Matrices

A matrix is one of the most powerful tools in maths - used to simplify ~~calulations~~ calculations, solve linear systems and handle data in a compact rectangular form.

Definition

A matrix is an ordered rectangular array of numbers or functions. The numbers (or functions) are called the elements or entries of the matrix.

$$A = \begin{bmatrix} 5 & -2 \\ 0 & 5 \\ 3 & 6 \end{bmatrix}$$

<-3 rows, 2 cols
<-order 3 x 2

Order of a Matrix

A matrix with m rows and n columns is said to be of order m x n (read m by n).

$$\boxed{A = [\,a_{ij}\,]_{m \times n}}$$

<-1 <= i <= m
<-1 <= j <= n

a__ij = element in i-th row , j-th column .

Total elements in m x n matrix = m n .

Types of Matrices

1. Row Matrix

Has only one row. Order is $1 \times n$.

$A = \begin{bmatrix} 2 & -3 & 5 & 7 \end{bmatrix}$ <-order 1×4

2. Column Matrix

Has only one column. Order is $m \times 1$.

$B = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$ <-order 3×1

3. Square Matrix

Number of rows = number of columns ;
order $n \times n$ (or simply order n).

$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ <-order 3 (square)
<-diag : 1, 5, 9

Elements a_{ii} (where $i = j$) form the diagonal.
Sum of diagonal elements = trace (A).
(Trace is not in syllabus but useful .)

More Types + Equality

4. Diagonal Matrix

A square matrix $B = [\, b_{ij} \,]$ is diagonal if $b_{ij} = 0$ whenever $i \,!\!= j$.

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ <-all off-diag = 0

5. Scalar Matrix

A diagonal matrix with all diagonal entries equal : i.e. $b_{ij} = k$ if $i = j$, else 0.

$$S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$ <-$k = 3$ (here)

6. Identity Matrix

Scalar matrix with $k = 1$. Denoted I_n .

$$I\_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ <-$A I = I A = A$

<-(identity prop.)

7. Zero / Null Matrix

All entries are 0 . Denoted 0 . Acts as additive identity : $A + 0 = A$.

Equality + Operations

Equality of Matrices

Two matrices A and B are equal iff :

(i) they are of the same order , and

(ii) each corresponding entry $a_{ij} = b_{ij}$.

$\boxed{A = B \iff a_{ij} = b_{ij}}$ <- for all i, j <-(same order)

Addition of Matrices

Only matrices of the SAME order can be added . If A , B are both $m \times n$ matrices ,

$\boxed{(A + B)_{ij} = a_{ij} + b_{ij}}$ <- entry-wise <- addition

Properties :

(i) Commutative : $A + B = B + A$

(ii) Associative : $(A + B) + C = A + (B + C)$

(iii) Additive identity : $A + 0 = A$

(iv) Additive inverse : $A + (-A) = 0$

Scalar Multiplication

If κ is a scalar , $\kappa A = [\kappa a_{ij}]$.

$\kappa (A + B) = \kappa A + \kappa B$; $(\kappa + 1) A = \kappa A + 1 A$.

Multiplication of Matrices

The product AB is defined ONLY when
no. of columns of A = no. of rows of B.
A is $m \times n$ and B is $n \times p$ => AB is

$$\boxed{(AB)_{ij} = \text{sum } a_{ik} \, b_{kj}}$$

<-k = 1 to n
<-row x col rule

Example :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ <-2 x 3

$$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \end{bmatrix}$$ <-3 x 2

$$AB = \begin{bmatrix} 5 & 11 \\ 14 & 23 \end{bmatrix}$$ <-(2x3)(3x2) = 2x2

Properties

(i) ~~Commutative~~ NOT commutative : $AB \neq BA$
(ii) Associative : $(AB)C = A(BC)$
(iii) Distributive : $A(B + C) = AB + AC$
(iv) Identity : $AI = IA = A$ (I of right order)
Note : $AB = 0$ does NOT imply $A = 0$ or $B = 0$.

Transpose of a Matrix

The transpose of A is obtained by inter-changing its rows and columns.
Denoted A' (or A^T).
If A is $m \times n$, then A' is $n \times m$.

$$\boxed{A = [\, a_{ij} \,] \Rightarrow A' = [\, a_{ji} \,]}$$

<-(i,j) of A
<-= (j,i) of A'

Example :

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \qquad A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

<-rows <-> cols

<-(3 x 2)

Properties of Transpose

(i) $(A')' = A$

(ii) $(kA)' = kA'$ (k scalar)

(iii) $(A + B)' = A' + B'$

$$\boxed{(AB)' = B'A'}$$

<-reverse order !
<-(important)

Useful : if A is square, AA' and $A'A$ are both symmetric matrices.

Symmetric & Skew-Symmetric

Symmetric Matrix

A square matrix A is symmetric if $A' = A$,
i.e. $a_{ij} = a_{ji}$ for all i, j.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

<-mirror across
<-the diagonal

Skew-Symmetric Matrix

A square matrix A is skew-symmetric if $A' = -A$,
i.e. $a_{ij} = -a_{ji}$ for all i, j.
Put $i = j$: $a_{ii} = -a_{ii} \Rightarrow a_{ii} = 0$.
(All diagonal entries are ~~equal~~ zero .)

$$Q = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

<-diag = 0

Decomposition Theorem

Every square matrix A can be written as the sum of a symmetric and a skew-sym. matrix :

$$\boxed{A = 1/2 (A + A') + 1/2 (A - A')}$$

<-P (symmetric) Q (skew-sym.)

Elementary Operations & Inverse

Elementary Operations

Six basic operations - three on rows (R), three on columns (C) :

(i) Interchange : $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(ii) Multiply by non-zero k :

$R_i \to k R_i$ or $C_i \to k C_i$, $k \neq 0$

(iii) Add a multiple of another row / col :

$R_i \to R_i + k R_j$ ($i \neq j$)

Invertible Matrix

A square matrix A of order n is invertible if there exists a square matrix B of order n such that

$$\boxed{AB = BA = I_n}$$

<-then B is the
<-inverse of A .

Written : $B = A^{-1}$ (and $A = B^{-1}$) .

Inverse is ~~not~~ UNIQUE when it exists .

Also : $(AB)^{-1} = B^{-1} A^{-1}$ (reverse order)

Only square matrices can be invertible.