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Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 2: Inverse Trigonometric Functions

What this chapter covers: how to invert the six trigonometric functions despite their periodicity, the *principal value branches* that make the inverses single-valued, the graphs of all six inverse functions, and the elementary identities. Compound-angle and sum/difference identities for inverse trig functions were removed from the rationalised NCERT but remain heavily tested at JEE/NEET — we include them clearly tagged as extensions.

1 Introduction and the Need for Restriction

In Class XI we treated the six trigonometric functions $\sin, \cos, \tan, \cot, \sec, \csc$ as machines that eat an angle and produce a ratio. Class XII reverses the question: given the ratio, what was the angle? That reverse direction is the subject of this entire chapter.

The trouble is that a trigonometric function takes the *same* value at infinitely many angles. $\sin 30° = \sin 150° = \sin 390° = \frac{1}{2}$. So $\sin^{-1}(\frac{1}{2})$ has no single answer — unless we agree, by convention, to pick one specific angle from the infinite list. That convention is the **principal value branch**.

1.1 Recap: Why a function needs to be one-one and onto for an inverse

From Chapter 1 we know an inverse f^{-1} exists only when f is bijective (one-one and onto). The six trigonometric functions, viewed on their natural domains, are far from one-one — they repeat every 2π (or π for tan and cot). To force a bijection we must

- shrink the domain to an interval on which the function is monotonic, and
- match the codomain to the actual range.

The shrunk domain is called a **branch**. The standard choice — the one used by calculators and by NCERT — is the **principal value branch**.

Why restriction is unavoidable

A trigonometric function is periodic, so it takes every value in its range infinitely often. No periodic function is one-one over its entire natural domain. Inversion requires a deliberate choice of a monotonic slice.

Contents

1.2 The six trigonometric functions: a quick reference

Before inverting, we list the trig functions with their natural domains and ranges (from Class XI).

Function	Domain	Range
$\sin x$	$\mathbb{R}$	$[-1, 1]$
$\cos x$	$\mathbb{R}$	$[-1, 1]$
$\tan x$	$\mathbb{R} \setminus \{(2n+1)\frac{\pi}{2}\}$	$\mathbb{R}$
$\cot x$	$\mathbb{R} \setminus \{n\pi\}$	$\mathbb{R}$
$\sec x$	$\mathbb{R} \setminus \{(2n+1)\frac{\pi}{2}\}$	$\mathbb{R} \setminus (-1, 1)$
$\csc x$	$\mathbb{R} \setminus \{n\pi\}$	$\mathbb{R} \setminus (-1, 1)$

Inverting these will swap the roles of domain and range. Notice the surprise: sec and csc never take values in $(-1, 1)$, so their inverses will not accept inputs from $(-1, 1)$.

Notation alert: $\sin^{-1} x$ is not $1/\sin x$

The superscript -1 here denotes *function inverse*, not reciprocal. $\sin^{-1} x$ is an angle, whereas $(\sin x)^{-1} = \dfrac{1}{\sin x} = \csc x$ is a ratio. Confusing the two leads to nonsensical equations. When in doubt, write $\arcsin x$ instead.

2 Principal Value Branches: Defining the Six Inverse Functions

For each trig function we now choose a monotonic interval that captures every range value exactly once. That interval becomes the range (output) of the inverse function. The interval is chosen so that, where possible, it contains 0 or sits on the positive side — this keeps formulas tidy and makes calculator behaviour predictable.

2.1 The inverse sine function

The sine function restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is strictly increasing, with range exactly $[-1, 1]$. So

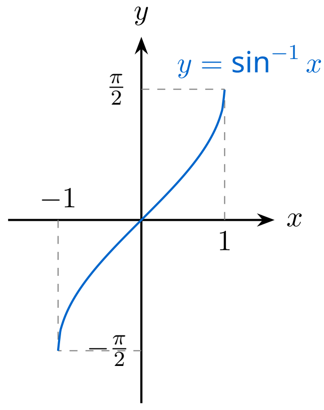
$$\sin^{-1} : [-1, 1] \longrightarrow \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right].$$

Every $y \in [-1, 1]$ has a unique pre-image in this interval, called the *principal value* of $\sin^{-1} y$.

Inverse Sine

$$y = \sin^{-1} x \iff \sin y = x, \quad x \in [-1, 1],\ y \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$

The graph of $\sin^{-1}$ is the graph of sin (restricted to $[-\pi/2, \pi/2]$) reflected across the line $y = x$.



Sign rule for $\sin^{-1}$

$\sin^{-1}$ is an **odd** function on its domain:

$$\sin^{-1}(-x) = -\sin^{-1} x, \quad x \in [-1, 1].$$

So $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$, not $\frac{11\pi}{6}$.

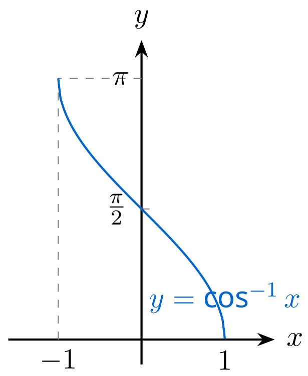
2.2 The inverse cosine function

The cosine function restricted to $[0, \pi]$ is strictly decreasing, with range $[-1, 1]$. So

$$\cos^{-1} : [-1, 1] \longrightarrow [0, \pi].$$

Inverse Cosine

$$y = \cos^{-1} x \iff \cos y = x, \quad x \in [-1, 1],\ y \in [0, \pi]$$



Unlike $\sin^{-1}$, the cosine inverse is *not* odd. The correct reflection law is:

$$\cos^{-1}(-x) = \pi - \cos^{-1} x, \quad x \in [-1, 1].$$

Quick Tip

When $x < 0$, do not flip the sign of $\cos^{-1} x$. Instead subtract from π. For instance, $\cos^{-1}(-\frac{1}{2}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

2.3 The inverse tangent function

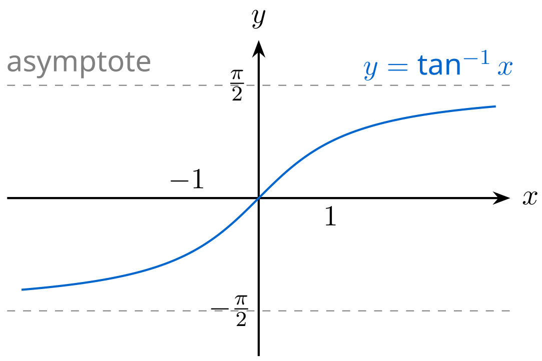
The tangent function restricted to the *open* interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is strictly increasing with range $\mathbb{R}$. Therefore

$$\tan^{-1} : \mathbb{R} \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Note the interval is open — tan blows up at $\pm\frac{\pi}{2}$, so $\tan^{-1}$ never attains those end-points.

Inverse Tangent

$$y = \tan^{-1} x \iff \tan y = x, \quad x \in \mathbb{R},\ y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



The graph approaches $\pm\frac{\pi}{2}$ as $x \to \pm\infty$, but never reaches. Like $\sin^{-1}$, $\tan^{-1}$ is an odd function: $\tan^{-1}(-x) = -\tan^{-1} x$.

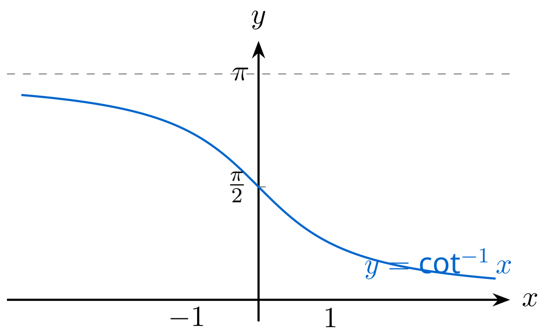
2.4 The inverse cotangent function

The cotangent function restricted to $(0, \pi)$ is strictly decreasing with range $\mathbb{R}$. So

$$\cot^{-1} : \mathbb{R} \longrightarrow (0, \pi).$$

Inverse Cotangent

$$y = \cot^{-1} x \iff \cot y = x, \quad x \in \mathbb{R},\ y \in (0, \pi)$$



The reflection law mirrors $\cos^{-1}$:

$$\cot^{-1}(-x) = \pi - \cot^{-1} x, \quad x \in \mathbb{R}.$$

$\cot^{-1}(-x) \neq -\cot^{-1} x$

A common slip is to treat $\cot^{-1}$ as odd. It is not. The principal branch $(0, \pi)$ lives entirely in the upper half — there are no negative outputs. So $\cot^{-1}(-\sqrt{3}) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$, not $-\frac{\pi}{6}$.

2.5 The inverse secant function

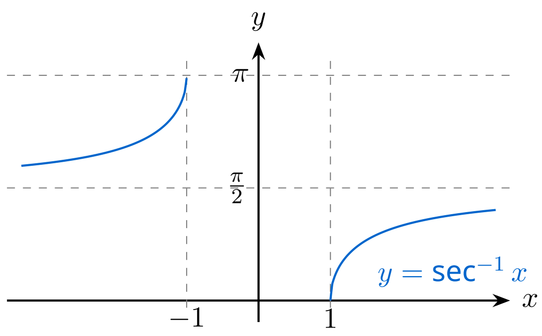
Because $\sec x = 1/\cos x$, the secant takes values $|y| \geq 1$. So $\sec^{-1}$ is defined only on $\mathbb{R} \setminus (-1, 1)$. The principal branch is sec restricted to $[0, \pi] \setminus \{\frac{\pi}{2}\}$:

$$\sec^{-1} : \mathbb{R} \setminus (-1, 1) \longrightarrow [0, \pi] \setminus \{\tfrac{\pi}{2}\}.$$

Inverse Secant

$$y = \sec^{-1} x \iff \sec y = x, \quad |x| \geq 1,\ y \in [0, \pi] \setminus \{\tfrac{\pi}{2}\}$$

The point $\frac{\pi}{2}$ is excluded because $\sec \frac{\pi}{2}$ is undefined — so no x maps there under the inverse.



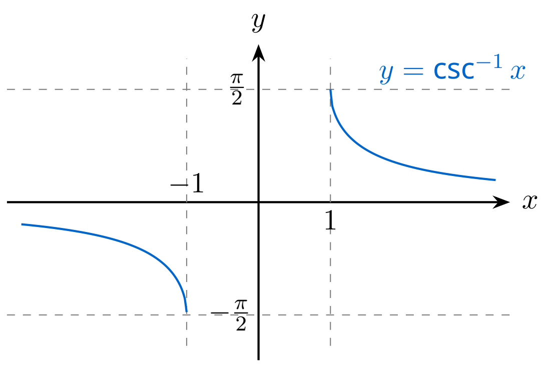
2.6 The inverse cosecant function

By analogy with secant, $\csc x = 1/\sin x$, so $|y| \geq 1$. The principal branch is csc restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$:

$$\csc^{-1} : \mathbb{R} \setminus (-1, 1) \longrightarrow [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \setminus \{0\}.$$

Inverse Cosecant

$$y = \csc^{-1} x \iff \csc y = x, \quad |x| \geq 1,\ y \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \setminus \{0\}$$



Three ranges, three families

The six principal-value ranges fall into just **three patterns**:

- $\sin^{-1}, \tan^{-1}, \csc^{-1}$ live near $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ — the "balanced" family.
- $\cos^{-1}, \cot^{-1}, \sec^{-1}$ live near $[0, \pi]$ — the "upper" family.
- "Balanced" inverses are odd; "upper" inverses obey $f(-x) = \pi - f(x)$.

3 The Principal-Value Range Table and How to Use It

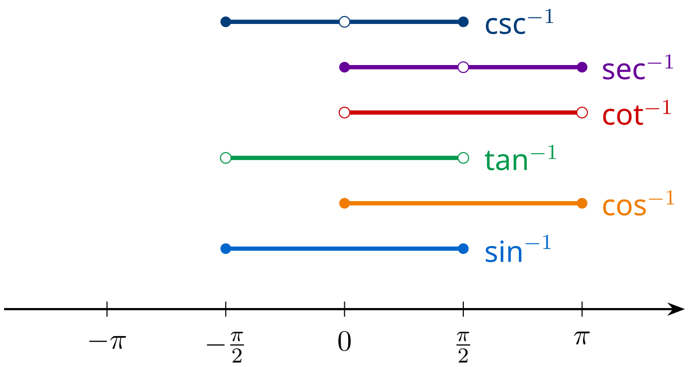
The six inverse trig functions, with their domains and principal-value ranges, sit at the heart of every problem in this chapter. Memorising this table is non-negotiable.

3.1 Master comparison table

Function	Domain (input)	Principal-Value Range (output)
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\cot^{-1} x$	$\mathbb{R}$	$(0, \pi)$
$\sec^{-1} x$	$\mathbb{R} \setminus (-1, 1)$	$[0, \pi] \setminus \{\frac{\pi}{2}\}$
$\csc^{-1} x$	$\mathbb{R} \setminus (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$

3.2 Visualising the six ranges on a number line

Sketching the six ranges on a single number line makes the structure obvious: three lie in $[-\pi/2, \pi/2]$, three lie in $[0, \pi]$.



A solid dot means the endpoint is included; an open dot means it is excluded.

Quick principal-value recipe

Step 1: Identify which trig function you are inverting. **Step 2:** Find the angle in the principal range whose trig value matches. **Step 3:** If the input is negative, apply the correct sign rule (odd for sin/tan/csc, $\pi-$ for cos/cot/sec). Never quote an angle outside the principal range.

3.3 Worked examples on principal values

Example 1. Find the principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

Let $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = y$. Then $\sin y = \frac{1}{\sqrt{2}}$ with $y \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The unique such angle is $y = \frac{\pi}{4}$.

Example 2. Find the principal value of $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

Let $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) = y$, so $\cot y = -\frac{1}{\sqrt{3}}$ with $y \in (0, \pi)$. Now $\cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$, and using $\cot(\pi-\theta) = -\cot\theta$,

$$\cot\left(\pi - \frac{\pi}{3}\right) = -\frac{1}{\sqrt{3}} \implies y = \frac{2\pi}{3}.$$

Example 3. Find $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$.

We need $y \in [0, \pi] \setminus \{\frac{\pi}{2}\}$ with $\sec y = \frac{2}{\sqrt{3}}$, i.e. $\cos y = \frac{\sqrt{3}}{2}$. So $y = \frac{\pi}{6}$.

Example 4. Evaluate $\tan^{-1}(1) + \cos^{-1}(-\frac{1}{2}) + \sin^{-1}(-\frac{1}{2})$.

Term by term:

- $\tan^{-1}(1) = \frac{\pi}{4}$ (since $\tan\frac{\pi}{4} = 1$ and $\frac{\pi}{4} \in (-\frac{\pi}{2}, \frac{\pi}{2})$).
- $\cos^{-1}(-\frac{1}{2}) = \pi - \cos^{-1}\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.
- $\sin^{-1}(-\frac{1}{2}) = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6}$.

Sum: $\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi+8\pi-2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$.

Example 5. Evaluate $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$.

$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. For $\sec^{-1}(-2)$, use $\sec^{-1}(-x) = \pi - \sec^{-1}x$: $\sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

$$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}.$$

Inverse trig in engineering and physics

Whenever a calculation produces a ratio — a slope, a refractive index sine, a phase shift cosine — and you need the corresponding angle, $\sin^{-1}, \cos^{-1}, \tan^{-1}$ deliver it. GPS satellites use $\tan^{-1}$ to convert direction-cosine ratios into latitude/longitude angles. Robotics inverse-kinematics solvers invert joint geometry the same way.

4 Elementary Properties of Inverse Trigonometric Functions

These identities follow directly from the definitions in Section 2 and from the symmetries of the original trig functions. They hold within the principal value branches; pushing them outside requires care.

4.1 Inverse–function inverse relations

If f and f^{-1} are genuine inverses, then $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$ — but only where both compositions are legal.

Composition identities

$$\sin(\sin^{-1} x) = x, \quad x \in [-1, 1]$$
$$\cos(\cos^{-1} x) = x, \quad x \in [-1, 1]$$
$$\tan(\tan^{-1} x) = x, \quad x \in \mathbb{R}$$

$$\sin^{-1}(\sin x) = x, \quad x \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]$$
$$\cos^{-1}(\cos x) = x, \quad x \in [0, \pi]$$
$$\tan^{-1}(\tan x) = x, \quad x \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right)$$

$\sin^{-1}(\sin x) \neq x$ for every x

The identity $\sin^{-1}(\sin x) = x$ holds *only* when $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Outside that range you must first fold x back into the principal interval. For example,

$$\sin^{-1}(\sin \tfrac{3\pi}{5}) = \sin^{-1}(\sin(\pi - \tfrac{3\pi}{5})) = \sin^{-1}(\sin \tfrac{2\pi}{5}) = \tfrac{2\pi}{5},$$

not $\frac{3\pi}{5}$.

4.2 Reciprocal relations

A consequence of $\sin\theta \cdot \csc\theta = 1$ and similar identities:

Reciprocal identities

$$\csc^{-1} x = \sin^{-1}\tfrac{1}{x}, \quad |x| \geq 1$$
$$\sec^{-1} x = \cos^{-1}\tfrac{1}{x}, \quad |x| \geq 1$$
$$\cot^{-1} x = \tan^{-1}\tfrac{1}{x}, \quad x > 0$$

For $x < 0$ in the third identity, the principal-range conventions force a correction:

$$\cot^{-1} x = \pi + \tan^{-1}\tfrac{1}{x}, \quad x < 0.$$

4.3 Negative-argument (reflection) identities

Direct from the parity of each trig function on its principal branch:

Negative argument identities

$$\begin{aligned}
\sin^{-1}(-x) &= -\sin^{-1} x, && x \in [-1, 1] \\
\tan^{-1}(-x) &= -\tan^{-1} x, && x \in \mathbb{R} \\
\csc^{-1}(-x) &= -\csc^{-1} x, && |x| \geq 1 \\
\cos^{-1}(-x) &= \pi - \cos^{-1} x, && x \in [-1, 1] \\
\cot^{-1}(-x) &= \pi - \cot^{-1} x, && x \in \mathbb{R} \\
\sec^{-1}(-x) &= \pi - \sec^{-1} x, && |x| \geq 1
\end{aligned}$$

Odd family

$\sin^{-1}, \tan^{-1}, \csc^{-1}$

$f(-x) = -f(x)$

symmetric about origin

$\pi-$ family

$\cos^{-1}, \cot^{-1}, \sec^{-1}$

$f(-x) = \pi - f(x)$

symmetric about $y = \pi/2$

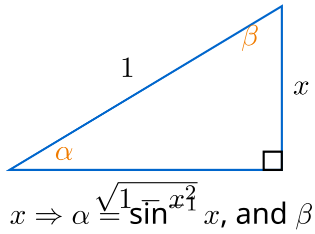
4.4 The complementary-angle identities

The cofunction relations of Class XI ($\sin\theta = \cos(\frac{\pi}{2} - \theta)$, etc.) translate into a beautiful trio of identities for the inverses.

Complementary-angle identities

$$\begin{aligned}
\sin^{-1} x + \cos^{-1} x &= \tfrac{\pi}{2}, && x \in [-1, 1] \\
\tan^{-1} x + \cot^{-1} x &= \tfrac{\pi}{2}, && x \in \mathbb{R} \\
\sec^{-1} x + \csc^{-1} x &= \tfrac{\pi}{2}, && |x| \geq 1
\end{aligned}$$

Why they hold. Let $\sin^{-1} x = \alpha$, so $\sin\alpha = x$ with $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then $\cos(\frac{\pi}{2} - \alpha) = \sin\alpha = x$, and $\frac{\pi}{2} - \alpha \in [0, \pi]$ — exactly the principal range of $\cos^{-1}$. So $\cos^{-1} x = \frac{\pi}{2} - \alpha$, giving the first identity.



Here $\sin\alpha = x \Rightarrow \alpha = \sin^{-1} x$, and $\beta = \cos^{-1} x$.
Since $\alpha + \beta = \frac{\pi}{2}$, the identity follows.

Quick Tip

Whenever a problem mixes $\sin^{-1}$ with $\cos^{-1}$ of the *same* x, replace one with $\frac{\pi}{2} -$ the other. The expression usually collapses. Same for tan / cot and sec / csc pairs.

4.5 Worked example: simplifying $\sin^{-1}(\sin x)$ outside the principal range

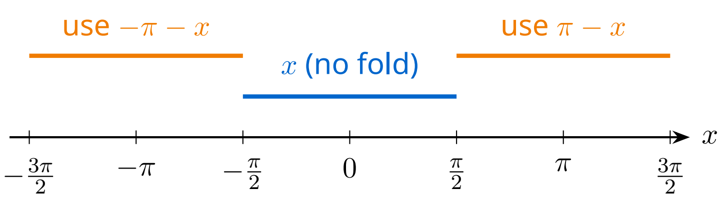
Find $\sin^{-1}(\sin \frac{3\pi}{5})$.

We cannot directly write $\frac{3\pi}{5}$ as the answer because $\frac{3\pi}{5} \notin [-\frac{\pi}{2}, \frac{\pi}{2}]$. Use $\sin(\pi - \theta) = \sin\theta$:

$$\sin \frac{3\pi}{5} = \sin\left(\pi - \frac{3\pi}{5}\right) = \sin \frac{2\pi}{5},$$

and $\frac{2\pi}{5} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Therefore $\sin^{-1}(\sin \frac{3\pi}{5}) = \frac{2\pi}{5}$.

Folding rule diagram. The folding-back procedure for $\sin^{-1}(\sin x)$ depends on which interval x lives in:

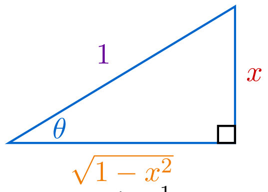


Example. Find $\tan^{-1}(\tan \frac{3\pi}{4})$. Since $\frac{3\pi}{4} \notin (-\frac{\pi}{2}, \frac{\pi}{2})$, use periodicity $\tan(\theta - \pi) = \tan\theta$: $\tan \frac{3\pi}{4} = \tan\left(\frac{3\pi}{4} - \pi\right) = \tan\left(-\frac{\pi}{4}\right)$. Then $\tan^{-1}\left(\tan \frac{3\pi}{4}\right) = -\frac{\pi}{4}$.

5 Geometric Interpretation via Right Triangles

A right-triangle picture is the single most useful tool for converting between inverse trig values. If $\theta = \sin^{-1} x$ with $x \in [0, 1]$, then θ is an angle whose opposite side is x and whose hypotenuse is 1. The remaining side, $\sqrt{1 - x^2}$, instantly gives all five other trig values.

5.1 The unit-hypotenuse triangle



$\theta = \sin^{-1} x$

From this triangle, with $\theta = \sin^{-1} x$:

$$\cos\theta = \sqrt{1-x^2}, \quad \tan\theta = \frac{x}{\sqrt{1-x^2}}, \quad \sec\theta = \frac{1}{\sqrt{1-x^2}}.$$

Triangle conversion shortcut

To evaluate $\cos(\sin^{-1} x)$, $\tan(\cos^{-1} x)$, etc., draw a right triangle with the relevant ratio matching the inner inverse, then read off the requested outer ratio. No identity gymnastics needed.

5.2 Worked example: converting between inverses

Show that $\tan^{-1}\frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$ for $x \in (-1, 1)$.

Put $x = \sin\theta$ with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $\sqrt{1-x^2} = \cos\theta > 0$ on this interval, so

$$\frac{x}{\sqrt{1-x^2}} = \frac{\sin\theta}{\cos\theta} = \tan\theta.$$

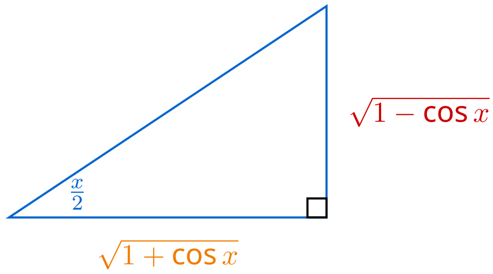
Applying $\tan^{-1}$, which returns θ from the same principal range, gives $\tan^{-1}\frac{x}{\sqrt{1-x^2}} = \theta = \sin^{-1} x$.

5.3 The half-angle simplification

This is one of the most often-asked board questions. Recall the half-angle formula $\tan^2\frac{x}{2} = \frac{1-\cos x}{1+\cos x}$. So

$$\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \tan^{-1}\left|\tan\frac{x}{2}\right|.$$

For $x \in (0, \pi)$, $\frac{x}{2} \in (0, \frac{\pi}{2})$, so $\tan\frac{x}{2} > 0$ and the absolute value is unnecessary. The result is simply $\frac{x}{2}$.



Triangle for $\tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}}$

5.4 Simplifying $\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right)$ for $x > 1$

Put $x = \sec\theta$ with $\theta \in [0, \frac{\pi}{2})$ (the part of the $\sec^{-1}$ principal branch where $x > 1$). Then

$$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \tan\theta,$$

so $\dfrac{1}{\sqrt{x^2-1}} = \cot\theta$. Hence

$$\cot^{-1}\left(\frac{1}{\sqrt{x^2-1}}\right) = \cot^{-1}(\cot\theta) = \theta = \sec^{-1} x.$$

The substitution dictionary

- For expressions involving $\sqrt{1-x^2}$: try $x = \sin\theta$ or $x = \cos\theta$.
- For $\sqrt{1+x^2}$: try $x = \tan\theta$.
- For $\sqrt{x^2-1}$: try $x = \sec\theta$.

The right substitution turns a messy radical into a clean trig identity.

Pick your substitution

"Sine for the small, tan for the sum, sec for the gap." If the radical is $\sqrt{1-x^2}$ (smaller than 1), use sine. If it is $\sqrt{1+x^2}$ (a sum), use tangent. If $\sqrt{x^2-1}$ (a gap above 1), use secant.

6 [JEE/NEET] Compound-Angle Identities

These identities — the sum, difference, and double-angle formulae for inverse trig functions — were rationalised out of the NCERT 2026–27 syllabus. They remain heavily tested in JEE Main, JEE Advanced, and NEET, and they unlock many Class 12 board questions in disguise. Learn them.

6.1 Sum and difference of two inverse tangents

Let $\tan^{-1} x = A$ and $\tan^{-1} y = B$. Then $\tan(A + B) = \dfrac{x+y}{1-xy}$. Taking $\tan^{-1}$ of both sides, with attention to principal-range corrections:

Sum/Difference: $\tan^{-1}$

$$\tan^{-1} x + \tan^{-1} y = \begin{cases} \tan^{-1} \dfrac{x+y}{1-xy}, & xy < 1 \\ \pi + \tan^{-1} \dfrac{x+y}{1-xy}, & x > 0,\ y > 0,\ xy > 1 \\ -\pi + \tan^{-1} \dfrac{x+y}{1-xy}, & x < 0,\ y < 0,\ xy > 1 \end{cases}$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}, \quad xy > -1$$

The $xy < 1$ condition is real

Many students apply $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$ blindly. When $xy > 1$, the RHS lies outside $(-\frac{\pi}{2}, \frac{\pi}{2})$, so the formula misses by $\pm\pi$. Always check the sign of $1 - xy$ first.

6.2 Sum and difference of two inverse sines and cosines

Sum/Difference: $\sin^{-1}$ and $\cos^{-1}$

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), \quad \text{if } x, y \geq 0 \text{ and } x^2 + y^2 \leq 1$$

$$\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$$

$$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), \quad x + y \geq 0$$

$$\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), \quad x \leq y$$

6.3 Double- and triple-angle identities

The case $x = y$ in the sum formula above produces the double-angle identities.

Multiple-angle identities

$$\begin{aligned}
2\sin^{-1}x &= \sin^{-1}\left(2x\sqrt{1-x^2}\right), \quad -\tfrac{1}{\sqrt{2}} \le x \le \tfrac{1}{\sqrt{2}}\\
2\cos^{-1}x &= \cos^{-1}(2x^2-1), \quad 0 \le x \le 1\\
2\tan^{-1}x &= \tan^{-1}\frac{2x}{1-x^2}, \quad -1 < x < 1\\
2\tan^{-1}x &= \sin^{-1}\frac{2x}{1+x^2}, \quad -1 \le x \le 1\\
2\tan^{-1}x &= \cos^{-1}\frac{1-x^2}{1+x^2}, \quad x \ge 0\\
3\sin^{-1}x &= \sin^{-1}(3x-4x^3), \quad -\tfrac{1}{2} \le x \le \tfrac{1}{2}\\
3\cos^{-1}x &= \cos^{-1}(4x^3-3x), \quad \tfrac{1}{2} \le x \le 1\\
3\tan^{-1}x &= \tan^{-1}\frac{3x-x^3}{1-3x^2}, \quad -\tfrac{1}{\sqrt{3}} < x < \tfrac{1}{\sqrt{3}}
\end{aligned}$$

6.4 Worked example: a famous JEE-style simplification

Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$ **in simplest form**, for $-\frac{3\pi}{2} < x < \frac{\pi}{2}$.

Write $\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2}$ and $1 - \sin x = \cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2} = \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2$. Then

$$\frac{\cos x}{1-\sin x} = \frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}} = \tan\left(\tfrac{\pi}{4}+\tfrac{x}{2}\right).$$

Hence $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \frac{\pi}{4} + \frac{x}{2}$.

Why JEE loves these

The identities compress a multi-step trig manipulation into a single inverse-function statement. JEE problems chain three or four of them together — if you can spot which identity applies, a five-minute problem becomes a thirty-second problem.

6.5 Equation-solving with inverse trig

Example. Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2\csc x)$.

Using $2\tan^{-1}u = \tan^{-1}\frac{2u}{1-u^2}$ for $|u| < 1$,

$$\tan^{-1}\frac{2\cos x}{1-\cos^2 x} = \tan^{-1}(2\csc x) \implies \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \implies \cos x = \sin x.$$

So $\tan x = 1$, giving $x = \frac{\pi}{4}$ (the principal solution in $(0, \pi/2)$).

Example. Solve $\tan^{-1}\dfrac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x$ for $x > 0$.

Note that $\tan^{-1}\dfrac{1-x}{1+x} = \tan^{-1}(1) - \tan^{-1}(x) = \frac{\pi}{4} - \tan^{-1}x$ (valid for $x > -1$). Substituting,

$$\frac{\pi}{4} - \tan^{-1}x = \frac{1}{2}\tan^{-1}x \implies \frac{\pi}{4} = \frac{3}{2}\tan^{-1}x \implies \tan^{-1}x = \frac{\pi}{6} \implies x = \frac{1}{\sqrt{3}}.$$

6.6 Proving a sum identity

Prove: $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$.

Let $\sin^{-1}\frac{3}{5} = \theta$. Then $\sin\theta = \frac{3}{5}$ with $\theta \in (0, \frac{\pi}{2})$, so $\cos\theta = \frac{4}{5}$ and $\tan\theta = \frac{3}{4}$. Now

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\cdot\frac{3}{4}}{1-\frac{9}{16}} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{24}{7}.$$

Since $2\theta \in (0, \pi)$ and the value $\tan(2\theta) = \frac{24}{7} > 0$ places $2\theta \in (0, \frac{\pi}{2})$, we get $2\theta = \tan^{-1}\frac{24}{7}$, as required.

The general problem-solving toolkit

- **Substitute trig for a radical** (the substitution dictionary from Section 5).
- **Use a right triangle** to convert one inverse into another.
- **Apply the sum/difference/double-angle formula** once you spot a matching pattern.
- **Verify the principal-range condition** on the resulting expression. This is where most marks are lost.

7 Quick Reference Summary

A condensed reference for the night before the exam.

7.1 Domains and principal-value ranges

Function	Domain	Range (Principal)	Family
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	Odd
$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$	$\pi-$
$\tan^{-1}$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	Odd
$\cot^{-1}$	$\mathbb{R}$	$(0, \pi)$	$\pi-$
$\sec^{-1}$	$\mathbb{R} \setminus (-1, 1)$	$[0, \pi] \setminus \{\frac{\pi}{2}\}$	$\pi-$
$\csc^{-1}$	$\mathbb{R} \setminus (-1, 1)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}$	Odd

7.2 Core identities to memorise

Top identities — board exam

$$\begin{aligned}
\sin^{-1} x + \cos^{-1} x &= \tfrac{\pi}{2}, && x \in [-1, 1] \\
\tan^{-1} x + \cot^{-1} x &= \tfrac{\pi}{2}, && x \in \mathbb{R} \\
\sec^{-1} x + \csc^{-1} x &= \tfrac{\pi}{2}, && |x| \geq 1 \\
\csc^{-1} x &= \sin^{-1} \tfrac{1}{x}, && |x| \geq 1 \\
\sec^{-1} x &= \cos^{-1} \tfrac{1}{x}, && |x| \geq 1 \\
\cot^{-1} x &= \tan^{-1} \tfrac{1}{x}, && x > 0 \\
\sin^{-1}(-x) &= -\sin^{-1} x, && x \in [-1, 1] \\
\cos^{-1}(-x) &= \pi - \cos^{-1} x, && x \in [-1, 1] \\
\tan^{-1}(-x) &= -\tan^{-1} x, && x \in \mathbb{R} \\
\cot^{-1}(-x) &= \pi - \cot^{-1} x, && x \in \mathbb{R}
\end{aligned}$$

7.3 Standard principal values

x	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	
0	0	$\frac{\pi}{2}$	0	$\frac{\pi}{2}$	
$\frac{1}{2}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$	—	—	
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	$\frac{\pi}{4}$	—	—	
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$	—	—	
1	$\frac{\pi}{2}$	0	$\frac{\pi}{4}$	$\frac{\pi}{4}$	
$\frac{1}{\sqrt{3}}$	—	—	$\frac{\pi}{6}$	$\frac{\pi}{3}$	
$\sqrt{3}$	—	—	$\frac{\pi}{3}$	$\frac{\pi}{6}$	

7.4 Final exam-day pointers

7.5 Common combined-evaluation patterns

A few patterns recur every year on board exams; recognising them shaves minutes.

Pattern A. $\tan(\sin^{-1} x + \cos^{-1} x)$ for $x \in [-1, 1]$. Using the complementary identity, the sum is $\frac{\pi}{2}$, and $\tan \frac{\pi}{2}$ is undefined. So this expression has no value — a classic "trick" answer.

Pattern B. $\sin(\tan^{-1} x)$ for $x \in \mathbb{R}$. Draw the right triangle with opposite $= x$ and adjacent $= 1$; the hypotenuse is $\sqrt{1+x^2}$. So

$$\sin(\tan^{-1} x) = \frac{x}{\sqrt{1+x^2}}.$$

Pattern C. $\cos(\sin^{-1} x)$ for $x \in [-1, 1]$. Draw the triangle with opposite $= x$, hypotenuse $= 1$; adjacent $= \sqrt{1-x^2}$. So

$$\cos(\sin^{-1} x) = \sqrt{1-x^2}.$$

Pattern D. $\sin^{-1} x + \sin^{-1} y$ when $x = \sin\alpha,\ y = \sin\beta$ are small. As long as $\alpha + \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the simple addition formula from the JEE/NEET extension applies.

Robotics inverse kinematics

A robotic arm with two joints reaching to point (a, b) has elbow angle θ_2 given by $\cos^{-1}\dfrac{a^2+b^2-L_1^2-L_2^2}{2L_1L_2}$ where L_1, L_2 are the link lengths. The principal-value branch $[0, \pi]$ corresponds exactly to the physical elbow range. Engineering uses inverse trig the same way calculators do.

Last-minute checklist

Before writing down any answer:

- Confirm the answer lies in the principal-value range of the relevant inverse function.
- For $\cos^{-1}$, $\cot^{-1}$, $\sec^{-1}$, negative inputs do not give negative outputs — subtract from π.
- For $\sin^{-1}(\sin x)$ when x is outside $[-\frac{\pi}{2}, \frac{\pi}{2}]$, fold using $\sin(\pi - \theta) = \sin\theta$ and re-check.
- For sums of two inverse tangents, check the sign of $1 - xy$ before applying the formula.

The one-line summary

Inverse trigonometric functions are the trigonometric functions *forced to be single-valued* by restricting to a principal-value branch. Every identity, graph, and computation in this chapter is just a careful bookkeeping exercise about which branch you are in.