Inverse Trigonometric Fns

Recap from Ch 1

Inverse of $f : X \to Y$ exists only if f is both ~~onto~~ one-one AND onto (bijective).
Then $g : Y \to X$ with $g(f(x)) = x$ is denoted by f^{-1}.

The Problem

Trig fns sin, cos, tan ... are periodic.
Over their NATURAL domain R they are NOT one-one $\Rightarrow$ no inverse exists.
Fix : restrict the domain to a smaller interval where the function becomes one-one AND onto.

Original Trig Fns (recall)

sin : $R \to [-1, 1]$

cos : $R \to [-1, 1]$

tan : $R - \{ (2n+1)\pi/2 \} \to R$

cot : $R - \{ n\pi \} \to R$

sec : $R - \{(2n+1)\pi/2\} \to R - (-1, 1)$

cosec : $R - \{ n\pi \} \to R - (-1, 1)$

Note : $(\sin x)^{-1} = 1 / \sin x$, NOT $\sin^{-1} x$.
Don't confuse the two notations !

Branch : sin inv & cos inv

Arc-sine (sin inv x)

Restrict sin to [-pi/2 , pi/2] => one-one, onto with range [-1, 1] . Inverse exists.

$$\boxed{\sin^{-1} : [-1, 1] \to [-\pi/2 ,} \; \pi/2]$$ <-principal value branch

*

Other branches : $[\pi/2 , 3\pi/2]$, $[-3\pi/2 , -\pi/2]$ etc. all give range $[-1, 1]$ -- but only the ~~first~~ principal one is the STANDARD convention.

Arc-cosine (cos inv x)

Restrict cos to [0 , pi] => one-one, onto.

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$$\boxed{\cos^{-1} : [-1, 1] \to [0} \; , \pi]$$ principal <-value branch

Graph Idea

Graph of y = sin inv x = reflection of y = sin x (restricted) in the line y = x .

Same trick works for every inverse :

swap the (x, y) coordinates of every point.

* dark portion of NCERT Fig 2.1 = principal branch

Other Four Inverses

cosec inv x

cosec x = 1/sin x => not defined when sin x = 0

i.e. when $x = n\pi$. Range $y >= 1$.

$$\boxed{\operatorname{cosec}^{-1} : R - (-1, 1) \to [-\pi/2 , \pi/2]} - \{0\}$$

sec inv x

sec x = 1/cos x => undefined at odd mult of pi/2 .

$$\boxed{\sec^{-1} : R - (-1, 1) \to [0 , \pi] -} \{ \pi/2 \}$$

tan inv x

Restrict tan to (-pi/2 , pi/2) -- open intvl !

$$\boxed{\tan^{-1} : R \to (-\pi/2 , \pi}/2$$ <-open, not <-closed !

cot inv x

Restrict cot to (0 , π) -- open intvl too.

$$\boxed{\cot^{-1} : R \to (0 , \pi})$$ <-open (0, pi)

Domain / Range Summary

Memorize this table -- examiners love it !

Function	Domain	Range (P.V.B.)
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\text{cosec}^{-1}x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$
$\tan^{-1}x$	R	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	R	$(0, \pi)$

Memory Tip

sin , cosec , tan -> ranges symmetric ABOUT 0

cos , sec , cot -> ranges from 0 to π (positive)

tan , cot -> OPEN brackets ($\pi/2$ excluded)

Principal value = value of the inverse fn lying inside its principal value branch.

Finding Principal Values

Eg 1 : sin inv (1/root2)

Let $\sin^{-1}(1/\text{root}2) = y$.

Then $\sin y = 1/\text{root}2$.

Range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$.

We know $\sin(\pi/4) = 1/\text{root}2$,

and $\pi/4$ lies inside the principal branch.

$\boxed{\text{Therefore } \sin^{-1}(1/\text{root}2) = \pi/4}$ <-Ans.

Eg 2 : cot inv (-1/root3)

Let $\cot^{-1}(-1/\text{root}3) = y$.

Then $\cot y = -1/\text{root}3$.

We know $\cot(\pi/3) = 1/\text{root}3$,

so $\cot y = -\cot(\pi/3) = \cot(\pi - \pi/3)$

$= \cot(2\pi/3)$.

Range of $\cot^{-1}$ is $(0, \pi)$ and $2\pi/3$ lies in it.

$\boxed{\text{Hence } \cot^{-1}(-1/\text{root}3) = 2\pi/3}$ <-Ans.

Property 1 : Self Inverse

For an invertible f, we always have

$f(f^{-1}(x)) = x$ AND $f^{-1}(f(x)) = x$.

For trig fns :

$$\boxed{\sin(\sin^{-1} x) = x, \quad x \text{ in } [-1, 1]}$$

<- outer fn
<- is trig

$$\boxed{\sin^{-1}(\sin x) = x, \quad x \text{ in } [-\pi/2, \pi/2]}$$

<- ONLY inside
<- principal !

Watch Out !

$\sin^{-1}(\sin x)$ is NOT equal to x
when x is OUTSIDE the principal branch.

Eg : $\sin^{-1}(\sin 3\pi/5) = ?$

$3\pi/5$ is NOT in $[-\pi/2, \pi/2]$.

Use $\sin(3\pi/5) = \sin(\pi - 3\pi/5) = \sin(2\pi/5)$

and $2\pi/5$ in $[-\pi/2, \pi/2]$

$$\boxed{\text{Therefore } \sin^{-1}(\sin 3\pi/5) = 2\pi/5}$$

<- Ans.

* Same gotcha for cos inv, tan inv etc.

ALWAYS check the principal range first !

Property 2 : Negative Arg

How does each inverse react to x -> -x ?

$$\boxed{\sin^{-1}(-x) = -\sin^{-1} x, \quad x \text{ in } [-1, 1]}$$

<- odd
<- function

$$\boxed{\tan^{-1}(-x) = -\tan^{-1} x, \quad x \text{ in } R}$$

<- odd

$$\boxed{\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, \quad x >= 1}$$

<- odd

Not odd (have pi shift)

$$\boxed{\cos^{-1}(-x) = \pi - \cos^{-1} x}$$

<- x in
<- [-1, 1]

$$\boxed{\sec^{-1}(-x) = \pi - \sec^{-1} x}$$

<- x >= 1

$$\boxed{\cot^{-1}(-x) = \pi - \cot^{-1} x}$$

<- x in R

Mnemonic : sin / tan / cosec are ODD ;
cos / sec / cot need π - (...) instead.

Reciprocal & Sum Identities

Reciprocal pair

$\boxed{\operatorname{cosec}^{-1} x = \sin^{-1}(1/x)}$ <- x >=1

$\boxed{\sec^{-1} x = \cos^{-1}(1/x)}$ <- x >=1

$\boxed{\cot^{-1} x = \tan^{-1}(1/x)}$ <- x > 0

Complementary sums

Each function + its co-function = pi/2 :

$\boxed{\sin^{-1} x + \cos^{-1} x = \pi/2}$ <- x in <- [-1, 1]

$\boxed{\tan^{-1} x + \cot^{-1} x = \pi/2}$ <- x in R

$\boxed{\operatorname{cosec}^{-1} x + \sec^{-1} x = \pi/2}$ <- x >=1

Proof idea : let $\sin^{-1} x = y$, so $\sin y = x$.
Then $\cos(\pi/2 - y) = \sin y = x \Rightarrow \cos^{-1} x = \pi/2$

Addition & Subtraction

tan inv x + tan inv y

$$\boxed{\tan^{-1} x + \tan^{-1} y = \tan^{-1} (x+y)/(1-xy)}$$

valid when $xy < 1$.

$$\boxed{\tan^{-1} x - \tan^{-1} y = \tan^{-1} (x-y)/(1+xy)}$$

valid when $xy > -1$.

If $xy > 1$ and $x,y > 0$: add +pi.

$$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} (x+y)/(1-xy)$$

If $xy > 1$ and $x,y < 0$: subtract -pi :

$$\tan^{-1} x + \tan^{-1} y = -\pi + \tan^{-1} (x+y)/(1-xy)$$

sin inv x + sin inv y

$$\boxed{\sin^{-1} x + \sin^{-1} y = \sin^{-1}(x \text{ root}(1-y2)+y \text{ root}(1-x2))}$$

valid when $-1 <= x,y <= 1$ and $x2 + y2 <= 1$.

$$\boxed{\cos^{-1} x + \cos^{-1} y = \cos^{-1}(xy - \text{root}((1-x2)(1-y2)))}$$

* watch xy term sign carefully !

2 theta (Double-Angle) Forms

2 tan inv x (three forms)

$$\boxed{2\tan^{-1}x = \sin^{-1}(2x/(1+x2))}$$ <- x <= 1

$$\boxed{2\tan^{-1}x = \cos^{-1}((1-x2)/(1+x2))}$$ <-x >= 0

$$\boxed{2\tan^{-1}x = \tan^{-1}(2x/(1-x2))}$$ <- -1<x<1

Double of sin inv , cos inv

$$\boxed{2\sin^{-1}x = \sin^{-1}(2x\ \mathrm{root}(1-x2))}$$ <- x <=1/r2

$$\boxed{2\cos^{-1}x = \cos^{-1}(2x2 - 1)}$$ <-x >= 1/r2

Triple-angle (Misc Ex)

$$\boxed{3\sin^{-1}x = \sin^{-1}(3x - 4x3)}$$ <-x in <-[-1/2,1/2]

$$\boxed{3\cos^{-1}x = \cos^{-1}(4x3 - 3x)}$$

Worked Eg : Proof of 2 sin inv

Show :

$$\boxed{\sin^{-1}(2x \text{ root}(1-x2)) = 2 \sin^{-1} x}$$

valid for -1/root2 <= x <= 1/root2 .

Proof (substitution trick)

(1) Let $x = \sin \Theta$ so that $\sin^{-1} x = \Theta$.

(2) Then root(1-x2) = root(1-sin2 th) = $\cos \Theta$.

(3) So $2x\cdot$ root(1-x2) = $2 \sin \Theta$ ~~cos2~~ $\cos \Theta$.

(4) Use $2 \sin\Theta \cos \Theta = \sin 2\Theta$.

(5) Therefore 2x root(1-x2) = $\sin 2\Theta$.

(6) Apply $\sin^{-1}$ on both sides :

$\sin^{-1}$(2x root(1-x2)) = $2\Theta = 2 \sin^{-1} x$.

$$\boxed{\text{Therefore } \sin^{-1}(2x \text{ root}(1-x2)) = 2 \sin^{-1} x}$$ <-QED.

* Same idea : x = cos th gives the 2 cos inv form.

Worked Eg : Simplify

Express $\tan^{-1}(\cos x / (1 - \sin x))$ in simplest form for $-3\pi/2 < x < \pi/2$.

Solution

(1) Use $\cos x = \cos2(x/2) - \sin2(x/2)$ and

$1 - \sin x = (\cos(x/2) - \sin(x/2))\ 2$.

(2) Numerator factors as

$(\cos(x/2)+\sin(x/2))(\cos(x/2)-\sin(x/2))$.

(3) Cancel one $(\cos(x/2)-\sin(x/2))$ factor :

$\tan^{-1}((\cos(x/2)+\sin(x/2)) / (\cos(x/2)-\sin(x/2)))$

(4) Divide num & denom by $\cos(x/2)$:

$= \tan^{-1}((1 + \tan(x/2)) / (1 - \tan(x/2)))$

(5) $(1+\tan\theta)/(1-\tan\theta) = \tan(\pi/4 + \theta)$.

(6) Hence answer $= \pi/4 + x/\cancel{3}2$.

$$\boxed{\tan^{-1}(\cos x/(1-\sin x)) = \pi/4 + x/2}$$ <-Ans.

Trick : convert to half-angle $x/2$ first .

More Worked Examples

Eg 5 : cot inv (1/root(x2-1)), x>1

(1) Let $x = \sec\theta$ (since $x > 1$).
(2) Then root(x2 - 1) = root(sec2θ - 1) = $\tan\theta$.
(3) So 1/root(x2-1) = 1/tan θ = $\cot\theta$.
(4) $\cot^{-1}(\cot\theta) = \theta = \sec^{-1} x$.

$\boxed{\cot^{-1}(1/\text{root}(x2-1)) = \sec^{-1} x}$ <-Ans.

Eg 6 : sin inv (sin 3pi/5)

First : $3\pi/5$ is NOT in $[-\pi/2 , \pi/2]$.
So we can NOT directly write $\sin^{-1}(\sin 3\pi/5) = 3\pi/5$.

Use $\sin(3\pi/5) = \sin(\pi - 3\pi/5) = \sin(2\pi/5)$.
Check : $2\pi/5$ in $[-\pi/2 , \pi/2]$? YES.

So $\sin^{-1}(\sin 3\pi/5) = \sin^{-1}(\sin 2\pi/5) =$ ~~3pi/5~~ $2\pi/5$.

$\boxed{\sin^{-1}(\sin 3\pi/5) = 2\pi/5}$ <-Ans.

* Always reduce to principal-branch form first.

Tips & Common Mistakes

Common Mistakes (avoid !)

(1) Writing $(\sin x)^{-1}$ for $\sin^{-1} x$.

They are DIFFERENT ! See pg 1.

(2) Assuming $\sin^{-1}(\sin x) = x$ for ALL x.

True only when x in $[-\pi/2, \pi/2]$.

(3) Forgetting the constraint $xy < 1$ when

using $\tan^{-1} x + \tan^{-1} y = \tan^{-1}(x+y))/(1-\cancel{x+y}xy)$.

(4) Confusing $\cos^{-1}(-x) = \pi - \cos^{-1} x$ with $-\cos^{-1} x$

(5) Picking ANY angle whose sin is given,

instead of the one INSIDE the principal

value branch.

Quick Roadmap

(a) Identify which inverse fn is asked.

(b) Substitute $x = \sin\theta$ / $\cos\theta$ / $\tan\theta$.

(c) Simplify using basic trig identities.

(d) Apply inverse on both sides.

(e) Check the θ lies in principal branch.

(f) If outside : use Property 1 / 2 to fix.