tan^{-1}\frac{2t}{1-t^2}$ for $|t| < 1$.

Step 1. $\cot^{-1} 3 = \tan^{-1}(1/3)$, so $2\cot^{-1} 3 = 2\tan^{-1}(1/3)$.

Step 2. Apply double-angle: $2\tan^{-1}(1/3) = \tan^{-1}\frac{2 \cdot 1/3}{1 - 1/9} = \tan^{-1}\frac{2/3}{8/9} = \tan^{-1}(3/4)$.

Step 3. Now the target angle is $\frac{\pi}{4} - \tan^{-1}(3/4)$. Its tangent is $\frac{1 - 3/4}{1 + 3/4} = \frac{1/4}{7/4} = \frac{1}{7}$.

Step 4. Cotangent is the reciprocal: $\frac{1}{1/7} = 7$.

Why this matters. Reducing $\cot^{-1}$ to $\tan^{-1}$ at the start unifies your identity toolkit and prevents sign mistakes that arise from the awkward $\cot^{-1}$ range $(0, \pi)$.

Final Answer: 7.

Q 2.4 **Find the value of** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.

SOLUTION

Concept used. The principal-range values: $\tan^{-1}$ returns an angle in $(-\pi/2, \pi/2)$; $\cot^{-1}$ returns an angle in $(0, \pi)$. Also, $\tan^{-1}(-x) = -\tan^{-1}x$ and $\cot^{-1}(1/\sqrt{3}) = \pi/3$ since $\cot(\pi/3) = 1/\sqrt{3}$.

Step 1. First term. $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$ (since $\tan(\pi/6) = 1/\sqrt{3}$).

Step 2. Second term. $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$ (since $\cot(\pi/3) = 1/\sqrt{3}$ and $\pi/3 \in (0, \pi)$).

Step 3. Third term. $\sin\left(-\frac{\pi}{2}\right) = -1$, so we need $\tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}$.

Step 4. Add.

$$-\frac{\pi}{6} + \frac{\pi}{3} - \frac{\pi}{4} = \frac{-2\pi + 4\pi - 3\pi}{12} = \frac{-\pi}{12}.$$

Final Answer: $-\frac{\pi}{12}$.

☞ Term-by-term tip

Evaluate each inverse-trig piece on its own using the principal-value table; combine only at the end. Identity chaining across mismatched inverse families almost always introduces sign errors.

EXPERT'S SOLUTION : *Pranav Kumar, M.Tech CS, IIT Madras*

Term-by-term angle. Reduce each piece to a standard reference angle, sign-correct, then add.

Concept used. Standard values $\tan(\pi/6) = 1/\sqrt{3}$, $\cot(\pi/3) = 1/\sqrt{3}$, $\tan(\pi/4) = 1$, combined with odd symmetry $\tan^{-1}(-x) = -\tan^{-1}x$.

Step 1. $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$.

Step 2. $\cot^{-1}(1/\sqrt{3}) = \pi/3$.

Step 3. $\sin(-\pi/2) = -1 \Rightarrow \tan^{-1}(-1) = -\pi/4$.

Step 4. Sum: take LCM 12. $-2/12 + 4/12 - 3/12 = -1/12$ (in units of π), i.e. $-\pi/12$.

Why this matters. Mixing inverse functions in one expression is a CBSE favourite. Always evaluate every term individually before combining; never try clever identity chains across mismatched inverse families.

Final Answer: $-\dfrac{\pi}{12}$.

Q 2.5 **Find the value of** $\tan^{-1}\left(\tan\dfrac{2\pi}{3}\right)$.

SOLUTION

Concept used. The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$. Since $\dfrac{2\pi}{3} \notin (-\pi/2, \pi/2)$, we must shift the inner argument by a multiple of π (the period of $\tan$) to bring it into the principal range.

Step 1. Subtract π from $\dfrac{2\pi}{3}$:

$$\frac{2\pi}{3} - \pi = \frac{2\pi - 3\pi}{3} = -\frac{\pi}{3}.$$

And $-\dfrac{\pi}{3} \in (-\pi/2, \pi/2)$.

Step 2. Use periodicity $\tan(\theta - \pi) = \tan\theta$:

$$\tan\frac{2\pi}{3} = \tan\left(-\frac{\pi}{3}\right).$$

Step 3. Apply the identity in the principal range:

$$\tan^{-1}\left(\tan\frac{2\pi}{3}\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}.$$

Final Answer: $\tan^{-1}\left(\tan\dfrac{2\pi}{3}\right) = -\dfrac{\pi}{3}$.

✗ Common Pitfall

The answer is not $\dfrac{2\pi}{3}$. The inverse function returns the principal value, so the result is always squeezed into $(-\pi/2, \pi/2)$ no matter what angle you started with.

EXPERT'S SOLUTION *: Aanya Gupta, Ph.D Pure Mathematics, IISc Bangalore*

Periodic-shift angle. The whole calculation is one period shift.
Concept used. $\tan$ has period π. Subtract one π from $2\pi/3$ to land on the equivalent principal-range angle $-\pi/3$.

Step 1. $2\pi/3 > \pi/2$, so we are outside the principal range.

Step 2. Shift: $2\pi/3 - \pi = -\pi/3 \in (-\pi/2, \pi/2)$.

Step 3. Cancel: $\tan^{-1}\tan(-\pi/3) = -\pi/3$.

Why this matters. Practise this shift on $5\pi/6$, $7\pi/6$, $-3\pi/4$ until it is instant; it appears in nearly every CBSE paper.

> **Final Answer:** $-\dfrac{\pi}{3}$.

Q 2.6 **Show that** $2\tan^{-1}(-3) = -\dfrac{\pi}{2} + \tan^{-1}\left(-\dfrac{4}{3}\right)$.

SOLUTION

Concept used. For $x > 1$ the double-angle identity is $2\tan^{-1}x = \pi + \tan^{-1}\dfrac{2x}{1-x^2}$ (because the right side is forced out of $(-\pi/2, \pi/2)$). Taking the odd-function rule $\tan^{-1}(-x) = -\tan^{-1}x$ on top, for $x < -1$ we get $2\tan^{-1}x = -\pi + \tan^{-1}\dfrac{2x}{1-x^2}$.

Step 1. Use $\tan^{-1}(-3) = -\tan^{-1}3$, so $2\tan^{-1}(-3) = -2\tan^{-1}3$.

Step 2. For $x = 3 > 1$, the corrected double-angle formula gives

$$2\tan^{-1}3 = \pi + \tan^{-1}\frac{2\cdot 3}{1-9} = \pi + \tan^{-1}\frac{6}{-8} = \pi + \tan^{-1}\left(-\frac{3}{4}\right).$$

Step 3. Therefore

$$2\tan^{-1}(-3) = -\pi - \tan^{-1}\left(-\frac{3}{4}\right) = -\pi + \tan^{-1}\left(\frac{3}{4}\right).$$

Step 4. Use the complementary identity $\tan^{-1}(3/4) + \tan^{-1}(4/3) = \pi/2$ (valid since both arguments are positive and reciprocal), so $\tan^{-1}(3/4) = \dfrac{\pi}{2} - \tan^{-1}(4/3)$. Substitute:

$$2\tan^{-1}(-3) = -\pi + \frac{\pi}{2} - \tan^{-1}(4/3) = -\frac{\pi}{2} - \tan^{-1}(4/3).$$

Step 5. Finally, $-\tan^{-1}(4/3) = \tan^{-1}(-4/3)$, hence

$$2\tan^{-1}(-3) = -\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right).$$

> **Final Answer:** $2\tan^{-1}(-3) = -\dfrac{\pi}{2} + \tan^{-1}\left(-\dfrac{4}{3}\right)$.

♥ **Why this matters**

The double-angle for $2\tan^{-1}x$ has three regimes: $|x| < 1$ (no correction), $x > 1$ (add π), $x < -1$ (subtract π). Forgetting the correction is the single biggest source of lost marks on this chapter.

EXPERT'S SOLUTION *: Aditi Mehta, M.Sc Applied Mathematics, IIT Kanpur*

Reciprocal-identity angle. Use $\tan^{-1}x + \tan^{-1}(1/x) = \pi/2$ for $x > 0$, $= -\pi/2$ for $x < 0$, to collapse the awkward $2\tan^{-1}(-3)$ in one step.

Concept used. For $x < 0$, $\tan^{-1}x + \tan^{-1}(1/x) = -\dfrac{\pi}{2}$. Also $\tan^{-1}(-3) = \tan^{-1}(-3)$ unchanged.

Step 1. $\tan^{-1}(-3) + \tan^{-1}(-1/3) = -\dfrac{\pi}{2}$, so

$$\tan^{-1}(-3) = -\frac{\pi}{2} - \tan^{-1}(-1/3) = -\frac{\pi}{2} + \tan^{-1}(1/3).$$

Step 2. Doubling: $2\tan^{-1}(-3) = -\pi + 2\tan^{-1}(1/3)$. Now $1/3 < 1$, so apply the unmodified double-angle: $2\tan^{-1}(1/3) = \tan^{-1}\dfrac{2/3}{1-1/9} = \tan^{-1}(3/4)$.

Step 3. Substitute: $2\tan^{-1}(-3) = -\pi + \tan^{-1}(3/4) = -\pi + (\pi/2 - \tan^{-1}(4/3)) = -\pi/2 - \tan^{-1}(4/3) = -\pi/2 + \tan^{-1}(-4/3)$.

Why this matters. Whenever $|x| > 1$ inside a $2\tan^{-1}$, the $\pm\pi$ correction is mandatory. Forgetting it changes the answer by $\pm\pi$ - the most common Exemplar mistake on this topic.

Final Answer: $2\tan^{-1}(-3) = -\dfrac{\pi}{2} + \tan^{-1}\left(-\dfrac{4}{3}\right)$.

Q 2.7 **Find the real solutions of the equation** $\tan^{-1}\sqrt{x(x+1)} + \sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2}$.

SOLUTION

Concept used. If $\sin^{-1}u = \dfrac{\pi}{2} - \tan^{-1}v$, then applying sin to both sides and using $\sin\left(\dfrac{\pi}{2} - \theta\right) = \cos\theta$ gives $u = \cos\theta$, where $\tan\theta = v$. Equivalently, $\sin^{-1}u + \cos^{-1}u = \dfrac{\pi}{2}$ shows $\sin^{-1}u = \dfrac{\pi}{2} - \cos^{-1}u$, so the equation forces $\tan^{-1}v = \cos^{-1}u$, i.e. u and v correspond to the same angle's cos and tan.

Step 1. Rearrange: $\sin^{-1}\sqrt{x^2+x+1} = \dfrac{\pi}{2} - \tan^{-1}\sqrt{x(x+1)}$.

Step 2. Apply sin to both sides. With $\theta = \tan^{-1}\sqrt{x(x+1)}$, the right side becomes

$\sin(\pi/2 - \theta) = \cos\theta$. From $\tan\theta = \sqrt{x(x+1)}$,

$$\cos\theta = \frac{1}{\sqrt{1 + x(x+1)}} = \frac{1}{\sqrt{x^2+x+1}}.$$

Step 3. So the equation simplifies to

$$\sqrt{x^2+x+1} = \frac{1}{\sqrt{x^2+x+1}}.$$

Squaring both sides: $x^2 + x + 1 = 1$, i.e. $x(x+1) = 0$.

Step 4. Solutions: $x = 0$ or $x = -1$. Check domain: $\sqrt{x(x+1)}$ requires $x(x+1) \geq 0$, satisfied by both $x = 0$ (gives 0) and $x = -1$ (gives 0). $\sqrt{x^2+x+1} = 1$ at both, which is in $[-1, 1]$.

Final Answer: $x = 0$ or $x = -1$.

☞ Tangent-to-cosine bridge

If $\tan\theta = v$ with $\theta \in (0, \pi/2)$, the right triangle has opposite v, adjacent 1, hypotenuse $\sqrt{1+v^2}$; hence $\cos\theta = 1/\sqrt{1+v^2}$.

EXPERT'S SOLUTION *: Karan Singh, B.Tech CSE, IIT Roorkee*

Triangle-substitution angle. Set $\theta = \tan^{-1}\sqrt{x(x+1)}$, read off all six trig ratios from the right triangle, then the equation collapses to an algebraic one.

Concept used. If $\tan\theta = t \geq 0$, draw a right triangle with opposite t, adjacent 1, hypotenuse $\sqrt{1+t^2}$. Then $\sin\theta = t/\sqrt{1+t^2}$ and $\cos\theta = 1/\sqrt{1+t^2}$.

Step 1. With $t = \sqrt{x(x+1)}$, hypotenuse $= \sqrt{1 + x(x+1)} = \sqrt{x^2+x+1}$.

Step 2. Equation: $\theta + \sin^{-1}\sqrt{x^2+x+1} = \pi/2$, i.e. $\sin^{-1}\sqrt{x^2+x+1} = \pi/2 - \theta$. Taking sin: $\sqrt{x^2+x+1} = \cos\theta = 1/\sqrt{x^2+x+1}$.

Step 3. Square: $x^2 + x + 1 = 1 \Rightarrow x(x+1) = 0 \Rightarrow x \in \{0, -1\}$.

Step 4. Both pass the surd-domain check.

Why this matters. "Set the inverse equal to θ and draw the triangle" is the universal first move on any inverse-trig equation.

Final Answer: $x \in \{0, -1\}$.

Q 2.8 **Find the value of the expression** $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1} 2\sqrt{2})$.

SOLUTION

Concept used. The double-angle identity $\sin(2\tan^{-1}x) = \dfrac{2x}{1+x^2}$ for $|x| \le 1$ comes from substituting $\tan\theta = x$ into $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$. For the second term, if $\tan\theta = y$ then in a right triangle opposite y, adjacent 1, hypotenuse $\sqrt{1+y^2}$, so $\cos(\tan^{-1}y) = \dfrac{1}{\sqrt{1+y^2}}$.

Step 1. First term. With $x = 1/3$:

$$\sin\left(2\tan^{-1}\frac{1}{3}\right) = \frac{2\cdot 1/3}{1+1/9} = \frac{2/3}{10/9} = \frac{2}{3}\cdot\frac{9}{10} = \frac{18}{30} = \frac{3}{5}.$$

Step 2. Second term. With $y = 2\sqrt{2}$:

$$\cos(\tan^{-1}2\sqrt{2}) = \frac{1}{\sqrt{1+8}} = \frac{1}{3}.$$

Step 3. Add.

$$\frac{3}{5} + \frac{1}{3} = \frac{9}{15} + \frac{5}{15} = \frac{14}{15}.$$

Final Answer: $\sin\left(2\tan^{-1}\frac{1}{3}\right) + \cos(\tan^{-1}2\sqrt{2}) = \dfrac{14}{15}$.

☞ Triangle shortcut

Every $\tan^{-1}t$ can be visualised on a right triangle with opposite t, adjacent 1, hypotenuse $\sqrt{1+t^2}$. Once you draw it, every other trig ratio is just a side ratio away.

EXPERT'S SOLUTION *: Riya Verma, M.Sc Mathematics, IIT Kanpur*

Two-triangle angle. Treat each inverse-trig as a right triangle, pick the required ratio, add.

Concept used. For $\tan^{-1}t$, build the right triangle with sides $(t, 1, \sqrt{1+t^2})$. From it, $\sin = t/\sqrt{1+t^2}$, $\cos = 1/\sqrt{1+t^2}$. For double angles, use $\sin 2\theta = 2\sin\theta\cos\theta$.

Step 1. Triangle for $\theta_1 = \tan^{-1}(1/3)$: sides $(1, 3, \sqrt{10})$. $\sin\theta_1 = 1/\sqrt{10}$, $\cos\theta_1 = 3/\sqrt{10}$. $\sin 2\theta_1 = 2(1/\sqrt{10})(3/\sqrt{10}) = 6/10 = 3/5$.

Step 2. Triangle for $\theta_2 = \tan^{-1}2\sqrt{2}$: sides $(2\sqrt{2}, 1, 3)$. $\cos\theta_2 = 1/3$.

Step 3. Add: $3/5 + 1/3 = 9/15 + 5/15 = 14/15$.

Why this matters. The triangle approach generalises to any $\sin(\cdot)$, $\cos(\cdot)$, $\sec(\cdot)$ of an inverse trig: build the triangle once, read every ratio.

Final Answer: $\frac{14}{15}$.

Q 2.9 **If $2\tan^{-1}(\cos\theta) = \tan^{-1}(2\csc\theta)$, then show that $\theta = \frac{\pi}{4}$, where θ lies in the relevant principal range.**

SOLUTION

Concept used. Apply the double-angle formula $2\tan^{-1} t = \tan^{-1}\frac{2t}{1-t^2}$ to the LHS (valid when $|t| < 1$, i.e. $|\cos\theta| < 1$, so $\theta \neq k\pi$). On the RHS the expression is already a single $\tan^{-1}$, so we equate the two arguments (since $\tan^{-1}$ is one-one on $(-\pi/2, \pi/2)$).

Step 1. Apply the LHS double-angle:

$$2\tan^{-1}(\cos\theta) = \tan^{-1}\frac{2\cos\theta}{1-\cos^2\theta} = \tan^{-1}\frac{2\cos\theta}{\sin^2\theta}.$$

Step 2. Equate to RHS $\tan^{-1}(2\csc\theta) = \tan^{-1}\frac{2}{\sin\theta}$:

$$\frac{2\cos\theta}{\sin^2\theta} = \frac{2}{\sin\theta}.$$

Step 3. Multiply both sides by $\sin^2\theta$ (assuming $\sin\theta \neq 0$):

$$2\cos\theta = 2\sin\theta \;\Rightarrow\; \tan\theta = 1.$$

Step 4. In the relevant principal range $(0, \pi/2)$ the solution is $\theta = \frac{\pi}{4}$.

Final Answer: $\theta = \frac{\pi}{4}$.

☞ **Equate arguments**

Whenever both sides of an equation are a single $\tan^{-1}(\cdots)$ with values in the same principal range, you may equate the arguments directly because $\tan^{-1}$ is strictly increasing and hence one-one.

EXPERT'S SOLUTION : *Neha Bhat, Ph.D Mathematics, IIT Delhi*

Identity-then-trig angle. Convert both sides to plain trig in θ using a single double-angle step on the LHS.

Concept used. $2\tan^{-1}(\cos\theta) = \tan^{-1}\frac{2\cos\theta}{\sin^2\theta}$ when $|\cos\theta| < 1$. The Pythagorean identity $1 - \cos^2\theta = \sin^2\theta$ is the only non-trivial input.

Step 1. LHS argument: $\dfrac{2\cos\theta}{1-\cos^2\theta} = \dfrac{2\cos\theta}{\sin^2\theta}$.

Step 2. RHS argument: $2\csc\theta = \dfrac{2}{\sin\theta}$.

Step 3. Equate and simplify: $\dfrac{\cos\theta}{\sin\theta} = 1 \Rightarrow \tan\theta = 1$.

Step 4. Principal-range solution: $\theta = \pi/4$.

Why this matters. The general family $\theta = \pi/4 + n\pi$ in the original equation reduces to $\pi/4$ when the principal range is restricted, mirroring how CBSE asks the question.

Final Answer: $\theta = \dfrac{\pi}{4}$.

Q 2.10 **Show that** $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(4\tan^{-1}\frac{1}{3}\right)$.

SOLUTION

Concept used. Two double-angle identities are needed: $2\tan^{-1}t = \tan^{-1}\frac{2t}{1-t^2}$ (for $|t| < 1$) and $\cos(2\tan^{-1}t) = \frac{1-t^2}{1+t^2}$, $\sin(2\tan^{-1}t) = \frac{2t}{1+t^2}$. We compute both sides explicitly and show they are equal.

Step 1. **LHS.** With $t = 1/7$:

$$\cos\left(2\tan^{-1}\frac{1}{7}\right) = \frac{1-1/49}{1+1/49} = \frac{48/49}{50/49} = \frac{48}{50} = \frac{24}{25}.$$

Step 2. **RHS, first double-angle.** With $t = 1/3$:

$$2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{2/3}{1-1/9} = \tan^{-1}\frac{2/3}{8/9} = \tan^{-1}\frac{3}{4}.$$

So $4\tan^{-1}(1/3) = 2\tan^{-1}(3/4)$.

Step 3. **RHS, $\sin$ of double angle.** With $u = 3/4$:

$$\sin\left(2\tan^{-1}\frac{3}{4}\right) = \frac{2\cdot 3/4}{1+9/16} = \frac{3/2}{25/16} = \frac{3}{2}\cdot\frac{16}{25} = \frac{48}{50} = \frac{24}{25}.$$

Step 4. LHS $= 24/25 =$ RHS. Hence the equality holds.

Final Answer: $\cos\left(2\tan^{-1}\frac{1}{7}\right) = \sin\left(4\tan^{-1}\frac{1}{3}\right) = \dfrac{24}{25}$.

☞ Chain double-angle

$4\tan^{-1}\frac{1}{3} = 2(2\tan^{-1}\frac{1}{3}) = 2\tan^{-1}\frac{3}{4}$. Chaining the double-angle formula is a CBSE favourite; it always reduces a $2^k\tan^{-1}$ to a single $\tan^{-1}$.

EXPERT'S SOLUTION *: Diya Reddy, M.Sc Mathematics, ISI Kolkata*

Compute-both-sides angle. Each side reduces to a clean rational; compare.

Concept used. $\cos(2\tan^{-1}t) = \dfrac{1-t^2}{1+t^2}$, $\sin(2\tan^{-1}t) = \dfrac{2t}{1+t^2}$, and the chain rule $4\tan^{-1}(1/3) = 2(2\tan^{-1}(1/3)) = 2\tan^{-1}(3/4)$.

Step 1. LHS: $\cos(2\tan^{-1}(1/7)) = \dfrac{1-1/49}{1+1/49} = \dfrac{48}{50} = 24/25$.

Step 2. Inner double-angle for RHS: $2\tan^{-1}(1/3) = \tan^{-1}(3/4)$.

Step 3. Outer double-angle: $\sin(2\tan^{-1}(3/4)) = \dfrac{2\cdot 3/4}{1+9/16} = \dfrac{3/2}{25/16} = 24/25$.

Step 4. Equal.

Why this matters. Doubling twice on $\tan^{-1}$ is exactly two applications of the double-angle formula. The intermediate $\tan^{-1}(3/4)$ is the famous 3-4-5 angle.

Final Answer: Both sides equal $\dfrac{24}{25}$.

Q 2.11 **Solve the equation** $\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\dfrac{3}{4}\right)$.

SOLUTION

Concept used. If $\theta = \tan^{-1}x$ then in a right triangle opposite x, adjacent 1, hypotenuse $\sqrt{1+x^2}$, hence $\cos\theta = \dfrac{1}{\sqrt{1+x^2}}$. If $\phi = \cot^{-1}(3/4)$ then $\cot\phi = 3/4$, so the triangle has adjacent 3, opposite 4, hypotenuse 5, giving $\sin\phi = 4/5$.

Step 1. Compute the RHS:

$$\sin\left(\cot^{-1}\frac{3}{4}\right) = \frac{4}{5}.$$

Step 2. Compute the LHS in terms of x:

$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}.$$

Step 3. Set equal and solve:

$$\frac{1}{\sqrt{1+x^2}} = \frac{4}{5} \Rightarrow \sqrt{1+x^2} = \frac{5}{4} \Rightarrow 1+x^2 = \frac{25}{16}.$$

Step 4. Hence $x^2 = \frac{25}{16} - 1 = \frac{9}{16}$, so $x = \pm\frac{3}{4}$.

Final Answer: $x = \pm\frac{3}{4}$.

☞ **3-4-5 triangle**

$\cot^{-1}(3/4)$ corresponds to the famous 3-4-5 right triangle: legs 3 and 4, hypotenuse 5. Every ratio you need (sin, cos, sec, csc) is on that triangle.

EXPERT'S SOLUTION : *Tara Joshi, M.Sc Mathematics, IIT Bombay*

Same-side-triangle angle. Read both sides off Pythagorean triangles, equate, solve.
Concept used. For an inverse-trig argument, draw the triangle that realises the inner ratio; every other ratio is then a side ratio on the same triangle.

Step 1. RHS triangle (3-4-5): $\cot\phi = 3/4 \Rightarrow \sin\phi = 4/5$.

Step 2. LHS triangle: $\tan\theta = x \Rightarrow \cos\theta = 1/\sqrt{1+x^2}$.

Step 3. Equation: $1/\sqrt{1+x^2} = 4/5 \Rightarrow 1+x^2 = 25/16$.

Step 4. $x^2 = 9/16 \Rightarrow x = \pm 3/4$.

Why this matters. Cosine being even means both signs of x satisfy the equation. The $\tan^{-1}$ wrapping does not narrow the solution to one sign.

Final Answer: $x = \pm\frac{3}{4}$.

Long Answer Type Questions

Q 2.12 **Prove that** $\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right] = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^2$**, for** $-1 < x < 1,\ x \neq 0$**.**

SOLUTION

Concept used. The substitution $x^2 = \cos 2\theta$ converts $\sqrt{1 \pm x^2}$ to clean trig forms: $1 + x^2 = 1 + \cos 2\theta = 2\cos^2\theta$ and $1 - x^2 = 1 - \cos 2\theta = 2\sin^2\theta$. Together with $\tan\left(\frac{\pi}{4} + \alpha\right) = \frac{1 + \tan\alpha}{1 - \tan\alpha}$, this collapses the messy radical expression to a single arctangent.

Step 1. Substitute $x^2 = \cos 2\theta$ with $2\theta = \cos^{-1} x^2 \in (0, \pi)$, i.e. $\theta \in (0, \pi/2)$. Then $\sqrt{1 + x^2} = \sqrt{2}\cos\theta$ and $\sqrt{1 - x^2} = \sqrt{2}\sin\theta$ (both positive on $\theta \in (0, \pi/2)$).

Step 2. The fraction inside the $\tan^{-1}$ becomes

$$\frac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

Step 3. Divide numerator and denominator by $\cos\theta$:

$$\frac{1 + \tan\theta}{1 - \tan\theta} = \frac{\tan\frac{\pi}{4} + \tan\theta}{1 - \tan\frac{\pi}{4}\tan\theta} = \tan\left(\frac{\pi}{4} + \theta\right).$$

Step 4. Therefore the LHS is

$$\tan^{-1}\left[\tan\left(\frac{\pi}{4} + \theta\right)\right] = \frac{\pi}{4} + \theta,$$

valid because $\theta \in (0, \pi/2)$ makes $\frac{\pi}{4} + \theta \in (\pi/4, 3\pi/4)$; specifically less than $\pi/2$ when $\theta < \pi/4$ and equal to $\pi/2-$ otherwise. Across this range the equivalent principal-value angle still evaluates to $\pi/4 + \theta$ within the chosen branch.

Step 5. Substitute $\theta = \frac{1}{2}\cos^{-1} x^2$:

$$\text{LHS} = \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2 = \text{RHS}.$$

Final Answer: LHS $= \frac{\pi}{4} + \frac{1}{2}\cos^{-1} x^2 =$ RHS.

☞ **Substitution insight**

Whenever you see $\sqrt{1 \pm x^2}$ together, try $x^2 = \cos 2\theta$ (or $x = \sin\theta$ for $\sqrt{1 - x^2}$ alone, or $x = \tan\theta$ for $\sqrt{1 + x^2}$ alone). The right substitution turns radicals into single trig terms.

EXPERT'S SOLUTION : *Krishna Kapoor, Ph.D Mathematics, IIT Delhi*

Direct-substitution angle. The half-angle substitution $x^2 = \cos 2\theta$ converts the entire radical expression to $\tan(\pi/4 + \theta)$ in three lines.

Concept used. Half-angle identities $1 + \cos 2\theta = 2\cos^2\theta$ and $1 - \cos 2\theta = 2\sin^2\theta$, plus the tangent addition formula $\tan(\pi/4 + \theta) = (1 + \tan\theta)/(1 - \tan\theta)$.

Step 1. Set $\theta = \frac{1}{2}\cos^{-1} x^2 \in (0, \pi/2)$, so $x^2 = \cos 2\theta$.

Step 2. $\sqrt{1+x^2} = \sqrt{2}\cos\theta$, $\sqrt{1-x^2} = \sqrt{2}\sin\theta$.

Step 3. Argument of $\tan^{-1}$: $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \dfrac{1 + \tan\theta}{1 - \tan\theta} = \tan\left(\frac{\pi}{4} + \theta\right)$.

Step 4. LHS $= \dfrac{\pi}{4} + \theta = \dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1} x^2$.

Step 5. Verify endpoints. At $x \to 0^+$, $x^2 \to 0$, so $\theta \to \frac{1}{2}\cos^{-1} 0 = \pi/4$, LHS $\to \pi/4 + \pi/4 = \pi/2$. The numerator and denominator of the original expression approach $\sqrt{2} + \sqrt{2}$ and 0 respectively, so $\tan^{-1}(\infty) = \pi/2$, matching the substitution result.

Why this matters. Recognising radicals as half-angle artefacts is the single most important trick in Chapter 2 LA problems; it appears verbatim in JEE Advanced.

Final Answer: $\dfrac{\pi}{4} + \dfrac{1}{2}\cos^{-1} x^2$.

Q 2.13 **Find the simplified form of** $\cos^{-1}\left(\dfrac{3}{5}\cos x + \dfrac{4}{5}\sin x\right)$**, where** $x \in \left(-\dfrac{3\pi}{4}, \dfrac{\pi}{4}\right)$**.**

SOLUTION

Concept used. An expression of the form $a\cos x + b\sin x$ (a, b constants with $a^2 + b^2 = 1$) can be rewritten as $\cos(x - \phi)$ where $\cos\phi = a$ and $\sin\phi = b$. Then $\cos^{-1}(\cos\alpha) = \alpha$ on the principal range $[0, \pi]$, so the final answer involves checking which equivalent principal-range angle $x - \phi$ corresponds to.

Step 1. Find ϕ with $\cos\phi = 3/5$ and $\sin\phi = 4/5$. Since both are positive, $\phi \in (0, \pi/2)$. Numerically, $\phi = \sin^{-1}(4/5)$. Equivalently $\phi = \tan^{-1}(4/3)$.

Step 2. Then $\dfrac{3}{5}\cos x + \dfrac{4}{5}\sin x = \cos\phi\cos x + \sin\phi\sin x = \cos(x - \phi)$.

Step 3. Hence the expression equals $\cos^{-1}(\cos(x - \phi))$.

Step 4. Domain check. With $x \in (-3\pi/4, \pi/4)$ and $\phi = \sin^{-1}(4/5) \in (0, \pi/2)$, we get $x - \phi \in (-3\pi/4 - \phi, \pi/4 - \phi) \subset (-\pi, \pi/4 - \phi)$. Since $\phi < \pi/2$, the upper end $\pi/4 - \phi$ is less than $\pi/4$. So $x - \phi$ may be negative or positive, but always in $(-\pi, \pi)$. The principal-range cosine retrieves $\cos^{-1}(\cos\alpha) = |\alpha|$ for $\alpha \in (-\pi, \pi)$. Specifically:

- If $x \geq \phi$, $\cos^{-1}(\cos(x - \phi)) = x - \phi$.

- If $x < \phi$, $\cos^{-1}(\cos(x-\phi)) = \phi - x$.

Step 5. So the simplified form is $|x-\phi|$ with $\phi = \sin^{-1}(4/5)$. Equivalently the answer is $x - \sin^{-1}(4/5)$ if $x \geq \sin^{-1}(4/5)$, else $\sin^{-1}(4/5) - x$.

Final Answer: $\cos^{-1}\left(\frac{3}{5}\cos x + \frac{4}{5}\sin x\right) = \left|x - \sin^{-1}\frac{4}{5}\right|$.

☞ Pattern recognition

Whenever you see $\frac{a}{c}\cos x + \frac{b}{c}\sin x$ with $a^2 + b^2 = c^2$, identify the Pythagorean triple $(3, 4, 5$ here) and rewrite it as $\cos(x-\phi)$ where ϕ is the corresponding acute angle. This is a CBSE classic.

EXPERT'S SOLUTION *: Ishaan Desai, M.Sc Mathematics, IIT Kanpur*

Cosine-of-difference angle. The whole simplification rides on recognising $3/5, 4/5$ as $\cos\phi, \sin\phi$ for some ϕ.

Concept used. Compound-angle identity $\cos(x-\phi) = \cos x\cos\phi + \sin x \sin\phi$ with ϕ chosen so the coefficients match.

Step 1. Identify $\cos\phi = 3/5, \sin\phi = 4/5$, so $\phi = \sin^{-1}(4/5)$ and $\phi \in (0, \pi/2)$.

Step 2. Inner expression: $\cos x\cos\phi + \sin x\sin\phi = \cos(x-\phi)$.

Step 3. Outer inverse: $\cos^{-1}(\cos(x-\phi))$. With the given domain, $x - \phi \in (-\pi, \pi/4 - \phi)$, so the answer is $|x-\phi|$.

Step 4. Domain check. The substitution requires $\cos\phi = 3/5, \sin\phi = 4/5$, valid since $(3/5)^2 + (4/5)^2 = 1$. So $\phi = \sin^{-1}(4/5) \in (0, \pi/2)$.

Step 5. Sign-split. For $x \geq \phi$, $x - \phi \geq 0$ and is in $[0, \pi]$, so $\cos^{-1}(\cos(x-\phi)) = x - \phi$. For $x < \phi$, $x - \phi < 0$, and using $\cos(-\theta) = \cos\theta$, the inverse returns $\phi - x$. Combined: $|x-\phi|$, the absolute value that appears in the final answer.

Why this matters. The Pythagorean-triple-to-angle conversion saves you from any 25-term expansion the brute force would require.

Final Answer: $\left|x - \sin^{-1}\frac{4}{5}\right|$.

Q 2.14 **Prove that** $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$.

SOLUTION

Concept used. The sum-of-arcsines formula is $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ provided $x, y \geq 0$ and $x^2 + y^2 \leq 1$ (so the right side stays in $[-1, 1]$ and the angle is in the principal range). We verify both conditions and apply the formula.

Step 1. Let $\alpha = \sin^{-1}(8/17)$ and $\beta = \sin^{-1}(3/5)$. Then $\sin\alpha = 8/17$, $\sin\beta = 3/5$, and both are in $(0, \pi/2)$ (arguments positive and less than 1).

Step 2. Compute the cosines. From the 8-15-17 triangle: $\cos\alpha = \sqrt{1-(8/17)^2} = \sqrt{1-64/289} = \sqrt{225/289} = 15/17$. From the 3-4-5 triangle: $\cos\beta = \sqrt{1-9/25} = \sqrt{16/25} = 4/5$.

Step 3. Sine of the sum:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta = \frac{8}{17}\cdot\frac{4}{5} + \frac{15}{17}\cdot\frac{3}{5} = \frac{32}{85} + \frac{45}{85} = \frac{77}{85}.$$

Step 4. Check that $\alpha + \beta$ is in the principal range of $\sin^{-1}$, i.e. $[-\pi/2, \pi/2]$. We have $\sin\alpha = 8/17 < 1/\sqrt{2}$, so $\alpha < \pi/4$. And $\sin\beta = 3/5 < 1/\sqrt{2}$, so $\beta < \pi/4$. Therefore $\alpha + \beta < \pi/2$, safely inside the principal range.

Step 5. Conclude:

$$\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}.$$

Final Answer: $\sin^{-1}\frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85}$.

☞ **Pythagorean triples**

Memorise: 3-4-5, 5-12-13, 8-15-17, 7-24-25, 20-21-29, 9-40-41. Every Exemplar inverse-trig sum question is built on one of these.

EXPERT'S SOLUTION : *Yash Banerjee, Ph.D Pure Mathematics, IISc Bangalore*

Pythagorean-triple angle. The two arguments come from the 8-15-17 and 3-4-5 triples; expand $\sin(\alpha+\beta)$ using those sides.

Concept used. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$. For $\alpha = \sin^{-1}(8/17)$ in $(0, \pi/2)$, use the 8-15-17 triangle. For $\beta = \sin^{-1}(3/5)$ in $(0, \pi/2)$, use the 3-4-5 triangle.

Step 1. Triangle 1: $\sin\alpha = 8/17, \cos\alpha = 15/17$.

Step 2. Triangle 2: $\sin\beta = 3/5, \cos\beta = 4/5$.

Step 3. Sum: $\sin(\alpha+\beta) = (8/17)(4/5) + (15/17)(3/5) = (32+45)/85 = 77/85$.

Step 4. Principal-range check: $\alpha + \beta < \pi/2$. Identity holds.

Step 5. Numerical check. $\sin^{-1}(8/17) \approx 0.490$ rad and $\sin^{-1}(3/5) \approx 0.644$ rad. Their sum is ≈ 1.134 rad. And $\sin^{-1}(77/85) \approx 1.134$ rad. They match to three decimal places, confirming the identity numerically.

Step 6. Symmetry remark. Both angles are acute; their sum is also acute since $\sin(\alpha + \beta) = 77/85 < 1$, so the principal-value identity holds without any π correction.

Why this matters. Memorising the 3-4-5, 5-12-13, and 8-15-17 Pythagorean triples lets you handle nearly every CBSE inverse-trig sum-identity question in seconds.

> **Final Answer:** $\sin^{-1}\dfrac{77}{85}$.

Q 2.15 **Show that** $\sin^{-1}\dfrac{5}{13} + \cos^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{63}{16}$.

SOLUTION

Concept used. Convert every term to a $\tan^{-1}$ using the right-triangle interpretation, then apply $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\dfrac{x+y}{1-xy}$ if $xy < 1$, else $\pi + \tan^{-1}\dfrac{x+y}{1-xy}$ if $xy > 1$ and both positive.

Step 1. Convert each. From the 5-12-13 triangle, $\sin^{-1}(5/13) = \tan^{-1}(5/12)$ (opposite 5, adjacent 12, hypotenuse 13).

Step 2. From the 3-4-5 triangle, $\cos^{-1}(3/5) = \tan^{-1}(4/3)$ (adjacent 3, opposite 4, hypotenuse 5).

Step 3. So we need $\tan^{-1}(5/12) + \tan^{-1}(4/3)$. Check $xy = \dfrac{5}{12} \cdot \dfrac{4}{3} = \dfrac{20}{36} = \dfrac{5}{9} < 1$, so the unmodified formula applies:

$$\tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3} = \tan^{-1}\frac{5/12 + 4/3}{1 - 5/9} = \tan^{-1}\frac{(5+16)/12}{4/9}.$$

Step 4. Simplify the fraction:

$$\frac{21/12}{4/9} = \frac{21}{12} \cdot \frac{9}{4} = \frac{189}{48} = \frac{63}{16}.$$

Step 5. Hence $\sin^{-1}\dfrac{5}{13} + \cos^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{63}{16}$.

> **Final Answer:** $\sin^{-1}\dfrac{5}{13} + \cos^{-1}\dfrac{3}{5} = \tan^{-1}\dfrac{63}{16}$.

☞ **Conversion formulas**

$\sin^{-1} p = \tan^{-1}\frac{p}{\sqrt{1-p^2}}$ and $\cos^{-1} q = \tan^{-1}\frac{\sqrt{1-q^2}}{q}$ for the right ranges. Always reduce to a single inverse family before adding.

EXPERT'S SOLUTION *: Ankit Verma, B.Tech Engineering Physics, IIT Bombay*

Reduce-to-tan angle. Inverse-trig sums become trivial once every term is a $\tan^{-1}$.
Concept used. Triangle conversion: from sides $(p, \sqrt{1-p^2}, 1)$ at appropriate positions, read off the $\tan^{-1}$ equivalent of any $\sin^{-1} p$ or $\cos^{-1} p$.

Step 1. $\sin^{-1}(5/13) \to$ 5-12-13 triangle $\to \tan^{-1}(5/12)$.

Step 2. $\cos^{-1}(3/5) \to$ 3-4-5 triangle $\to \tan^{-1}(4/3)$.

Step 3. $\tan^{-1}(5/12) + \tan^{-1}(4/3)$ with $xy = 5/9 < 1$ gives
$\tan^{-1}\frac{5/12 + 4/3}{1 - 5/9} = \tan^{-1}\frac{21/12}{4/9} = \tan^{-1}(63/16)$.

Why this matters. The conversion to $\tan^{-1}$ is universally useful and reuses the same Pythagorean-triple library you built for Q14.

Final Answer: $\tan^{-1}\frac{63}{16}$.

Q 2.16 **Prove that** $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \sin^{-1}\frac{1}{\sqrt{5}}$.

SOLUTION

Concept used. Use the $\tan^{-1}$ sum formula (valid since $xy = (1/4)(2/9) = 1/18 < 1$), then convert the resulting $\tan^{-1}$ to a $\sin^{-1}$ using the right-triangle interpretation.

Step 1. Apply the sum formula:

$$\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\frac{1/4 + 2/9}{1 - (1/4)(2/9)}.$$

Step 2. Simplify the numerator: $\frac{1}{4} + \frac{2}{9} = \frac{9+8}{36} = \frac{17}{36}$. And the denominator: $1 - \frac{2}{36} = 1 - \frac{1}{18} = \frac{17}{18}$.

Step 3. Divide: $\frac{17/36}{17/18} = \frac{17}{36} \cdot \frac{18}{17} = \frac{18}{36} = \frac{1}{2}$.

Step 4. So $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \tan^{-1}\frac{1}{2}$.

Step 5. Convert: if $\theta = \tan^{-1}(1/2)$ with $\theta \in (0, \pi/2)$, the right triangle has opposite 1, adjacent 2, hypotenuse $\sqrt{1+4} = \sqrt{5}$. So $\sin\theta = \dfrac{1}{\sqrt{5}}$, i.e. $\theta = \sin^{-1}(1/\sqrt{5})$.

Step 6. Therefore $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \sin^{-1}\dfrac{1}{\sqrt{5}}$.

Final Answer: $\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9} = \sin^{-1}\dfrac{1}{\sqrt{5}}$.

☞ **Conversion strategy**

When the question requires a $\sin^{-1}$ in the final answer but the natural route produces $\tan^{-1}$, use $\tan^{-1} t = \sin^{-1}(t/\sqrt{1+t^2})$ for $t > 0$. Always check the sign convention.

EXPERT'S SOLUTION *: Aditya Pillai, M.Sc Mathematics, IIT Bombay*

Two-step reduction angle. Sum first, convert second.

Concept used. $\tan^{-1}$ addition with $xy < 1$, then $\tan^{-1} t = \sin^{-1}\dfrac{t}{\sqrt{1+t^2}}$ for $t > 0$.

Step 1. Sum: $(1/4 + 2/9)/(1 - 1/18) = (17/36)/(17/18) = 1/2$. So the sum is $\tan^{-1}(1/2)$.

Step 2. Convert: with $t = 1/2$, $\sin^{-1}\dfrac{1/2}{\sqrt{1+1/4}} = \sin^{-1}\dfrac{1/2}{\sqrt{5}/2} = \sin^{-1}\dfrac{1}{\sqrt{5}}$.

Step 3. Cross-check by computing sin of the result. $\sin^{-1}(1/\sqrt{5}) = \theta$ means $\sin\theta = 1/\sqrt{5}$, so on a right triangle with opposite 1 and hypotenuse $\sqrt{5}$, the adjacent side is 2, and $\tan\theta = 1/2$. This matches our $\tan^{-1}(1/2)$ exactly. Identity verified.

Why this matters. The $\tan^{-1} \to \sin^{-1}$ conversion via $\sqrt{1+t^2}$ is asked in CBSE every few years; reading it off the right triangle prevents memorisation errors.

Final Answer: $\sin^{-1}\dfrac{1}{\sqrt{5}}$.

Q 2.17 **Find the value of** $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239}$.

SOLUTION

Concept used. The classical Machin formula. We apply $2\tan^{-1} t = \tan^{-1}\dfrac{2t}{1-t^2}$ twice on $\tan^{-1}(1/5)$ to convert $4\tan^{-1}(1/5)$ into a single $\tan^{-1}$, then subtract using

$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\dfrac{x-y}{1+xy}$ (valid here since $xy > -1$).

Step 1. First double-angle (with $t = 1/5$):

$$2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{2/5}{1-1/25} = \tan^{-1}\frac{2/5}{24/25} = \tan^{-1}\frac{2}{5}\cdot\frac{25}{24} = \tan^{-1}\frac{5}{12}.$$

Step 2. Second double-angle (with $t = 5/12$):

$$2\tan^{-1}\frac{5}{12} = \tan^{-1}\frac{2\cdot 5/12}{1-25/144} = \tan^{-1}\frac{5/6}{119/144} = \tan^{-1}\frac{5}{6}\cdot\frac{144}{119} = \tan^{-1}\frac{120}{119}.$$

Step 3. Thus $4\tan^{-1}(1/5) = \tan^{-1}(120/119)$.

Step 4. Subtract $\tan^{-1}(1/239)$. With $x = 120/119, y = 1/239$:

$$\frac{x-y}{1+xy} = \frac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}.$$

Numerator: common denominator $119\cdot 239 = 28441$. $\dfrac{120\cdot 239-119}{28441} = \dfrac{28680-119}{28441} = \dfrac{28561}{28441}$. Denominator: $1+\dfrac{120}{28441} = \dfrac{28441+120}{28441} = \dfrac{28561}{28441}$.

Step 5. Therefore $\dfrac{x-y}{1+xy} = \dfrac{28561}{28441}\cdot\dfrac{28441}{28561} = 1$, so the answer is $\tan^{-1} 1 = \dfrac{\pi}{4}$.

Final Answer: $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$.

♥ Machin's formula

This is John Machin's 1706 formula, the first practical way to compute π to many digits. He used it to compute π to 100 decimal places by hand. The same identity is still the backbone of several modern π-computation programs.

EXPERT'S SOLUTION : *Rahul Nair, Ph.D Mathematics, IIT Bombay*

Double-then-double angle. Two applications of $2\tan^{-1} t = \tan^{-1}\dfrac{2t}{1-t^2}$ collapse $4\tan^{-1}(1/5)$ to $\tan^{-1}(120/119)$.

Concept used. Double-angle on $\tan^{-1}$ chained twice, then the subtraction formula on $\tan^{-1}$.

Step 1. $2\tan^{-1}(1/5) = \tan^{-1}(5/12)$ (from $2/5 \div 24/25$).

Step 2. $2\tan^{-1}(5/12) = \tan^{-1}(120/119)$ (from $5/6 \div 119/144$).

Step 3. $\tan^{-1}(120/119) - \tan^{-1}(1/239)$: the fraction $(120/119 - 1/239)/(1 + 120/(119 \cdot 239))$ equals 1 exactly.

Step 4. Result: $\tan^{-1} 1 = \pi/4$.

Step 5. Sanity check. After two double-angles, $4\tan^{-1}(1/5) \approx 4 \times 0.1974 = 0.7897$ rad and $\tan^{-1}(120/119) \approx 0.7897$ rad. Agreement to four decimals.

Step 6. Why 239? The final fraction's numerator and denominator both equal $28561/28441$ because $120 \cdot 239 = 28680 = 28561 + 119$ and $28441 + 120 = 28561$. The cancellation that gives exactly $\pi/4$ is engineered into the choice of 239.

Why this matters. 239 is not a coincidence: it is chosen so the denominator of the final fraction matches the numerator. The problem is engineered to give $\pi/4$ exactly.

Final Answer: $\frac{\pi}{4}$.

Q 2.18 **Show that** $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$**, and justify why the other value** $\frac{4+\sqrt{7}}{3}$ **is ignored.**

SOLUTION

Concept used. If $\sin^{-1}(3/4) = 2\alpha$ then $\sin 2\alpha = 3/4$ with $2\alpha \in [0, \pi/2]$ (positive argument), so $\alpha \in [0, \pi/4]$. Use the half-angle identity $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$ to set up a quadratic in $t = \tan\alpha$.

Step 1. Let $\alpha = \frac{1}{2}\sin^{-1}(3/4)$, $t = \tan\alpha$. From $\sin^{-1}(3/4) = 2\alpha$:

$$\sin 2\alpha = \frac{3}{4} \;\Rightarrow\; \frac{2t}{1+t^2} = \frac{3}{4}.$$

Step 2. Cross-multiply: $8t = 3(1+t^2)$, i.e. $3t^2 - 8t + 3 = 0$.

Step 3. Quadratic formula: $t = \frac{8 \pm \sqrt{64-36}}{6} = \frac{8 \pm \sqrt{28}}{6} = \frac{8 \pm 2\sqrt{7}}{6} = \frac{4 \pm \sqrt{7}}{3}$.

Step 4. **Which root?** Since $2\alpha = \sin^{-1}(3/4) \in [0, \pi/2]$, we have $\alpha \in [0, \pi/4]$, so $t = \tan\alpha \in [0, 1]$. Compare the two candidates:

$$\frac{4-\sqrt{7}}{3} \approx \frac{4-2.6458}{3} \approx \frac{1.354}{3} \approx 0.451 \in [0,1]. \checkmark$$

$$\frac{4+\sqrt{7}}{3} \approx \frac{6.646}{3} \approx 2.215 \notin [0,1]. \text{ rejected.}$$

Step 5. Therefore the only admissible root is $\tan\alpha = \frac{4-\sqrt{7}}{3}$.

Final Answer: $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$.

✗ Common Pitfall

The quadratic produces two algebraic roots, but only one corresponds to the principal half-angle. Always go back to the range of the original inverse to pick the right root. Forgetting this gives partial marks.

EXPERT'S SOLUTION : *Pooja Chatterjee, M.Sc Mathematics, IIT Bombay*

Half-angle quadratic angle. Set $\sin 2\alpha = 3/4$ and solve the resulting quadratic in $\tan\alpha$; reject the root outside the admissible range.

Concept used. $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$; principal range constraints on $\sin^{-1}$ force $2\alpha \in [-\pi/2, \pi/2]$, hence $\alpha \in [-\pi/4, \pi/4]$.

Step 1. Quadratic: $3t^2 - 8t + 3 = 0 \Rightarrow t = (4 \pm \sqrt{7})/3$.

Step 2. Range: $\sin^{-1}(3/4) > 0$, so $\alpha \in (0, \pi/4)$, hence $t \in (0, 1)$.

Step 3. $(4-\sqrt{7})/3 \approx 0.45$ - admissible. $(4+\sqrt{7})/3 \approx 2.22$ - rejected.

Step 4. Numerical cross-check. $(4-\sqrt{7})/3 \approx (4-2.6458)/3 \approx 0.4514$. And $\sin^{-1}(3/4) \approx 0.8481$ rad, half is ≈ 0.4240 rad, $\tan(0.4240) \approx 0.4515$. The roots match to three decimals.

Why this matters. The rejection step is worth half the marks in many board papers; never write both roots as the final answer.

Final Answer: $\frac{4-\sqrt{7}}{3}$.

Q 2.19 **If $a_1, a_2, a_3, \ldots, a_n$ is an arithmetic progression with common difference d, then evaluate** $\tan\left[\tan^{-1}\frac{d}{1+a_1a_2} + \tan^{-1}\frac{d}{1+a_2a_3} + \cdots + \tan^{-1}\frac{d}{1+a_{n-1}a_n}\right]$.

SOLUTION

Concept used. The telescoping identity $\tan^{-1} a_{k+1} - \tan^{-1} a_k = \tan^{-1}\frac{a_{k+1}-a_k}{1+a_ka_{k+1}} = \tan^{-1}\frac{d}{1+a_ka_{k+1}}$ because the AP has $a_{k+1} - a_k = d$. Summing this from $k=1$ to $k=n-1$ telescopes into a single difference.

Step 1. Recognise the general term:

$$\tan^{-1}\frac{d}{1+a_k a_{k+1}} = \tan^{-1}\frac{a_{k+1}-a_k}{1+a_k a_{k+1}} = \tan^{-1} a_{k+1} - \tan^{-1} a_k.$$

(Provided each pair (a_k, a_{k+1}) satisfies the $\tan^{-1}$ difference-formula condition $a_k a_{k+1} > -1$, which holds generically.)

Step 2. Sum from $k = 1$ to $k = n - 1$:

$$S = \sum_{k=1}^{n-1}(\tan^{-1} a_{k+1} - \tan^{-1} a_k) = \tan^{-1} a_n - \tan^{-1} a_1.$$

(All intermediate $\tan^{-1} a_2, \ldots, \tan^{-1} a_{n-1}$ cancel.)

Step 3. Apply the outer $\tan$:

$$\tan S = \tan(\tan^{-1} a_n - \tan^{-1} a_1) = \frac{a_n - a_1}{1 + a_1 a_n}.$$

Step 4. Use $a_n = a_1 + (n-1)d$, so $a_n - a_1 = (n-1)d$:

$$\tan S = \frac{(n-1)d}{1 + a_1 a_n}.$$

Final Answer: $\tan\left[\sum_{k=1}^{n-1} \tan^{-1}\frac{d}{1+a_k a_{k+1}}\right] = \frac{(n-1)d}{1+a_1 a_n}.$

☞ Telescoping pattern

Any sum that can be rewritten as $\sum_k (f(k+1) - f(k))$ collapses to $f(n) - f(1)$. Spotting that pattern inside a $\tan^{-1}$ sum is the single most powerful trick on this chapter.

EXPERT'S SOLUTION *: Dev Rao, M.Tech CS, IIT Madras*

Telescoping angle. Each term is the $\tan^{-1}$ difference of consecutive AP elements; the sum collapses to two boundary terms.

Concept used. $\tan^{-1} p - \tan^{-1} q = \tan^{-1}\frac{p-q}{1+pq}$ for $pq > -1$. Reading right-to-left identifies each summand as such a difference.

Step 1. $\tan^{-1}\frac{d}{1+a_k a_{k+1}} = \tan^{-1} a_{k+1} - \tan^{-1} a_k$ because $d = a_{k+1} - a_k$.

Step 2. Sum telescopes to $\tan^{-1} a_n - \tan^{-1} a_1$.

Step 3. $\tan$ of the difference: $\frac{a_n - a_1}{1 + a_1 a_n} = \frac{(n-1)d}{1 + a_1 a_n}$.

Step 4. Boundary verification. For $n = 2$ the sum has one term:

$\tan^{-1}\dfrac{d}{1+a_1a_2} = \tan^{-1}a_2 - \tan^{-1}a_1$, and tan of this is $\dfrac{a_2-a_1}{1+a_1a_2} = \dfrac{(2-1)d}{1+a_1a_2}$, matching the general formula $(n-1)d/(1+a_1a_n)$ with $a_n = a_2$. The formula passes its own boundary test.

Why this matters. Series of inverse trig terms almost always hide a telescoping structure. When the kth term involves a_ka_{k+1} in the denominator and the AP difference d in the numerator, you know exactly what to do.

Final Answer: $\dfrac{(n-1)d}{1+a_1a_n}$.

Objective Type Questions (MCQ)

Q 2.20 **Which of the following is the principal value branch of $\cos^{-1}x$?**

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **(B)** $(0, \pi)$

(C) $[0, \pi]$ **(D)** $(0, \pi) - \left\{\frac{\pi}{2}\right\}$

SOLUTION

Correct option: (C) $[0, \pi]$.

Concept used. The principal value branch of an inverse trig function is the range of y-values that the inverse outputs. For $\cos^{-1} : [-1, 1] \to [0, \pi]$, the range is the closed interval $[0, \pi]$. This is the standard convention adopted by NCERT and CBSE.

Step 1. $\cos$ is one-one on $[0, \pi]$ (strictly decreasing from 1 to -1), hence invertible on that interval.

Step 2. The inverse $\cos^{-1} : [-1, 1] \to [0, \pi]$ has range $[0, \pi]$, i.e. a closed interval including the endpoints.

Step 3. Compare with the options. Only (C) gives this exact set. Option (A) is the $\sin^{-1}$ range; (B) is the $\cot^{-1}$ range (open); (D) is the $\sec^{-1}$ range with the wrong endpoint open/closed style.

Final Answer: Option **(C)**: $[0, \pi]$.

☞ **Six principal ranges**

$\sin^{-1} : [-\pi/2, \pi/2]$, $\cos^{-1} : [0, \pi]$, $\tan^{-1} : (-\pi/2, \pi/2)$, $\cot^{-1} : (0, \pi)$, $\sec^{-1} : [0, \pi]\setminus\{\pi/2\}$, $\csc^{-1} : [-\pi/2, \pi/2]\setminus\{0\}$.

EXPERT'S SOLUTION : *Siddharth Mehta, M.Sc Mathematics, IIT Bombay*

Memorise-the-table angle. Each inverse trig has a unique principal range; the only way to win this question is to know the table cold.
Concept used. $\cos$ is monotone decreasing on $[0, \pi]$ from 1 to -1, covering $[-1, 1]$ exactly. Inverting gives the range $[0, \pi]$.

Step 1. $\cos^{-1} : [-1, 1] \to [0, \pi]$ is the standard branch.

Step 2. Cross-check distractors: (A) belongs to $\sin^{-1} / \csc^{-1}$; (B), (D) are open or modified - both wrong.

Why this matters. CBSE often tests one of the six branches per paper; (C) is the only closed interval $[0, \pi]$ in the table.

> **Final Answer:** Option **(C)**.

Q 2.21 **Which of the following is the principal value branch of $\csc^{-1} x$?**

(A) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **(B)** $[0, \pi] - \left\{\frac{\pi}{2}\right\}$

(C) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ **(D)** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

SOLUTION

Correct option: (D) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.
Concept used. $\csc x = 1/\sin x$. Where $\sin x = 0$ (i.e. at $x = 0$ inside the would-be branch), $\csc x$ is undefined. So the principal range for $\csc^{-1}$ is $[-\pi/2, \pi/2]$ with 0 excluded. The domain of $\csc^{-1}$ is $\mathbb{R} \setminus (-1, 1)$.

Step 1. Start with the $\sin^{-1}$ principal range $[-\pi/2, \pi/2]$.

Step 2. Remove $\{0\}$ (where $\sin$ is zero, so $\csc$ is undefined).

Step 3. Result: $[-\pi/2, \pi/2] \setminus \{0\}$, which is option (D).

Step 4. Option (A) is the $\tan^{-1}$ range; (B) is for $\sec^{-1}$; (C) is the $\sin^{-1}$ range without removing 0.

> **Final Answer:** Option **(D)**.

☞ **Six principal ranges**

$\sin^{-1}$, $\csc^{-1}$ centre around 0; $\cos^{-1}$, $\sec^{-1}$, $\cot^{-1}$ centre around $\pi/2$. $\csc^{-1}$ excludes 0; $\sec^{-1}$ excludes $\pi/2$. $\cot^{-1}$ is open at both endpoints.

EXPERT'S SOLUTION *: Aanya Iyer, M.Sc Mathematics, IIT Bombay*

Reciprocal-then-restrict angle. Take the $\sin^{-1}$ range, delete the singularity of csc.
Concept used. Whenever you build the inverse of a reciprocal trig function, start from the base principal range and remove the zero of the underlying trig (which becomes the pole of the reciprocal).

Step 1. Base range from $\sin^{-1}$: $[-\pi/2, \pi/2]$.

Step 2. Pole of csc inside the range: $x = 0$.

Step 3. Delete it: $[-\pi/2, \pi/2] \setminus \{0\}$.

Why this matters. The same logic gives the $\sec^{-1}$ range: $[0, \pi] \setminus \{\pi/2\}$. Same recipe, different base range.

> **Final Answer:** Option **(D)**.

Q 2.22 **If $3\tan^{-1} x + \cot^{-1} x = \pi$, then x equals**

(A) 0 **(B)** 1 **(C)** -1 **(D)** $\frac{1}{2}$

SOLUTION

Correct option: (B) 1.
Concept used. The complementary identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$ for all $x \in \mathbb{R}$. We rewrite the equation in terms of $\tan^{-1} x$ alone using this relation.

Step 1. Write $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$.

Step 2. Substitute:

$$3\tan^{-1} x + \frac{\pi}{2} - \tan^{-1} x = \pi \Rightarrow 2\tan^{-1} x = \frac{\pi}{2}.$$

Step 3. Solve: $\tan^{-1} x = \frac{\pi}{4}$, so $x = \tan\frac{\pi}{4} = 1$.

Step 4. Verify: $3 \cdot \frac{\pi}{4} + \frac{\pi}{4} = \pi$. ✓

> **Final Answer:** Option **(B)**: $x = 1$.

☞ Eliminate one inverse

Whenever a sum/multiple of $\tan^{-1} x$ and $\cot^{-1} x$ appears, eliminate $\cot^{-1} x$ via the complementary identity. The equation collapses to a linear equation in $\tan^{-1} x$.

EXPERT'S SOLUTION : *Rohit Sharma, M.Sc Mathematics, IIT Bombay*

Reduce-to-one-inverse angle. Use the complementary identity to eliminate $\cot^{-1}$.
Concept used. $\tan^{-1} x + \cot^{-1} x = \pi/2$ everywhere on $\mathbb{R}$.

Step 1. $3\tan^{-1} x + (\pi/2 - \tan^{-1} x) = \pi \Rightarrow 2\tan^{-1} x = \pi/2$.

Step 2. $\tan^{-1} x = \pi/4 \Rightarrow x = 1$.

Why this matters. The identity holds for all real x, unlike the analogous $\sin^{-1} x + \cos^{-1} x = \pi/2$ which needs $x \in [-1, 1]$.

> **Final Answer:** Option **(B)**.

Q 2.23 **The value of** $\sin^{-1}\left[\cos\left(\frac{33\pi}{5}\right)\right]$ **is**

(A) $\frac{3\pi}{5}$ **(B)** $-\frac{7\pi}{5}$ **(C)** $\frac{\pi}{10}$ **(D)** $-\frac{\pi}{10}$

SOLUTION

Correct option: (D) $-\frac{\pi}{10}$.
Concept used. Reduce the inner cosine angle modulo 2π, then convert $\cos$ to $\sin$ via $\cos\theta = \sin(\pi/2 - \theta)$ so we can apply $\sin^{-1}(\sin\alpha) = \alpha$ within the principal range $[-\pi/2, \pi/2]$.

Step 1. $\frac{33\pi}{5} = \frac{30\pi + 3\pi}{5} = 6\pi + \frac{3\pi}{5}$. Since $\cos$ has period 2π:

$$\cos\frac{33\pi}{5} = \cos\left(6\pi + \frac{3\pi}{5}\right) = \cos\frac{3\pi}{5}.$$

Step 2. Convert to sin: $\cos\frac{3\pi}{5} = \sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right) = \sin\left(-\frac{\pi}{10}\right)$.

Step 3. Since $-\frac{\pi}{10} \in [-\pi/2, \pi/2]$, we may cancel:

$$\sin^{-1}\left[\sin\left(-\frac{\pi}{10}\right)\right] = -\frac{\pi}{10}.$$

> **Final Answer:** Option **(D)**: $-\frac{\pi}{10}$.

☞ **Co-function bridge**

When the outer inverse is $\sin^{-1}$ but the inner trig is $\cos$ (or vice versa), bridge via the co-function identity $\cos\theta = \sin(\pi/2 - \theta)$. The inverse then cancels cleanly.

EXPERT'S SOLUTION *: Priya Singh, M.Sc Mathematics, ISI Kolkata*

Modulo-then-co-function angle. Take the inner angle modulo the period, then bridge to the outer inverse's family.

Concept used. $\cos\theta = \sin(\pi/2 - \theta)$ converts cosines to sines, letting us use $\sin^{-1}(\sin\alpha) = \alpha$ on $[-\pi/2, \pi/2]$.

Step 1. Reduce: $33\pi/5 \bmod 2\pi = 3\pi/5$.

Step 2. $\cos(3\pi/5) = \sin(\pi/2 - 3\pi/5) = \sin(-\pi/10)$.

Step 3. $\sin^{-1}(\sin(-\pi/10)) = -\pi/10$ since $-\pi/10 \in [-\pi/2, \pi/2]$.

Why this matters. Bridging $\sin \leftrightarrow \cos$ via the co-function identity is essential when the outer inverse is from a different family than the inner trig.

Final Answer: Option **(D)**.

Q 2.24 **The domain of the function $\cos^{-1}(2x-1)$ is**

(A) $[0, 1]$ **(B)** $[-1, 1]$ **(C)** $(-1, 1)$ **(D)** $[0, \pi]$

SOLUTION

Correct option: (A) $[0, 1]$.

Concept used. The domain of $\cos^{-1} u$ is $u \in [-1, 1]$. Substituting $u = 2x - 1$, we need $-1 \le 2x - 1 \le 1$, then solve for x.

Step 1. Set up: $-1 \le 2x - 1 \le 1$.

Step 2. Add 1: $0 \le 2x \le 2$.

Step 3. Divide by 2: $0 \le x \le 1$.

Step 4. So the domain is $[0, 1]$. Option (D) $[0, \pi]$ is the range, not the domain - distractor trap.

Final Answer: Option **(A)**: $[0, 1]$.

✗ Domain vs range

The domain is the set of permissible x-values (inputs), measured on the x-axis. The range $[0, \pi]$ of $\cos^{-1}$ is the set of output angles, measured on the y-axis. Never confuse the two.

EXPERT'S SOLUTION *: Ananya Patel, M.Sc Mathematics, IIT Bombay*

Solve-the-inequality angle. Domain conditions are pure linear algebra.
Concept used. $\cos^{-1}$ accepts inputs in $[-1, 1]$, so the inner expression must lie there.

Step 1. $-1 \leq 2x - 1 \leq 1$.

Step 2. $0 \leq 2x \leq 2$, i.e. $0 \leq x \leq 1$.

Why this matters. The trap is to confuse domain (input set, on the x-axis) with range (output set, on the y-axis).

Final Answer: Option **(A)**.

Q 2.25 **The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is**

(A) $[1, 2]$ **(B)** $[-1, 1]$ **(C)** $[0, 1]$ **(D) none of these**

SOLUTION

Correct option: (A) $[1, 2]$.
Concept used. Two constraints must hold simultaneously: (i) the radicand $x - 1 \geq 0$ for the square root, and (ii) the argument of $\sin^{-1}$ must lie in $[-1, 1]$, so $\sqrt{x-1} \in [0, 1]$, i.e. $0 \leq x - 1 \leq 1$.

Step 1. Surd: $x - 1 \geq 0 \Rightarrow x \geq 1$.

Step 2. $\sin^{-1}$ argument: $0 \leq \sqrt{x-1} \leq 1$. Squaring: $0 \leq x - 1 \leq 1 \Rightarrow 1 \leq x \leq 2$.

Step 3. Intersect: $x \in [1, 2]$.

Final Answer: Option **(A)**: $[1, 2]$.

✗ Common Pitfall

Forgetting the surd domain $x - 1 \geq 0$ leads people to write $[-1, 1]$ mechanically. Both constraints (the surd AND the $\sin^{-1}$ argument) must hold.

EXPERT'S SOLUTION : *Meera Joshi, Ph.D Pure Mathematics, IISc Bangalore*

Two-constraint angle. Intersect the surd domain with the $\sin^{-1}$ domain.

Concept used. Composite functions need every layer's domain condition simultaneously.

Step 1. Surd: $x \geq 1$.

Step 2. Outer $\sin^{-1}$: $\sqrt{x-1} \leq 1 \Rightarrow x \leq 2$. Lower bound $\sqrt{x-1} \geq 0$ is automatic.

Step 3. $x \in [1, 2]$.

Why this matters. Composite-domain questions are a CBSE favourite. Always list every layer's constraint and intersect.

> **Final Answer:** Option **(A)**.

Q 2.26 **If** $\cos\left(\sin^{-1}\frac{2}{5} + \cos^{-1} x\right) = 0$, **then** x **is equal to**

(A) $\frac{1}{5}$ **(B)** $\frac{2}{5}$ **(C)** 0 **(D)** 1

SOLUTION

Correct option: (B) $\frac{2}{5}$.

Concept used. $\cos\theta = 0 \Leftrightarrow \theta = \frac{\pi}{2} + k\pi$. On the relevant range (since both inverses give angles in $[0, \pi]$ when their arguments are non-negative), the inner expression equals $\frac{\pi}{2}$. Combined with $\sin^{-1} u + \cos^{-1} u = \frac{\pi}{2}$, this forces $u = 2/5 = x$.

Step 1. Set $\sin^{-1}(2/5) + \cos^{-1} x = \frac{\pi}{2}$ (the only value of the inner sum in $[0, \pi]$ giving $\cos = 0$).

Step 2. Use the identity $\sin^{-1}(2/5) + \cos^{-1}(2/5) = \frac{\pi}{2}$. Subtracting from the equation: $\cos^{-1} x - \cos^{-1}(2/5) = 0$, hence $\cos^{-1} x = \cos^{-1}(2/5)$.

Step 3. Since $\cos^{-1}$ is one-one, $x = \frac{2}{5}$.

> **Final Answer:** Option **(B)**: $x = \frac{2}{5}$.

☞ **Match-to-identity**

Whenever an equation reduces to $\sin^{-1} u + \cos^{-1} v = \pi/2$, force $u = v$ via the complementary identity and you're done. This shortcut beats expanding $\cos$ of a sum.

EXPERT'S SOLUTION : *Aaditi Banerjee, M.Sc Mathematics, IIT Bombay*

Zero-of-cosine angle. Set the inner sum equal to $\pi/2$ and use the complementary identity.

Concept used. $\sin^{-1} u + \cos^{-1} u = \pi/2$ for $u \in [-1, 1]$.

Step 1. $\cos(\cdot) = 0 \Rightarrow \sin^{-1}(2/5) + \cos^{-1} x = \pi/2$.

Step 2. Compare with identity: must have $x = 2/5$.

Step 3. Cross-check by plugging $x = 2/5$ back. $\sin^{-1}(2/5) + \cos^{-1}(2/5) = \pi/2$ identically, so $\cos(\pi/2) = 0$ as required. The unique x satisfying the equation in the natural domain $[-1, 1]$ is therefore $x = 2/5$, no spurious solutions.

Step 4. Distractor check. Options (A)$= 1/5$ and (D)$= 1$ would give $\sin^{-1}(2/5) + \cos^{-1}(1/5) \neq \pi/2$, since $\sin^{-1}(2/5) \neq \sin^{-1}(1/5)$.

Why this matters. Matching to a known identity beats brute trig expansion every time.

> **Final Answer:** Option **(B)**.

Q 2.27 **The value of** $\sin(2\tan^{-1}(0.75))$ **is equal to**

(A) 0.75 **(B)** 1.5 **(C)** 0.96 **(D)** $\sin 1.5$

SOLUTION

Correct option: (C) 0.96.

Concept used. The double-angle identity $\sin(2\tan^{-1} t) = \dfrac{2t}{1+t^2}$.

Step 1. With $t = 0.75 = 3/4$:

$$\sin(2\tan^{-1}(0.75)) = \frac{2(3/4)}{1+(3/4)^2} = \frac{3/2}{1+9/16}.$$

Step 2. Simplify denominator: $1 + 9/16 = 25/16$.

Step 3. Divide: $\dfrac{3/2}{25/16} = \dfrac{3}{2} \cdot \dfrac{16}{25} = \dfrac{48}{50} = \dfrac{24}{25} = 0.96$.

> **Final Answer:** Option **(C)**: 0.96.

☞ **Double-angle shortcut**

$\sin(2\tan^{-1} t) = \dfrac{2t}{1+t^2}$, $\cos(2\tan^{-1} t) = \dfrac{1-t^2}{1+t^2}$. These two formulas pop up on nearly every paper.

EXPERT'S SOLUTION : *Dev Kapoor, M.Sc Mathematics, IIT Bombay*

Plug-and-go angle. Direct application of the formula.
Concept used. Double-angle on $\tan^{-1}$ for sine.

Step 1. $t = 3/4$. Formula gives $2t/(1+t^2) = (3/2)/(25/16) = 24/25$.

Step 2. $24/25 = 0.96$.

Why this matters. Recognising $3/4$ as a clean rational that gives $24/25$ via this formula is a fingerprint of the 3-4-5 triangle.

> **Final Answer:** Option **(C)**.

Q 2.28 **The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is equal to**

(A) $\frac{\pi}{2}$ **(B)** $\frac{3\pi}{2}$ **(C)** $\frac{5\pi}{2}$ **(D)** $\frac{7\pi}{2}$

SOLUTION

Correct option: (A) $\frac{\pi}{2}$.

Concept used. $\cos^{-1}(\cos\theta) = \theta$ only when $\theta \in [0, \pi]$. Since $\frac{3\pi}{2} \notin [0, \pi]$, reduce it into the principal range using the periodicity $\cos(\theta - 2\pi) = \cos\theta$ and the property $\cos(-\theta) = \cos\theta$.

Step 1. Period shift: $\cos\frac{3\pi}{2} = \cos\left(\frac{3\pi}{2} - 2\pi\right) = \cos\left(-\frac{\pi}{2}\right)$.

Step 2. Even-function: $\cos\left(-\frac{\pi}{2}\right) = \cos\frac{\pi}{2}$.

Step 3. Cancel: $\cos^{-1}\left(\cos\frac{\pi}{2}\right) = \frac{\pi}{2} \in [0, \pi]$.

> **Final Answer:** Option **(A)**: $\frac{\pi}{2}$.

✗ Period vs reflection

Reducing $3\pi/2$ via period gives $-\pi/2$; reflecting via $\cos(-\theta) = \cos\theta$ flips the sign. Use both, not just one. The principal-range angle is $\pi/2$, not $-\pi/2$.

EXPERT'S SOLUTION : *Ishita Nair, M.Sc Mathematics, IIT Bombay*

Period-and-parity angle. Two reductions: period of $\cos$ is 2π and $\cos$ is even.
Concept used. Reduce inner angle to $[0, \pi]$ before cancelling.

Step 1. $3\pi/2 - 2\pi = -\pi/2$.

Step 2. $\cos$ is even: $\cos(-\pi/2) = \cos(\pi/2) = 0$.

Step 3. $\cos^{-1}(0) = \pi/2$.

Why this matters. The answer is not $3\pi/2$. The inverse always lives in the principal range.

> **Final Answer:** Option **(A)**.

Q 2.29 **The value of the expression** $2\sec^{-1} 2 + \sin^{-1}\frac{1}{2}$ **is**

(A) $\frac{\pi}{6}$ **(B)** $\frac{5\pi}{6}$ **(C)** $\frac{7\pi}{6}$ **(D)** 1

SOLUTION

Correct option: (B) $\frac{5\pi}{6}$.
Concept used. $\sec^{-1} x = \cos^{-1}(1/x)$ for $|x| \geq 1$. Specifically $\sec^{-1} 2 = \cos^{-1}(1/2) = \pi/3$. And $\sin^{-1}(1/2) = \pi/6$.

Step 1. $\sec^{-1} 2 = \cos^{-1}(1/2) = \frac{\pi}{3}$.

Step 2. $2\sec^{-1} 2 = 2 \cdot \frac{\pi}{3} = \frac{2\pi}{3}$.

Step 3. $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

Step 4. Add: $\frac{2\pi}{3} + \frac{\pi}{6} = \frac{4\pi + \pi}{6} = \frac{5\pi}{6}$.

> **Final Answer:** Option **(B)**: $\frac{5\pi}{6}$.

☞ **Reciprocal trick**

$\sec^{-1} x = \cos^{-1}(1/x)$ and $\csc^{-1} x = \sin^{-1}(1/x)$ for $|x| \geq 1$. These conversions plug into the standard 30-60-90 table directly.

EXPERT'S SOLUTION : *Sanya Reddy, M.Sc Mathematics, IIT Bombay*

Reciprocal-conversion angle. Replace $\sec^{-1}$ by $\cos^{-1}$ of the reciprocal, then plug in standard values.

Concept used. $\sec^{-1} x = \cos^{-1}(1/x)$.

Step 1. $\sec^{-1} 2 = \cos^{-1}(1/2) = \pi/3$.

Step 2. Double: $2\pi/3$. Add $\sin^{-1}(1/2) = \pi/6$.

Step 3. Sum: $2\pi/3 + \pi/6 = 5\pi/6$.

Why this matters. The reciprocal-conversion formula $\sec^{-1} \to \cos^{-1}$ (and similarly $\csc^{-1} \to \sin^{-1}$, $\cot^{-1} \to \tan^{-1}$ for positive arguments) lets you reuse the standard 30-60-90 reference values.

Final Answer: Option **(B)**.

Q 2.30 If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ equals

(A) $\frac{\pi}{5}$ **(B)** $\frac{2\pi}{5}$ **(C)** $\frac{3\pi}{5}$ **(D)** π

SOLUTION

Correct option: (A) $\frac{\pi}{5}$.

Concept used. The complementary identity $\tan^{-1} t + \cot^{-1} t = \frac{\pi}{2}$ for all $t \in \mathbb{R}$. Therefore $\cot^{-1} t = \frac{\pi}{2} - \tan^{-1} t$.

Step 1. $\cot^{-1} x + \cot^{-1} y = \left(\frac{\pi}{2} - \tan^{-1} x\right) + \left(\frac{\pi}{2} - \tan^{-1} y\right)$.

Step 2. Simplify: $= \pi - (\tan^{-1} x + \tan^{-1} y) = \pi - \frac{4\pi}{5} = \frac{5\pi - 4\pi}{5} = \frac{\pi}{5}$.

Final Answer: Option **(A)**: $\frac{\pi}{5}$.

♥ Pair-sum identity

$\tan^{-1} x + \cot^{-1} x = \pi/2$ doubled gives $\sum_i(\tan^{-1} x_i + \cot^{-1} x_i) = n\pi/2$. This pair-sum trick converts unknown $\cot^{-1}$ sums into known $\tan^{-1}$ sums in one move.

EXPERT'S SOLUTION *: Aditi Pillai, M.Sc Mathematics, IIT Bombay*

Pair-substitution angle. Replace each $\cot^{-1}$ by $\pi/2 - \tan^{-1}$.
Concept used. $\tan^{-1} t + \cot^{-1} t = \pi/2$.

Step 1. Sum: $\cot^{-1} x + \cot^{-1} y = \pi - (\tan^{-1} x + \tan^{-1} y)$.

Step 2. Substitute: $= \pi - 4\pi/5 = \pi/5$.

Why this matters. The pair-sum identity reduces an unknown expression to a known one in one move.

> **Final Answer:** Option **(A)**.

Q 2.31 **If** $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, **where** $a, x \in (0,1)$, **then the value of** x **is**

(A) 0 **(B)** $\frac{a}{2}$ **(C)** a **(D)** $\frac{2a}{1-a^2}$

SOLUTION

Correct option: (D) $\frac{2a}{1-a^2}$.
Concept used. The half-angle / double-angle identities for $\tan^{-1}$:
$\sin^{-1}\frac{2a}{1+a^2} = 2\tan^{-1} a$ (for $a \in (0,1)$) and $\cos^{-1}\frac{1-a^2}{1+a^2} = 2\tan^{-1} a$ (for $a \geq 0$).
Similarly $\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1} x$ (for $|x| < 1$).

Step 1. Convert LHS: $2\tan^{-1} a + 2\tan^{-1} a = 4\tan^{-1} a$.

Step 2. Convert RHS: $2\tan^{-1} x$.

Step 3. Equate: $4\tan^{-1} a = 2\tan^{-1} x$, i.e. $\tan^{-1} x = 2\tan^{-1} a$.

Step 4. By double-angle (since $a \in (0,1)$, $|a| < 1$): $\tan^{-1} x = \tan^{-1}\frac{2a}{1-a^2}$, hence
$$x = \frac{2a}{1-a^2}.$$

> **Final Answer:** Option **(D)**: $x = \frac{2a}{1-a^2}$.

☞ **Trio of identities**
$\sin^{-1}\frac{2a}{1+a^2} = 2\tan^{-1} a$, $\cos^{-1}\frac{1-a^2}{1+a^2} = 2\tan^{-1} a$, $\tan^{-1}\frac{2a}{1-a^2} = 2\tan^{-1} a$. All three express the same double-angle in different families.

EXPERT'S SOLUTION : *Karan Gupta, M.Sc Mathematics, IIT Bombay*

Triple-identity angle. Recognise each piece as $2\tan^{-1}$ in disguise.
Concept used. All three Pythagorean-fraction forms collapse to the same $2\tan^{-1}$.

Step 1. LHS $= 2\tan^{-1} a + 2\tan^{-1} a = 4\tan^{-1} a$.

Step 2. RHS $= 2\tan^{-1} x$.

Step 3. Equation: $\tan^{-1} x = 2\tan^{-1} a$, so $x = \dfrac{2a}{1-a^2}$.

Why this matters. The Pythagorean-fraction pattern marks double-angles instantly. Train your eye for $2a/(1+a^2)$, $(1-a^2)/(1+a^2)$, $2a/(1-a^2)$ - they are everywhere.

Final Answer: Option **(D)**.

Q 2.32 **The value of** $\cot\left(\cos^{-1}\dfrac{7}{25}\right)$ **is**

(A) $\dfrac{25}{24}$ **(B)** $\dfrac{25}{7}$ **(C)** $\dfrac{24}{25}$ **(D)** $\dfrac{7}{24}$

SOLUTION

Correct option: (D) $\dfrac{7}{24}$.
Concept used. If $\theta = \cos^{-1}(7/25)$ then in a right triangle with adjacent 7, hypotenuse 25, the opposite side is $\sqrt{25^2-7^2} = \sqrt{625-49} = \sqrt{576} = 24$. So $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{7}{24}$.

Step 1. Build triangle: adj $= 7$, hyp $= 25$, opp $= \sqrt{625-49} = 24$. This is the 7-24-25 Pythagorean triple.

Step 2. Read $\cot\theta = \dfrac{\cos\theta}{\sin\theta} = \dfrac{7/25}{24/25} = \dfrac{7}{24}$.

Final Answer: Option **(D)**: $\dfrac{7}{24}$.

☞ **7-24-25 triangle**

$7^2 + 24^2 = 49 + 576 = 625 = 25^2$. The 7-24-25 right triangle is one of the basic Pythagorean triples that appears in this chapter.

EXPERT'S SOLUTION : *Vivaan Bhat, M.Sc Mathematics, IIT Bombay*

Triangle-side angle. Recognise the 7-24-25 triple, read $\cot$ off it.

Concept used. The 7-24-25 Pythagorean triple: $7^2 + 24^2 = 49 + 576 = 625 = 25^2$.

Step 1. adj $= 7$, hyp $= 25$, opp $= 24$.

Step 2. $\cot =$ adj/opp $= 7/24$.

Why this matters. Standard Pythagorean triples (3-4-5, 5-12-13, 8-15-17, 7-24-25, 20-21-29) save dozens of seconds per question.

> **Final Answer:** Option **(D)**.

Q 2.33 **The value of the expression** $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ **is**

(A) $2+\sqrt{5}$ **(B)** $\sqrt{5}-2$ **(C)** $\frac{\sqrt{5}+2}{2}$ **(D)** $\sqrt{5}+2$

SOLUTION

Correct option: (B) $\sqrt{5}-2$.

Concept used. Half-angle identity $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$ for $\theta \in [0,\pi]$ (so $\theta/2 \in [0,\pi/2]$ and the principal value is positive).

Step 1. Let $\theta = \cos^{-1}\frac{2}{\sqrt{5}}$, so $\cos\theta = \frac{2}{\sqrt{5}}$ and $\theta \in [0,\pi/2]$ (positive cosine).

Step 2. Apply the half-angle formula:

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-2/\sqrt{5}}{1+2/\sqrt{5}}} = \sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+2}}.$$

Step 3. Rationalise: multiply numerator and denominator under the radical by $\sqrt{5}-2$:

$$\frac{\sqrt{5}-2}{\sqrt{5}+2}\cdot\frac{\sqrt{5}-2}{\sqrt{5}-2} = \frac{(\sqrt{5}-2)^2}{5-4} = (\sqrt{5}-2)^2.$$

Step 4. Take square root: $\tan\frac{\theta}{2} = \sqrt{5}-2$ (positive since $\theta/2 \in [0,\pi/4]$).

> **Final Answer:** Option **(B)**: $\sqrt{5}-2$.

☞ **Rationalising squares**

If $\sqrt{(a-b)/(a+b)}$ appears in the answer, multiplying numerator and denominator by $(a-b)$ inside the radical often yields $(a-b)^2/(a^2-b^2)$. Take the square root and the radical disappears.

EXPERT'S SOLUTION *: Sneha Mehta, M.Sc Mathematics, IIT Kanpur*

Half-angle direct angle. Plug into the half-angle formula and rationalise the surd.
Concept used. $\tan(\theta/2) = \sqrt{(1-\cos\theta)/(1+\cos\theta)}$ for $\theta \in [0, \pi]$.

Step 1. $\cos\theta = 2/\sqrt{5}$.

Step 2. Half-angle: $\tan(\theta/2) = \sqrt{(\sqrt{5}-2)/(\sqrt{5}+2)} = \sqrt{(\sqrt{5}-2)^2/(5-4)} = \sqrt{5}-2$.

Step 3. Sanity-check numerically. $\cos^{-1}(2/\sqrt{5}) \approx 0.4636$ rad, so half-angle ≈ 0.2318 rad, and $\tan(0.2318) \approx 0.2361$. Compare with $\sqrt{5}-2 \approx 2.236-2 = 0.236$. Match to three decimal places.

Why this matters. Rationalising the conjugate $(\sqrt{5} \pm 2)$ yields a perfect square, eliminating the radical.

Final Answer: Option **(B)**.

Q 2.34 **If** $|x| \leq 1$**, then** $2\tan^{-1}x + \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ **is equal to**

(A) $4\tan^{-1}x$ **(B)** 0 **(C)** $\dfrac{\pi}{2}$ **(D)** π

SOLUTION

Correct option: (A) $4\tan^{-1}x$.
Concept used. For $|x| \leq 1$ (equivalently $-1 \leq x \leq 1$), the double-angle identity $\sin^{-1}\dfrac{2x}{1+x^2} = 2\tan^{-1}x$ holds without any $\pm\pi$ correction.

Step 1. Apply the identity: $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = 2\tan^{-1}x$.

Step 2. Add: $2\tan^{-1}x + 2\tan^{-1}x = 4\tan^{-1}x$.

Final Answer: Option **(A)**: $4\tan^{-1}x$.

✗ Common Pitfall

For $|x| > 1$ the identity $\sin^{-1}\dfrac{2x}{1+x^2} = 2\tan^{-1}x$ fails; you must add π (if $x > 1$) or subtract π (if $x < -1$). Always check $|x| \le 1$ before applying.

EXPERT'S SOLUTION *: Karan Verma, M.Sc Mathematics, IIT Bombay*

Identity-recognition angle. The expression is literally $2\tan^{-1}x + 2\tan^{-1}x$ in disguise.
Concept used. $\sin^{-1}(2x/(1+x^2)) = 2\tan^{-1}x$ for $|x| \le 1$.

Step 1. Identity replaces second term by $2\tan^{-1}x$.

Step 2. Sum $= 4\tan^{-1}x$.

Why this matters. Knowing the identity range $|x| \le 1$ prevents wrong $\pm\pi$ choices.

> **Final Answer:** Option **(A)**.

Q 2.35 If $\cos^{-1}\alpha + \cos^{-1}\beta + \cos^{-1}\gamma = 3\pi$, **then** $\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta)$ **equals**

(A) 0 **(B)** 1 **(C)** 6 **(D)** 12

SOLUTION

Correct option: (C) 6.
Concept used. Since $\cos^{-1}$ ranges over $[0, \pi]$, each term is at most π. The only way three such terms can sum to 3π is if each equals π, which forces $\alpha = \beta = \gamma = \cos\pi = -1$.

Step 1. Maximum of $\cos^{-1}$ is π. Sum equals 3π only when each addend equals π.

Step 2. So $\cos^{-1}\alpha = \cos^{-1}\beta = \cos^{-1}\gamma = \pi$, giving $\alpha = \beta = \gamma = -1$.

Step 3. Evaluate target expression with $\alpha = \beta = \gamma = -1$:
$$\alpha(\beta+\gamma) + \beta(\gamma+\alpha) + \gamma(\alpha+\beta) = -1(-2) - 1(-2) - 1(-2) = 2 + 2 + 2 = 6.$$

> **Final Answer:** Option **(C)**: 6.

☞ Extremal-value trick

When a sum equals the maximum possible, each addend must equal its own maximum. This converts a hard equation into three trivial ones.

EXPERT'S SOLUTION : *Rahul Iyer, M.Sc Mathematics, IIT Bombay*

Boundary-saturation angle. Three numbers in $[0, \pi]$ that sum to 3π must each be π.
Concept used. If $x_i \in [0, M]$ and $\sum x_i = nM$, then each $x_i = M$.

Step 1. Each $\cos^{-1} = \pi \Rightarrow \alpha = \beta = \gamma = -1$.

Step 2. Plug into target: $3 \times [(-1)(-2)] = 6$.

Why this matters. The boundary saturation argument shows up in JEE Advanced regularly; recognising it instantly is worth easy marks.

> **Final Answer:** Option **(C)**.

Q 2.36 **The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is**

(A) 0 **(B)** 1 **(C)** 2 **(D) infinite**

SOLUTION

Correct option: **(A)** 0.
Concept used. $1 + \cos 2x = 2\cos^2 x$, so the LHS becomes $\sqrt{2\cos^2 x} = \sqrt{2}\,|\cos x|$. On $[\pi/2, \pi]$, $\cos x \le 0$, so $|\cos x| = -\cos x$. The RHS uses $\cos^{-1}(\cos x) = x$ on $[0, \pi]$, so on $[\pi/2, \pi]$ that gives $\cos^{-1}(\cos x) = x$.

Step 1. LHS: $\sqrt{1+\cos 2x} = \sqrt{2}\,|\cos x| = -\sqrt{2}\cos x$ on $[\pi/2, \pi]$.

Step 2. RHS: $\sqrt{2}\cos^{-1}(\cos x) = \sqrt{2}\,x$ on $[\pi/2, \pi]$.

Step 3. Equation: $-\sqrt{2}\cos x = \sqrt{2}\,x$, i.e. $-\cos x = x$, or $x + \cos x = 0$.

Step 4. Analyse $f(x) = x + \cos x$ on $[\pi/2, \pi]$: $f(\pi/2) = \pi/2 + 0 = \pi/2 > 0$ and $f(\pi) = \pi + (-1) = \pi - 1 > 0$. The derivative $f'(x) = 1 - \sin x \ge 0$ (with $= 0$ only at $\pi/2$), so f is strictly positive on $[\pi/2, \pi]$, hence never zero.

Step 5. No solutions.

> **Final Answer:** Option **(A)**: 0 solutions.

♥ Absolute-value radicals

$\sqrt{u^2} = |u|$, not u. On intervals where the inner expression is negative, the absolute value flips the sign. Always honour the absolute value before simplifying.

EXPERT'S SOLUTION : *Aarav Joshi, Ph.D Mathematics, IIT Delhi*

Reduce-then-test angle. Simplify both sides on the given interval, then look for sign agreement.

Concept used. $\sqrt{1+\cos 2x} = \sqrt{2}|\cos x|$; $\cos^{-1}(\cos x) = x$ on $[0, \pi]$.

Step 1. On $[\pi/2, \pi]$, LHS $= -\sqrt{2}\cos x \geq 0$ and RHS $= \sqrt{2}x > 0$.

Step 2. Equation: $-\cos x = x$, equivalently $x + \cos x = 0$.

Step 3. $f(x) = x + \cos x > 0$ on $[\pi/2, \pi]$ since $f(\pi/2) = \pi/2 > 0$ and f is non-decreasing there.

Step 4. Zero solutions.

Why this matters. The classical trick is to simplify radicals of $1 \pm \cos 2x$ to $\sqrt{2}|\cos x|$ or $\sqrt{2}|\sin x|$. Always honour the absolute value on the interval of interest.

Final Answer: Option **(A)**.

Q 2.37 If $\cos^{-1} x > \sin^{-1} x$, then

(A) $\frac{1}{\sqrt{2}} < x \leq 1$ **(B)** $0 \leq x < \frac{1}{\sqrt{2}}$

(C) $-1 \leq x < \frac{1}{\sqrt{2}}$ **(D)** $x > 0$

SOLUTION

Correct option: (C) $-1 \leq x < \frac{1}{\sqrt{2}}$.

Concept used. The functions $\cos^{-1} x$ (strictly decreasing from π to 0 on $[-1, 1]$) and $\sin^{-1} x$ (strictly increasing from $-\pi/2$ to $\pi/2$) cross exactly where they are equal, which is where $x = \sin\alpha = \cos\alpha$, i.e. $\alpha = \pi/4$, so $x = 1/\sqrt{2}$.

Step 1. At $x = 1/\sqrt{2}$: $\cos^{-1}(1/\sqrt{2}) = \sin^{-1}(1/\sqrt{2}) = \pi/4$. Equal.

Step 2. For $x < 1/\sqrt{2}$: $\cos^{-1} x > \pi/4 > \sin^{-1} x$, since $\cos^{-1}$ decreases (as x decreases from $1/\sqrt{2}$, $\cos^{-1} x$ increases above $\pi/4$) and $\sin^{-1}$ increases (as x decreases, $\sin^{-1} x$ decreases below $\pi/4$).

Step 3. For $x > 1/\sqrt{2}$: opposite inequality, $\cos^{-1} x < \sin^{-1} x$.

Step 4. Combined with the domain $[-1, 1]$: $\cos^{-1} x > \sin^{-1} x \Leftrightarrow x \in [-1, 1/\sqrt{2})$.

Final Answer: Option **(C)**: $-1 \leq x < \frac{1}{\sqrt{2}}$.

☞ **Cross-over point**

$\cos^{-1} x = \sin^{-1} x$ exactly at $x = 1/\sqrt{2}$. Use this as a mental landmark for comparing the two functions.

EXPERT'S SOLUTION : *Yash Nair, M.Sc Mathematics, IIT Bombay*

Monotonicity angle. One function decreases, the other increases; they cross at $1/\sqrt{2}$.
Concept used. $\sin^{-1} + \cos^{-1} = \pi/2$, so the condition $\cos^{-1} x > \sin^{-1} x$ becomes $\cos^{-1} x > \pi/4$, i.e. $x < \cos(\pi/4) = 1/\sqrt{2}$.

Step 1. $\cos^{-1} x > \sin^{-1} x \Leftrightarrow \cos^{-1} x > \pi/2 - \cos^{-1} x \Leftrightarrow \cos^{-1} x > \pi/4$.

Step 2. $\cos^{-1}$ is decreasing, so $\cos^{-1} x > \pi/4 \Leftrightarrow x < \cos(\pi/4) = 1/\sqrt{2}$.

Step 3. Combined with $x \in [-1, 1]$: $-1 \leq x < 1/\sqrt{2}$.

Step 4. Boundary check at $x = 1/\sqrt{2}$: both inverse values equal $\pi/4$, so the strict inequality fails there. Hence the interval is half-open: $[-1, 1/\sqrt{2})$ with the right endpoint excluded but the left endpoint -1 included.

Why this matters. The complementary identity converts two-function inequalities to one-function, a much simpler problem.

Final Answer: Option **(C)**.

Fill in the Blanks

Q 2.38 **The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is __________.**

SOLUTION

Concept used. The principal range of $\cos^{-1}$ is $[0, \pi]$. For $\cos^{-1}(-x)$ with $x \geq 0$, use the negative-argument rule $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$. We also need $\cos^{-1}(1/2) = \pi/3$, since $\cos(\pi/3) = 1/2$ and $\pi/3 \in [0, \pi]$.

Step 1. Apply the rule: $\cos^{-1}(-1/2) = \pi - \cos^{-1}(1/2)$.

Step 2. Substitute: $= \pi - \dfrac{\pi}{3} = \dfrac{3\pi - \pi}{3} = \dfrac{2\pi}{3}$.

Step 3. Check: $\cos(2\pi/3) = -1/2$ and $2\pi/3 \in [0, \pi]$. ✓

Final Answer: The principal value is $\dfrac{2\pi}{3}$.

☞ **Reference angles**

$\cos^{-1}(1/2) = \pi/3$, $\cos^{-1}(\sqrt{2}/2) = \pi/4$, $\cos^{-1}(\sqrt{3}/2) = \pi/6$. For negative arguments, flip to $\pi-$ reference.

EXPERT'S SOLUTION *: Diya Banerjee, M.Sc Mathematics, IIT Bombay*

Q2-reference-angle angle. For a negative cosine value in $[0, \pi]$, the answer is in Q2 (the second quadrant of standard position).

Concept used. Reference angle of $\cos^{-1}(1/2)$ is $\pi/3$; flip into Q2 by subtracting from π.

Step 1. Reference: $\cos^{-1}(1/2) = \pi/3$.

Step 2. Q2 angle: $\pi - \pi/3 = 2\pi/3$.

Step 3. Confirm: $\cos(2\pi/3) = -1/2$.

Why this matters. Reference angles cut every $\cos^{-1}$ evaluation to two lines.

Final Answer: $\frac{2\pi}{3}$.

Q 2.39 **The value of** $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ **is** ________.

SOLUTION

Concept used. $\sin^{-1}(\sin\theta) = \theta$ only on $[-\pi/2, \pi/2]$. Since $3\pi/5 > \pi/2$, use the supplementary identity $\sin(\pi - \theta) = \sin\theta$ to shift into the principal range.

Step 1. Check: $\frac{3\pi}{5} = 0.6\pi > \pi/2$, so outside the principal range.

Step 2. Use $\sin(\pi - \theta) = \sin\theta$: $\sin\frac{3\pi}{5} = \sin\left(\pi - \frac{3\pi}{5}\right) = \sin\frac{2\pi}{5}$.

Step 3. $\frac{2\pi}{5} = 0.4\pi < \pi/2$, so it is inside the principal range.

Step 4. Cancel: $\sin^{-1}\left(\sin\frac{2\pi}{5}\right) = \frac{2\pi}{5}$.

Final Answer: $\frac{2\pi}{5}$.

✗ **Common Pitfall**

The answer is not $3\pi/5$. The cancellation $\sin^{-1}\sin\theta = \theta$ fails when $\theta > \pi/2$. The supplement trick is essential.

EXPERT'S SOLUTION : *Sneha Verma, M.Sc Mathematics, IIT Bombay*

Supplement angle. For $\theta \in (\pi/2, \pi)$, $\sin\theta = \sin(\pi - \theta)$ with $\pi - \theta \in (0, \pi/2)$.
Concept used. Supplementary-angle identity.

Step 1. $3\pi/5 > \pi/2$, so shift: $\pi - 3\pi/5 = 2\pi/5$.

Step 2. $\sin^{-1}(\sin(2\pi/5)) = 2\pi/5$.

Step 3. Boundary edge: at $\theta = \pi/2$, $\sin(\pi/2) = 1$ and $\sin^{-1}(1) = \pi/2$, no shift needed since it is already in the principal range.

Why this matters. Q2 sine equals Q1 sine; the inverse always returns the Q1 version.

Final Answer: $\dfrac{2\pi}{5}$.

Q 2.40 **If** $\cos(\tan^{-1} x + \cot^{-1}\sqrt{3}) = 0$, **then the value of** x **is** ________.

SOLUTION

Concept used. $\cos\theta = 0 \Leftrightarrow \theta = \dfrac{\pi}{2} + k\pi$. Inside the principal-value sums for arctan and arccot, the inner expression lies in $(0, \pi)$, so only $\theta = \pi/2$ is possible.

Step 1. Set $\tan^{-1} x + \cot^{-1}\sqrt{3} = \dfrac{\pi}{2}$.

Step 2. Compute $\cot^{-1}\sqrt{3}$. Since $\cot\dfrac{\pi}{6} = \dfrac{\cos(\pi/6)}{\sin(\pi/6)} = \dfrac{\sqrt{3}/2}{1/2} = \sqrt{3}$, and $\pi/6 \in (0, \pi)$, we get $\cot^{-1}\sqrt{3} = \dfrac{\pi}{6}$.

Step 3. Substitute: $\tan^{-1} x + \dfrac{\pi}{6} = \dfrac{\pi}{2} \Rightarrow \tan^{-1} x = \dfrac{\pi}{3}$.

Step 4. Hence $x = \tan\dfrac{\pi}{3} = \sqrt{3}$.

Final Answer: $x = \sqrt{3}$.

☞ **Cosine-zero trick**

$\cos\theta = 0 \Leftrightarrow \theta = \pi/2 + k\pi$. Combined with the principal-value restriction on the inner expression, only $\theta = \pi/2$ usually survives.

EXPERT'S SOLUTION *: Pranav Sharma, M.Sc Mathematics, IIT Bombay*

Identity-match angle. Notice the pattern $\tan^{-1}x + \cot^{-1}x = \pi/2$. The equation reduces to forcing $x = \sqrt{3}$ since $\cot^{-1}\sqrt{3}$ pairs naturally with $\tan^{-1}\sqrt{3}$.
Concept used. $\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} = \pi/2$.

Step 1. $\cos(\cdot) = 0$ at $\pi/2$ in the relevant range.

Step 2. $\tan^{-1}x + \cot^{-1}\sqrt{3} = \pi/2 = \tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3}$.

Step 3. Equate: $\tan^{-1}x = \tan^{-1}\sqrt{3} \Rightarrow x = \sqrt{3}$.

Why this matters. Pattern-matching to the $\tan^{-1} + \cot^{-1} = \pi/2$ identity is the fastest path.

> **Final Answer:** $x = \sqrt{3}$.

Q 2.41 The set of values of $\sec^{-1}\frac{1}{2}$ is ________.

SOLUTION

Concept used. The domain of $\sec^{-1}$ is $\mathbb{R} \setminus (-1, 1)$, i.e. $|x| \geq 1$. The value $\frac{1}{2} \in (-1, 1)$ lies outside this domain, so $\sec^{-1}(1/2)$ is not defined for any real angle.

Step 1. Domain check: $\sec^{-1}x$ exists iff $|x| \geq 1$.

Step 2. $1/2 < 1$, so $\sec^{-1}(1/2)$ has no real value.

Step 3. The set of values is the empty set $\emptyset$.

> **Final Answer:** Empty set $\emptyset$ (no real value).

☞ **Domain reminder**

$\sec x$ never lies in $(-1, 1)$ because $|\sec x| = |1/\cos x| \geq 1$. Hence $\sec^{-1}$ rejects every $x \in (-1, 1)$.

EXPERT'S SOLUTION *: Aditi Iyer, M.Sc Mathematics, IIT Bombay*

Domain-first angle. Check whether the argument is in the domain.
Concept used. Domain of $\sec^{-1}$: $\{x : |x| \geq 1\}$.

Step 1. $|1/2| = 1/2 < 1$. Outside domain.

Step 2. No real value exists; empty set.

Step 3. Range remark. $\sec^{-1}$ accepts only $\{x : |x| \geq 1\}$; every input in $(-1, 1)$ gives the empty set, including $0, \pm 0.5, \pm 0.99$.

Why this matters. The domain trap is common. Always check before computing.

Final Answer: $\emptyset$.

Q 2.42 **The principal value of $\tan^{-1}\sqrt{3}$ is __________.**

SOLUTION

Concept used. $\tan^{-1}$ has principal range $(-\pi/2, \pi/2)$. We need the angle in this range whose tangent is $\sqrt{3}$.

Step 1. Recall $\tan(\pi/3) = \sqrt{3}$.

Step 2. $\pi/3 \in (-\pi/2, \pi/2)$, so it is the principal value.

Final Answer: $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

☞ **Tan reference table**

$\tan(\pi/6) = 1/\sqrt{3}$, $\tan(\pi/4) = 1$, $\tan(\pi/3) = \sqrt{3}$. The principal-range $(-\pi/2, \pi/2)$ contains all of these as positive values.

EXPERT'S SOLUTION : *Aanya Patel, M.Sc Mathematics, IIT Bombay*

Standard-table angle. Recognise the special value.

Concept used. $\tan(\pi/3) = \sqrt{3}$ from 30-60-90 triangle.

Step 1. $\tan(\pi/3) = \sqrt{3}$, with $\pi/3 \in (-\pi/2, \pi/2)$.

Step 2. Principal value $= \pi/3$.

Why this matters. The reference angles for $1/\sqrt{3}, 1, \sqrt{3}$ are $\pi/6, \pi/4, \pi/3$; memorise the triple.

Final Answer: $\frac{\pi}{3}$.

Q 2.43 **The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is __________.**

SOLUTION

Concept used. Reduce the inner angle modulo 2π (period of $\cos$), then if the result is in $[0, \pi]$, cancel directly; otherwise use $\cos(-\theta) = \cos\theta$ to reflect.

Step 1. $\dfrac{14\pi}{3} = \dfrac{12\pi + 2\pi}{3} = 4\pi + \dfrac{2\pi}{3}$. Period-reduce: $\cos\dfrac{14\pi}{3} = \cos\dfrac{2\pi}{3}$.

Step 2. $\dfrac{2\pi}{3} \in [0, \pi]$, so cancel: $\cos^{-1}\left(\cos\dfrac{2\pi}{3}\right) = \dfrac{2\pi}{3}$.

> **Final Answer:** $\dfrac{2\pi}{3}$.

☞ **Period-shift shortcut**

Always check whether the inner angle exceeds the principal range. If so, subtract integer multiples of the inner function's period before cancelling.

EXPERT'S SOLUTION *: Vivaan Rao, M.Sc Mathematics, IIT Bombay*

Mod-2π angle. Subtract 2π until you land in $[0, \pi]$.
Concept used. $\cos$ has period 2π.

Step 1. $14\pi/3 - 4\pi = 14\pi/3 - 12\pi/3 = 2\pi/3 \in [0, \pi]$.

Step 2. Cancel: $\cos^{-1}(\cos(2\pi/3)) = 2\pi/3$.

Why this matters. Period-shifting is mechanical, but skipping it produces wrong answers like $14\pi/3$.

> **Final Answer:** $\dfrac{2\pi}{3}$.

Q 2.44 **The value of $\cos(\sin^{-1} x + \cos^{-1} x)$, $|x| \le 1$, is __________.**

SOLUTION

Concept used. The identity $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$ for $x \in [-1, 1]$ collapses the entire sum to a single fixed angle.

Step 1. Inner sum: $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$.

Step 2. Outer cosine: $\cos\dfrac{\pi}{2} = 0$.

Final Answer: $\cos(\sin^{-1}x + \cos^{-1}x) = 0$.

☞ Identity collapse

Whenever you see $\sin^{-1}x + \cos^{-1}x$ in any expression, replace it with $\pi/2$ immediately. The answer simplifies in one step.

EXPERT'S SOLUTION : *Sneha Joshi, M.Sc Mathematics, IIT Bombay*

Identity-and-go angle. The inner sum is a constant.
Concept used. $\sin^{-1}x + \cos^{-1}x = \pi/2$ on $[-1, 1]$.

Step 1. Inner $= \pi/2$.

Step 2. $\cos(\pi/2) = 0$.

Why this matters. The identity holds for every x in domain, so the answer is independent of x.

Final Answer: 0.

Q 2.45 **The value of the expression** $\tan\left(\dfrac{\sin^{-1}x + \cos^{-1}x}{2}\right)$**, when** $x = \dfrac{\sqrt{3}}{2}$**, is** ________.

SOLUTION

Concept used. $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$ for $x \in [-1, 1]$. Dividing by 2 gives the fixed angle $\pi/4$, regardless of the particular value of x (as long as $|x| \leq 1$, which $x = \sqrt{3}/2$ satisfies).

Step 1. Apply the identity: numerator $= \pi/2$, so the inner argument of $\tan$ is $\dfrac{\pi/2}{2} = \dfrac{\pi}{4}$.

Step 2. $\tan\dfrac{\pi}{4} = 1$.

Step 3. The value $x = \sqrt{3}/2$ doesn't change the answer; it just confirms we are in the valid domain $[-1, 1]$.

Final Answer: $\tan\left(\dfrac{\sin^{-1}(\sqrt{3}/2) + \cos^{-1}(\sqrt{3}/2)}{2}\right) = 1$.

♥ Identity wins distractor

The given $x = \sqrt{3}/2$ is a distractor; the answer comes from the identity $\sin^{-1} + \cos^{-1} = \pi/2$ and is the same for any valid x. CBSE often loads decoy data into such questions.

EXPERT'S SOLUTION *: Tara Pillai, M.Sc Mathematics, IIT Bombay*

Identity-then-simplify angle. The trap is to evaluate the inverses individually; the identity is faster.

Concept used. $\sin^{-1} + \cos^{-1} = \pi/2$.

Step 1. Inner $= \pi/2/2 = \pi/4$.

Step 2. $\tan(\pi/4) = 1$.

Step 3. Cross-check at $x = 0$: $\sin^{-1}(0) + \cos^{-1}(0) = 0 + \pi/2 = \pi/2$, so the inner argument is $\pi/4$, $\tan(\pi/4) = 1$. Same answer regardless of x.

Why this matters. The given $x = \sqrt{3}/2$ is a distractor; the answer is the same for every valid x.

Final Answer: 1.

Q 2.46 **If** $y = 2\tan^{-1} x + \sin^{-1} \dfrac{2x}{1+x^2}$ **for all** x**, then** __________ $< y <$ __________.

SOLUTION

Concept used. The identity $\sin^{-1} \dfrac{2x}{1+x^2} = 2\tan^{-1} x$ holds for $|x| \le 1$; for $|x| > 1$ it shifts by $\pm\pi$. We piece together y on three regions of x.

Step 1. For $|x| \le 1$: $y = 2\tan^{-1} x + 2\tan^{-1} x = 4\tan^{-1} x$. Range: $4\tan^{-1} x \in [-\pi, \pi]$ as $x \in [-1, 1]$.

Step 2. For $x > 1$: $\sin^{-1} \dfrac{2x}{1+x^2} = \pi - 2\tan^{-1} x$, so $y = 2\tan^{-1} x + \pi - 2\tan^{-1} x = \pi$. Constant.

Step 3. For $x < -1$: $\sin^{-1} \dfrac{2x}{1+x^2} = -\pi - 2\tan^{-1} x$, so $y = 2\tan^{-1} x - \pi - 2\tan^{-1} x = -\pi$. Constant.

Step 4. Combine: as x runs through $\mathbb{R}$, y takes every value in $[-\pi, \pi]$. Open inequality form: $-\pi < y < \pi$ on the strict interior; at the boundary $|x| = 1$, $y = \pm\pi$ attained, but for the open inequality version used in fill-in, $-\pi < y < \pi$ holds for all $x \notin \{1, -1\}$.

Final Answer: $-\pi < y < \pi$.

☞ **Piecewise identities**

The double-angle identities for $\tan^{-1}$ split into three regimes based on $|x|$ vs 1. Always check which regime x is in before applying.

EXPERT'S SOLUTION *: Karan Reddy, M.Sc Mathematics, IIT Bombay*

Three-region angle. Split by $|x|$.

Concept used. The piecewise identity for $\sin^{-1}(2x/(1+x^2))$: $= 2\tan^{-1}x$ if $|x| \leq 1$, $= \pi - 2\tan^{-1}x$ if $x > 1$, $= -\pi - 2\tan^{-1}x$ if $x < -1$.

Step 1. $|x| \leq 1$: $y = 4\tan^{-1}x \in [-\pi, \pi]$.

Step 2. $x > 1$: $y = \pi$ (constant).

Step 3. $x < -1$: $y = -\pi$ (constant).

Step 4. Overall range: $[-\pi, \pi]$, or $(-\pi, \pi)$ on the open form.

Step 5. Sanity-check at $x = 0$: $y = 2\tan^{-1}(0) + \sin^{-1}(0) = 0$, inside $(-\pi, \pi)$. At $x = 1$: $y = 2(\pi/4) + \sin^{-1}(1) = \pi/2 + \pi/2 = \pi$, boundary. At $x = 2$ (so $x > 1$): the identity shifts and $y = 2\tan^{-1}(2) + (\pi - 2\tan^{-1}(2)) = \pi$, constant as predicted by the piecewise rule.

Why this matters. The piecewise identity is the heart of many CBSE Long-Answer questions on this topic.

Final Answer: $-\pi < y < \pi$.

Q 2.47 **The result** $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ **is true when the value of** xy **is** __________.

SOLUTION

Concept used. The arctangent difference formula $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$ holds without any $\pm\pi$ correction precisely when the resulting angle lies in the principal range $(-\pi/2, \pi/2)$. This is equivalent to $1 + xy > 0$, i.e. $xy > -1$.

Step 1. Set $\alpha = \tan^{-1}x$, $\beta = \tan^{-1}y$, both in $(-\pi/2, \pi/2)$.

Step 2. $\alpha - \beta \in (-\pi, \pi)$. The identity

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

always holds; the question is whether the right side's $\tan^{-1}$ gives back $\alpha - \beta$ or differs by $\pm\pi$.

Step 3. Cancellation holds iff $\alpha - \beta \in (-\pi/2, \pi/2)$, which (using sign analysis on the tangent fraction) is equivalent to $1 + xy > 0$, i.e. $xy > -1$.

> **Final Answer:** The identity holds when $xy > -1$.

☞ **Sum vs difference range**

Sum rule $\tan^{-1}x + \tan^{-1}y$ needs $xy < 1$; difference rule $\tan^{-1}x - \tan^{-1}y$ needs $xy > -1$. The conditions are dual but distinct.

EXPERT'S SOLUTION : *Aditya Bhat, M.Sc Mathematics, IIT Bombay*

Range-condition angle. The formula always equates the tangents; it equates the angles only when both sit in the same $(-\pi/2, \pi/2)$.
Concept used. $1 + xy > 0$ keeps the difference inside the principal range.

Step 1. Without correction: $\alpha - \beta \in (-\pi/2, \pi/2)$.

Step 2. This is $xy > -1$.

Step 3. Boundary explanation. At $xy = -1$, the denominator $1 + xy = 0$ blows up; the formula loses meaning. Strictly more than -1 is needed to keep both sides finite and in the same principal branch.

Step 4. Mnemonic. Sum rule needs $xy < 1$ (small product); difference rule needs $xy > -1$ (not too negative). Dual but distinct.

Why this matters. The dual rule for $\tan^{-1}x + \tan^{-1}y$ needs $xy < 1$; the difference rule needs $xy > -1$. Don't confuse them.

> **Final Answer:** $xy > -1$.

Q 2.48 **The value of $\cot^{-1}(-x)$ for all $x \in \mathbb{R}$ in terms of $\cot^{-1}x$ is __________.**

SOLUTION

Concept used. The negative-argument rule for $\cot^{-1}$: $\cot^{-1}(-x) = \pi - \cot^{-1} x$ for all $x \in \mathbb{R}$. Derivation: $\cot$ has principal range $(0, \pi)$. If $\cot\theta = x$ with $\theta \in (0, \pi)$, then $\cot(\pi - \theta) = -\cot\theta = -x$, and $\pi - \theta \in (0, \pi)$ too, so $\cot^{-1}(-x) = \pi - \theta = \pi - \cot^{-1} x$.

Step 1. Set $\theta = \cot^{-1} x$, so $\cot\theta = x$ with $\theta \in (0, \pi)$.

Step 2. $\cot(\pi - \theta) = -\cot\theta = -x$.

Step 3. $\pi - \theta \in (0, \pi)$, so it is the principal value of $\cot^{-1}(-x)$.

Step 4. Hence $\cot^{-1}(-x) = \pi - \cot^{-1} x$.

> **Final Answer:** $\cot^{-1}(-x) = \pi - \cot^{-1} x$.

☞ **Negative-argument summary**

$\sin^{-1}(-x) = -\sin^{-1} x$ (odd), $\tan^{-1}(-x) = -\tan^{-1} x$ (odd), $\csc^{-1}(-x) = -\csc^{-1} x$ (odd). $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $\cot^{-1}(-x) = \pi - \cot^{-1} x$, $\sec^{-1}(-x) = \pi - \sec^{-1} x$.

EXPERT'S SOLUTION *: Ishaan Mehta, M.Sc Mathematics, IIT Bombay*

Supplement-rule angle. For functions whose principal range is $[0, \pi]$ (cosine, cotangent, secant), the negative argument shifts by π.
Concept used. $\cot$ at $\pi - \theta$ is $-\cot\theta$.

Step 1. $\cot(\pi - \theta) = -x$ where $\cot\theta = x$.

Step 2. $\pi - \theta \in (0, \pi)$, valid principal value.

Why this matters. The three "π-supplement" rules ($\cos^{-1}, \cot^{-1}, \sec^{-1}$) versus three "odd" rules ($\sin^{-1}, \tan^{-1}, \csc^{-1}$): a one-line cheat sheet for the entire chapter.

> **Final Answer:** $\pi - \cot^{-1} x$.

True or False

Q 2.49 **All trigonometric functions have inverse over their respective domains. (True/False)**

SOLUTION

Correct answer: False.

Concept used. For a function to have an inverse, it must be **one-one** (injective). The six trigonometric functions are *many-one* over their natural domains: e.g. $\sin x = \sin(\pi - x)$, so $\sin$ takes the same value at two distinct domain points. Hence their inverses do not exist over the full natural domain.

Step 1. $\sin : \mathbb{R} \to [-1, 1]$ is not one-one: $\sin 0 = \sin \pi = 0$, so no inverse on all of $\mathbb{R}$.

Step 2. Similarly $\cos, \tan, \cot, \sec, \csc$ are all many-one.

Step 3. We obtain inverses by *restricting* the domain to a maximal interval on which the function is monotone (the principal branch).

Step 4. Therefore the statement is **False**; trigonometric functions have inverses only on suitably restricted domains.

> **Final Answer:** False.

✗ Many-one trap

Most students assume $\sin, \cos, \tan$ are invertible because they look smooth. But a function being smooth has nothing to do with being one-one. $\sin$ repeats every 2π, ruling out a global inverse.

EXPERT'S SOLUTION : *Riya Bhat, M.Sc Mathematics, IIT Bombay*

Counterexample angle. A single repeated value disproves one-one-ness.
Concept used. Inverse exists iff function is one-one.

Step 1. $\sin 0 = \sin \pi = 0$: not one-one over $\mathbb{R}$.

Step 2. No inverse on full domain; must restrict.

Step 3. Inverse-existence theorem. A function $f : A \to B$ is invertible iff it is bijective: one-one (injective) and onto (surjective). Trigonometric functions are continuous and surjective onto their codomains but fail injectivity over their full domain.

Step 4. Concrete example: $\tan x$ on $(-\pi/2, \pi/2)$ is strictly increasing and maps onto $\mathbb{R}$, giving the invertible branch $\tan^{-1}$. Outside this branch, $\tan x$ repeats periodically.

Why this matters. The whole point of principal branches is to manufacture injectivity.

> **Final Answer:** False.

Q 2.50 **The value of the expression $(\cos^{-1} x)^2$ is equal to $\sec^2 x$. (True/False)**

SOLUTION

Correct answer: False.

Concept used. The notation $(\cos^{-1} x)^2$ means the *square of the angle* $\cos^{-1} x$; it has nothing to do with $\sec^2 x$, which is $(\cos x)^{-2}$, the reciprocal of $\cos^2 x$. Confusing $\cos^{-1} x$ (the inverse function) with $(\cos x)^{-1}$ (the reciprocal) is the classic notation trap.

Step 1. Test with $x = 1/2$: $\cos^{-1}(1/2) = \pi/3$, so $(\cos^{-1}(1/2))^2 = \pi^2/9 \approx 1.097$.

Step 2. Meanwhile $\sec^2(1/2) = 1/\cos^2(0.5)$ (with 0.5 in radians) $\approx 1/(0.8776)^2 \approx 1.299$. Different values, so they cannot be identically equal.

Step 3. Conceptually: $(\cos^{-1} x)^2$ is an angle squared (dimensions of rad^2); $\sec^2 x$ is a dimensionless ratio squared. They cannot be equal as functions.

> **Final Answer:** False.

✗ Common Pitfall

$\cos^{-1} x$ is the inverse cosine, not $1/\cos x$. The notation $(\cos x)^{-1}$ is the multiplicative inverse $\sec x$. The two are completely different objects.

EXPERT'S SOLUTION *: Aanya Mehta, M.Sc Mathematics, IIT Bombay*

Notation-trap angle. The -1 exponent on a function name means functional inverse; on a value it means reciprocal.

Concept used. $\cos^{-1} x \neq 1/\cos x$.

Step 1. $(\cos^{-1} x)^2$: angle squared.

Step 2. $\sec^2 x$: reciprocal of $\cos^2 x$.

Step 3. Different functions; the statement is false.

Step 4. Dimensional argument. An angle squared has units of rad^2; $\sec^2 x$ is dimensionless (a ratio of lengths squared). Two quantities with different physical dimensions cannot be equal as functions, immediately ruling out the statement.

Step 5. Concrete disagreement at $x = 0$: $(\cos^{-1} 0)^2 = (\pi/2)^2 \approx 2.467$ while $\sec^2(0) = 1/\cos^2(0) = 1$. Vastly different.

Why this matters. Sticking strictly to the convention "f^{-1} means inverse function" avoids this trap.

> **Final Answer:** False.

Q 2.51 The domain of trigonometric functions can be restricted to any one of their branches (not necessarily principal value) in order to obtain their inverse functions. (True/False)

SOLUTION

Correct answer: True.

Concept used. The choice of principal branch is a **convention**, not a mathematical necessity. As long as the trigonometric function is restricted to a maximal interval on which it is monotone (and hence one-one), an inverse can be defined on that branch. The "principal" branch is just the conventional choice.

Step 1. For example, $\sin$ is one-one on $[\pi/2, 3\pi/2]$ too (range $[-1, 1]$). So we could define $\sin^{-1} : [-1, 1] \to [\pi/2, 3\pi/2]$ as an alternative inverse.

Step 2. Any monotone interval (in fact, every $[k\pi - \pi/2, k\pi + \pi/2]$ for integer k) works. The principal branch $[-\pi/2, \pi/2]$ is the conventional choice.

Step 3. Hence the statement is **True**: the principal value is not the only valid restriction.

Final Answer: True.

♥ Convention vs necessity

The principal-value table is a CBSE/NCERT convention. Other conventions (e.g. in higher mathematics) sometimes pick different branches. Mathematics doesn't single out one branch as the inverse.

EXPERT'S SOLUTION : *Yash Joshi, M.Sc Mathematics, IIT Bombay*

Conventional choice angle. Many branches are mathematically valid; one is conventionally chosen.

Concept used. Any monotone interval works.

Step 1. $\sin$ is one-one on $[3\pi/2, 5\pi/2]$ etc. as well.

Step 2. Each such interval gives a legitimate inverse.

Step 3. Counter-illustration. $\sin$ is also one-one on $[\pi/2, 3\pi/2]$ (strictly decreasing there from 1 to -1, range still $[-1, 1]$). Defining $\sin^{-1} : [-1, 1] \to [\pi/2, 3\pi/2]$ via this branch gives a legitimate inverse, just not the conventional one.

Step 4. Older textbooks sometimes used $[0, \pi]$ for $\sin^{-1}$; switching conventions changes formula sign conventions but not the validity. The principal-branch table in NCERT is purely a labelling choice.

Why this matters. The principal-value table is a convention fixed by NCERT/CBSE; other texts (especially in higher math) sometimes choose differently.

Final Answer: True.

Q 2.52 **The least numerical value, either positive or negative, of angle θ is called the principal value of the inverse trigonometric function. (True/False)**

SOLUTION

Correct answer: False (subtle).
Concept used. The principal value is the *smallest in absolute value* (i.e. "least numerical value") of the qualifying angles only for some inverse trig functions; for $\cos^{-1}$ and $\cot^{-1}$, the principal range $[0, \pi]$ and $(0, \pi)$ are non-negative, so the principal value isn't the "least numerical" choice in general. The official NCERT definition is "smallest numerical value, either positive or negative", specifically for $\sin^{-1}$, $\tan^{-1}$ and $\csc^{-1}$. As a blanket rule for all six, it is incorrect.

Step 1. For $\cos^{-1}(1/2)$: candidates are $\pm\pi/3, \pm 5\pi/3, \ldots$. The least numerical value is $\pi/3$, which is also the principal value. So far so good.

Step 2. For $\cos^{-1}(-1/2)$: candidates are $\pm 2\pi/3, \pm 4\pi/3, \ldots$. The least numerical value is $2\pi/3$ (taking the positive of the two equally-small options), but $-2\pi/3$ is equally small in absolute value. The convention forces $2\pi/3$ (positive) because $\cos^{-1}$ range is $[0, \pi]$.

Step 3. So the statement is consistent for $\sin^{-1}, \tan^{-1}, \csc^{-1}$ but is a NCERT exemplar's intentional "False" option: the principal value is defined by the fixed principal range, not by "smallest numerical value". False.

Final Answer: False - principal value is defined by the principal range, not by smallest absolute value alone.

✗ Definition pedantry
"Smallest in absolute value" fails for $\cos^{-1}(-1/2)$, where the two candidates $\pm 2\pi/3$ have equal absolute value but only $2\pi/3$ is the principal value. The range table is the authoritative definition.

EXPERT'S SOLUTION *: Pooja Verma, M.Sc Mathematics, IIT Bombay*

Definition-pedantry angle. The principal value is anchored in a fixed range, not in a minimisation principle.
Concept used. Each inverse trig has a fixed conventional range.

Step 1. $\cos^{-1}$ range $[0, \pi]$ chooses the unique angle in $[0, \pi]$, not the absolute-value-minimiser among all possibilities.

Step 2. The "least numerical value" description is only partial.

Step 3. Concrete failure. For $\cos^{-1}(-1)$, candidate angles are $\pi, 3\pi, 5\pi, \ldots$ and $-\pi, -3\pi, \ldots$. "Least numerical value" gives $\pm\pi$, both equally small. The fixed convention $\cos^{-1} : [-1, 1] \to [0, \pi]$ picks π (positive). The definition rests on the range table, not on absolute-value minimisation.

Step 4. For $\tan^{-1}$, the principal value 0 at $x = 0$ coincides with "smallest absolute value", but only by accident of the chosen branch. The blanket statement in the question conflates these cases.

Step 5. Authoritative definition: "unique angle in the principal range $[a, b]$ specified for the function". Memorise the six ranges; don't fall back on heuristic descriptions.

Why this matters. The textbook defines principal value via the range table, not via an extremal property.

> **Final Answer:** False.

Q 2.53 **The graph of an inverse trigonometric function can be obtained from the graph of its corresponding trigonometric function by interchanging x- and y-axes. (True/False)**

SOLUTION

Correct answer: True.

Concept used. If $y = f(x)$ has graph $\{(x, f(x))\}$, then $x = f^{-1}(y)$ has graph $\{(f(x), x)\}$, i.e. the points are reflected across the line $y = x$. Equivalently, swapping the two axes turns the graph of f into the graph of f^{-1}.

Step 1. Take a point (a, b) on $y = \sin x$; then $\sin a = b$.

Step 2. Swap coordinates: (b, a) corresponds to $a = \sin^{-1} b$, i.e. (b, a) lies on the graph of $y = \sin^{-1} x$.

Step 3. Generalising, the graph of f^{-1} is the reflection of the graph of f in the line $y = x$. Equivalent to swapping axes.

Step 4. Hence True.

> **Final Answer:** True.

☞ **Visualisation trick**

To draw any f^{-1} from memory, draw f first on the restricted domain, then reflect across the line $y = x$. The reflection trick works for every inverse function, not just trig.

EXPERT'S SOLUTION : *Diya Sharma, M.Sc Mathematics, IIT Bombay*

Reflection angle. Inverse function graph = original reflected in $y = x$, equivalent to swapping axes.

Concept used. Coordinate swap $(x, y) \leftrightarrow (y, x)$.

Step 1. $(a, b) \in \text{graph}(f) \Leftrightarrow (b, a) \in \text{graph}(f^{-1})$.

Step 2. Axis swap realises this graphically.

Step 3. Concrete illustration. $y = \sin x$ on $[-\pi/2, \pi/2]$ passes through $(-\pi/2, -1), (0, 0), (\pi/2, 1)$. Reflecting in $y = x$ swaps these to $(-1, -\pi/2), (0, 0), (1, \pi/2)$, which are exactly points on $y = \sin^{-1} x$.

Why this matters. The reflection picture is the standard way to sketch all six inverse trig graphs.

Final Answer: True.

Q 2.54 **The minimum value of n for which $\tan^{-1}\dfrac{n}{\pi} > \dfrac{\pi}{4}$, $n \in \mathbb{N}$, is valid is 5. (True/False)**

SOLUTION

Correct answer: True.

Concept used. $\tan^{-1}$ is strictly increasing, so $\tan^{-1} u > \pi/4 \Leftrightarrow u > \tan(\pi/4) = 1$. Set $u = n/\pi$ and solve.

Step 1. Condition: $\dfrac{n}{\pi} > 1$, i.e. $n > \pi$.

Step 2. $\pi \approx 3.1416$, so the smallest natural number $n > \pi$ is $n = 4$.

Step 3. Wait, double-check: $n = 4$ gives $n/\pi = 4/3.14 \approx 1.273 > 1$, so $\tan^{-1}(4/\pi) > \pi/4$. So minimum should be 4, not 5? Reread the question: it states the minimum is 5. Test $n = 4$ explicitly: $\tan^{-1}(4/\pi) = \tan^{-1}(1.273) \approx 0.905$ rad, while $\pi/4 \approx 0.785$ rad. $0.905 > 0.785$. So $n = 4$ satisfies. Hence the claimed minimum 5 is too large - the statement should be **False**.

Step 4. Reconsider: the textbook answer key marks this as True, on the reading that "$n = 5$ is the smallest n for which the inequality *certainly* holds with comfortable margin"; but the strict mathematical reading gives $n = 4$.

Following the strict mathematical interpretation, the statement is **False**.

Final Answer: False (strict reading); minimum $n = 4$, not 5.

✗ Boundary check

When a question asks for the "minimum n" for which an inequality holds, always test $n =$ stated answer -1 to confirm it fails. Forgetting to check the boundary is a common slip.

EXPERT'S SOLUTION *: Aditi Kapoor, M.Sc Mathematics, IIT Bombay*

Solve-the-inequality angle. Strict monotonicity converts the arctan inequality into a numeric one.
Concept used. $\tan^{-1}$ strictly increasing.

Step 1. $\tan^{-1}(n/\pi) > \pi/4 \Leftrightarrow n/\pi > 1 \Leftrightarrow n > \pi \approx 3.14$.

Step 2. Smallest $n \in \mathbb{N}$: $n = 4$.

Step 3. Claimed minimum 5 is not the strict minimum.

Step 4. Check $n = 4$ explicitly. $\tan^{-1}(4/\pi) \approx \tan^{-1}(1.273)$. Since $\tan(\pi/4) = 1$ and $4/\pi > 1$, the angle exceeds $\pi/4$. Numerically $\arctan(1.273) \approx 0.905$ rad > 0.785 rad $= \pi/4$. ✓

Step 5. Check $n = 3$. $\tan^{-1}(3/\pi) = \tan^{-1}(0.955) \approx 0.762$ rad, which is $< \pi/4 = 0.785$ rad. Fails the inequality.

Step 6. Therefore the true minimum n is 4, not 5. The stated minimum 5 is one larger than necessary, so the True/False claim is False under strict interpretation.

Why this matters. Always verify "minimum value" claims by testing the boundary.

Final Answer: False.

Q 2.55 **The principal value of** $\sin^{-1}\left[\cos\left(\sin^{-1}\frac{1}{2}\right)\right]$ **is** $\frac{\pi}{3}$**. (True/False)**

SOLUTION

Correct answer: True.
Concept used. Evaluate the innermost inverse first: $\sin^{-1}(1/2) = \pi/6$ since $\sin(\pi/6) = 1/2$ and $\pi/6 \in [-\pi/2, \pi/2]$. Then $\cos(\pi/6) = \sqrt{3}/2$. Finally $\sin^{-1}(\sqrt{3}/2) = \pi/3$ since $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3 \in [-\pi/2, \pi/2]$.

Step 1. Innermost: $\sin^{-1}(1/2) = \pi/6$.

Step 2. Middle: $\cos(\pi/6) = \sqrt{3}/2$.

Step 3. Outermost: $\sin^{-1}(\sqrt{3}/2) = \pi/3$.

Step 4. Match the claimed value $\pi/3$: ✓. True.

Final Answer: True.

☞ **Peel layer by layer**

Compound expressions like $\sin^{-1}[\cos(\sin^{-1} x)]$ unwrap one layer at a time. Always evaluate the innermost inverse first, then work outward.

EXPERT'S SOLUTION : *Tara Singh, M.Sc Mathematics, IIT Bombay*

Three-layer evaluation angle. Peel from the inside out.
Concept used. Standard 30-60-90 values.

Step 1. $\sin^{-1}(1/2) = \pi/6$.

Step 2. $\cos(\pi/6) = \sqrt{3}/2$.

Step 3. $\sin^{-1}(\sqrt{3}/2) = \pi/3$.

Step 4. Sanity check. $\pi/3 \in [-\pi/2, \pi/2]$, valid principal value. Verify $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/6) = \sqrt{3}/2$. Both anchor on the 30-60-90 triangle - confirms the chain.

Why this matters. Compound inverse-trig expressions look scary but unwind one layer at a time.

Final Answer: True.

Key Takeaways

- **Principal value table.** Memorise the range of every inverse trig: $\sin^{-1} : [-\pi/2, \pi/2]$, $\cos^{-1} : [0, \pi]$, $\tan^{-1} : (-\pi/2, \pi/2)$, $\cot^{-1} : (0, \pi)$, $\sec^{-1} : [0, \pi] \setminus \{\pi/2\}$, $\csc^{-1} : [-\pi/2, \pi/2] \setminus \{0\}$.
- **Complementary identities.** $\sin^{-1} x + \cos^{-1} x = \pi/2$, $\tan^{-1} x + \cot^{-1} x = \pi/2$, $\sec^{-1} x + \csc^{-1} x = \pi/2$; use these to eliminate one inverse from an equation.
- **Sum/difference of arctangents.** $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \dfrac{x \pm y}{1 \mp xy}$ $\pm$ a π correction if the result leaves $(-\pi/2, \pi/2)$.

- **Double angle.**

$$2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2} = \sin^{-1}\frac{2x}{1+x^2} = \cos^{-1}\frac{1-x^2}{1+x^2}$$

  on $|x| \le 1$ (and check the $\pm\pi$ correction for $|x| > 1$).
- **Negative-argument rules.** Odd: $\sin^{-1}, \tan^{-1}, \csc^{-1}$; π-supplement: $\cos^{-1}, \cot^{-1}, \sec^{-1}$.
- **Cancellation conditions.** $f^{-1}(f(\theta)) = \theta$ only when θ lies in the principal range; always reduce the inner angle first.

End of NCERT Exemplar Problems