Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 3: Matrices

What this chapter covers: the matrix — a rectangular grid of numbers — and the algebra that goes with it. We cover order, the eight standard types, equality, the three core operations (addition, scalar multiplication, multiplication), transpose, and the two symmetry classes (symmetric and skew-symmetric). The rationalised NCERT stops at invertibility as a definition; we add *elementary row operations* and the *inverse via row operations* as JEE/NEET extensions because they remain heavily tested.

1 Matrices: Order, Notation, and Types

A **matrix** is a rectangular arrangement of numbers (or functions) into rows and columns. The single most important habit when working with matrices is to think of them not as a bag of numbers but as a structured object: each number has an address (i, j) giving its row and column.

Matrices arise the moment you have data with two indices — production levels at multiple factories for multiple products, marks of multiple students in multiple subjects, coefficients of a system of linear equations. The algebra we develop in this chapter is the language for transforming such data efficiently.

1.1 Definition and notation

A matrix A of **order** $m \times n$ has m rows and n columns. We write

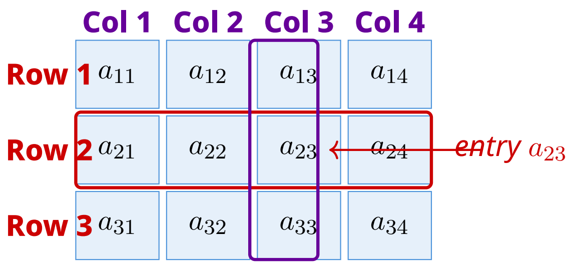
$$A = [a_{ij}]_{m\times n}, \qquad 1 \le i \le m,\ 1 \le j \le n,$$

where a_{ij} is the entry in the i-th row and j-th column. The pair (i, j) is read row-first, column-second — always.

Order and Element Count

A matrix of order $m \times n$ has exactly mn entries. Conversely, if a matrix has N entries, its possible orders are precisely the factor pairs of N.

$$\text{Order: } m \times n \quad \Longleftrightarrow \quad m \text{ rows, } n \text{ columns}, \quad mn \text{ entries.}$$



A 3×4 matrix: 3 rows (red), 4 columns (purple); element a_{ij} at intersection.

Worked observation. If a matrix has 24 elements, its possible orders are $1 \times 24,\ 2 \times 12,\ 3 \times 8,\ 4 \times 6,\ 6 \times 4,\ 8 \times 3,\ 12 \times 2,\ 24 \times 1$ — eight orders in total. If it has 13 (prime) elements, only 1×13 and 13×1 are possible.

> **Factor Pairs $\to$ Possible Orders**
>
> For "a matrix has N elements, list possible orders" questions: count the divisors of N. Each divisor d gives the order $d \times (N/d)$.

1.2 Construction from a rule

Many problems ask you to *construct* a matrix from a formula like $a_{ij} = \frac{1}{2}|i - 3j|$. The procedure is mechanical: walk through each (i, j) in row-major order, substitute, and write the answer in its slot.

For $a_{ij} = \frac{1}{2}|i - 3j|$ with order 3×2:

$$A = \begin{bmatrix} \frac{1}{2}|1-3| & \frac{1}{2}|1-6| \\ \frac{1}{2}|2-3| & \frac{1}{2}|2-6| \\ \frac{1}{2}|3-3| & \frac{1}{2}|3-6| \end{bmatrix} = \begin{bmatrix} 1 & \frac{5}{2} \\ \frac{1}{2} & 2 \\ 0 & \frac{3}{2} \end{bmatrix}.$$

> **Index order matters**
>
> a_{23} means *row 2, column 3* — not column 2, row 3. Get this wrong once at the start of a problem and every subsequent calculation is off. Always whisper "row first, column second" as you write the index.

1.3 Types of matrices

NCERT introduces eight standard types. Most are defined by relationships between m, n and the entries a_{ij} themselves.

Row matrix: only one row. Order $1 \times n$, e.g. $[5 \ \ -2 \ \ 7]$.

Column matrix: only one column. Order $m \times 1$.

Square matrix: same number of rows as columns. Order $n \times n$, called a square matrix of order n. The entries $a_{11}, a_{22}, \ldots, a_{nn}$ form the **principal diagonal**.

Diagonal matrix: square, with every off-diagonal entry zero. $a_{ij} = 0$ for $i \neq j$.

Scalar matrix: diagonal, with all diagonal entries equal. $a_{ij} = 0$ for $i \neq j$ and $a_{ii} = k$ for some constant k.

Identity matrix I_n: scalar with $k = 1$. The diagonal is all ones, everything else zero.

Zero (null) matrix O: every entry is zero. Can be of any order.

Upper triangular (JEE-flagged): square, with all entries below the diagonal zero. **Lower triangular**: all entries above the diagonal zero.

Square (3×3)
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Diagonal
$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Scalar ($k = 4$)
$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Identity I_3
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Upper △
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Lower △
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Hierarchy of square types

Every *identity* matrix is a *scalar*, every scalar is a *diagonal*, every diagonal is a *square*. So as you specialise — by adding constraints — you climb a nested ladder: Square $\supset$ Diagonal $\supset$ Scalar $\supset$ Identity.

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Type	Defining condition	Example
Row matrix	Order $1 \times n$	$\begin{bmatrix} 2 & -1 & 5 \end{bmatrix}$
Column matrix	Order $m \times 1$	$\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$
Square matrix	$m = n$	$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$
Diagonal	Square, $a_{ij} = 0$ for $i \neq j$	$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$
Scalar	Diagonal, all $a_{ii} = k$	$\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
Identity I_n	Scalar with $k = 1$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
Zero / null O	All $a_{ij} = 0$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Upper triangular	Square, $a_{ij} = 0$ for $i > j$	$\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$
Lower triangular	Square, $a_{ij} = 0$ for $i < j$	$\begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$

Square $\supset$ Diagonal $\supset$ Scalar $\supset$ Identity

Remember the four-tier ladder **S-D-S-I**: **S**quare, **D**iagonal, **S**calar, **I**dentity. Each step adds a constraint; identity is the most specialised. Reverse-implication is always true: every I_n is square, but a square is not necessarily I_n.

1.4 Equality of matrices

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are **equal** when

1. they have the same order, and
2. $a_{ij} = b_{ij}$ for every (i, j).

Both conditions must hold. Same numbers in a different order is not enough; even "rearranged" matrices are unequal if their orders differ.

Equality

$$A = B \iff \dim A = \dim B \text{ and } a_{ij} = b_{ij}\ \forall\, i, j.$$

A single matrix equation in $m \times n$ matrices is shorthand for mn scalar equations.

This last point is the workhorse of solving equations using equality. If

$$\begin{bmatrix} 2a + b & a - 2b \\ 5c - d & 4c + 3d \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 11 & 24 \end{bmatrix},$$

we instantly read off four scalar equations: $2a + b = 4$, $a - 2b = -3$, $5c - d = 11$, $4c + 3d = 24$, which solve to $a = 1,\ b = 2,\ c = 3,\ d = 4$.

Matrix equation → system of scalars

A single matrix equation between two $m \times n$ matrices is exactly mn ordinary scalar equations. Write them all out before solving — ignoring even one will make the system inconsistent.

2 Operations on Matrices

We define four operations on matrices: addition, scalar multiplication, subtraction (a derived combination of the first two), and matrix multiplication. The first three behave like ordinary arithmetic. Matrix multiplication has surprises that the rest of the chapter will pay off.

2.1 Addition and subtraction

For matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the *same* order $m \times n$,

$$A + B = [a_{ij} + b_{ij}]_{m \times n}.$$

If A and B have different orders, $A + B$ is simply **undefined** — not zero, not anything — the operation has no meaning.

Subtraction is just addition with the negative: $A - B = A + (-1)B$.

Matrix Addition

$(A + B)_{ij} = a_{ij} + b_{ij}$, provided A and B have the same order.

Properties of matrix addition. For any matrices A, B, C of the same order:

- **Commutative**: $A + B = B + A$.
- **Associative**: $(A + B) + C = A + (B + C)$.
- **Additive identity**: $A + O = A$, where O is the zero matrix of the same order.
- **Additive inverse**: $A + (-A) = O$.

These four facts together mean: the set of all $m \times n$ matrices forms an abelian (commutative) group under addition. You don't need that vocabulary for the board exam, but it's why matrix addition feels exactly like ordinary number addition.

Worked example. Find matrix X such that $2A + 3X = 5B$, where $A = \begin{bmatrix} 8 & 0 \\ 4 & -2 \\ 3 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -2 \\ 4 & 2 \\ -5 & 1 \end{bmatrix}$.

Rearranging, $3X = 5B - 2A$, so $X = \frac{1}{3}(5B - 2A)$. Compute $5B - 2A$:

$$5B = \begin{bmatrix} 10 & -10 \\ 20 & 10 \\ -25 & 5 \end{bmatrix}, \quad 2A = \begin{bmatrix} 16 & 0 \\ 8 & -4 \\ 6 & 12 \end{bmatrix}, \quad 5B - 2A = \begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix}.$$

Hence $X = \frac{1}{3}\begin{bmatrix} -6 & -10 \\ 12 & 14 \\ -31 & -7 \end{bmatrix}$.

Data tables in the wild

Spreadsheets are matrices in disguise. Merging two months of sales data — one matrix per month, same products in rows, same channels in columns — is exactly matrix addition. Applying a 2% profit margin uniformly is scalar multiplication by 0.02.

2.2 Scalar multiplication

For a real number k and matrix $A = [a_{ij}]_{m \times n}$,

$$kA = [k\, a_{ij}]_{m \times n}.$$

The scalar multiplies every entry. The order is unchanged.

Properties: for scalars k, l and matrices A, B of the same order,

- $k(A + B) = kA + kB$,
- $(k + l)A = kA + lA$,
- $k(lA) = (kl)A$,
- $1 \cdot A = A, \quad 0 \cdot A = O$.

$$3 \cdot \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 4 \\ 2 & 5 & -3 \end{bmatrix} \xrightarrow{\text{scale}} \begin{bmatrix} 3 & 6 & 0 \\ -3 & 9 & 12 \\ 6 & 15 & -9 \end{bmatrix}$$

Scalar multiplication: every entry is multiplied by the scalar (here, $k = 3$).

2.3 Matrix multiplication

This is the most consequential operation in the chapter, and it does *not* behave like ordinary multiplication. Read carefully.

When is AB defined? Only when the number of columns of A equals the number of rows of B. Specifically, if A is $m \times n$ and B is $n \times p$, then AB is defined and has order $m \times p$.

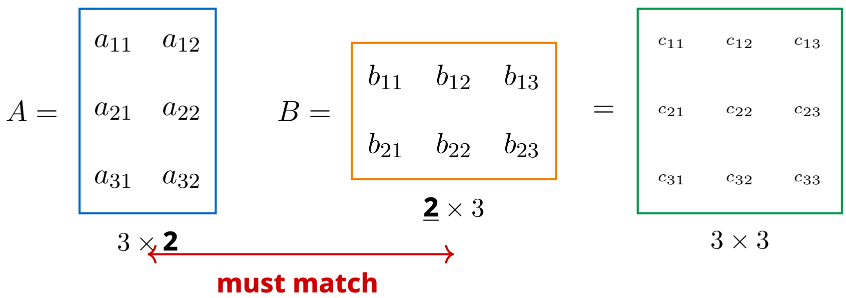
How is AB computed? The (i,k)-entry of AB is the *dot product* of the i-th row of A with the k-th column of B:

$$(AB)_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}.$$

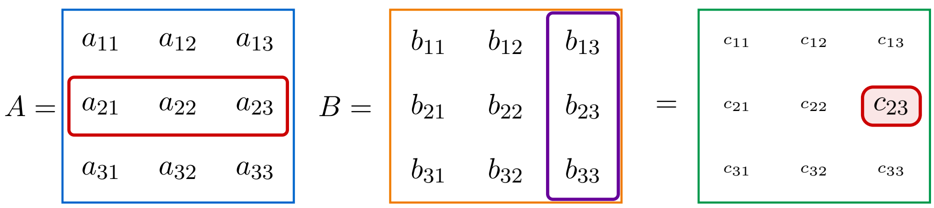
Matrix Product

$$A_{m\times n}\, B_{n\times p} = (AB)_{m\times p}, \qquad (AB)_{ik} = \sum_{j=1}^{n} a_{ij}\, b_{jk}.$$

The middle dimension n must match; the outer dimensions m, p become the order of the product.



Visualising one entry. The dashed lines below show how the $(2,3)$-entry of AB is computed: dot product of row 2 of A with column 3 of B.



$c_{23} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$ — *row 2 of A · column 3 of B.*

Worked example (non-commutativity). Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Both are 2×2, so AB and BA exist and are both 2×2. Compute:

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + 0\cdot 1 & 1\cdot 1 + 0\cdot 0 \\ 0\cdot 0 + (-1)\cdot 1 & 0\cdot 1 + (-1)\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

So $AB \neq BA$. Matrix multiplication is **not commutative**, even when both products exist and have the same order.

The Big Fact: $AB \neq BA$ in general

Unlike numbers, matrices generally satisfy $AB \neq BA$. Equality *can* happen for special pairs (e.g., A and I, or two diagonal matrices, or A and A^{-1}) — but you must verify it; never assume it.

"Cancellation" fails

For numbers, $ab = ac$ with $a \neq 0$ implies $b = c$. For matrices, $AB = AC$ does *not* imply $B = C$, even if A is non-zero. The matrix analogue of cancellation requires A to be *invertible*, not merely non-zero.

2.4 Properties of matrix multiplication

When the relevant products are defined, matrix multiplication satisfies:

- **Associative**: $(AB)C = A(BC)$.
- **Distributive over addition**: $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.
- **Multiplicative identity**: $AI_n = I_mA = A$ if A is $m \times n$ (each I has the matching order).
- **Not commutative** (as shown).
- $AB = O$ does *not* imply $A = O$ or $B = O$. Matrices can have "zero divisors".

The associativity is what lets us write ABC without parentheses; the distributivity is what justifies expanding $(A+B)^2$ — but be careful: $(A+B)^2 = A^2+AB+BA+B^2$, *not* $A^2 + 2AB + B^2$ unless A and B commute.

$(A + B)^2 \neq A^2 + 2AB + B^2$

The middle terms are AB and BA, which are different in general. Always expand carefully:

$$(A + B)^2 = A^2 + AB + BA + B^2.$$

The $2AB$ identity is valid only when $AB = BA$.

Computer graphics

Every rotation, scaling, or translation applied to a 3D object in a video game is a matrix. Combining multiple transformations is just multiplying their matrices — in the correct order, because of non-commutativity. Rotate-then-translate gives a different result than translate-then-rotate.

2.5 Powers of a square matrix

If A is square, $A^2 = AA$, $A^3 = A \cdot A^2$, etc., are well-defined. By convention $A^0 = I$ for any non-zero square A.

A common identity worth memorising: for the rotation-like matrix

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \quad A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}.$$

A clean proof is by induction; for the exam, recognising the pattern saves time.

Powers of triangular matrices

If A is upper (or lower) triangular, so is every A^n, and the diagonal entries of A^n are just the n-th powers of A's diagonal entries. Useful for shortcut multiple-choice elimination.

Worked example (computing AB). With $A = \begin{bmatrix} 6 & 9 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 6 & 0 \\ 7 & 9 & 8 \end{bmatrix}$, A is 2×2 and B is 2×3, so AB is defined and has order 2×3:

$$AB = \begin{bmatrix} 6\cdot2+9\cdot7 & 6\cdot6+9\cdot9 & 6\cdot0+9\cdot8 \\ 2\cdot2+3\cdot7 & 2\cdot6+3\cdot9 & 2\cdot0+3\cdot8 \end{bmatrix} = \begin{bmatrix} 75 & 117 & 72 \\ 25 & 39 & 24 \end{bmatrix}.$$

Note: BA is *not* defined here, since B has 3 columns but A has 2 rows.

2.6 Application: matrices model real systems

The reason matrices were invented is that "data with two indices" is everywhere. A single matrix product can replace dozens of scalar arithmetic operations.

Worked example. Two farmers, Ramkishan and Gurcharan, sell three rice varieties (Basmati, Permal, Naura). The September sales matrix (rows: farmers, columns: varieties) is

$$A = \begin{bmatrix} 10000 & 20000 & 30000 \\ 50000 & 30000 & 10000 \end{bmatrix} \text{ (rupees)}.$$

If both farmers earn 2% profit on gross sales, the profit per (farmer, variety) pair is just $0.02\,A$:

$$0.02\,A = \begin{bmatrix} 200 & 400 & 600 \\ 1000 & 600 & 200 \end{bmatrix}.$$

Scalar multiplication has replaced six separate percentage calculations.

Markov chains, weather, web search

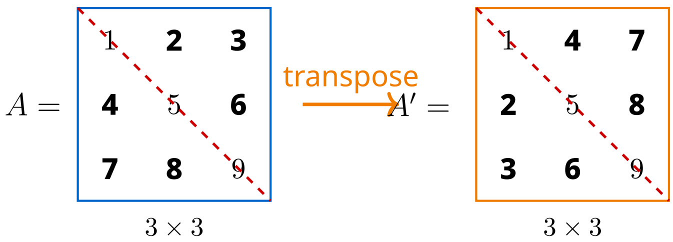
A matrix P where p_{ij} is the probability of moving from state i to state j in one step is called a *transition matrix*. The n-step probabilities are simply P^n. Weather models, board-game analyses, and Google's PageRank all rest on this single observation.

3 Transpose of a Matrix

The **transpose** of $A = [a_{ij}]_{m\times n}$ is the matrix $A' = A^T = [a_{ji}]_{n\times m}$ obtained by swapping rows and columns: row 1 of A becomes column 1 of A', row 2 becomes column 2, and so on. The order flips from $m \times n$ to $n \times m$.

3.1 Visualising transpose

Think of transpose as reflecting the matrix across its principal diagonal. Diagonal entries stay put; off-diagonal entries swap with their mirror partners across the diagonal.



Reflection across the principal diagonal (red dashed). Diagonal entries $1, 5, 9$ are fixed.

3.2 Properties of transpose

For matrices A, B of suitable orders and scalar k:

- $(A')' = A$
- $(kA)' = k\,A'$
- $(A + B)' = A' + B'$
- $(AB)' = B'\,A'$ **(reverses the order!)**

Transpose of a Product

$$(AB)' = B'\,A'.$$

The order *reverses*. By induction, $(A_1A_2\cdots A_k)' = A'_k \cdots A'_2A'_1$.

The order reversal is the only non-trivial item on the list; the other three are obvious from definitions. The reversal happens because of how rows-of-A pair with columns-of-B when you transpose — those roles flip.

Verification on a small example. Let $A = \begin{bmatrix}-2\\4\\5\end{bmatrix}$ (a 3×1 column) and $B = [1\ \ 3\ \ -6]$ (a 1×3 row). Then

$$AB = \begin{bmatrix}-2\\4\\5\end{bmatrix}[1\ \ 3\ \ -6] = \begin{bmatrix}-2 & -6 & 12\\4 & 12 & -24\\5 & 15 & -30\end{bmatrix},$$

so $(AB)'$ is $\begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix}$. Independently $A' = [-2 \;\; 4 \;\; 5]$ and $B' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$, giving

$$B'A' = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} [-2 \;\; 4 \;\; 5] = \begin{bmatrix} -2 & 4 & 5 \\ -6 & 12 & 15 \\ 12 & -24 & -30 \end{bmatrix},$$

which agrees with $(AB)'$ exactly.

$(AB)' \neq A'B'$

The product transposes *in reverse order*: $(AB)' = B'A'$, not $A'B'$. Forgetting the reversal is the single most common matrix-algebra slip on board and JEE papers. The pattern echoes inverses: $(AB)^{-1} = B^{-1}A^{-1}$.

"Socks and Shoes"

To get dressed, you put socks on *then* shoes. To get undressed, you reverse: *shoes off, then socks*. Transpose and inverse both "undress" a product, so they reverse the order: $(AB)' = B'A'$ and $(AB)^{-1} = B^{-1}A^{-1}$.

3.3 A useful trigonometric example

The matrix $M(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ satisfies $M'M = I$, i.e., $M^{-1} = M'$. Such matrices are called **orthogonal**, and they describe rotations in the plane.

Verification.

$$M'M = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 \\ 0 & \sin^2\alpha + \cos^2\alpha \end{bmatrix} = I.$$

Why rotations preserve length

For a rotation matrix M, $M'M = I$ means M preserves the dot product: $(Mu) \cdot (Mv) = u'M'Mv = u'v$. Hence lengths and angles are preserved — exactly what rotation should do. The same property makes orthogonal matrices central to crystallography and quantum mechanics.

4 Symmetric and Skew-Symmetric Matrices

Among square matrices, two symmetry classes are especially important. They are defined by how the matrix relates to its own transpose.

4.1 Definitions

A square matrix A is

- **symmetric** if $A' = A$, i.e., $a_{ji} = a_{ij}$ for all i, j;
- **skew-symmetric** if $A' = -A$, i.e., $a_{ji} = -a_{ij}$ for all i, j.

Setting $i = j$ in the skew condition gives $a_{ii} = -a_{ii}$, hence $a_{ii} = 0$: **every diagonal entry of a skew-symmetric matrix is zero**.

Symmetric vs Skew-Symmetric

$A' = A$ (symmetric) $\qquad A' = -A$ (skew-symmetric, diagonal $= 0$).

Symmetric ($A' = A$)

$$\begin{pmatrix} 3 & 2 & 7 \\ 2 & -1 & 4 \\ 7 & 4 & 5 \end{pmatrix}$$

Mirror pairs (bold) equal; diagonal free.

Skew-symmetric ($A' = -A$)

$$\begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix}$$

Mirror pairs negate; diagonal must be zero.

Detect quickly

For a small square A, write A' next to it. If $A' = A$, symmetric. If $A' = -A$ (all entries negate), skew-symmetric. If the diagonal of A has any non-zero entry, A cannot be skew-symmetric — you can stop checking.

4.2 Two foundational theorems

Theorem 1. For any square matrix A with real entries, $A + A'$ is symmetric and $A - A'$ is skew-symmetric.

Proof sketch. Let $B = A + A'$. Then $B' = (A + A')' = A' + (A')' = A' + A = A + A' = B$. So $B' = B$, i.e., B is symmetric. The skew case is analogous: $(A - A')' = A' - A = -(A - A')$.

Theorem 2. Every square matrix A can be written uniquely as the sum of a symmetric and a skew-symmetric matrix:

$$A = \underbrace{\tfrac{1}{2}(A + A')}_{\text{symmetric part}} + \underbrace{\tfrac{1}{2}(A - A')}_{\text{skew-symmetric part}} .$$

Symmetric–Skew Decomposition

$$A = P + Q, \quad P = \tfrac{1}{2}(A + A'), \; Q = \tfrac{1}{2}(A - A').$$

P is symmetric, Q is skew-symmetric, and the decomposition is unique.

Worked example. Decompose $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$.

First, $B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$. Then

$$P = \tfrac{1}{2}(B + B') = \begin{bmatrix} 2 & -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 3 & 1 \\ -\frac{3}{2} & 1 & -3 \end{bmatrix}, \quad Q = \tfrac{1}{2}(B - B') = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}.$$

P is visibly symmetric (mirror entries match); Q has zero diagonal and mirror entries negated, so it is skew-symmetric. Their sum recovers B.

Two halves of every square matrix

Just as every real-valued function $f(x)$ splits into an even part $\frac{1}{2}(f(x) + f(-x))$ and odd part $\frac{1}{2}(f(x) - f(-x))$, every square matrix splits into a symmetric and a skew-symmetric piece. Same algebraic trick; different setting.

4.3 A neat result

If A, B are symmetric of the same order, then $AB - BA$ is skew-symmetric.

Proof. $(AB - BA)' = (AB)' - (BA)' = B'A' - A'B' = BA - AB = -(AB - BA)$. Hence skew. This pops up as a JEE multiple-choice favourite.

$AB - BA$ on symmetric matrices is always skew

Memorise the result: symmetric $A, B \Rightarrow AB - BA$ skew-symmetric. Conversely, $AB + BA$ is symmetric. These two facts answer at least one JEE-Main MCQ most years.

4.4 Counting independent entries

How many independent entries does an $n \times n$ symmetric matrix have? Diagonal entries (there are n of them) are free, and the upper-triangular off-diagonal entries (there are $\binom{n}{2}$) determine the lower-triangular ones by symmetry. Total: $n + \binom{n}{2} = \frac{n(n+1)}{2}$.

For a skew-symmetric $n \times n$ matrix the diagonal is forced to zero, and the upper triangle (still $\binom{n}{2}$ entries) determines the lower by negation. Total free entries: $\frac{n(n-1)}{2}$.

Degrees of freedom

$$\#\text{symmetric}(n) = \frac{n(n+1)}{2}, \qquad \#\text{skew-symmetric}(n) = \frac{n(n-1)}{2}.$$

Sum equals n^2 — matching Theorem 2's decomposition.

Counting trick

The two counts sum to $n^2 = n^2/2 \cdot 2$ because every $n \times n$ matrix splits uniquely as (symmetric) + (skew). The split is dimension-preserving — a neat consistency check.

5 Invertible Matrices

A square matrix A of order n is **invertible** if there exists a square matrix B of the same order with

$$AB = BA = I_n.$$

When such a B exists, it is unique, and we denote it A^{-1}.

5.1 Definition and uniqueness

Theorem 3 (Uniqueness). If A has an inverse, it has only one.

Proof. Suppose B and C are both inverses of A. Then $B = BI = B(AC) = (BA)C = IC = C$. Done.

Theorem 4 (Inverse of a product). If A, B are invertible of the same order, $(AB)^{-1} = B^{-1}A^{-1}$.

Properties of inverse

$$(A^{-1})^{-1} = A, \quad (AB)^{-1} = B^{-1}A^{-1}, \quad (A')^{-1} = (A^{-1})'.$$

A rectangular (non-square) matrix has no inverse: both products AB and BA must be defined and equal I, which forces square shape.

Verification. Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$. Compute $AB = \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. By similar arithmetic $BA = I$ too. So $B = A^{-1}$.

"Inverse" captures division

Number division: $a/b = a \cdot b^{-1}$. Matrix division doesn't exist as a standalone symbol, but AB^{-1} (post-multiply) or $B^{-1}A$ (pre-multiply) plays the same role — with the caveat that the two are different in general. Always specify the side.

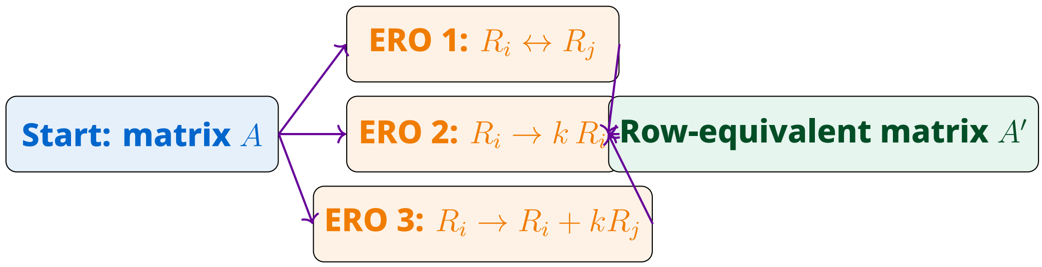
5.2 Elementary row operations [JEE/NEET Extension]

The rationalised NCERT 2026–27 syllabus removes this topic, but it remains a standard JEE-Main question type and a useful computational tool.

There are three **elementary row operations** (EROs) on a matrix:

1. **Row swap**: $R_i \leftrightarrow R_j$ (interchange row i and row j).
2. **Row scaling**: $R_i \to k\,R_i$ with $k \neq 0$ (multiply a row by a non-zero scalar).
3. **Row addition**: $R_i \to R_i + k\,R_j$ (add k times row j to row i).

The same three operations are also defined for columns (C_i, C_j). Each ERO is reversible: every row operation has an inverse row operation.



Each ERO transforms A into a row-equivalent matrix; repeat to reach the desired form.

5.3 Inverse via EROs [JEE/NEET Extension]

To find A^{-1} using row operations:

1. Write A next to I to form the augmented block $[\,A \mid I\,]$.
2. Apply a sequence of EROs to the whole block, with the goal of converting the left side A into I.
3. When the left side becomes I, the right side — which started as I — has become A^{-1}. So the final block is $[\,I \mid A^{-1}\,]$.

If at any stage a row of zeros appears on the left side, A is *not* invertible (also called *singular*).

Worked example. Find A^{-1} for $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$.

Set up $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right]$.

$R_2 \to R_2 - 2R_1$ gives $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right]$.

$R_2 \to -\frac{1}{5}R_2$ gives $\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$.

$R_1 \to R_1 - 2R_2$ gives $\left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} \end{array}\right]$.

Reading off, $A^{-1} = \frac{1}{5}\begin{bmatrix}1 & 2 \\ 2 & -1\end{bmatrix}$. A quick check: $A\,A^{-1} = \frac{1}{5}\begin{bmatrix}1+4 & 2-2 \\ 2-2 & 4+1\end{bmatrix} = I_2$.

Don't mix row and column operations

When computing A^{-1} by EROs, use *only* row operations. Mixing column operations into the same procedure produces a different (and wrong) answer. There is a parallel column-operation method, but the two should not be combined in one solve.

Solving linear systems at scale

Engineering simulations (structural analysis, circuit solvers, machine learning) routinely solve $A\vec{x} = \vec{b}$ with A having millions of rows. The underlying algorithm is a numerically robust descendant of the ERO method — Gauss elimination plus partial pivoting — still rooted in the three row operations.

5.4 Second worked example with EROs [JEE/NEET Extension]

Find A^{-1} for $A = \begin{bmatrix}2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & -1 & 1\end{bmatrix}$ by elementary row operations.

Start with $[\,A \mid I_3\,]$:

$$\left[\begin{array}{ccc|ccc}2 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1\end{array}\right].$$

Step 1. $R_1 \leftrightarrow R_2$ (place a 1 in the top-left):

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1\end{array}\right].$$

Step 2. $R_2 \to R_2 - 2R_1$:

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1\end{array}\right].$$

Step 3. $R_3 \to R_3 + R_2$:

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1\end{array}\right].$$

Step 4. $R_3 \to \frac{1}{2}R_3$:

$$\left[\begin{array}{ccc|ccc}1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -2 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2}\end{array}\right].$$

Step 5. $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right].$$

So $A^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 4 & -1 \\ 1 & -2 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

The discipline is: only operate on rows, work one column at a time (left to right), and check the answer by computing AA^{-1} at the end.

> **Pivot strategy**
>
> For each column, first put a 1 in the pivot position (row swap or scaling), then zero out every other entry in that column. Move to the next column. This always works when A is invertible.

6 Quick Reference Summary

A one-page rapid-revision sheet of every formula, definition, and trap from the chapter.

6.1 Definitions and notation

- Matrix: $A = [a_{ij}]_{m\times n}$. Order = (rows) $\times$ (columns) = $m \times n$.
- Element count: mn. Possible orders of an N-entry matrix = factor pairs of N.
- Equality $A = B$ requires both same order *and* entrywise equality.

6.2 Operations

- Addition (same order): $(A+B)_{ij} = a_{ij} + b_{ij}$. Commutative, associative.
- Scalar multiplication: $(kA)_{ij} = k\,a_{ij}$. Order unchanged.
- Matrix product (compatibility $A_{m\times n} \cdot B_{n\times p} = (AB)_{m\times p}$): $(AB)_{ik} = \sum_j a_{ij}b_{jk}$. Associative, distributive, **not commutative**.
- Identity: $AI = IA = A$ when orders match. $A^0 = I$.
- $AB = O \not\Rightarrow A = O$ or $B = O$.

6.3 Transpose

- $A' = [a_{ji}]$, swaps order from $m \times n$ to $n \times m$.
- $(A')' = A$, $(kA)' = kA'$, $(A+B)' = A' + B'$.
- $(AB)' = B'A'$ — order reverses (socks-and-shoes).

6.4 Symmetry

- Symmetric: $A' = A$. Skew-symmetric: $A' = -A$ (diagonal all zero).
- $A + A'$ is symmetric; $A - A'$ is skew. Decomposition $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$.
- Symmetric $A, B \Rightarrow AB - BA$ skew, $AB + BA$ symmetric.

6.5 Invertibility

- A invertible $\Leftrightarrow \exists\, B$ with $AB = BA = I$. If exists, unique. Square only.
- $(A^{-1})^{-1} = A$, $(AB)^{-1} = B^{-1}A^{-1}$, $(A')^{-1} = (A^{-1})'$.
- *JEE method:* reduce $[\,A \mid I\,] \to [\,I \mid A^{-1}\,]$ via EROs. Row of zeros $\Rightarrow$ singular.

6.6 Standard types — one-line reminders

Type	One-line definition
Row	Single row, $1 \times n$.
Column	Single column, $m \times 1$.
Square	$m = n$.
Diagonal	Square; $a_{ij} = 0$ for $i \neq j$.
Scalar	Diagonal with constant diagonal value k.
Identity I_n	Scalar with $k = 1$.
Zero O	Every entry zero.
Upper triangular	Square; $a_{ij} = 0$ for $i > j$.
Lower triangular	Square; $a_{ij} = 0$ for $i < j$.
Symmetric	$A' = A$.
Skew-symmetric	$A' = -A$; zero diagonal.
Invertible (non-singular)	$\exists\, A^{-1}$ with $AA^{-1} = A^{-1}A = I$.

6.7 Top traps to dodge

- "a_{23}" is row 2, column 3 — not the other way.
- AB and BA generally differ.
- $(A + B)^2 \neq A^2 + 2AB + B^2$ in general.
- $(AB)' = B'A'$ and $(AB)^{-1} = B^{-1}A^{-1}$ — both reverse the order.
- Skew-symmetric matrices always have zero diagonal — spot it instantly.
- Cancellation $AB = AC \not\Rightarrow B = C$ unless A is invertible.

End of Chapter 3 — Matrices

Next: Chapter 4 — Determinants, which builds the numerical tool for testing whether A^{-1} actually exists, and the explicit formula for it.