NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 3 — Representative Set

Chapter 3: Matrices

About this Chapter

A **matrix** is an ordered rectangular array of numbers (or functions) arranged in rows and columns. A matrix with m rows and n columns is said to have **order** $m \times n$. This Exemplar set drills the operational and structural ideas of the chapter: matrix construction from a rule $a_{ij} = f(i,j)$, equality, scalar and matrix multiplication, the transpose A^T, **symmetric** and **skew-symmetric** matrices, the splitting $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, the Cayley–Hamilton style identities that lead to A^{-1}, and the use of **elementary row operations** to invert a square matrix.

Topics covered: Order of a matrix • Construction from $a_{ij} = f(i,j)$ • Equality of matrices • Matrix addition and scalar multiplication • Matrix multiplication • Transpose A^T • Symmetric & skew-symmetric • Symmetric + skew decomposition • Matrix polynomial identities • Inverse by row operations

Quick Formula Sheet

Order:
$A = [a_{ij}]_{m \times n}$ has mn entries

Transpose:
$(A^T)^T = A,\ (AB)^T = B^T A^T$

Symmetric / Skew:
$A^T = A$ / $A^T = -A$ (diag. $= 0$)

Decomposition:
$A = \frac{A+A^T}{2} + \frac{A-A^T}{2}$

Inverse:
$AB = BA = I \Rightarrow B = A^{-1}$;
$(AB)^{-1} = B^{-1}A^{-1}$

Row reduction:
$A = IA \xrightarrow{\text{row ops}} I = BA$, so $B = A^{-1}$

I. Short Answer (S.A.)

Q 3.1 **If a matrix has 28 elements, what are the possible orders it can have? What if it has 13 elements?**

SOLUTION

Concept used. A matrix of order $m \times n$ has exactly mn elements, where $m, n \in \mathbb{N}$. Hence the possible orders of a matrix with a given number of elements N are obtained by listing all ordered pairs (m, n) of positive integers whose product is N, i.e. by listing the **ordered factor pairs** of N.

Step 1. *For* $N = 28$*:* factor $28 = 2^2 \cdot 7$, divisors are $\{1, 2, 4, 7, 14, 28\}$. The ordered pairs (m, n) with $mn = 28$ are obtained by pairing each divisor with 28 divided by it:

$$(1, 28),\ (2, 14),\ (4, 7),\ (7, 4),\ (14, 2),\ (28, 1).$$

That gives **six** possible orders.

Step 2. *For* $N = 13$*:* 13 is prime, so its only divisors are $\{1, 13\}$. The ordered pairs are

$$(1, 13) \text{ and } (13, 1),$$

i.e. **two** possible orders.

> **Final Answer:** 28 elements: 6 orders — $1\times28,\ 2\times14,\ 4\times7,\ 7\times4,\ 14\times2,\ 28\times1$.
> 13 elements: 2 orders — $1\times13,\ 13\times1$.

☞ Why prime N gives only two orders

A prime number has no non-trivial factorisation, so a matrix with a prime number of elements is always a single row or a single column. This is the standard quick check before you start listing pairs.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Divisor-list angle. The number of valid orders equals the total number of divisors of N (each divisor d gives the order $d \times (N/d)$).

Concept used. If $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ then the number of divisors is $\tau(N) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$.

Step 1. For $N = 28 = 2^2 \cdot 7^1$: $\tau(28) = (2 + 1)(1 + 1) = 3 \cdot 2 = 6$. So 6 orders.

Step 2. Listing them: divisor list $\{1, 2, 4, 7, 14, 28\}$ paired as $(d, 28/d)$ produces the six orders above.

Step 3. For $N = 13 = 13^1$: $\tau(13) = 1 + 1 = 2$. So 2 orders, namely 1×13 and 13×1.

Step 4. Sanity check: a 4×7 matrix and a 7×4 matrix are different objects (different shapes), so order is an *ordered* pair, not an unordered one.

Final Answer: 6 orders for 28; 2 orders for 13.

Q 3.2 **Construct a 2×2 matrix where $a_{ij} = \dfrac{(i-2j)^2}{2}$.**

SOLUTION

Concept used. To construct a matrix from a rule $a_{ij} = f(i, j)$, substitute every valid pair (i, j) with $1 \le i \le m$ and $1 \le j \le n$, then place each computed value at the (i, j)-position of the matrix. Here $m = n = 2$, so we compute four entries.

Step 1. $a_{11} = \dfrac{(1-2\cdot 1)^2}{2} = \dfrac{(-1)^2}{2} = \dfrac{1}{2}$.

Step 2. $a_{12} = \dfrac{(1-2\cdot 2)^2}{2} = \dfrac{(-3)^2}{2} = \dfrac{9}{2}$.

Step 3. $a_{21} = \dfrac{(2-2\cdot 1)^2}{2} = \dfrac{0^2}{2} = 0$.

Step 4. $a_{22} = \dfrac{(2-2\cdot 2)^2}{2} = \dfrac{(-2)^2}{2} = 2$.

Step 5. Assemble the matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}.$$

Final Answer: $A = \begin{bmatrix} \frac{1}{2} & \frac{9}{2} \\ 0 & 2 \end{bmatrix}$.

☞ **Index convention**

a_{ij} means "row i, column j". Read the first subscript as the row, the second as the column — never reverse them.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Tabulate-then-assemble. Build a 2×2 table of values of $f(i, j) = (i - 2j)^2/2$ first, then drop them into matrix form.

Concept used. $(i - 2j)^2 \ge 0$, so every entry is a non-negative rational; the rule is symmetric under $j \to j$ but not under $(i, j) \to (j, i)$, so the matrix need not be symmetric.

Step 1. $(1, 1) \to (1-2)^2/2 = 1/2$. $(1, 2) \to (1-4)^2/2 = 9/2$.

Step 2. $(2,1) \to (2-2)^2/2 = 0$. $(2,2) \to (2-4)^2/2 = 4/2 = 2$.

Step 3. Place the row-1 pair $(1/2, 9/2)$ on top, the row-2 pair $(0,2)$ below:

$$A = \begin{bmatrix} 1/2 & 9/2 \\ 0 & 2 \end{bmatrix}.$$

Step 4. Cross-check: A is not symmetric ($a_{12} = 9/2 \neq 0 = a_{21}$), consistent with the rule.

> **Final Answer:** $A = \begin{bmatrix} 1/2 & 9/2 \\ 0 & 2 \end{bmatrix}$.

Q 3.3 **Find values of a and b if $A = B$, where** $A = \begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 2a+2 & b^2+2 \\ 8 & b^2-5b \end{bmatrix}.$

SOLUTION

Concept used. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** iff (a) they have the same order, and (b) $a_{ij} = b_{ij}$ for every (i,j). Both matrices here are 2×2, so we just equate corresponding entries.

Step 1. *Entry* $(1,1)$*:* $a + 4 = 2a + 2 \Rightarrow 4 - 2 = 2a - a \Rightarrow a = 2$.

Step 2. *Entry* $(2,2)$*:* $-6 = b^2 - 5b \Rightarrow b^2 - 5b + 6 = 0$. Factor: $(b-2)(b-3) = 0$, so $b = 2$ or $b = 3$.

Step 3. *Entry* $(1,2)$*:* $3b = b^2 + 2 \Rightarrow b^2 - 3b + 2 = 0 \Rightarrow (b-1)(b-2) = 0$, so $b = 1$ or $b = 2$.

Step 4. *Common value of* b (must satisfy both equations)*:* intersection of $\{2,3\}$ and $\{1,2\}$ is $\{2\}$. So $b = 2$.

Step 5. *Entry* $(2,1)$*:* $8 = 8$ holds automatically.

> **Final Answer:** $a = 2, \; b = 2$.

✗ Intersect, do not union

A common slip is to give every value of b from any one entry. Equality of matrices requires *all* entries to match, so b must satisfy *every* entry-level equation simultaneously — take the intersection of the candidate sets.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Delhi*

System-of-equations angle. Stack the four entry-wise equations into a system and solve.

Concept used. Matrix equality is a coordinate-wise condition; a single value of (a, b) must satisfy every entry equation simultaneously.

Step 1. Entry $(1,1)$: $a + 4 = 2a + 2 \Rightarrow a = 2$ (unique).

Step 2. Entry $(1,2)$ and $(2,2)$ both give quadratics in b. From $b^2 - 3b + 2 = 0$: $b \in \{1, 2\}$. From $b^2 - 5b + 6 = 0$: $b \in \{2, 3\}$. Intersection: $b = 2$.

Step 3. Verify: $A = \begin{bmatrix} 6 & 6 \\ 8 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 2(2)+2 & 4+2 \\ 8 & 4-10 \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ 8 & -6 \end{bmatrix}$. They match.

Final Answer: $a = 2,\ b = 2$.

Q 3.4 **If** $X = \begin{bmatrix} 3 & 1 & -1 \\ 5 & -2 & -3 \end{bmatrix}$ **and** $Y = \begin{bmatrix} 2 & 1 & -1 \\ 7 & 2 & 4 \end{bmatrix}$**, find**

(i) $X + Y$

(ii) $2X - 3Y$

(iii) A matrix Z such that $X + Y + Z$ is a zero matrix.

SOLUTION

Concept used. For matrices of the *same order*, addition and scalar multiplication are performed **element-wise**: $(A + B)_{ij} = a_{ij} + b_{ij}$ and $(kA)_{ij} = k\,a_{ij}$. The **zero matrix** O has every entry 0, and $Z = -(X + Y)$ is the unique matrix with $X + Y + Z = O$.

Step 1. *(i)* $X + Y$: add entry-by-entry:

$$X + Y = \begin{bmatrix} 3+2 & 1+1 & -1-1 \\ 5+7 & -2+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & -2 \\ 12 & 0 & 1 \end{bmatrix}.$$

Step 2. *(ii)* $2X - 3Y$: compute $2X$ and $3Y$ separately, then subtract entry-by-entry.

$$2X = \begin{bmatrix} 6 & 2 & -2 \\ 10 & -4 & -6 \end{bmatrix}, \quad 3Y = \begin{bmatrix} 6 & 3 & -3 \\ 21 & 6 & 12 \end{bmatrix}.$$

$$2X - 3Y = \begin{bmatrix} 6-6 & 2-3 & -2+3 \\ 10-21 & -4-6 & -6-12 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 1 \\ -11 & -10 & -18 \end{bmatrix}.$$

Step 3. *(iii)* Z: from $X + Y + Z = O$ we get $Z = -(X + Y)$. Negate every entry of $X + Y$:

$$Z = \begin{bmatrix} -5 & -2 & 2 \\ -12 & 0 & -1 \end{bmatrix}.$$

Final Answer: (i) $\begin{bmatrix} 5 & 2 & -2 \\ 12 & 0 & 1 \end{bmatrix}$; (ii) $\begin{bmatrix} 0 & -1 & 1 \\ -11 & -10 & -18 \end{bmatrix}$; (iii) $Z = \begin{bmatrix} -5 & -2 & 2 \\ -12 & 0 & -1 \end{bmatrix}$.

EXPERT'S SOLUTION *: Vivaan Patel, M.Sc Mathematics, BHU Varanasi*

Linear-combination angle. Treat $2X - 3Y$ as a single linear combination of X and Y rather than as two separate scalar multiples followed by a subtraction; compute one entry at a time using the unified rule $c_{ij} = 2x_{ij} - 3y_{ij}$. The single-pass view halves the bookkeeping and avoids sign slips that arise from holding $2X$ and $3Y$ as separate intermediate matrices.

Concept used. Linear combinations $\alpha A + \beta B$ on same-order matrices commute and distribute exactly as on real numbers, because each matrix operation is just element-wise; the space of 2×3 matrices is an honest $\mathbb{R}$-vector space of dimension 6, with element-wise addition and scaling.

Step 1. *(i) $X + Y$.* Add corresponding entries pair-by-pair: $(3+2,\ 1+1,\ -1-1) = (5, 2, -2)$ on row 1, and $(5+7,\ -2+2,\ -3+4) = (12, 0, 1)$ on row 2. Assemble: $X + Y = \begin{bmatrix} 5 & 2 & -2 \\ 12 & 0 & 1 \end{bmatrix}$.

Step 2. *(ii) $2X - 3Y$, entry-by-entry.* Row 1: $(1,1) \to 2(3) - 3(2) = 0$; $(1,2) \to 2(1) - 3(1) = -1$; $(1,3) \to 2(-1) - 3(-1) = 1$. Row 2: $(2,1) \to 2(5) - 3(7) = 10 - 21 = -11$; $(2,2) \to 2(-2) - 3(2) = -10$; $(2,3) \to 2(-3) - 3(4) = -18$. Assemble: $2X - 3Y = \begin{bmatrix} 0 & -1 & 1 \\ -11 & -10 & -18 \end{bmatrix}$.

Step 3. *(iii) The matrix Z.* Since $X + Y$ is known from (i), the requirement $X + Y + Z = O$ forces $Z = -(X + Y)$. Negating each entry gives the boxed answer below.

Step 4. *Consistency check:* $Z + (X + Y)$ adds entry-by-entry to $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$, as required.

Final Answer: $Z = -(X + Y) = \begin{bmatrix} -5 & -2 & 2 \\ -12 & 0 & -1 \end{bmatrix}$.

Q 3.5 **Find non-zero values of x satisfying the matrix equation** $x\begin{bmatrix}2x & 2\\3 & x\end{bmatrix} + 2\begin{bmatrix}8 & 5x\\4 & 4x\end{bmatrix} = 2\begin{bmatrix}x^2+8 & 24\\10 & 6x\end{bmatrix}$.

SOLUTION

Concept used. Distribute scalar multiplication across each matrix, then equate corresponding entries of the LHS and RHS. The resulting algebraic equations in x must all be satisfied simultaneously, so we take the common solution (excluding $x = 0$ as the question demands).

Step 1. Compute the LHS:

$$\begin{bmatrix}2x^2 & 2x\\3x & x^2\end{bmatrix} + \begin{bmatrix}16 & 10x\\8 & 8x\end{bmatrix} = \begin{bmatrix}2x^2+16 & 12x\\3x+8 & x^2+8x\end{bmatrix}.$$

Step 2. Compute the RHS:

$$\begin{bmatrix}2x^2+16 & 48\\20 & 12x\end{bmatrix}.$$

Step 3. Equate entry $(1,2)$: $12x = 48 \Rightarrow x = 4$.

Step 4. Equate entry $(2,1)$: $3x + 8 = 20 \Rightarrow 3x = 12 \Rightarrow x = 4$.

Step 5. Equate entry $(2,2)$: $x^2 + 8x = 12x \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0$, so $x = 0$ or $x = 4$. Excluding $x = 0$, we get $x = 4$.

Step 6. Entry $(1,1)$: $2x^2 + 16 = 2x^2 + 16$ is identically true. All entries agree at $x = 4$.

Final Answer: $x = 4$.

☞ Pick the linear entry first

Three entries here give equations in x. Always solve the linear one first — it pins down x uniquely, after which the quadratic serves as a confirmation, not a hunt.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Linear-pin angle. Spot the easiest entry first: among the four entry-wise equations, entry $(1,2)$ is purely linear in x. That equation alone pins down x uniquely; the other three then serve as confirmations, not as a search. This "pick the easiest constraint first" habit saves time on exam-style multi-entry matrix equations.

Concept used. A matrix equation in one unknown is equivalent to a system of scalar equations, one per entry. The unknown must satisfy every equation simultaneously, so if any single equation is linear and gives a unique value, the other equations must agree on that same value.

Step 1. Compute the LHS and RHS once: LHS $= \begin{bmatrix} 2x^2+16 & 12x \\ 3x+8 & x^2+8x \end{bmatrix}$ and RHS $= \begin{bmatrix} 2x^2+16 & 48 \\ 20 & 12x \end{bmatrix}$.

Step 2. Entry $(1,2)$ is linear: $12x = 48 \Rightarrow x = 4$ (unique).

Step 3. Verify entry $(2,1)$ at $x = 4$: LHS $= 3(4) + 8 = 20$ vs. RHS $= 20$. They match.

Step 4. Verify entry $(2,2)$ at $x = 4$: LHS $= 16 + 32 = 48$ and RHS $= 12(4) = 48$. They match.

Step 5. Entry $(1,1)$: LHS and RHS are identically $2x^2 + 16$, so no information; $x = 4$ is consistent.

Step 6. Finally $x = 4 \neq 0$, satisfying the problem's non-zero constraint. No other root survives all four checks.

Final Answer: $x = 4$.

Q 3.6 **If** $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$**, show that** $(A+B)(A-B) \neq A^2 - B^2$**.**

SOLUTION

Concept used. The identity $(a+b)(a-b) = a^2 - b^2$ holds for real numbers because multiplication is commutative. For matrices, in general $AB \neq BA$, so

$$(A+B)(A-B) = A^2 - AB + BA - B^2,$$

which equals $A^2 - B^2$ *only if* $AB = BA$. We show this fails here by directly computing both sides.

Step 1. Compute $A + B$ and $A - B$:

$$A + B = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad A - B = \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}.$$

Step 2. Compute $(A+B)(A-B)$. Each entry is row-of-LHS dotted with column-of-RHS:

$$(A+B)(A-B) = \begin{bmatrix} 0\cdot 0 + 0\cdot 0 & 0\cdot 2 + 0\cdot 1 \\ 2\cdot 0 + 1\cdot 0 & 2\cdot 2 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}.$$

Step 3. Compute A^2:

$$A^2 = \begin{bmatrix} 0\cdot 0+1\cdot 1 & 0\cdot 1+1\cdot 1 \\ 1\cdot 0+1\cdot 1 & 1\cdot 1+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

Step 4. Compute B^2:

$$B^2 = \begin{bmatrix} 0\cdot 0+(-1)\cdot 1 & 0\cdot(-1)+(-1)\cdot 0 \\ 1\cdot 0+0\cdot 1 & 1\cdot(-1)+0\cdot 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Step 5. Therefore $A^2 - B^2 = \begin{bmatrix} 1-(-1) & 1-0 \\ 1-0 & 2-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$.

Step 6. Compare: $(A+B)(A-B) = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A^2 - B^2$. The identity **fails**.

Final Answer: $(A+B)(A-B) \neq A^2 - B^2$ since matrix multiplication is not commutative.

♥ Why commutativity matters

The school-algebra identity $(a+b)(a-b) = a^2 - b^2$ depends on $ab = ba$. The same algebraic step on matrices produces $A^2 - AB + BA - B^2$. The two "cross terms" cancel *only when* $AB = BA$. Whenever you see $(A \pm B)^n$ in a problem, check commutativity first.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Structural angle. Compute the "defect" $BA - AB$ directly. If the commutator is non-zero, the school-algebra identity $(A+B)(A-B) = A^2 - B^2$ automatically fails, with the discrepancy exactly equal to the commutator. This view replaces six entry computations with two and turns the question into a single "is $AB = BA$?" check.

Concept used. Expand the LHS: $(A+B)(A-B) = A^2 - AB + BA - B^2$. Subtract $A^2 - B^2$ from both sides: $(A+B)(A-B) - (A^2 - B^2) = BA - AB$. So the two sides differ by exactly the **commutator** $[B, A] = BA - AB$, and equality holds iff $AB = BA$.

Step 1. Compute AB. Entry $(1,1)$: $0(0) + 1(1) = 1$; $(1,2)$: $0(-1) + 1(0) = 0$; $(2,1)$: $1(0) + 1(1) = 1$; $(2,2)$: $1(-1) + 1(0) = -1$. So $AB = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

Step 2. Compute BA. Entry $(1,1)$: $0(0) + (-1)(1) = -1$; $(1,2)$: $0(1) + (-1)(1) = -1$; $(2,1)$: $1(0) + 0(1) = 0$; $(2,2)$: $1(1) + 0(1) = 1$. So $BA = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$.

Step 3. Subtract: $BA - AB = \begin{bmatrix} -1-1 & -1-0 \\ 0-1 & 1-(-1) \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} \neq O.$

Step 4. Since the commutator is non-zero, $(A+B)(A-B) \neq A^2 - B^2$. Indeed $A^2 - B^2 - (A+B)(A-B) = AB - BA = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$, the entry-wise discrepancy between the two computed matrices from the main solution.

Final Answer: $BA - AB \neq O$, so the identity does not hold.

Q 3.7 **Find the value of x if** $\begin{bmatrix} 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = O.$

SOLUTION

Concept used. A 1×3 row times a 3×3 matrix times a 3×1 column produces a 1×1 matrix, i.e. a single scalar. Setting it to the zero matrix means setting that scalar to 0. We multiply left-to-right.

Step 1. Multiply the 1×3 row with the 3×3 matrix. The $(1,1)$ entry is $1(1) + x(2) + 1(15) = 16 + 2x$. The $(1,2)$ entry is $1(3) + x(5) + 1(3) = 6 + 5x$. The $(1,3)$ entry is $1(2) + x(1) + 1(2) = 4 + x$. So the intermediate row is

$$\begin{bmatrix} 16 + 2x, & 6 + 5x, & 4 + x \end{bmatrix}.$$

Step 2. Multiply this 1×3 row with the 3×1 column $\begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}$:

$$(16 + 2x)(1) + (6 + 5x)(2) + (4 + x)(x).$$

Step 3. Expand each piece: $(16 + 2x) \cdot 1 = 16 + 2x$; $(6 + 5x) \cdot 2 = 12 + 10x$; $(4 + x) \cdot x = 4x + x^2$.

Step 4. Sum: $16 + 2x + 12 + 10x + 4x + x^2 = x^2 + 16x + 28$.

Step 5. Set the scalar to 0: $x^2 + 16x + 28 = 0$. Apply the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1,\ b = 16,\ c = 28$:

$$x = \frac{-16 \pm \sqrt{256 - 112}}{2} = \frac{-16 \pm \sqrt{144}}{2} = \frac{-16 \pm 12}{2}.$$

So $x = \dfrac{-16 + 12}{2} = -2$ or $x = \dfrac{-16 - 12}{2} = -14$.

Final Answer: $x = -2$ or $x = -14$.

☞ **Compatible orders**

$(1{\times}3)(3{\times}3)(3{\times}1)$ collapses left-to-right to $(1{\times}3)(3{\times}1) = (1{\times}1)$. Always check inner dimensions match before multiplying.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Delhi*

Right-to-left angle. Multiply the rightmost two factors first; that turns the triple product into a 1×3 row times a 3×1 column, which is a single inner product. The arithmetic is identical to the main solution, but the bookkeeping is cleaner because we never carry an intermediate row containing the unknown x scattered across all three slots.

Concept used. Matrix multiplication is associative: $(uM)v = u(Mv)$ whenever the dimensions match. We may choose whichever grouping reduces effort or makes the expression in x cleaner. Here, computing Mv first leaves x in only one slot of each column entry, simplifying the next step.

Step 1. $Mv = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix}$. Row 1: $1(1) + 3(2) + 2(x) = 7 + 2x$. Row 2: $2(1) + 5(2) + 1(x) = 12 + x$. Row 3: $15(1) + 3(2) + 2(x) = 21 + 2x$. So $Mv = \begin{bmatrix} 7+2x \\ 12+x \\ 21+2x \end{bmatrix}$.

Step 2. $u\,(Mv) = [1\ x\ 1] \begin{bmatrix} 7+2x \\ 12+x \\ 21+2x \end{bmatrix} = 1(7+2x) + x(12+x) + 1(21+2x)$.

Step 3. Expand each piece: $1 \cdot (7+2x) = 7+2x$; $x \cdot (12+x) = 12x + x^2$; $1 \cdot (21+2x) = 21 + 2x$. Sum: $x^2 + (2+12+2)x + (7+21) = x^2 + 16x + 28$.

Step 4. Set to zero: $x^2 + 16x + 28 = 0$. Discriminant $\Delta = 16^2 - 4(28) = 256 - 112 = 144$, so $\sqrt{\Delta} = 12$ and roots $x = (-16 \pm 12)/2$, giving $x = -2$ or $x = -14$. Same quadratic and same roots as in the main solution, computed via the alternative grouping.

Final Answer: $x \in \{-2,\ -14\}$.

Q 3.8 **Show that** $A = \begin{bmatrix} 5 & 3 \\ -1 & -2 \end{bmatrix}$ **satisfies the equation** $A^2 - 3A - 7I = O$ **and hence**

find A^{-1}.

SOLUTION

Concept used. A square matrix A **satisfies** a polynomial equation $p(A) = O$ when substituting A into the polynomial (with the constant term multiplied by I) yields the zero matrix. Once this is shown, we can rearrange to express I as a polynomial in A, and pre-multiply by A^{-1} to read off A^{-1} as a polynomial in A — no row reduction required.

Step 1. Compute A^2. Each entry is a row-column dot product:

$$A^2 = \begin{bmatrix} 5(5)+3(-1) & 5(3)+3(-2) \\ -1(5)+(-2)(-1) & -1(3)+(-2)(-2) \end{bmatrix} = \begin{bmatrix} 25-3 & 15-6 \\ -5+2 & -3+4 \end{bmatrix} = \begin{bmatrix} 22 & 9 \\ -3 & 1 \end{bmatrix}.$$

Step 2. Compute $3A = \begin{bmatrix} 15 & 9 \\ -3 & -6 \end{bmatrix}$ and $7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

Step 3. Compute $A^2 - 3A - 7I$ entry by entry:

$$\begin{bmatrix} 22-15-7 & 9-9-0 \\ -3+3-0 & 1+6-7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O.$$

So the identity $A^2 - 3A - 7I = O$ holds.

Step 4. Rearrange to isolate I: $A^2 - 3A = 7I$, i.e. $A(A - 3I) = 7I$, so $A \cdot \dfrac{A-3I}{7} = I$. Hence

$$A^{-1} = \frac{1}{7}(A - 3I).$$

Step 5. Compute $A - 3I = \begin{bmatrix} 5-3 & 3 \\ -1 & -2-3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$. Hence

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix} = \begin{bmatrix} 2/7 & 3/7 \\ -1/7 & -5/7 \end{bmatrix}.$$

Final Answer: $A^{-1} = \dfrac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$.

☞ Cayley–Hamilton in disguise

For any 2×2 matrix, $A^2 - (\operatorname{tr} A)\,A + (\det A)I = O$. Here $\operatorname{tr} A = 5 + (-2) = 3$ and $\det A = 5(-2) - 3(-1) = -7$, giving $A^2 - 3A - 7I = O$ *without* computing A^2 at all.

EXPERT'S SOLUTION : *Vivaan Patel, M.Sc Mathematics, BHU Varanasi*

Trace-determinant shortcut. Recognise that the matrix identity $A^2 - 3A - 7I = O$ is the Cayley–Hamilton theorem for 2×2 matrices written out explicitly. The trace and determinant of A encode the only coefficients we need, so the inverse can be read off in two lines without ever computing A^2. This is the recommended exam strategy when the matrix itself is given in the problem.

Concept used. For any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det A \neq 0$, $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A) = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The adjugate is obtained by swapping diagonal entries and negating off-diagonal entries.

Step 1. Compute the determinant: $\det A = 5(-2) - 3(-1) = -10 + 3 = -7$. Since $\det A \neq 0$, A^{-1} exists.

Step 2. Build the adjugate. Swap the diagonal entries ($5 \leftrightarrow -2$) and negate the off-diagonal entries ($3 \to -3,\ -1 \to 1$): $\operatorname{adj} A = \begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix}$.

Step 3. Divide by $\det A = -7$ to get $A^{-1} = \frac{1}{-7}\begin{bmatrix} -2 & -3 \\ 1 & 5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$. Identical to the polynomial-method answer.

Step 4. Sanity check:

$$A \cdot A^{-1} = \frac{1}{7}\begin{bmatrix} 5(2) + 3(-1) & 5(3) + 3(-5) \\ -1(2) + (-2)(-1) & -1(3) + (-2)(-5) \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = I. \text{ OK.}$$

Final Answer: $A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & 3 \\ -1 & -5 \end{bmatrix}$.

Q 3.9 **If** $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & 1 \\ 2 & 3 \\ 1 & 2 \end{bmatrix}$**, find** BA **and** AB**.**

SOLUTION

Concept used. AB is defined iff the number of columns of A equals the number of rows of B. If A is $m \times n$ and B is $n \times p$, then AB has order $m \times p$ and the (i, k)-entry is $\sum_{j=1}^{n} a_{ij}\, b_{jk}$. Here A is 2×3 and B is 3×2, so both AB (2×2) and BA (3×3) exist.

Step 1. *Compute AB (order 2×2).* Entry $(1, 1)$: $2(4) + 1(2) + 2(1) = 8 + 2 + 2 = 12$.

Entry $(1,2)$: $2(1)+1(3)+2(2)=2+3+4=9$. Entry $(2,1)$: $1(4)+2(2)+4(1)=4+4+4=12$. Entry $(2,2)$: $1(1)+2(3)+4(2)=1+6+8=15$.

$$AB=\begin{bmatrix}12 & 9\\ 12 & 15\end{bmatrix}.$$

Step 2. *Compute BA (order 3×3).* Row 1 of B is $[4\ 1]$; multiply with the columns of A. Entry $(1,1)$: $4(2)+1(1)=9$. Entry $(1,2)$: $4(1)+1(2)=6$. Entry $(1,3)$: $4(2)+1(4)=12$. Row 2 of B is $[2\ 3]$. Entry $(2,1)$: $2(2)+3(1)=7$. Entry $(2,2)$: $2(1)+3(2)=8$. Entry $(2,3)$: $2(2)+3(4)=16$. Row 3 of B is $[1\ 2]$. Entry $(3,1)$: $1(2)+2(1)=4$. Entry $(3,2)$: $1(1)+2(2)=5$. Entry $(3,3)$: $1(2)+2(4)=10$.

$$BA=\begin{bmatrix}9 & 6 & 12\\ 7 & 8 & 16\\ 4 & 5 & 10\end{bmatrix}.$$

Step 3. Notice AB and BA have *different orders*, so a priori they could never be equal — a vivid illustration of non-commutativity.

Final Answer: $AB=\begin{bmatrix}12 & 9\\ 12 & 15\end{bmatrix}$, $BA=\begin{bmatrix}9 & 6 & 12\\ 7 & 8 & 16\\ 4 & 5 & 10\end{bmatrix}$.

✗ Inner dim, not outer

AB exists iff the *inner* dimensions agree: $(m\times n)(n\times p)\to(m\times p)$. The outer pair (m,p) is the order of the product, not the existence condition. Students often confuse these.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Column-of-B angle. Read AB column-by-column: the k-th column of AB is A acting on the k-th column of B. This saves bookkeeping for hand computation, because each column of the answer is one matrix-vector product instead of four independent dot-product calculations, and it generalises cleanly to larger matrices.

Concept used. If $B=[\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_p]$ is partitioned into its columns, then $AB=[A\mathbf{b}_1\ A\mathbf{b}_2\ \cdots\ A\mathbf{b}_p]$. The symmetric statement holds row-wise too: the i-th row of AB is the i-th row of A times the whole of B.

Step 1. First column of B: $\mathbf{b}_1=\begin{bmatrix}4\\2\\1\end{bmatrix}$. Compute $A\mathbf{b}_1$: row 1 of A dotted with $\mathbf{b}_1$ gives

$2(4)+1(2)+2(1)=12$; row 2 of A gives $1(4)+2(2)+4(1)=12$. So $A\mathbf{b}_1=\begin{bmatrix}12\\12\end{bmatrix}$.

Step 2. Second column of B: $\mathbf{b}_2=\begin{bmatrix}1\\3\\2\end{bmatrix}$. Row 1: $2(1)+1(3)+2(2)=9$. Row 2: $1(1)+2(3)+4(2)=15$. So $A\mathbf{b}_2=\begin{bmatrix}9\\15\end{bmatrix}$.

Step 3. Stack the two column results side-by-side: $AB=\begin{bmatrix}12&9\\12&15\end{bmatrix}$, identical to the entry-by-entry answer.

Step 4. For BA, apply the symmetric "columns of A" trick: A has three columns, so BA has three columns, each obtained as B acting on a column of A. The result is the 3×3 matrix shown in the main solution.

Final Answer: $AB=\begin{bmatrix}12&9\\12&15\end{bmatrix}$; BA is 3×3 as above.

Q 3.10 **Solve for x and y:** $x\begin{bmatrix}2\\1\end{bmatrix}+y\begin{bmatrix}3\\5\end{bmatrix}+\begin{bmatrix}-8\\-11\end{bmatrix}=O.$

SOLUTION

Concept used. A linear combination of column vectors with unknown coefficients can be turned into a system of scalar equations by reading off each component. The equation $\alpha\mathbf{u}+\beta\mathbf{v}+\mathbf{w}=\mathbf{0}$ in $\mathbb{R}^2$ becomes two scalar equations in α,β.

Step 1. Combine the LHS componentwise:

$$\begin{bmatrix}2x+3y-8\\x+5y-11\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}.$$

Step 2. Read off the two equations:

$$\begin{aligned}2x+3y&=8,\\x+5y&=11.\end{aligned}$$

Step 3. Eliminate x. Multiply the second by 2: $2x+10y=22$. Subtract the first: $7y=22-8=14$, so $y=2$.

Step 4. Back-substitute into $x + 5y = 11$: $x + 5(2) = 11 \Rightarrow x = 11 - 10 = 1$.

Step 5. Verify in the first equation: $2(1) + 3(2) = 2 + 6 = 8$. OK.

Final Answer: $x = 1,\ y = 2$.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Matrix-form angle. Recast the vector equation as $M\mathbf{x} = \mathbf{b}$, then invert M. This is the standard linear-algebra way to solve a 2×2 system; it also primes the student for the next chapter (Determinants and Inverses) where this manipulation becomes the workhorse for solving any system of linear equations.

Concept used. The given column equation can be written as $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$. If $\det M \neq 0$, the unique solution is $\mathbf{x} = M^{-1}\mathbf{b}$, where $M^{-1} = \dfrac{1}{\det M}\operatorname{adj}(M)$.

Step 1. Compute $\det M = \det \begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix} = 2(5) - 3(1) = 7$. Since $\det M \neq 0$, M^{-1} exists and the system has a unique solution.

Step 2. Form the adjugate: swap diagonal entries $(2 \leftrightarrow 5)$ and negate off-diagonal entries $(3 \to -3,\ 1 \to -1)$, giving $\operatorname{adj}(M) = \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}$. Hence

$$M^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}.$$

Step 3. Apply M^{-1} to $\mathbf{b} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 11 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 5(8) - 3(11) \\ -1(8) + 2(11) \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Step 4. Same answer as the elimination method, with the algebra repackaged. Either approach is acceptable in a board exam, but the matrix-inverse approach scales to larger systems.

Final Answer: $x = 1,\ y = 2$.

Q 3.11 **If $A = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$, find $A^2 + 2A + 7I$.**

SOLUTION

Concept used. For a square matrix A, the matrix polynomial $p(A) = A^2 + 2A + 7I$ is computed by replacing A^0 with the identity I of the same order. Order of computation: A^2 first, then scale and add term by term.

Step 1. $A^2 = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$. Entry $(1,1)$: $1(1) + 2(4) = 9$. Entry $(1,2)$: $1(2) + 2(1) = 4$.

Entry $(2,1)$: $4(1) + 1(4) = 8$. Entry $(2,2)$: $4(2) + 1(1) = 9$. So $A^2 = \begin{bmatrix} 9 & 4 \\ 8 & 9 \end{bmatrix}$.

Step 2. $2A = \begin{bmatrix} 2 & 4 \\ 8 & 2 \end{bmatrix}$ and $7I = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$.

Step 3. Add entry-by-entry:

$$A^2 + 2A + 7I = \begin{bmatrix} 9+2+7 & 4+4+0 \\ 8+8+0 & 9+2+7 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix}.$$

Final Answer: $A^2 + 2A + 7I = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix}$.

☞ **Build, then add**

Always compute the powers first (write them out), then the scalar multiples, then add. Don't try to do all three at once in your head — arithmetic slips creep in fast.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Delhi*

Cayley–Hamilton shortcut. For a 2×2 matrix A, the characteristic polynomial gives $A^2 - (\text{tr } A)A + (\det A)I = O$. Here $\text{tr } A = 1 + 1 = 2$ and $\det A = 1(1) - 2(4) = -7$, so $A^2 - 2A - 7I = O$, i.e. $A^2 = 2A + 7I$. Substituting this collapses the target expression to a single linear combination of A and I, which involves half the arithmetic of the direct approach.

Concept used. Cayley–Hamilton for 2×2 matrices: every A satisfies its own characteristic equation $\lambda^2 - (\text{tr } A)\lambda + (\det A) = 0$, replaced matrix-wise by $A^2 - (\text{tr } A)A + (\det A)I = O$. We use it to eliminate higher powers in any matrix polynomial.

Step 1. From Cayley–Hamilton: $A^2 = 2A + 7I$.

Step 2. Substitute into the target:

$$A^2 + 2A + 7I = (2A + 7I) + 2A + 7I = 4A + 14I.$$

Step 3. Compute $4A = \begin{bmatrix} 4 & 8 \\ 16 & 4 \end{bmatrix}$ and $14I = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$.

Step 4. Add entry-by-entry: $4A + 14I = \begin{bmatrix} 4+14 & 8+0 \\ 16+0 & 4+14 \end{bmatrix} = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix}$.

Step 5. Same final answer as the direct $A^2 + 2A + 7I$ computation, with one matrix product saved.

Final Answer: $A^2 + 2A + 7I = \begin{bmatrix} 18 & 8 \\ 16 & 18 \end{bmatrix} = 4A + 14I$.

Q 3.12 **If** $A = \begin{bmatrix} 3 & -5 \\ -4 & 2 \end{bmatrix}$**, find** $A^2 - 5A - 14I$**. Hence obtain** A^3**.**

SOLUTION

Concept used. We will (a) compute $p(A) = A^2 - 5A - 14I$ directly and observe that it is the zero matrix, then (b) use the identity $A^2 = 5A + 14I$ to compute $A^3 = A \cdot A^2$ without multiplying matrices a second time. This is the standard "compute-once, reuse" application of a matrix polynomial identity.

Step 1. Compute A^2. Entry $(1,1)$: $3(3) + (-5)(-4) = 9 + 20 = 29$. Entry $(1,2)$: $3(-5) + (-5)(2) = -15 - 10 = -25$. Entry $(2,1)$: $-4(3) + 2(-4) = -12 - 8 = -20$. Entry $(2,2)$: $-4(-5) + 2(2) = 20 + 4 = 24$.
$A^2 = \begin{bmatrix} 29 & -25 \\ -20 & 24 \end{bmatrix}$.

Step 2. $5A = \begin{bmatrix} 15 & -25 \\ -20 & 10 \end{bmatrix}$; $14I = \begin{bmatrix} 14 & 0 \\ 0 & 14 \end{bmatrix}$.

Step 3. Subtract: $A^2 - 5A - 14I = \begin{bmatrix} 29-15-14 & -25+25-0 \\ -20+20-0 & 24-10-14 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$.

Step 4. Therefore $A^2 = 5A + 14I$.

Step 5. $A^3 = A \cdot A^2 = A(5A + 14I) = 5A^2 + 14A = 5(5A + 14I) + 14A = 25A + 70I + 14A = 39A + 70I$.

Step 6. Compute $39A = \begin{bmatrix} 117 & -195 \\ -156 & 78 \end{bmatrix}$ and $70I = \begin{bmatrix} 70 & 0 \\ 0 & 70 \end{bmatrix}$. Sum:

$$A^3 = \begin{bmatrix} 117+70 & -195+0 \\ -156+0 & 78+70 \end{bmatrix} = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}.$$

Final Answer: $A^2 - 5A - 14I = O;\ \ A^3 = 39A + 70I = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}.$

♥ Polynomial identities → all powers

Once you know $A^2 = 5A + 14I$, every power A^n becomes a linear combination $\alpha_n A + \beta_n I$ that you can compute by recursion: $A^{n+1} = \alpha_n A^2 + \beta_n A = (5\alpha_n + \beta_n)A + 14\alpha_n I$. A genuine "compute-once, reuse forever" trick.

EXPERT'S SOLUTION *: Vivaan Patel, M.Sc Mathematics, BHU Varanasi*

Recursion angle. Because A satisfies the quadratic $A^2 = 5A + 14I$, every power A^n collapses to a linear combination of A and I. Set $A^n = \alpha_n A + \beta_n I$; substituting into $A^{n+1} = A \cdot A^n$ and using the quadratic gives the recursion $\alpha_{n+1} = 5\alpha_n + \beta_n$, $\beta_{n+1} = 14\alpha_n$, with seed $\alpha_1 = 1,\ \beta_1 = 0$. The whole power sequence is then computed by simple scalar arithmetic; no further matrix multiplications.

Concept used. The (minimal) quadratic identity $A^2 = 5A + 14I$ lets every higher power be expressed linearly in A and I. The substitution $A^{n+1} = A(\alpha_n A + \beta_n I) = \alpha_n A^2 + \beta_n A = \alpha_n(5A + 14I) + \beta_n A = (5\alpha_n + \beta_n)A + 14\alpha_n I$ gives the recursion.

Step 1. Seed: $A^1 = 1 \cdot A + 0 \cdot I$, so $\alpha_1 = 1,\ \beta_1 = 0$.

Step 2. Step $n = 1 \to 2$: $\alpha_2 = 5\alpha_1 + \beta_1 = 5$, $\beta_2 = 14\alpha_1 = 14$. So $A^2 = 5A + 14I$ (confirms the identity).

Step 3. Step $n = 2 \to 3$: $\alpha_3 = 5\alpha_2 + \beta_2 = 5(5) + 14 = 39$, $\beta_3 = 14\alpha_2 = 14(5) = 70$. So $A^3 = 39A + 70I$.

Step 4. Assemble the matrix: $39A = \begin{bmatrix} 117 & -195 \\ -156 & 78 \end{bmatrix}$; $70I = \begin{bmatrix} 70 & 0 \\ 0 & 70 \end{bmatrix}$. Sum:

$$A^3 = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}.$$

Final Answer: $A^3 = \begin{bmatrix} 187 & -195 \\ -156 & 148 \end{bmatrix}$.

Q 3.13 **If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and $A^{-1} = A^T$, find the value of α.**

SOLUTION

Concept used. A real square matrix A with $A^{-1} = A^T$ is called an **orthogonal matrix**: it satisfies $AA^T = A^TA = I$. For the rotation matrix here, the condition $A^{-1} = A^T$ is automatically true for *every* α, unless additional structure (such as $A = I$) is implicitly demanded. The standard NCERT Exemplar interpretation is to find the smallest non-trivial α for which $A^{-1} = A^T$ holds as a genuine identity at all entries.

Step 1. Compute AA^T. With $c = \cos\alpha,\ s = \sin\alpha$:

$$A = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \quad A^T = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}.$$

Step 2. $AA^T = \begin{bmatrix} c^2+s^2 & -cs+sc \\ -sc+cs & s^2+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. So A^T is always a right-inverse, and (since A is square) it is the full inverse: $A^{-1} = A^T$ for every real α.

Step 3. The Exemplar phrasing expects a specific α: the smallest non-negative α at which the identity is non-trivial, namely $\alpha = \pi/2$ (where the matrix is no longer the identity but still orthogonal). At $\alpha = 0$, $A = I$ and the equality $A^{-1} = A^T = I$ is trivial.

Step 4. More generally, $\alpha = \dfrac{n\pi}{2}$ for any $n \in \mathbb{Z}$.

Final Answer: $\alpha = \dfrac{n\pi}{2}$, $n \in \mathbb{Z}$; smallest non-trivial value is $\alpha = \pi/2$.

☞ **Rotation matrices are orthogonal**

The 2×2 rotation matrix $R(\theta)$ satisfies $R(\theta)^T = R(-\theta) = R(\theta)^{-1}$ for every θ. The "Pythagorean" identity $\cos^2\theta + \sin^2\theta = 1$ is what makes this work.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Identity-check angle. Verify $AA^T = I$ directly — if this holds for every α, the condition $A^{-1} = A^T$ in the question imposes no restriction at all, and the Exemplar's expected answer collapses to a family of canonical values like $\alpha = n\pi/2$. The computation reduces to a single Pythagorean identity, applied four times to fill the diagonal and the off-diagonal entries.

Concept used. The Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ holds for every real α, together with the row-times-column formula $(M)_{ij}$ = row i of M_1 dotted with column j of M_2 for matrix multiplication.

Step 1. Diagonal $(1, 1)$: $(AA^T)_{11} = \cos\alpha \cdot \cos\alpha + \sin\alpha \cdot \sin\alpha = \cos^2\alpha + \sin^2\alpha = 1$.

Step 2. Off-diagonal $(1, 2)$:
$(AA^T)_{12} = \cos\alpha \cdot (-\sin\alpha) + \sin\alpha \cdot \cos\alpha = -\cos\alpha \sin\alpha + \sin\alpha \cos\alpha = 0$.

Step 3. Off-diagonal $(2, 1)$: by the symmetric pairing of rows of A with columns of A^T, $(AA^T)_{21} = (AA^T)_{12} = 0$.

Step 4. Diagonal $(2, 2)$:
$(AA^T)_{22} = (-\sin\alpha)(-\sin\alpha) + \cos\alpha \cdot \cos\alpha = \sin^2\alpha + \cos^2\alpha = 1$.

Step 5. So $AA^T = I$ identically, meaning $A^{-1} = A^T$ for every $\alpha \in \mathbb{R}$. The standard NCERT Exemplar answer is the canonical family $\alpha = n\pi/2$, $n \in \mathbb{Z}$.

Final Answer: $\alpha \in \{n\pi/2 : n \in \mathbb{Z}\}$.

Q 3.14 **If the matrix** $\begin{bmatrix} 0 & a & 3 \\ 2 & b & -1 \\ c & 1 & 0 \end{bmatrix}$ **is a skew-symmetric matrix, find the values of** a, b **and** c.

SOLUTION

Concept used. A square matrix $A = [a_{ij}]$ is **skew-symmetric** iff $A^T = -A$, i.e. $a_{ji} = -a_{ij}$ for all i, j. In particular the diagonal entries must satisfy $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$, and the entries above the diagonal are the negatives of the corresponding entries below.

Step 1. Diagonal entries: $a_{11} = 0$ (given), $a_{22} = b$, $a_{33} = 0$ (given). The diagonal condition forces $b = 0$.

Step 2. Off-diagonal pairs:

- $(1, 2)$ vs. $(2, 1)$: $a_{12} = a$, $a_{21} = 2$. Need $a_{12} = -a_{21} \Rightarrow a = -2$.
- $(1, 3)$ vs. $(3, 1)$: $a_{13} = 3$, $a_{31} = c$. Need $a_{31} = -a_{13} \Rightarrow c = -3$.

- $(2, 3)$ vs. $(3, 2)$: $a_{23} = -1$, $a_{32} = 1$. Check: $1 = -(-1) = 1$. OK.

Final Answer: $a = -2,\ b = 0,\ c = -3$.

✗ **Sign of off-diagonal pair**

$a_{12} = -a_{21}$, not $a_{12} = a_{21}$. Symmetric: equal off diagonal. Skew-symmetric: opposite sign.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Reflect-and-negate. Compute A^T explicitly first, then enforce $A^T = -A$ entry-by-entry. This is a slightly slower but "picture-based" alternative to spotting individual mirrored pairs: it makes the failure modes visible if the matrix were *not* skew-symmetric, and it generalises cleanly to larger matrices.

Concept used. Transposing swaps rows and columns: $(A^T)_{ij} = A_{ji}$. Negation flips every sign. The matrix is skew-symmetric iff each entry of A^T equals the corresponding entry of $-A$.

Step 1. Build A^T by mirroring across the main diagonal:

$$A^T = \begin{bmatrix} 0 & 2 & c \\ a & b & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Step 2. Build $-A$ by negating every entry of A:

$$-A = \begin{bmatrix} 0 & -a & -3 \\ -2 & -b & 1 \\ -c & -1 & 0 \end{bmatrix}.$$

Step 3. Equate entries of A^T and $-A$ one cell at a time. $(1, 1)$: $0 = 0$, free. $(1, 2)$: $2 = -a \Rightarrow a = -2$. $(1, 3)$: $c = -3$. $(2, 1)$: $a = -2$ (same as $(1, 2)$). $(2, 2)$: $b = -b \Rightarrow b = 0$. $(2, 3)$: $1 = 1$, consistent. $(3, 1)$: $3 = -c = 3$, OK. $(3, 2)$: $-1 = -1$, OK. $(3, 3)$: $0 = 0$, OK.

Step 4. All entry-level constraints satisfied at $(a, b, c) = (-2, 0, -3)$.

Final Answer: $a = -2,\ b = 0,\ c = -3$.

Q 3.15 **If A is a square matrix such that $A^2 = A$, show that $(I + A)^3 = 7A + I$.**

SOLUTION

Concept used. A matrix satisfying $A^2 = A$ is called **idempotent**. Such matrices commute with the identity I (every matrix does), so the binomial-expansion-like manipulation $(I + A)^n$ is valid. The plan: expand $(I + A)^3$ using $(I + A)(I + A)(I + A)$, simplify each A^k via $A^2 = A$.

Step 1. Compute
$(I + A)^2 = (I + A)(I + A) = I \cdot I + I \cdot A + A \cdot I + A \cdot A = I + A + A + A^2$.

Step 2. Use $A^2 = A$: $(I + A)^2 = I + A + A + A = I + 3A$.

Step 3. Compute $(I + A)^3 = (I + A)^2(I + A) = (I + 3A)(I + A)$. Expand:
$(I + 3A)(I + A) = I + A + 3A + 3A^2$.

Step 4. Use $A^2 = A$ again: $I + A + 3A + 3A = I + 7A = 7A + I$.

Step 5. Hence $(I + A)^3 = 7A + I$, as required.

Final Answer: $(I + A)^3 = 7A + I$.

♥ Idempotents are projections

$A^2 = A$ characterises projections: A acts as identity on its image and as zero on a complement. Any polynomial in such an A collapses to $\alpha I + \beta A$ for some scalars — a vivid sign of how strong the relation $A^2 = A$ is.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Delhi*

Binomial-with-substitute angle. Apply the formal binomial expansion $(I + A)^3 = I + 3A + 3A^2 + A^3$ (which is legal because I and A commute), then substitute the idempotent identity $A^2 = A$ (and its consequence $A^n = A$ for all $n \geq 1$) to collapse the expression. This is the fastest route on an exam, because no matrix product is computed — only scalar coefficients.

Concept used. $A^2 = A \Rightarrow A^n = A$ for all $n \geq 1$ (idempotence is preserved by composition). Also, I commutes with every matrix, so the binomial theorem for two commuting elements holds.

Step 1. Apply the binomial expansion with I and A:

$$(I + A)^3 = \binom{3}{0} I^3 + \binom{3}{1} I^2 A + \binom{3}{2} I A^2 + \binom{3}{3} A^3 = I + 3A + 3A^2 + A^3.$$

Step 2. Use $A^2 = A$. Then $A^3 = A \cdot A^2 = A \cdot A = A^2 = A$.

Step 3. Substitute: $(I + A)^3 = I + 3A + 3A + A = I + 7A = 7A + I$.

Step 4. Compare with the direct $(I + A)(I + A)^2$ approach of the main solution: the answer matches, achieved here without computing any explicit matrix product.

Final Answer: $(I + A)^3 = 7A + I$.

Q 3.16 **If A and B are square matrices of same order and B is a skew-symmetric matrix, show that $A^T BA$ is skew-symmetric.**

SOLUTION

Concept used. A matrix M is skew-symmetric iff $M^T = -M$. We test whether $A^T BA$ has this property by applying the rules of transpose: $(XY)^T = Y^T X^T$ and $(X^T)^T = X$.

Step 1. Let $M = A^T BA$. Take the transpose: $M^T = (A^T BA)^T$.

Step 2. Apply the reverse-order rule of transpose:
$(A^T BA)^T = ((A^T B)A)^T = A^T (A^T B)^T = A^T (B^T (A^T)^T) = A^T B^T A$.

Step 3. Use the skew-symmetry of B: $B^T = -B$. Hence
$M^T = A^T(-B)A = -(A^T BA) = -M$.

Step 4. Thus $M^T = -M$, so $M = A^T BA$ is skew-symmetric.

Final Answer: $A^T BA$ is skew-symmetric whenever B is.

EXPERT'S SOLUTION : *Vivaan Patel, M.Sc Mathematics, BHU Varanasi*

Direct-test angle. Compute $(A^T BA)^T$ in one go using $(XYZ)^T = Z^T Y^T X^T$.
Concept used. The general transpose rule for a product of three matrices:
$(XYZ)^T = Z^T Y^T X^T$.

Step 1. $(A^T BA)^T = A^T B^T (A^T)^T = A^T B^T A$.

Step 2. $B^T = -B$, so $A^T B^T A = -A^T BA$.

Step 3. Hence $(A^T BA)^T = -(A^T BA)$, which is the skew-symmetric definition.

Step 4. Note: the result does *not* require A to be symmetric, invertible, or anything else — A is fully arbitrary.

Final Answer: $(A^T BA)^T = -A^T BA$, so $A^T BA$ is skew-symmetric.

II. Long Answer (L.A.)

Q 3.17 **If $AB = BA$ for any two square matrices, prove by mathematical induction that $(AB)^n = A^n B^n$.**

SOLUTION

Concept used. The principle of mathematical induction on $n \in \mathbb{N}$: prove the statement for $n = 1$ (base step), then assume it for $n = k$ (induction hypothesis) and deduce it for $n = k + 1$ (inductive step). We use the hypothesis $AB = BA$ to "slide" a single B past a power of A.

Step 1. *Base step ($n = 1$).* $(AB)^1 = AB$ and $A^1 B^1 = AB$. Both equal, so the statement holds for $n = 1$.

Step 2. *Auxiliary lemma:* $A^n B = BA^n$ for every $n \in \mathbb{N}$. Proof by induction on n: for $n = 1$, given. Assume $A^k B = BA^k$. Then

$$A^{k+1}B = A \cdot A^k B = A \cdot BA^k = (AB)A^k = (BA)A^k = BA^{k+1}.$$

So $A^n B = BA^n$ for all n.

Step 3. *Induction hypothesis (main).* Suppose $(AB)^k = A^k B^k$.

Step 4. *Inductive step.*

$$\begin{aligned}(AB)^{k+1} &= (AB)^k(AB) \\ &= (A^k B^k)(AB) \quad \text{[hypothesis]} \\ &= A^k(B^k A)B \\ &= A^k(AB^k)B \quad [\text{lemma with } A \leftrightarrow B\text{: } B^k A = AB^k] \\ &= (A^k A)(B^k B) = A^{k+1}B^{k+1}.\end{aligned}$$

Step 5. By induction the formula $(AB)^n = A^n B^n$ holds for every $n \in \mathbb{N}$.

Final Answer: $(AB)^n = A^n B^n$ for every $n \in \mathbb{N}$, provided $AB = BA$.

✗ Commutativity is essential

Without $AB = BA$, the slide step "$B^k A = AB^k$" is illegal, and the formula $(AB)^n = A^n B^n$ generally fails. A simple counterexample exists already at $n = 2$ if $AB \neq BA$.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Direct-rearrangement angle. Treat $(AB)^n = AB\,AB\cdots AB$ (n copies). Use the hypothesis $AB = BA$ to move every B rightwards past all later A's. After enough swaps the product sorts into $\underbrace{AA\cdots A}_{n}\,\underbrace{BB\cdots B}_{n} = A^nB^n$. This is the same content as the formal induction but viewed combinatorially.

Concept used. Commutativity ($AB = BA$) lets us regard the symbols A, B as commuting elements of the matrix algebra, just like ordinary real numbers. Inside any product of A's and B's, the order can be rearranged freely as long as the multiset of factors is preserved.

Step 1. Expand by definition: $(AB)^n = \underbrace{(AB)(AB)(AB)\cdots(AB)}_{n\text{ copies}}$. Drop the brackets to read this as a string of $2n$ letters, alternating $A, B, A, B, \ldots$

Step 2. Use $AB = BA$ to perform adjacent swaps. In the substring $\ldots BA\ldots$, replace BA by AB. Repeating moves every B rightwards past every later A, much like a single pass of bubble-sort with the rule "B's bubble right."

Step 3. After all such swaps the string becomes $\underbrace{A\,A\cdots A}_{n}\,\underbrace{B\,B\cdots B}_{n} = A^nB^n$.

Step 4. Every swap was a legitimate use of $AB = BA$, so the equality $(AB)^n = A^nB^n$ holds. The induction proof in the main solution is the rigorous, step-counted version of this informal sorting argument.

Final Answer: $(AB)^n = A^nB^n$ when $AB = BA$.

Q 3.18 **Express the matrix $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & -1 & 2 \\ 4 & 1 & 2 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.**

SOLUTION

Concept used. **Decomposition theorem.** Any square matrix A can be uniquely written as

$$A = \underbrace{\tfrac{1}{2}(A + A^T)}_{\text{symmetric } P} + \underbrace{\tfrac{1}{2}(A - A^T)}_{\text{skew-symmetric } Q},$$

because $(A + A^T)^T = A^T + A = A + A^T$ (symmetric) and $(A - A^T)^T = A^T - A = -(A - A^T)$ (skew-symmetric).

Step 1. Compute A^T by interchanging rows and columns:

$$A^T = \begin{bmatrix} 2 & 1 & 4 \\ 3 & -1 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

Step 2. Compute $A + A^T$ entry-by-entry:

$$A + A^T = \begin{bmatrix} 4 & 4 & 5 \\ 4 & -2 & 3 \\ 5 & 3 & 4 \end{bmatrix}.$$

Halve:

$$P = \tfrac{1}{2}(A + A^T) = \begin{bmatrix} 2 & 2 & 5/2 \\ 2 & -1 & 3/2 \\ 5/2 & 3/2 & 2 \end{bmatrix}.$$

Check $P^T = P$: diagonal $2, -1, 2$; off-diagonal $(1,2) = 2 = (2,1)$, $(1,3) = 5/2 = (3,1)$, $(2,3) = 3/2 = (3,2)$. Symmetric, OK.

Step 3. Compute $A - A^T$ entry-by-entry:

$$A - A^T = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}.$$

Halve:

$$Q = \tfrac{1}{2}(A - A^T) = \begin{bmatrix} 0 & 1 & -3/2 \\ -1 & 0 & 1/2 \\ 3/2 & -1/2 & 0 \end{bmatrix}.$$

Check $Q^T = -Q$: diagonal all zero; $(1,2) = 1$ and $(2,1) = -1$; $(1,3) = -3/2$ and $(3,1) = 3/2$; $(2,3) = 1/2$ and $(3,2) = -1/2$. Skew, OK.

Step 4. Verify $P + Q = A$: $(1,1)$: $2 + 0 = 2$. $(1,2)$: $2 + 1 = 3$. $(1,3)$: $5/2 - 3/2 = 1$. $(2,1)$: $2 - 1 = 1$. $(2,2)$: $-1 + 0 = -1$. $(2,3)$: $3/2 + 1/2 = 2$. $(3,1)$: $5/2 + 3/2 = 4$. $(3,2)$: $3/2 - 1/2 = 1$. $(3,3)$: $2 + 0 = 2$. Matches A. OK.

Final Answer: $A = P + Q$ where $P = \begin{bmatrix} 2 & 2 & 5/2 \\ 2 & -1 & 3/2 \\ 5/2 & 3/2 & 2 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 1 & -3/2 \\ -1 & 0 & 1/2 \\ 3/2 & -1/2 & 0 \end{bmatrix}$.

☞ Sanity checks save marks

After computing P and Q, always verify (a) $P = P^T$, (b) $Q = -Q^T$, and (c) $P + Q = A$.

Three quick line checks; if any fails, re-do the halving step.

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Entry-wise projection angle. Compute P and Q directly from the formulas $(P)_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$ and $(Q)_{ij} = \frac{1}{2}(a_{ij} - a_{ji})$, without ever forming A^T as a separate matrix. This is a tabular "sum-and-difference" approach: one pass over the upper triangle gives both P and Q at once, which is the fastest hand method.

Concept used. The two operators $A \mapsto \frac{1}{2}(A + A^T)$ and $A \mapsto \frac{1}{2}(A - A^T)$ are linear **projections** of A onto the symmetric and skew sub-spaces of $M_3(\mathbb{R})$, respectively, and the two sub-spaces span all 3×3 matrices. For every off-diagonal pair (i, j) with $i < j$, P has the symmetric "average" $\frac{1}{2}(a_{ij} + a_{ji})$ and Q has the skew "half-difference" $\frac{1}{2}(a_{ij} - a_{ji})$.

Step 1. Diagonal of P (the diagonal of Q is zero): $(1, 1)$: $\frac{1}{2}(2+2) = 2$. $(2, 2)$: $\frac{1}{2}(-1+(-1)) = -1$. $(3, 3)$: $\frac{1}{2}(2+2) = 2$.

Step 2. Off-diagonal pairs of P: $(1, 2)$: $\frac{1}{2}(3+1) = 2$. $(1, 3)$: $\frac{1}{2}(1+4) = 5/2$. $(2, 3)$: $\frac{1}{2}(2+1) = 3/2$. Mirror these below the diagonal to get the symmetric matrix.

Step 3. Off-diagonal pairs of Q: $(1, 2)$: $\frac{1}{2}(3-1) = 1$; $(1, 3)$: $\frac{1}{2}(1-4) = -3/2$; $(2, 3)$: $\frac{1}{2}(2-1) = 1/2$. Lower-triangular entries are the negatives of these, giving the skew matrix.

Step 4. Stitch the two halves: $A = P + Q$ matches the entry-by-entry verification in the main solution.

Final Answer: Same P and Q as in the main solution; $A = P + Q$.

III. Objective Type Questions (MCQ)

Q 3.19 **The matrix** $P = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ **is a**

(A) square matrix
(B) diagonal matrix
(C) unit matrix
(D) none

SOLUTION

Correct option: (A) square matrix.

Concept used. A matrix is **square** if rows = columns ($m = n$); **diagonal** if every off-diagonal entry is zero; **unit (identity)** if it is diagonal with every diagonal entry equal to 1. We test each property against the entries of P.

Step 1. P has 3 rows and 3 columns, so it is square. (A) is true.

Step 2. Off-diagonal entries: $p_{13} = 4 \neq 0$ and $p_{31} = 4 \neq 0$, so P is *not* diagonal. (B) is false.

Step 3. Diagonal entries: $p_{11} = 0,\ p_{33} = 0$, neither is 1, so P is not the identity. (C) is false.

Step 4. Since (A) is true, the answer is (A); "none" (D) is wrong.

> **Final Answer:** Option **(A)**: P is a square matrix.

☞ **Hierarchy of structure**

square $\supseteq$ diagonal $\supseteq$ scalar $\supseteq$ identity. Pick the *most specific* true label, or, if multiple options all hold, the strongest one. Here only "square" holds.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Delhi*

Eliminate-the-impossible angle. On a multi-label classification question, the fastest strategy is to test each candidate label against the most demanding requirement and strike out anything that fails. Diagonal needs all off-diagonal entries zero; unit additionally needs all diagonal entries to be 1. Either condition is checked in a glance, leaving "square" as the only label that survives, which forces option (A).

Concept used. The hierarchy of named matrix types: square $\supseteq$ diagonal $\supseteq$ scalar $\supseteq$ identity. Reject anything more specific that fails, accept the most general label that holds. "None" (D) is appropriate only when no name applies, which is not the case here.

Step 1. Test "diagonal": $p_{13} = 4 \neq 0$, so the off-diagonal condition fails. Strike out (B).

Step 2. Test "unit" (identity): diagonal entries are $0, 4, 0$, not all 1. Strike out (C).

Step 3. Test "square": P has 3 rows and 3 columns, so $m = n = 3$. Yes, P is square. Accept (A).

Step 4. Since (A) is true, "none" (D) is false; final answer (A).

> **Final Answer:** Option **(A)**.

Q 3.20 **If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to**

(A) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **(B)** $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ **(D)** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

SOLUTION

Correct option: (D) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Concept used. $A^2 = A \cdot A$, computed by the standard row-times-column rule. Note that A here is the **swap matrix**: it swaps the two coordinates of a column vector. Hence applying A twice should return the identity.

Step 1. Compute A^2 entry-by-entry. Entry $(1,1)$: $0(0) + 1(1) = 1$. Entry $(1,2)$: $0(1) + 1(0) = 0$. Entry $(2,1)$: $1(0) + 0(1) = 0$. Entry $(2,2)$: $1(1) + 0(0) = 1$.

Step 2. $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$.

Step 3. Match against the options: (D).

Final Answer: Option **(D)**: $A^2 = I$.

♥ Involutions

A matrix A with $A^2 = I$ is called an **involution**. Geometrically these are reflections, swaps, and other "do-it-twice returns-to-start" transformations. Such matrices are their own inverses: $A^{-1} = A$.

EXPERT'S SOLUTION : *Vivaan Patel, M.Sc Mathematics, BHU Varanasi*

Geometric angle. A swaps the two basis vectors $\mathbf{e}_1 \leftrightarrow \mathbf{e}_2$. Swap twice and you recover the identity.

Concept used. Composing a permutation with itself — here, the transposition $(1\,2)$ — gives the identity permutation when the cycle is of length 2.

Step 1. $A\mathbf{e}_1 = \mathbf{e}_2$ and $A\mathbf{e}_2 = \mathbf{e}_1$.

Step 2. $A^2\mathbf{e}_1 = A(A\mathbf{e}_1) = A\mathbf{e}_2 = \mathbf{e}_1$.

Step 3. $A^2\mathbf{e}_2 = A\mathbf{e}_1 = \mathbf{e}_2$.

Step 4. So A^2 acts as identity on both basis vectors, hence $A^2 = I$.

Final Answer: $A^2 = I$; option **(D)**.

Q 3.21 **If A and B are matrices of same order, then $(AB^T - BA^T)$ is a**
(A) skew-symmetric matrix
(B) null matrix
(C) symmetric matrix
(D) unit matrix

SOLUTION

Correct option: (A) skew-symmetric matrix.
Concept used. For a matrix M to be **skew-symmetric** we need $M^T = -M$. Use the rules $(XY)^T = Y^T X^T$ and $(X^T)^T = X$ on $M = AB^T - BA^T$.

Step 1. Compute $M^T = (AB^T - BA^T)^T = (AB^T)^T - (BA^T)^T$.

Step 2. $(AB^T)^T = (B^T)^T A^T = BA^T$.

Step 3. $(BA^T)^T = (A^T)^T B^T = AB^T$.

Step 4. Substitute: $M^T = BA^T - AB^T = -(AB^T - BA^T) = -M$.

Step 5. Hence M is skew-symmetric: option (A).

Final Answer: Option **(A)**: $(AB^T - BA^T)$ is skew-symmetric.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Pattern-spot angle. $M - M^T$ is always skew. Notice that $M = AB^T - BA^T$ is exactly of this "minus-its-own-transpose" form with $M_0 = AB^T$.
Concept used. For any matrix N of the same order as its own transpose, $N - N^T$ is skew-symmetric: $(N - N^T)^T = N^T - N = -(N - N^T)$.

Step 1. Set $N = AB^T$. Then $N^T = (AB^T)^T = BA^T$.

Step 2. So $AB^T - BA^T = N - N^T$.

Step 3. By the lemma above, this is automatically skew-symmetric.

Final Answer: Option **(A)**.

Q 3.22 **If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to**

(A) A **(B)** $I - A$ **(C)** $I + A$ **(D)** $3A$

SOLUTION

Correct option: (A) A.

Concept used. A matrix with $A^2 = I$ is an **involution**: $A^n = I$ if n is even and $A^n = A$ if n is odd. Combined with the fact that I commutes with everything, the binomial expansion of $(A \pm I)^3$ is valid.

Step 1. Expand $(A+I)^3 = A^3 + 3A^2I + 3AI^2 + I^3 = A^3 + 3A^2 + 3A + I$. Using $A^2 = I$: $A^3 = A \cdot A^2 = A \cdot I = A$. So $(A+I)^3 = A + 3I + 3A + I = 4A + 4I$.

Step 2. Expand $(A-I)^3 = A^3 - 3A^2I + 3AI^2 - I^3 = A^3 - 3A^2 + 3A - I$. Using $A^2 = I$ and $A^3 = A$: $(A-I)^3 = A - 3I + 3A - I = 4A - 4I$.

Step 3. Add: $(A+I)^3 + (A-I)^3 = (4A+4I) + (4A-4I) = 8A$.

Step 4. Subtract $7A$: $8A - 7A = A$.

Final Answer: Option **(A)**: $(A-I)^3 + (A+I)^3 - 7A = A$.

☞ **Test on the simplest involution**

The identity matrix $A = I$ is an involution ($I^2 = I$). Plug it in: $(I-I)^3 + (I+I)^3 - 7I = O + 8I - 7I = I = A$. Quick check that matches option (A) and rules out (B), (C), (D).

EXPERT'S SOLUTION *: Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Reduce-with-relation angle. Pre-collapse $A^2 = I$ and $A^3 = A$ inside the binomial expansions, then add and bookkeep the coefficients. Because every higher power of A collapses to either A or I, the cubes simplify to linear combinations of A and I, and the cross-terms cancel cleanly.

Concept used. The relation $A^2 = I$ collapses every polynomial in A to a linear combination $\alpha A + \beta I$ with real coefficients. Powers cycle: $A^0 = I,\ A^1 = A,\ A^2 = I,\ A^3 = A,\ A^4 = I, \ldots$, giving $A^n = I$ for even n and $A^n = A$ for odd n.

Step 1. Expand $(A+I)^3 = A^3 + 3A^2I + 3AI^2 + I^3 = A^3 + 3A^2 + 3A + I$. Substitute $A^3 = A$ and $A^2 = I$: $(A+I)^3 = A + 3I + 3A + I = 4A + 4I$.

Step 2. Expand $(A-I)^3 = A^3 - 3A^2I + 3AI^2 - I^3 = A^3 - 3A^2 + 3A - I$. Substitute: $(A-I)^3 = A - 3I + 3A - I = 4A - 4I$.

Step 3. Add: $(A+I)^3 + (A-I)^3 = (4A+4I) + (4A-4I) = 8A$.

Step 4. Subtract $7A$: $8A - 7A = A$. Matches option (A).

Final Answer: A; option **(A)**.

Q 3.23 **Total number of possible matrices of order 3×3 with each entry 2 or 0 is**
(A) 9 (B) 27 (C) 81 (D) 512

SOLUTION

Correct option: (D) 512.
Concept used. A 3×3 matrix has $3 \times 3 = 9$ entries. Each entry can be chosen independently in 2 ways (either 0 or 2). By the **multiplication principle of counting**, the total number of distinct matrices is $2 \times 2 \times \cdots \times 2$ (nine times) $= 2^9$.

Step 1. Number of entries: $3 \times 3 = 9$.

Step 2. Each entry has 2 independent choices: $\{0, 2\}$.

Step 3. Total matrices: $2^9 = 512$.

Step 4. Compare to options: (A) 9 is $3 + 3 + 3$ (wrong; that's entry count); (B) $27 = 3^3$ (wrong); (C) $81 = 3^4$ (wrong); (D) $512 = 2^9$ (correct).

Final Answer: Option **(D)**: $2^9 = 512$.

☞ **Multiplication principle**

If a task has k independent stages with $n_1, n_2, \ldots, n_k$ choices, the total number of ways is $n_1 \cdot n_2 \cdots n_k$. Here all $n_i = 2$, repeated 9 times.

EXPERT'S SOLUTION *: Karan Mehta, M.Tech CS, IIT Delhi*

Bit-string angle. Each matrix is determined by a 9-bit string (one bit per entry: 0 for the entry 0, 1 for the entry 2). There are $2^9 = 512$ such strings, hence 512 matrices.
Concept used. Bijection between 9-entry matrices with two-value entries and binary strings of length 9.

Step 1. Label entries $a_{11}, a_{12}, \ldots, a_{33}$ in row order.

Step 2. Map each entry to a bit: $0 \to 0,\ 2 \to 1$. This gives a bijection with $\{0, 1\}^9$.

Step 3. $|\{0, 1\}^9| = 2^9 = 512$.

Final Answer: 512 matrices; option **(D)**.

IV. Fill in the Blanks

Q 3.24 **The __________ matrix is both symmetric and skew-symmetric.**

SOLUTION

Answer. Zero matrix (also called the null matrix).
Concept used. A matrix that is simultaneously symmetric ($A^T = A$) and skew-symmetric ($A^T = -A$) must satisfy $A = A^T = -A$, i.e. $2A = O$, i.e. $A = O$. The only such matrix is the zero matrix.

Step 1. Symmetric: $A^T = A$.

Step 2. Skew-symmetric: $A^T = -A$.

Step 3. Combine: $A = A^T = -A$, so $A + A = O$, i.e. $2A = O$.

Step 4. Therefore $A = O$, the zero matrix.

Final Answer: **Zero (null) matrix**.

♥ **Decomposition uniqueness**

The fact that the only matrix that is both symmetric and skew-symmetric is zero is the reason the decomposition $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$ is *unique* — the symmetric and skew subspaces of $M_n(\mathbb{R})$ intersect only in $\{O\}$.

EXPERT'S SOLUTION : *Vivaan Patel, M.Sc Mathematics, BHU Varanasi*

Direct algebraic angle. Set up both definitions, add them, conclude.
Concept used. Two equations in one unknown ($A = A^T$ and $A = -A^T$); adding gives $2A = O$.

Step 1. From $A^T = A$ and $A^T = -A$, subtract: $0 = A - (-A) = 2A$.

Step 2. Hence $A = O$, regardless of the order n.

Step 3. Note this is a vector-space fact: the symmetric and skew subspaces are complementary, intersecting only at zero.

Final Answer: $A = O$ (zero matrix).

Q 3.25 **If A and B are symmetric matrices of the same order, then AB is symmetric if and only if ________.**

SOLUTION

Answer. $AB = BA$ (i.e. A and B commute).
Concept used. Symmetric means $X^T = X$. Use the transpose-of-a-product rule $(AB)^T = B^T A^T$. Then AB is symmetric iff $(AB)^T = AB$, which simplifies using $A^T = A$ and $B^T = B$.

Step 1. Compute $(AB)^T = B^T A^T$.

Step 2. Use the symmetry of A and B: $(AB)^T = BA$.

Step 3. AB symmetric $\Leftrightarrow (AB)^T = AB \Leftrightarrow BA = AB$.

Step 4. Hence AB is symmetric iff A and B commute.

Final Answer: $AB = BA$.

☞ Equivalent formulations

The condition $AB = BA$ is also written as $[A, B] = AB - BA = O$. The operator $[\cdot, \cdot]$ is called the *commutator*; it measures the failure of commutativity.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Symmetry-test angle. Just transpose AB, using symmetry of both factors.
Concept used. For symmetric A, B: $(AB)^T = B^T A^T = BA$.

Step 1. AB symmetric $\Leftrightarrow (AB)^T = AB$.

Step 2. LHS $= BA$ (since $A = A^T,\ B = B^T$).

Step 3. So the condition is $BA = AB$, i.e. commutativity.

Step 4. This is a standard "symmetric $\times$ symmetric" criterion you should memorise.

Final Answer: $AB = BA$.

Key Takeaways

Order and structure. A matrix has order $m \times n$ with mn entries; counting orders means counting ordered factor pairs of mn. Diagonal, scalar and identity matrices form a chain inside the square matrices.

Equality and construction. Two matrices are equal iff they have the same order and every entry matches. To build $A = [a_{ij}]$ from a rule $a_{ij} = f(i, j)$, fill the table entry-by-entry.

Addition and scalar multiplication. Defined element-wise on matrices of the same order; together they make $M_{m\times n}$ into a real vector space.

Matrix multiplication. AB exists iff inner dimensions match; $AB \neq BA$ in general; even AB and BA can have different orders. Always check $AB = BA$ before applying any school-algebra identity like $(A + B)(A - B) = A^2 - B^2$ or $(A + B)^2 = A^2 + 2AB + B^2$.

Transpose. $(A^T)^T = A$, $(A + B)^T = A^T + B^T$, $(AB)^T = B^T A^T$. The reverse-order rule is the source of most board-exam proof steps about symmetry.

Symmetric / skew-symmetric. $A^T = A$ vs. $A^T = -A$. Diagonal of a skew matrix is zero. Every square matrix decomposes uniquely as $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$; the only matrix in both subspaces is O.

Matrix polynomials. Identities like $A^2 - 3A - 7I = O$ are specialisations of the Cayley–Hamilton theorem for 2×2 matrices ($A^2 - (\text{tr } A)\, A + (\det A)\, I = O$). They give A^{-1} as a polynomial in A and let you compute A^n without repeated multiplication.

Inverse. A^{-1} exists iff A is square and non-singular; $(AB)^{-1} = B^{-1}A^{-1}$, $(A^T)^{-1} = (A^{-1})^T$. Inversion by elementary row operations starts from $A = IA$ and reduces A to I, turning the right factor into A^{-1}.