NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 4 — Determinants

Chapter 4: Determinants

About this Chapter

A **determinant** is a scalar associated with every square matrix; for a 2×2 matrix it equals $ad - bc$, and for a 3×3 matrix it is the alternating expansion along any row or column using **minors** and **cofactors**. This Exemplar set drills the seven **properties of determinants** (row/column swap, identical rows, scalar multiplication, sum splitting, $R_i \to R_i + kR_j$), the **area of a triangle** formula, the **adjoint** and **inverse** via $A^{-1} = \frac{1}{|A|}\text{adj } A$, and the **matrix method** for solving systems $AX = B$ via $X = A^{-1}B$.

Topics covered: 2×2 and 3×3 determinants • Expansion by cofactors • Seven properties of determinants • Area of a triangle • Minors and cofactors • Adjoint and inverse of a matrix • Singular vs. non-singular • Solution of $AX = B$ by $X = A^{-1}B$

Quick Formula Sheet

2×2:
$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Row swap / identical rows:
sign flips / $\Delta = 0$

$|kA|$ for $n \times n$:
$|kA| = k^n |A|$

Area of triangle:
$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Cofactor:
$A_{ij} = (-1)^{i+j} M_{ij}$

Adjoint / Inverse:
$A(\text{adj } A) = |A| I$;
$A^{-1} = \frac{1}{|A|}\text{adj } A$

$|\text{adj } A|$:
$|\text{adj } A| = |A|^{n-1}$

System $AX = B$:
$|A| \neq 0 \Rightarrow X = A^{-1}B$

I. Short Answer (S.A.)

Q 4.1 **Using the properties of determinants, evaluate** $\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$.

SOLUTION

Concept used. For a 2×2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. We can also use the column operation $C_1 \to C_1 - C_2$ (which leaves the value unchanged) to simplify before expanding.

Step 1. Apply $C_1 \to C_1 - C_2$:

$$\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix} = \begin{vmatrix} (x^2 - x + 1) - (x - 1) & x - 1 \\ (x + 1) - (x + 1) & x + 1 \end{vmatrix}.$$

Step 2. Simplify the new first column:

$$(x^2 - x + 1) - (x - 1) = x^2 - 2x + 2, \qquad (x + 1) - (x + 1) = 0.$$

So the determinant becomes

$$\begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix}.$$

Step 3. Expand: $ad - bc = (x^2 - 2x + 2)(x + 1) - (x - 1) \cdot 0 = (x^2 - 2x + 2)(x + 1)$.

Step 4. Multiply out:

$$\begin{aligned}(x^2 - 2x + 2)(x + 1) &= x^3 + x^2 - 2x^2 - 2x + 2x + 2 \\ &= x^3 - x^2 + 2.\end{aligned}$$

Final Answer: $\Delta = x^3 - x^2 + 2$.

☞ **Column operation rule**

Adding a scalar multiple of one column (or row) to another column (row) never changes the value of a determinant. Use it freely to get a 0 in a convenient slot.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Direct-expansion angle. For a 2×2 we can also just expand by $ad - bc$ and simplify at the end — the column operation above is a one-line shortcut, but the direct computation is just as quick once you keep the algebra tidy.

Step 1. Expand directly:

$$\Delta = (x^2 - x + 1)(x + 1) - (x - 1)(x + 1).$$

Step 2. First product: $(x^2 - x + 1)(x + 1) = x^3 + x^2 - x^2 - x + x + 1 = x^3 + 1$. (This is the standard identity $a^2 - a + 1 = (a^3 + 1)/(a + 1)$ rewritten.)

Step 3. Second product: $(x-1)(x+1) = x^2 - 1$.

Step 4. Subtract:

$$\Delta = (x^3+1) - (x^2-1) = x^3 - x^2 + 2.$$

Step 5. Cross-check at $x = 1$: original $\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2$; our formula gives $1 - 1 + 2 = 2$. Match. ✓

Final Answer: $\Delta = x^3 - x^2 + 2$.

Q 4.2 Using the properties of determinants, evaluate $\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$.

SOLUTION

Concept used. The operation $C_1 \to C_1 + C_2 + C_3$ adds the second and third columns to the first column. Determinant value is unchanged. After this, every entry in column 1 will be $a+x+y+z$, allowing us to factor $(a+x+y+z)$ out of C_1.

Step 1. Apply $C_1 \to C_1 + C_2 + C_3$. New first-column entries:

$$(a+x)+y+z = a+x+y+z,$$
$$x+(a+y)+z = a+x+y+z,$$
$$x+y+(a+z) = a+x+y+z.$$

So

$$\Delta = \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix}.$$

Step 2. Factor $(a+x+y+z)$ out of C_1:

$$\Delta = (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix}.$$

Step 3. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}.$$

Step 4. The right matrix is upper-triangular, so its determinant equals the product of diagonal entries: $1 \cdot a \cdot a = a^2$.

Final Answer: $\Delta = a^2(a + x + y + z)$.

☞ Spot the "column sum" pattern

Whenever every column sums to the same expression (or every row sums to the same expression), apply $C_1 \to C_1 + C_2 + \cdots$ (or $R_1 \to R_1 + R_2 + \cdots$). This is the single most common exam-trick for 3×3 determinants.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

Symmetry angle. Notice that every row of the determinant has the structure (something plus a) + (rest of the row), and the three columns play symmetric roles. The column-sum trick exploits this symmetry to create a column of identical entries.

Step 1. After $C_1 \to C_1 + C_2 + C_3$, every entry of C_1 is $S = a + x + y + z$. Pull S out.

Step 2. The reduced determinant has $C_1 = (1, 1, 1)^T$. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to wipe everything below the top-left 1.

Step 3. The result is $\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$, upper triangular with diagonal $1, a, a$. Determinant $= a^2$.

Step 4. Combine: $\Delta = S \cdot a^2 = a^2(a + x + y + z)$.

Step 5. Sanity check at $x = y = z = 0$: original $\begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a^3$. Formula: $a^2(a + 0 + 0 + 0) = a^3$. Match. ✓

Final Answer: $\Delta = a^2(a + x + y + z)$.

Q 4.3 **Using the properties of determinants, evaluate** $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$.

SOLUTION

Concept used. A common factor can be pulled out of any row or column: if every entry of column j contains a factor k, the determinant equals k times the determinant with that factor removed. We use this repeatedly to extract x, y, z.

Step 1. Look at column 1: entries are 0, x^2y, x^2z. Factor x^2 out of C_1 (the 0 stays a 0). Actually a cleaner path is to factor row-by-row. *Row 1:* $(0, xy^2, xz^2) = x \cdot (0, y^2, z^2)$. Pull out x. *Row 2:* $(x^2y, 0, yz^2) = y \cdot (x^2, 0, z^2)$. Pull out y. *Row 3:* $(x^2z, zy^2, 0) = z \cdot (x^2, y^2, 0)$. Pull out z.

$$\Delta = xyz \begin{vmatrix} 0 & y^2 & z^2 \\ x^2 & 0 & z^2 \\ x^2 & y^2 & 0 \end{vmatrix}.$$

Step 2. Now factor by column: C_1 entries are 0, x^2, x^2 — pull x^2 from C_1. Likewise y^2 from C_2, and z^2 from C_3:

$$\Delta = xyz \cdot x^2y^2z^2 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}.$$

Step 3. Evaluate the 3×3 with all-zero diagonal. Expand along R_1:

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix}$$
$$= 0 - (0 - 1) + (1 - 0) = 1 + 1 = 2.$$

Step 4. Combine: $\Delta = xyz \cdot x^2y^2z^2 \cdot 2 = 2x^3y^3z^3$.

Final Answer: $\Delta = 2x^3y^3z^3$.

☞ Row vs. column factor pull

You can pull a common factor from a row *or* from a column. Each pull multiplies the running determinant by that factor. In this problem we pull x from row 1 *and* x^2 from column 1, giving total power x^3 — matching the final $x^3y^3z^3$.

EXPERT'S SOLUTION *: Vivaan Mehta, Ph.D Mathematics, IIT Delhi*

Power-counting angle. Total degree of every term in the expansion of Δ is 9 (each diagonal product has three factors, each of total degree 3). And the answer must be symmetric in (x, y, z) up to a sign because the determinant has a nice symmetric pattern. Both observations point at $2x^3y^3z^3$.

Step 1. Pull x, y, z from rows $1, 2, 3$: factor xyz.

Step 2. Pull x^2, y^2, z^2 from columns $1, 2, 3$: factor $x^2y^2z^2$.

Step 3. The leftover is the constant determinant $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$.

Step 4. Multiply: $\Delta = xyz \cdot x^2y^2z^2 \cdot 2 = 2x^3y^3z^3$.

Step 5. Cross-check the sign at $x = y = z = 1$: original is $\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$, formula gives $2 \cdot 1 = 2$. Match. ✓

Final Answer: $\Delta = 2x^3y^3z^3$.

Why this matters. The pattern "every row has a common factor, every column also has a common factor" is the standard factorisation trick for symmetric polynomial determinants. Whenever the final answer ought to be a monomial of total degree $\deg(\text{row factors}) \cdot 3 + \deg(\text{column factors}) \cdot 3$, spotting it in advance lets you skip the full 3×3 expansion.

Common mistake. Students often forget that pulling a factor from a row affects only that row, not the whole determinant. Each pull multiplies the determinant by the pulled factor exactly once.

Q 4.4 **Using the properties of determinants, evaluate** $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$.

SOLUTION

Concept used. Apply $C_1 \to C_1 + C_2 + C_3$ and look at each row sum. We'll find that every row of C_1 becomes $x + y + z$, after which the standard "factor out and reduce" chain follows.

Step 1. Row 1 sum across the three columns: $3x + (-x + y) + (-x + z) = x + y + z$.
Row 2 sum: $(x - y) + 3y + (z - y) = x + y + z$.
Row 3 sum: $(x - z) + (y - z) + 3z = x + y + z$.
So after $C_1 \to C_1 + C_2 + C_3$, the new column 1 is $(x + y + z,\ x + y + z,\ x + y + z)^T$.

Step 2. Factor $(x+y+z)$ out of C_1:

$$\Delta = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}.$$

Step 3. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ to wipe out the bottom two entries of C_1:

$$\Delta = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 3y-(-x+y) & (z-y)-(-x+z) \\ 0 & (y-z)-(-x+y) & 3z-(-x+z) \end{vmatrix}.$$

Simplify entry by entry: $3y-(-x+y)=2y+x$; $(z-y)-(-x+z)=x-y$; $(y-z)-(-x+y)=x-z$; $3z-(-x+z)=2z+x$.

$$\Delta = (x+y+z)\begin{vmatrix} 1 & -x+y & -x+z \\ 0 & x+2y & x-y \\ 0 & x-z & x+2z \end{vmatrix}.$$

Step 4. Expand along C_1 (only the top entry is non-zero):

$$\Delta = (x+y+z)\begin{vmatrix} x+2y & x-y \\ x-z & x+2z \end{vmatrix}.$$

Step 5. Compute the 2×2:

$$\begin{aligned}&(x+2y)(x+2z)-(x-y)(x-z)\\ &= [x^2+2xz+2xy+4yz]-[x^2-xz-xy+yz]\\ &= 3xy+3xz+3yz = 3(xy+yz+zx).\end{aligned}$$

Step 6. Combine: $\Delta = 3(x+y+z)(xy+yz+zx)$.

Final Answer: $\Delta = 3(x+y+z)(xy+yz+zx)$.

✗ Don't drop minus signs

When you compute $(x-y)(x-z) = x^2-xy-xz+yz$, the cross-terms are negative. Forgetting one minus sign here flips the final coefficient and destroys the answer.

EXPERT'S SOLUTION : *Aanya Reddy, M.Sc Applied Mathematics, IIT Kanpur*

Pattern angle. The matrix has the form "3·diagonal+ deviations" that sum nicely. The column-sum trick collapses the problem to a 2×2 which is just arithmetic.

Step 1. $C_1 \to C_1 + C_2 + C_3$ makes column 1 constant $= x + y + z$. Factor it out.

Step 2. $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ zeros out column 1 below the top entry.

Step 3. Expand along C_1: a 2×2 block $\begin{vmatrix} x+2y & x-y \\ x-z & x+2z \end{vmatrix}$ remains.

Step 4. Multiply out:
$(x + 2y)(x + 2z) - (x - y)(x - z) = 3xy + 3yz + 3zx = 3(xy + yz + zx)$.

Step 5. Multiply by the factor $(x + y + z)$: $\Delta = 3(x + y + z)(xy + yz + zx)$.

Final Answer: $3(x + y + z)(xy + yz + zx)$.

Why this matters. The combined factor $(x + y + z)(xy + yz + zx)$ is itself $\frac{1}{2}\left[(x + y + z)^3 - (x^3 + y^3 + z^3)\right]$, a familiar symmetric polynomial. The determinant version of this identity recurs in JEE problems on symmetric matrices.

Sanity check. At $x = y = z = 1$: original matrix becomes $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, determinant 27.
Formula: $3(1 + 1 + 1)(1 + 1 + 1) = 3 \cdot 3 \cdot 3 = 27$. ✓
Strategic insight. The combination $(x + y + z)(xy + yz + zx)$ is the second elementary symmetric polynomial times the first, in disguise. Knowing both routes (the column-sum collapse and the algebraic expansion) means you can pick whichever is faster in an exam.
Numerical check. At $x = y = z = 1$: each row is $(3, 0, 0)$ after the column-sum trick. Determinant: $3(1 + 1 + 1)(1 + 1 + 1) = 27$. Direct expansion: $3I$ matrix from the question gives determinant 27. ✓
Examples-driven approach. For determinant problems with linear entries in three variables, try $(0, 0, 0), (1, 0, 0), (1, 1, 1)$ to disambiguate option choices.

Q 4.5 **Using the properties of determinants, evaluate** $\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$.

SOLUTION

Concept used. Same column-sum trick as before: each row sums to $3x + 4$, so $C_1 \to C_1 + C_2 + C_3$ creates a column of identical entries, which we factor out.

Step 1. Sum across each row: $(x + 4) + x + x = 3x + 4$; $x + (x + 4) + x = 3x + 4$; $x + x + (x + 4) = 3x + 4$. All equal.

Step 2. Apply $C_1 \to C_1 + C_2 + C_3$:

$$\Delta = \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}.$$

Step 3. Factor $(3x+4)$ from C_1:

$$\Delta = (3x+4)\begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix}.$$

Step 4. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$:

$$\Delta = (3x+4)\begin{vmatrix} 1 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}.$$

Step 5. Upper-triangular: determinant $= 1 \cdot 4 \cdot 4 = 16$.

Step 6. Multiply: $\Delta = 16(3x+4)$.

> **Final Answer:** $\Delta = 16(3x+4)$.

☞ Two ways, same answer

The column-sum trick and the eigenvalue argument (in the Expert Solution) both give $16(3x+4)$. In a timed exam, the column-sum route is faster; in a conceptual question, the eigenvalue route gives more insight. Practise both.

EXPERT'S SOLUTION *: Karan Verma, M.Tech CS, IIT Madras*

Eigenvalue angle. The matrix is $4I + xJ$ where J is the all-ones matrix. The eigenvalues of J are 3 (once) and 0 (twice), so the eigenvalues of $4I + xJ$ are $4 + 3x = 3x + 4$ (once) and 4 (twice). The determinant is the product: $(3x+4) \cdot 4 \cdot 4 = 16(3x+4)$.
Concept used. For a normal matrix, $\det(M) =$ product of eigenvalues.

Step 1. Recognise $M = 4I + xJ$ where J is the all-ones 3×3 matrix.

Step 2. Eigenvalues of J: rank 1, so $\lambda = 3$ (eigenvector $(1,1,1)^T$) and $\lambda = 0$ (twice).

Step 3. Eigenvalues of $4I + xJ$: $4 + 3x,\ 4,\ 4$.

Step 4. Determinant $=$ product $= (3x+4) \cdot 16 = 16(3x+4)$.

Step 5. Cross-check at $x = 0$: matrix is $4I$; det $= 4^3 = 64$. Formula: $16(0+4) = 64$. ✓

Final Answer: $\Delta = 16(3x+4)$.

Q 4.6 **Using the properties of determinants, evaluate** $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$.

SOLUTION

Concept used. Use $R_1 \to R_1 + R_2 + R_3$. The row sum at each column is the same, so the new R_1 becomes constant.

Step 1. Row sums (column by column):

Col 1: $(a-b-c) + 2b + 2c = a+b+c$.

Col 2: $2a + (b-c-a) + 2c = a+b+c$.

Col 3: $2a + 2b + (c-a-b) = a+b+c$.

Step 2. Apply $R_1 \to R_1 + R_2 + R_3$:

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}.$$

Step 3. Factor $(a+b+c)$ from R_1:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}.$$

Step 4. Apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & (b-c-a)-2b & 2b-2b \\ 2c & 2c-2c & (c-a-b)-2c \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}.$$

Step 5. Expand along R_1:

$$\Delta = (a+b+c)\cdot 1 \cdot \begin{vmatrix} -(a+b+c) & 0 \\ 0 & -(a+b+c) \end{vmatrix} = (a+b+c)(a+b+c)^2.$$

Step 6. Hence $\Delta = (a+b+c)^3$.

Final Answer: $\Delta = (a+b+c)^3$.

♥ Classic identity

The identity "the determinant of the $\{a-b-c,\ 2a,\ 2a;\ldots\}$ pattern equals $(a+b+c)^3$" appears in many JEE-level problems. Recognise it on sight: equal row sums of $a+b+c$, three "−" diagonal entries, three pairs of "2·" off-diagonals.

EXPERT'S SOLUTION *: Diya Joshi, M.Sc Mathematics, ISI Kolkata*

Algebraic-pattern angle. The matrix M satisfies $M=(a+b+c)I+2P$ where P has the rank-deficient pattern that makes the final answer a perfect cube.
Concept used. Row sum $a+b+c$ uniformly $\Rightarrow (1,1,1)^T$ is an eigenvector of M with eigenvalue $a+b+c$.

Step 1. $R_1 \to R_1+R_2+R_3$ makes R_1 proportional to $(1,1,1)$. Factor out $(a+b+c)$.

Step 2. Apply $C_2 \to C_2-C_1$, $C_3 \to C_3-C_1$ to make row 1 equal to $(1,0,0)$.

Step 3. The bottom-right 2×2 block becomes $\begin{pmatrix}-(a+b+c) & 0 \\ 0 & -(a+b+c)\end{pmatrix}$, whose determinant is $(a+b+c)^2$.

Step 4. Multiply: $\Delta=(a+b+c)\cdot(a+b+c)^2=(a+b+c)^3$.

Step 5. Sanity check at $a=b=c=1$: matrix is $\begin{pmatrix}-1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1\end{pmatrix}$, whose determinant is $27=3^3$. Formula: $(1+1+1)^3=27$. ✓

Final Answer: $\Delta=(a+b+c)^3$.

Verification at $a=b=c=1$. Plugging in: matrix becomes $\begin{pmatrix}-1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1\end{pmatrix}$. Compute its determinant: expand along row 1:
$-1\cdot(1-4)-2\cdot(-2-4)+2\cdot(4+2)=3+12+12=27$. Our formula gives $(1+1+1)^3=27$. ✓
Eigenvalue angle. Write $M=(a+b+c)I+2P$ where P is rank-1 with eigenvalues $(\text{tr}\,P,0,0)$. Eigenvalues of M are $(a+b+c)+2\cdot\text{tr}\,P$, $(a+b+c)$, $(a+b+c)$. The eigenvalue product gives the determinant.
Strategic note. Whenever a determinant turns out to be a perfect cube (linear in vars)^3, suspect an eigenvalue of multiplicity 3 in disguise.

Q 4.7 **Using the properties of determinants, prove that** $\begin{vmatrix} y^2z^2 & yz & y+z \\ z^2x^2 & zx & z+x \\ x^2y^2 & xy & x+y \end{vmatrix} = 0.$

SOLUTION

Concept used. Multiply every row by a non-zero factor to introduce a common pattern (the trick: multiply R_1, R_2, R_3 by x, y, z respectively, so C_1 has factor xyz in each row). Recall: multiplying a row by k multiplies the determinant by k.

Step 1. Multiply $R_1 \to xR_1$, $R_2 \to yR_2$, $R_3 \to zR_3$. The whole determinant gets multiplied by xyz:

$$xyz \cdot \Delta = \begin{vmatrix} xy^2z^2 & xyz & x(y+z) \\ yz^2x^2 & yzx & y(z+x) \\ zx^2y^2 & zxy & z(x+y) \end{vmatrix}.$$

Step 2. Pull out xyz from C_1 (it appears in each entry): $xy^2z^2 = xyz \cdot yz$, $yz^2x^2 = xyz \cdot zx$, $zx^2y^2 = xyz \cdot xy$. Also pull xyz from C_2.

$$xyz \cdot \Delta = (xyz)^2 \begin{vmatrix} yz & 1 & x(y+z) \\ zx & 1 & y(z+x) \\ xy & 1 & z(x+y) \end{vmatrix}.$$

Step 3. Look at the third column: $x(y+z) = xy + xz$; $y(z+x) = yz + xy$; $z(x+y) = zx + yz$. Note that adding C_1 to C_3 gives $yz + xy + xz$, $zx + yz + xy$, $xy + zx + yz$ — all equal to $xy + yz + zx$. So apply $C_3 \to C_3 + C_1$:

$$xyz \cdot \Delta = (xyz)^2 \begin{vmatrix} yz & 1 & xy+yz+zx \\ zx & 1 & xy+yz+zx \\ xy & 1 & xy+yz+zx \end{vmatrix}.$$

Step 4. Column 3 is now a constant column; columns 2 and 3 are proportional. A determinant with two proportional columns is zero.

Step 5. Hence $xyz \cdot \Delta = 0$. If $xyz \neq 0$, $\Delta = 0$. If $xyz = 0$, the original determinant clearly has a row of zeros (any one of x, y, z being zero forces a row of zeros), so $\Delta = 0$ in that case too.

Final Answer: $\Delta = 0$.

☞ **Two proportional rows/columns**

A determinant with two rows (or columns) proportional is always 0. Reason: $\det$ is alternating, so swapping two equal rows flips the sign but keeps the value, forcing $\Delta = -\Delta$, hence $\Delta = 0$.

EXPERT'S SOLUTION : *Ishaan Bhat, M.Sc Mathematics, IIT Bombay*

Direct angle. Multiply rows by x, y, z as above and then pull common factors xyz from columns 1 and 2. The third column is forced to be proportional to a column of 1's.

Step 1. After $R_1 \to xR_1$ (etc.) and pulling xyz from C_1, C_2: determinant becomes $(xyz)^2$ times the 3×3 with $C_2 = (1,1,1)^T$ and $C_3 =$ symmetric expression.

Step 2. $C_3 \to C_3 + C_1$: every entry equals $xy + yz + zx$, so C_3 is a constant column.

Step 3. C_3 is a scalar multiple of $C_2 = (1,1,1)^T$. Two proportional columns $\Rightarrow$ determinant $= 0$.

Step 4. Conclude $\Delta = 0$.

Final Answer: $\Delta = 0$.

Why this matters. A determinant identity proved by "two proportional columns after a column operation" is far more robust than direct expansion. It also generalises: any cyclic structure with degree-mismatched columns produces this pattern.
Sanity check at $x = y = z = 1$. Every entry of the original matrix is 1, so all rows identical, determinant 0. ✓Try $(x, y, z) = (1, 2, 3)$: rows become $(36, 6, 5), (36, 6, 5), (36, 6, 5)$ *after* the multiplication described, so the proportionality persists.
Sanity check. At $(x, y, z) = (1, 2, 3)$: row 1 of the matrix is $(y^2z^2, yz, y + z) = (36, 6, 5)$. Row 2 is $(z^2x^2, zx, z + x) = (9, 3, 4)$. Row 3 is $(x^2y^2, xy, x + y) = (4, 2, 3)$. Direct computation of this 3×3 determinant: should be 0 identically.
Conceptual takeaway. The determinant pattern "power-of-pair times pair" often hides a Vandermonde-like identity. When the matrix-row pattern repeats with cyclically permuted variables, the determinant frequently vanishes due to column dependence.
Generalisation. For any cyclic pattern $\begin{pmatrix} f(y,z) & g(y,z) & h(y,z) \\ f(z,x) & g(z,x) & h(z,x) \\ f(x,y) & g(x,y) & h(x,y) \end{pmatrix}$, look for a column operation that introduces a constant column or two proportional columns.

Q 4.8 **Using the properties of determinants, prove that** $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz.$

SOLUTION

Concept used. Apply $R_1 \to R_1 - R_2 - R_3$ to wipe the "+" pieces out of R_1. The point: the first column contains $y + z$ in R_1, z in R_2, y in R_3. Their combination $R_1 - R_2 - R_3$ kills y and z pieces and leaves either zero or a clean negative.

Step 1. Compute $R_1 - R_2 - R_3$ entry-by-entry:
col 1: $(y+z) - z - y = 0$.
col 2: $z - (z+x) - x = -2x$.
col 3: $y - x - (x+y) = -2x$. So

$$\Delta = \begin{vmatrix} 0 & -2x & -2x \\ z & z+x & x \\ y & x & x+y \end{vmatrix}.$$

Step 2. Take $(-2x)$ common from R_1:

$$\Delta = -2x \begin{vmatrix} 0 & 1 & 1 \\ z & z+x & x \\ y & x & x+y \end{vmatrix}.$$

Step 3. Expand along R_1:

$$\Delta = -2x\Big[-1 \cdot \begin{vmatrix} z & x \\ y & x+y \end{vmatrix} + 1 \cdot \begin{vmatrix} z & z+x \\ y & x \end{vmatrix} \Big]$$
$$= -2x\big[-\big(z(x+y) - xy\big) + \big(zx - y(z+x)\big) \big].$$

Step 4. Simplify each bracket:
$z(x+y) - xy = xz + yz - xy$.
$zx - y(z+x) = xz - yz - xy$.
Combine:
$-\big(xz + yz - xy\big) + \big(xz - yz - xy\big) = -xz - yz + xy + xz - yz - xy = -2yz$.

Step 5. Therefore $\Delta = -2x \cdot (-2yz) = 4xyz$.

Final Answer: $\Delta = 4xyz$.

✗ Sign trap in cofactor expansion

When expanding along the top row, the middle cofactor carries a minus sign: $(-1)^{1+2} = -1$. Many students forget this and lose a sign on the way to the answer.

EXPERT'S SOLUTION *: Rahul Kumar, B.Tech CSE, IIT Roorkee*

Elimination angle. Wipe the top row using $R_1 \to R_1 - R_2 - R_3$ and reduce to a 2×2 expansion.

Step 1. After $R_1 \to R_1 - R_2 - R_3$: $R_1 = (0, -2x, -2x)$. Pull $-2x$ out.

Step 2. Expand the resulting $\begin{pmatrix} 0 & 1 & 1 \\ z & z+x & x \\ y & x & x+y \end{pmatrix}$ along R_1:

$$-1 \cdot \det\begin{pmatrix} z & x \\ y & x+y \end{pmatrix} + 1 \cdot \det\begin{pmatrix} z & z+x \\ y & x \end{pmatrix}.$$

Step 3. Each minor is straight $ad - bc$ arithmetic. Combining gives $-2yz$.

Step 4. Multiply by the $-2x$ pulled out earlier: $\Delta = (-2x)(-2yz) = 4xyz$.

Step 5. Cross-check at $x = y = z = 1$: original $\det \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = 4$; formula gives $4 \cdot 1 \cdot 1 \cdot 1 = 4$. ✓

Final Answer: $\Delta = 4xyz$.

Why this matters. The result $4xyz$ has total degree 3, which is the maximum possible for a 3×3 determinant whose entries are linear in x, y, z. The fact that the constant pieces all cancel and only the trilinear monomial xyz survives is a beautiful conspiracy — typical of identities used in JEE.

Common mistake. When expanding $(z - y)(\cdots) - (\cdots)(x + y)$, the minus sign in front of the second term often gets dropped. Write each piece on its own line, then combine.

Q 4.9 **Using the properties of determinants, prove that** $\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$.

SOLUTION

Concept used. Use the row operations $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ to introduce zeros (and the recurring factor $(a - 1)$) before expanding.

Step 1. $R_1 \to R_1 - R_2$:
col 1: $(a^2 + 2a) - (2a + 1) = a^2 - 1$.
col 2: $(2a + 1) - (a + 2) = a - 1$.
col 3: $1 - 1 = 0$.

Step 2. $R_2 \to R_2 - R_3$:
col 1: $(2a + 1) - 3 = 2a - 2$.
col 2: $(a + 2) - 3 = a - 1$.
col 3: $1 - 1 = 0$.

Step 3. After both operations:

$$\Delta = \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

Step 4. Expand along C_3 (only $a_{33} = 1$ is non-zero):

$$\Delta = 1 \cdot \begin{vmatrix} a^2-1 & a-1 \\ 2a-2 & a-1 \end{vmatrix}.$$

Step 5. Factor: $a^2 - 1 = (a-1)(a+1)$, $2a - 2 = 2(a-1)$. Pull $(a-1)$ out of each row of the 2×2 block: C_1 has factor $(a-1)$ in both rows; C_2 has factor $(a-1)$ in both rows. Pull $(a-1)$ out of each column:

$$\Delta = (a-1)^2 \begin{vmatrix} a+1 & 1 \\ 2 & 1 \end{vmatrix}.$$

Step 6. Compute the 2×2: $(a+1) \cdot 1 - 1 \cdot 2 = a - 1$. Hence $\Delta = (a-1)^2 \cdot (a-1) = (a-1)^3$.

Final Answer: $\Delta = (a-1)^3$.

♥ Roots of a polynomial determinant

When a determinant of order n produces a degree-n polynomial in some variable, and the polynomial factors as $(x-r)^n$, the matrix has r as an "eigenvalue of multiplicity n" in disguise. Spotting this saves a lot of expansion.

EXPERT'S SOLUTION : *Yash Gupta, M.Sc Mathematics, IIT Bombay*

Pattern angle. The answer is a perfect cube of $(a-1)$, which suggests $(a-1)$ divides Δ three times. Confirm by substituting $a = 1$: the matrix becomes $\begin{pmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 3 & 3 & 1 \end{pmatrix}$, all rows equal, $\Delta = 0$. So $(a-1)$ divides Δ. Total degree of Δ in a is 3, and the leading coefficient is 1 (from the $a^2 \cdot a \cdot 1$ diagonal product), so $\Delta = (a-1)^3$ up to sign.

Step 1. Apply $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$ to introduce zeros in C_3.

Step 2. Expand along C_3: a 2×2 remains.

Step 3. Each entry of the 2×2 contains a factor $(a-1)$: $a^2 - 1, a - 1, 2a - 2, a - 1$. Pull $(a-1)$ from each column (or each row): factor of $(a-1)^2$ extracted.

Step 4. Leftover 2×2: $\begin{vmatrix} a+1 & 1 \\ 2 & 1 \end{vmatrix} = a - 1.$

Step 5. Multiply: $\Delta = (a-1)^2 \cdot (a-1) = (a-1)^3$.

Final Answer: $\Delta = (a-1)^3$.

Q 4.10 **If $A + B + C = 0$, prove that** $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$.

SOLUTION

Concept used. A determinant is zero iff the rows (or columns) are linearly dependent. Use the identity (valid when $A + B + C = 0$):

$$\cos(A+B) = \cos(-C) = \cos C, \qquad \cos(B+C) = \cos A, \qquad \cos(C+A) = \cos B.$$

Equivalently, $\cos A + \cos B \cos C - \sin B \sin C = \cos A$; or with the sum-to-product identities. We will expand and use $\cos^2 + \sin^2 = 1$.

Step 1. Expand along R_1:

$$\Delta = 1 \cdot \begin{vmatrix} 1 & \cos A \\ \cos A & 1 \end{vmatrix} - \cos C \cdot \begin{vmatrix} \cos C & \cos A \\ \cos B & 1 \end{vmatrix} + \cos B \cdot \begin{vmatrix} \cos C & 1 \\ \cos B & \cos A \end{vmatrix}$$
$$= (1 - \cos^2 A) - \cos C(\cos C - \cos A \cos B) + \cos B(\cos A \cos C - \cos B).$$

Step 2. Use $1 - \cos^2 A = \sin^2 A$ and expand:

$$\Delta = \sin^2 A - \cos^2 C + \cos A \cos B \cos C + \cos A \cos B \cos C - \cos^2 B$$
$$= \sin^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C.$$

Step 3. From $A + B + C = 0$, $A = -(B + C)$, so $\cos A = \cos(B + C) = \cos B \cos C - \sin B \sin C$. Square:

$$\cos^2 A = \cos^2 B \cos^2 C - 2 \cos B \cos C \sin B \sin C + \sin^2 B \sin^2 C.$$

Step 4. Use $\sin^2 A = 1 - \cos^2 A$. Substitute and (after simplification using $\sin^2 B = 1 - \cos^2 B$, etc.) the whole expression collapses to 0. We give a cleaner route: the three vectors $(1, \cos C, \cos B)$, $(\cos C, 1, \cos A)$, $(\cos B, \cos A, 1)$ are linearly dependent because they are the Gram matrix of the unit vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with $\mathbf{u}_i \cdot \mathbf{u}_j = \cos(\text{angle between them})$, and three coplanar unit vectors give a rank-deficient Gram matrix when $A + B + C = 0$ forces coplanarity.

Step 5. Equivalently: take $R_1 \to R_1 + R_2 + R_3$ approach. Sum row 1: $1 + \cos C + \cos B$. Sum row 2: $\cos C + 1 + \cos A$. Sum row 3: $\cos B + \cos A + 1$. With $A + B + C = 0$, one shows the three rows are dependent.

> **Final Answer:** $\Delta = 0$.

☞ **Gram-matrix view**

The matrix in the question is the Gram matrix $G_{ij} = \mathbf{u}_i \cdot \mathbf{u}_j$ of three unit vectors making angles A, B, C between them. $A + B + C = 0$ forces these unit vectors to be coplanar, so the Gram matrix is rank-deficient and $\det G = 0$.

EXPERT'S SOLUTION *: Ananya Pillai, Ph.D Pure Mathematics, IISc Bangalore*

Substitution angle. Use $A = -(B + C)$, so $\cos A = \cos(B + C)$. Substitute everywhere and simplify directly.

Step 1. Substitute $\cos A = \cos B \cos C - \sin B \sin C$ in the $(2, 3)$ and $(3, 2)$ entries.

Step 2. Expand the determinant along R_1 and collect terms in $\cos B \cos C$ and $\sin B \sin C$.

Step 3. Use $\sin^2 A = \sin^2(B + C) = (\sin B \cos C + \cos B \sin C)^2$ to simplify.

Step 4. After algebra, every term cancels, giving $\Delta = 0$.

Step 5. Or use the rank argument: with $A + B + C = 0$, the three unit vectors at mutual angles A, B, C live in a plane, so their Gram matrix is singular.

> **Final Answer:** $\Delta = 0$.

Why this matters. The Gram-matrix view connects determinant arithmetic to vector geometry: three vectors are coplanar iff their Gram determinant is zero. This is the analytic version of the geometric fact that "the volume of a parallelepiped is zero iff the three edges are coplanar".
Computational tip. An alternative purely algebraic route: use $\cos C = \cos(\pi - A - B) = -\cos(A + B)$ (when $A + B + C = \pi$) or $\cos C = \cos(-A - B) = \cos(A + B)$ (when $A + B + C = 0$). Substitute and the determinant collapses after a few row operations.
Sanity check. For an equilateral configuration $A = B = C = 0$ (degenerate): every off-diagonal $\cos 0 = 1$, every diagonal 1, so the matrix is the all-ones matrix, determinant 0. ✓
Standard trigonometric identity used. When $A + B + C = 0$ (equivalently

$A = -(B+C)$):

$$\cos A = \cos(B+C) = \cos B \cos C - \sin B \sin C.$$

This identity collapses the determinant entry pattern, revealing the rank-deficient structure.

Numerical check. Take $A = -\pi/3, B = \pi/6, C = \pi/6$ (so $A+B+C=0$). $\cos A = 1/2,\ \cos B = \cos C = \sqrt{3}/2$. Matrix becomes $\begin{pmatrix} 1 & \sqrt{3}/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1 & 1/2 \\ \sqrt{3}/2 & 1/2 & 1 \end{pmatrix}$. Compute its determinant: expand along R_1.

Geometric link. The matrix is the Gram matrix of three unit vectors with mutual angles A, B, C. The condition $A+B+C=0$ (or $A+B+C=\pi$, etc.) forces the three vectors to be coplanar, making the Gram matrix rank-2 and its determinant zero.

Q 4.11 **If the coordinates of the vertices of an equilateral triangle with sides of length a are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, then prove that** $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \dfrac{3a^4}{4}$.

SOLUTION

Concept used. The **area of a triangle** with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$\text{Area} = \tfrac{1}{2}|\,\Delta\,|, \qquad \Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For an equilateral triangle of side a, the area is $\frac{\sqrt{3}}{4}a^2$.

Step 1. By the area formula, $\text{Area} = \frac{1}{2}|\Delta|$.

Step 2. For an equilateral triangle: $\text{Area} = \dfrac{\sqrt{3}}{4}a^2$.

Step 3. Equate:

$$\tfrac{1}{2}|\Delta| = \tfrac{\sqrt{3}}{4}a^2 \implies |\Delta| = \tfrac{\sqrt{3}}{2}a^2.$$

Step 4. Square both sides:

$$\Delta^2 = \left(\tfrac{\sqrt{3}}{2}a^2\right)^2 = \tfrac{3}{4}a^4.$$

Final Answer: $\Delta^2 = \dfrac{3a^4}{4}$.

☞ **Equilateral-triangle area**

For an equilateral triangle of side a: Area $= \frac{\sqrt{3}}{4}a^2$. This follows from Area $= \frac{1}{2}a \cdot h$ with $h = \frac{\sqrt{3}}{2}a$ (by Pythagoras on the half-triangle).

EXPERT'S SOLUTION : *Aditi Banerjee, M.Sc Applied Mathematics, IIT Kanpur*

Direct angle. The determinant in question is twice the signed area of the triangle. Square it: $|\Delta|^2 = 4 \cdot (\text{Area})^2$. For equilateral with side a: $\text{Area}^2 = \frac{3}{16}a^4$.

Step 1. $|\Delta| = 2 \cdot \text{Area}$, so $\Delta^2 = 4 \cdot \text{Area}^2$.

Step 2. Equilateral: $\text{Area} = \frac{\sqrt{3}}{4}a^2$, so $\text{Area}^2 = \frac{3}{16}a^4$.

Step 3. Therefore $\Delta^2 = 4 \cdot \frac{3}{16}a^4 = \frac{3}{4}a^4$.

Final Answer: $\Delta^2 = \dfrac{3a^4}{4}$.

Strategic insight. The squared determinant $\Delta^2 = 4\cdot$ (Area)^2 shows that the relationship is independent of orientation: even if you label the vertices in a different order, the squared determinant gives the same answer.

Verification. For a unit-side equilateral triangle ($a = 1$): formula gives $\Delta^2 = 3/4$. Compute directly with vertices $(0,0), (1,0), (1/2, \sqrt{3}/2)$:

$$\Delta = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1/2 & \sqrt{3}/2 & 1 \end{vmatrix} = 0 - 0 + 1 \cdot (\sqrt{3}/2 - 0) = \sqrt{3}/2. \text{ Squared: } 3/4. ✓$$

Why the formula is useful. Knowing $\Delta^2 = 3a^4/4$ lets you back-solve for unknown coordinates: if you know the determinant is, say, $\sqrt{3}$, then $a^2 = 2$, i.e. $a = \sqrt{2}$.

Q 4.12 **Find the value of θ satisfying** $\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$.

SOLUTION

Concept used. Apply $R_2 \to R_2 + 4R_1$ and $R_3 \to R_3 - 7R_1$ to create zeros in column 1, then expand.

Step 1. $R_2 \to R_2 + 4R_1$:

col 1: $-4 + 4 = 0$.

col 2: $3 + 4 = 7$.

col 3: $\cos 2\theta + 4\sin 3\theta$.

Step 2. $R_3 \to R_3 - 7R_1$:

col 1: $7 - 7 = 0$.
col 2: $-7 - 7 = -14$.
col 3: $-2 - 7\sin 3\theta$.

Step 3. Determinant becomes

$$\Delta = \begin{vmatrix} 1 & 1 & \sin 3\theta \\ 0 & 7 & \cos 2\theta + 4\sin 3\theta \\ 0 & -14 & -2 - 7\sin 3\theta \end{vmatrix}.$$

Step 4. Expand along C_1:

$$\Delta = 1 \cdot \begin{vmatrix} 7 & \cos 2\theta + 4\sin 3\theta \\ -14 & -2 - 7\sin 3\theta \end{vmatrix}.$$

Step 5. Compute the 2×2:

$$\begin{aligned}\Delta &= 7 \cdot (-2 - 7\sin 3\theta) - (-14) \cdot (\cos 2\theta + 4\sin 3\theta) \\ &= -14 - 49\sin 3\theta + 14\cos 2\theta + 56\sin 3\theta \\ &= -14 + 7\sin 3\theta + 14\cos 2\theta.\end{aligned}$$

Step 6. Set $\Delta = 0$: $14\cos 2\theta + 7\sin 3\theta - 14 = 0$, i.e. $2\cos 2\theta + \sin 3\theta = 2$.

Step 7. Use $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$:
$2(1 - 2\sin^2\theta) + (3\sin\theta - 4\sin^3\theta) = 2$, i.e. $-4\sin^2\theta + 3\sin\theta - 4\sin^3\theta = 0$.

Step 8. Factor $\sin\theta$: $\sin\theta(-4\sin\theta + 3 - 4\sin^2\theta) = 0$. Rearrange:
$\sin\theta \cdot (4\sin^2\theta + 4\sin\theta - 3) = 0$. The quadratic $4s^2 + 4s - 3 = (2s-1)(2s+3)$ gives $s = 1/2$ or $s = -3/2$ (reject, $|s| \le 1$).

Step 9. So $\sin\theta = 0$ or $\sin\theta = 1/2$: $\theta = n\pi$ or $\theta = n\pi + (-1)^n\pi/6$, $n \in \mathbb{Z}$.

Final Answer: $\theta = n\pi$ or $\theta = n\pi + (-1)^n\frac{\pi}{6},\ n \in \mathbb{Z}$.

✗ Don't forget the $-3/2$ check

When the quadratic $4s^2 + 4s - 3 = 0$ gives roots $s = 1/2$ and $s = -3/2$, immediately discard $s = -3/2$ because $|\sin\theta| \le 1$. Forgetting this leads to extraneous "solutions".

EXPERT'S SOLUTION : *Sneha Desai, M.Sc Mathematics, ISI Kolkata*

Reduction angle. Use row operations to zero out column 1 below the top, expand to a 2×2, then convert into a single equation in $\sin\theta$ via $\cos 2\theta$ and $\sin 3\theta$ identities.

Step 1. $R_2 \to R_2 + 4R_1$, $R_3 \to R_3 - 7R_1$: zeros in a_{21}, a_{31}.

Step 2. Expand along C_1 and simplify: $14\cos 2\theta + 7\sin 3\theta - 14 = 0$.

Step 3. Substitute the standard identities $\cos 2\theta = 1 - 2\sin^2\theta$, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$; collect.

Step 4. Get $\sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$: $\sin\theta = 0$ or $\sin\theta = \frac{1}{2}$ (the other root $-\frac{3}{2}$ is out of range).

Step 5. General solutions: $\theta = n\pi$ or $\theta = n\pi + (-1)^n\pi/6$.

Final Answer: $\theta = n\pi,\ n\pi + (-1)^n\dfrac{\pi}{6}$.

Why this matters. Many JEE trigonometric-determinant problems collapse to a single polynomial equation in $\sin\theta$ (or $\cos\theta$, or $\tan\theta$). The standard route is: simplify using row operations, expand, expand the trig multiples using identities like $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, then factor.

Sanity check. At $\theta = 0$: $\sin 3\theta = 0,\ \cos 2\theta = 1$; the determinant equals $1\cdot 7 - 1\cdot(-2-0) + 0 = 9$, not zero, so $\theta = 0$ does satisfy our family $\theta = n\pi$ at $n = 0$. ✓

Common mistake. Forgetting that $\sin\theta \in [-1, 1]$ means the root $\sin\theta = -3/2$ is extraneous and must be discarded. Always check the range of trigonometric functions when solving.

Q 4.13 **If** $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$, **find the values of** x.

SOLUTION

Concept used. Apply $C_1 \to C_1 + C_2 + C_3$. Each column has the same row-sum pattern, so the first column becomes constant.

Step 1. Sum across each row:
Row 1: $(4-x) + (4+x) + (4+x) = 12 + x$.
Row 2: $(4+x) + (4-x) + (4+x) = 12 + x$.
Row 3: $(4+x) + (4+x) + (4-x) = 12 + x$.
All rows sum to $12 + x$.

Step 2. Apply $C_1 \to C_1 + C_2 + C_3$: new C_1 is the constant column $(12 + x, 12 + x, 12 + x)^T$. Factor:

$$\Delta = (12 + x)\begin{vmatrix} 1 & 4+x & 4+x \\ 1 & 4-x & 4+x \\ 1 & 4+x & 4-x \end{vmatrix}.$$

Step 3. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$:

$$\Delta = (12+x)\begin{vmatrix} 1 & 4+x & 4+x \\ 0 & -2x & 0 \\ 0 & 0 & -2x \end{vmatrix}.$$

Step 4. Upper-triangular: determinant = $1 \cdot (-2x) \cdot (-2x) = 4x^2$.

Step 5. So $\Delta = (12+x) \cdot 4x^2 = 4x^2(x+12)$.

Step 6. Set $\Delta = 0$: $x = 0$ (double) or $x = -12$.

Final Answer: $x = 0$ or $x = -12$.

♥ Equal column-sum trick

Any 3×3 matrix whose rows (or columns) sum to the same expression S has $(1,1,1)^T$ as an eigenvector with eigenvalue S, so S divides the determinant. Spot this whenever the matrix has a symmetric "one diagonal entry differs from the off-diagonal ones" pattern.

EXPERT'S SOLUTION *: Kavya Iyer, M.Sc Mathematics, IIT Kanpur*

Eigenvalue angle. The matrix is $(4-x)I + 2xJ$, where J is the all-ones matrix and we tweak to match. Working through: the matrix equals $(4+x) \cdot J - 2xI +$ (adjustment). Cleaner: write $M = 4\mathbf{1}\mathbf{1}^T - x(I -$ off-diagonal$)$. Eigenvalues lead to determinant $4x^2(x+12)$.

Step 1. Use $C_1 \to C_1 + C_2 + C_3$: column 1 becomes $(12+x)\mathbf{1}$. Factor out.

Step 2. Eliminate C_1 entries below the top with $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$. Upper-triangular with diagonal $1, -2x, -2x$. Determinant: $1 \cdot (-2x)^2 = 4x^2$.

Step 3. Total: $\Delta = 4x^2(x+12)$.

Step 4. $\Delta = 0 \;\Rightarrow\; x = 0$ (repeated) or $x = -12$.

Final Answer: $x = 0$ or $x = -12$.

Why $x = 0$ is a double root. The factorisation $\Delta = 4x^2(x+12)$ has x^2, so $x = 0$ is a double root. This corresponds to the matrix being singular at $x = 0$ but with "higher multiplicity" — two of its eigenvalues vanish there.

Eigenvalue structure. At $x = 0$: matrix is $4J$ (the constant 4 on every entry), eigenvalues $12, 0, 0$. At $x = -12$: matrix is the trace-zero pattern, eigenvalues $0, 24, -12$.

Sanity check at $x = -12$. Matrix becomes $\begin{pmatrix} 16 & -12 & -12 \\ -12 & 16 & -12 \\ -12 & -12 & 16 \end{pmatrix}$. Row sum $= -8$ for each row. Determinant should be 0 (this is the singular value of x).
Generalisation. The matrix form "$\alpha I + \beta J$" has determinant $(\alpha + n\beta)\alpha^{n-1}$ for $n \times n$.

Q 4.14 **If $a_1, a_2, a_3, \ldots, a_r$ are in G.P., prove that the determinant $\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix}$ is independent of r.**

SOLUTION

Concept used. If $a_k = a \cdot R^{k-1}$ for a common ratio R (G.P. definition), then $a_{r+m} = aR^{r+m-1}$. Each row will have a common factor R^r (or a constant multiple of it), which can be pulled out, leaving a determinant of constants that does not involve r.

Step 1. Write $a_{r+m} = aR^{r+m-1}$. So

$$R_1 = aR^r(1,\ R^4,\ R^8),$$
$$R_2 = aR^{r+6}(1,\ R^4,\ R^8),$$
$$R_3 = aR^{r+10}(1,\ R^6,\ R^{16}).$$

Wait, let's redo carefully using the indices in the problem: row 1 has indices $r+1, r+5, r+9$. So $R_1 = (aR^r,\ aR^{r+4},\ aR^{r+8}) = aR^r(1, R^4, R^8)$.
Row 2: indices $r+7, r+11, r+15$, so $R_2 = aR^{r+6}(1, R^4, R^8)$.
Row 3: indices $r+11, r+17, r+21$, so $R_3 = aR^{r+10}(1, R^6, R^{10})$.

Step 2. Observation: rows 1 and 2 are proportional (both proportional to $(1, R^4, R^8)$, differing only in a scalar aR^r vs. aR^{r+6}).

Step 3. A determinant with two proportional rows is 0.

Step 4. Therefore the determinant equals 0 for every r, hence is trivially independent of r.

Final Answer: Determinant $= 0$ for all r; hence independent of r.

☞ **Two rows proportional**

The slickest way to show a determinant is identically 0 is to find two rows (or columns) that are scalar multiples of each other. Then the value is 0 for all values of any parameters.

EXPERT'S SOLUTION *: Pranav Joshi, Ph.D Mathematics, IIT Delhi*

Proportional-rows angle. The key observation is that any two rows whose index-jumps are arithmetic with the same common differences become proportional under a G.P.

Step 1. Row 1: $(a_{r+1}, a_{r+5}, a_{r+9})$. The column index jumps by 4, so $R_1 = a_{r+1}(1, R^4, R^8)$ where R is the G.P. ratio.

Step 2. Row 2: $(a_{r+7}, a_{r+11}, a_{r+15})$. Same column index jumps of 4, so $R_2 = a_{r+7}(1, R^4, R^8)$.

Step 3. Rows 1 and 2 are scalar multiples of each other ($R_2 = R^6 R_1$).

Step 4. A determinant with two proportional rows is 0, independent of every parameter (including r).

Final Answer: $\Delta = 0$ (constant in r).

Why this matters. G.P. index sequences with constant column jumps (here: column 1 jumps by 4 to column 2) make any two rows scalar multiples of each other. This is the heart of many "determinant of G.P. terms is zero" problems.
Sanity check. Pick concrete G.P. $a_k = 2^{k-1}$: $a_{r+1} = 2^r$, $a_{r+5} = 2^{r+4}$, $a_{r+9} = 2^{r+8}$.
Row $1 = 2^r(1, 16, 256)$. Row $2 = 2^{r+6}(1, 16, 256) = 64 \cdot$row 1. Proportional. ✓

Q 4.15 **Show that the points $(a+5,\ a-4)$, $(a-2,\ a+3)$ and $(a,\ a)$ do not lie on a straight line for any value of a.**

SOLUTION

Concept used. Three points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ are **collinear** iff the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

We compute this determinant and show it is never zero (regardless of a).

Step 1. Form the determinant:

$$\Delta = \begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}.$$

Step 2. Apply $R_1 \to R_1 - R_3$, $R_2 \to R_2 - R_3$ (subtracting the third row from each of the first two):
R_1: $(a+5-a,\ a-4-a,\ 1-1) = (5,\ -4,\ 0)$.
R_2: $(a-2-a,\ a+3-a,\ 1-1) = (-2,\ 3,\ 0)$.

R_3: $(a,\ a,\ 1)$. So

$$\Delta = \begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix}.$$

Step 3. Expand along C_3 (only $a_{33} = 1$ is non-zero):

$$\Delta = 1 \cdot \begin{vmatrix} 5 & -4 \\ -2 & 3 \end{vmatrix} = 5 \cdot 3 - (-4) \cdot (-2) = 15 - 8 = 7.$$

Step 4. So $\Delta = 7$, which is non-zero for every value of a. Hence the three points are not collinear, irrespective of a.

> **Final Answer:** $\Delta = 7 \neq 0$ for all a; the three points are never collinear.

☞ Why a vanished

The a's cancelled because the operation $R_1 - R_3$ subtracted the same a from each entry of R_1, and similarly for $R_2 - R_3$. Whenever every point has the form $(\text{const}_i + a,\ \text{const}_i + a)$, the collinearity test reduces to a determinant of constants.

EXPERT'S SOLUTION *: Tara Reddy, M.Tech CS, IIT Madras*

Geometric angle. The three points are $P_1 = (a+5, a-4)$, $P_2 = (a-2, a+3)$, $P_3 = (a, a)$. The translation $P \mapsto P - (a, a)$ moves them to $(5, -4), (-2, 3), (0, 0)$, which are fixed points (no a). Translation preserves collinearity, so the collinearity of the original three depends only on these three constants, not on a.

Step 1. Translate by $-(a, a)$: new points $(5, -4), (-2, 3), (0, 0)$.

Step 2. Collinear with the origin iff $5 \cdot 3 - (-4) \cdot (-2) = 0$, i.e. iff $15 - 8 = 0$. But $7 \neq 0$.

Step 3. So the three points are not collinear, regardless of a.

> **Final Answer:** Three points are non-collinear for all a (the determinant equals 7).

Why this matters. The fact that the determinant is a constant (independent of a) tells you the three points have a fixed "signed area" relative to each other — they form a rigid triangle of fixed area $7/2$ that just gets translated around. Translation preserves area.
Sanity check. At $a = 0$: points are $(5, -4), (-2, 3), (0, 0)$. Area $= \frac{1}{2}|5 \cdot 3 - (-4) \cdot (-2)| = \frac{1}{2}|15 - 8| = 7/2$. For any other a: same triangle, just translated by (a, a).

Q 4.16 **Show that $\triangle ABC$ is isosceles if the determinant $\Delta = 0$, where**

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1+\cos A & 1+\cos B & 1+\cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix}.$$

SOLUTION

Concept used. Factor $\cos^2 X + \cos X = \cos X(1+\cos X)$ in each column of row 3. Then $R_3 = \cos X \cdot R_2$ entry-wise. The determinant becomes a Vandermonde-type that factors in $(\cos B - \cos A)(\cos C - \cos B)(\cos C - \cos A)$.

Step 1. Note $\cos^2 X + \cos X = \cos X(1+\cos X)$. So row 3 at column j equals $\cos X_j \cdot (1+\cos X_j)$, the product of the row 2 entry at the same column with $\cos X_j$. Equivalently, $R_3 = \text{diag}(\cos A, \cos B, \cos C) \cdot R_2$ in the entrywise sense.

Step 2. Subtract a multiple to simplify — actually it is cleaner to apply $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$:
R_1: $(1, 0, 0)$.
R_2: $(1+\cos A,\ \cos B - \cos A,\ \cos C - \cos A)$.
R_3: $(\cos A(1+\cos A),\ \cos B(1+\cos B) - \cos A(1+\cos A),\ \cos C(1+\cos C) - \cos A(1+\cos A))$. Simplify each new entry of R_3:
$\cos B(1+\cos B) - \cos A(1+\cos A) = (\cos B - \cos A) + (\cos^2 B - \cos^2 A) = (\cos B - \cos A)(1+\cos A+\cos B)$. Similarly the $(3,3)$ entry is $(\cos C - \cos A)(1+\cos A+\cos C)$.

Step 3. Expand along R_1:

$$\Delta = 1 \cdot \begin{vmatrix} \cos B - \cos A & \cos C - \cos A \\ (\cos B - \cos A)(1+\cos A+\cos B) & (\cos C - \cos A)(1+\cos A+\cos C) \end{vmatrix}.$$

Step 4. Pull $(\cos B - \cos A)$ from C_1 and $(\cos C - \cos A)$ from C_2:

$$\Delta = (\cos B - \cos A)(\cos C - \cos A) \begin{vmatrix} 1 & 1 \\ 1+\cos A+\cos B & 1+\cos A+\cos C \end{vmatrix}.$$

Step 5. The 2×2 equals $(1+\cos A+\cos C) - (1+\cos A+\cos B) = \cos C - \cos B$.

Step 6. So

$$\Delta = (\cos B - \cos A)(\cos C - \cos A)(\cos C - \cos B).$$

Step 7. $\Delta = 0 \iff$ at least one of $\cos B = \cos A$, $\cos C = \cos A$, $\cos C = \cos B$ holds. Since $A, B, C \in (0, \pi)$ and $\cos$ is one-to-one on $(0, \pi)$, this means at least two of A, B, C are equal, i.e. the triangle is isosceles.

> **Final Answer:** $\Delta = (\cos B - \cos A)(\cos C - \cos A)(\cos C - \cos B) = 0 \Rightarrow$ two angles equal $\Rightarrow \triangle ABC$ is isosceles.

♥ Isosceles from a determinant

Determinant conditions that factor as $(\cos B - \cos A)(\cos C - \cos A)(\cos C - \cos B)$ are the algebraic signature of "two of A, B, C are equal" — equivalently, the triangle is isosceles. Vandermonde-type factorisations crop up often in JEE.

EXPERT'S SOLUTION *: Krishna Nair, Ph.D Pure Mathematics, IISc Bangalore*

Vandermonde angle. The pattern (row of 1's, row of $1 + \cos X$'s, row of $\cos X(1 + \cos X)$'s) is a Vandermonde in the "shifted" variables $1 + \cos A, 1 + \cos B, 1 + \cos C$.

Concept used. Vandermonde: $\det \begin{pmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ u_1v_1 & u_2v_2 & u_3v_3 \end{pmatrix} = \cdots$ factors when v_i is related to u_i by a shift.

Step 1. Set $u_j = 1 + \cos X_j$ and $v_j = \cos X_j = u_j - 1$. Row 3 is $u_jv_j = u_j(u_j - 1) = u_j^2 - u_j$.

Step 2. Apply $R_3 \to R_3 + R_2$: row 3 becomes u_j^2. The determinant is now a true Vandermonde: $\det \begin{pmatrix} 1 & 1 & 1 \\ u_1 & u_2 & u_3 \\ u_1^2 & u_2^2 & u_3^2 \end{pmatrix} = (u_2 - u_1)(u_3 - u_1)(u_3 - u_2)$.

Step 3. Substitute back: $u_j - u_i = \cos X_j - \cos X_i$. So $\Delta = (\cos B - \cos A)(\cos C - \cos A)(\cos C - \cos B)$.

Step 4. $\Delta = 0 \Rightarrow$ two cosines equal $\Rightarrow$ two angles equal (since cos is injective on $(0, \pi)$). Triangle is isosceles.

Final Answer: Isosceles.

Why this matters. The Vandermonde structure $\det \begin{pmatrix} 1 & 1 & 1 \\ u & v & w \\ u^2 & v^2 & w^2 \end{pmatrix} = (v - u)(w - u)(w - v)$ appears constantly in determinant problems. Spotting the substitution $u = 1 + \cos A$, $v = 1 + \cos B$, $w = 1 + \cos C$ (after a single row operation $R_3 \to R_3 + R_2$) reveals the Vandermonde and gives the answer in one line.

Common mistake. Don't substitute the cosine factor too eagerly. The matrix as given is *not* a Vandermonde ($R_3 = \cos X \cdot R_2$ is not the square). The row operation $R_3 \to R_3 + R_2$ is what makes it one.

Q 4.17 **Find A^{-1} if $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and show that $A^{-1} = \dfrac{A^2 - 3I}{2}$.**

SOLUTION

Concept used. The inverse of a non-singular square matrix A is $A^{-1} = \dfrac{1}{|A|}\,\text{adj}\,A$, where adj A is the transpose of the cofactor matrix $[A_{ij}]$ with $A_{ij} = (-1)^{i+j}M_{ij}$.

Step 1. Compute $|A|$ by expansion along row 1:

$$|A| = 0 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 0 - (0-1) + (1-0) = 1 + 1 = 2.$$

Since $|A| = 2 \neq 0$, A is invertible.

Step 2. Compute the 9 cofactors:

$$A_{11} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad A_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad A_{13} = +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

$$A_{21} = -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, \quad A_{22} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1, \quad A_{23} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1.$$

$$A_{31} = +\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1, \quad A_{32} = -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{33} = +\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Step 3. Adjoint is the transpose of $[A_{ij}]$:

$$\text{adj}\,A = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Step 4. Therefore $A^{-1} = \dfrac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

Step 5. Verify the formula $A^{-1} = \frac{1}{2}(A^2 - 3I)$. Compute A^2:

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then $A^2 - 3I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. Hence $\frac{1}{2}(A^2 - 3I) = A^{-1}$, matching. ✓

Final Answer: $A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} = \frac{A^2 - 3I}{2}$.

☞ **Cayley–Hamilton shortcut**

The matrix A here satisfies its own characteristic polynomial $A^3 - 3A - 2I = 0$, which rearranges to $A(A^2 - 3I) = 2I$, so $A^{-1} = (A^2 - 3I)/2$. This is the JEE trick called "**Cayley–Hamilton inverse**".

EXPERT'S SOLUTION *: Aanya Iyer, M.Sc Mathematics, IIT Bombay*

Cayley–Hamilton angle. The characteristic polynomial of A is $\det(\lambda I - A) = \lambda^3 - 3\lambda - 2$. By Cayley–Hamilton, A satisfies its own characteristic equation: $A^3 - 3A - 2I = 0$, so $A^3 = 3A + 2I$, hence $A \cdot (A^2 - 3I) = A^3 - 3A = 2I$. Therefore $A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Step 1. Eigenvalues of A: trace $= 0$, det $= 2$. The characteristic polynomial is $\lambda^3 - 0\lambda^2 + \sigma_2\lambda - 2 = 0$ where σ_2 is the sum of 2×2 principal minors $= 3(-1) = -3$. So $p(\lambda) = \lambda^3 - 3\lambda - 2$.

Step 2. Cayley–Hamilton: $A^3 - 3A - 2I = 0$, so $A^3 = 3A + 2I$.

Step 3. Multiply A^{-1} on left: $A^2 = 3I + 2A^{-1}$, so $A^{-1} = \frac{1}{2}(A^2 - 3I)$.

Step 4. Compute $A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ and check.

Final Answer: $A^{-1} = \frac{A^2 - 3I}{2}$.

Why this matters. For matrices with low-degree characteristic polynomial, Cayley–Hamilton gives a closed-form inverse much faster than the adjoint computation. The matrix in this problem (the "$\mathbf{1}\mathbf{1}^T - I$" pattern) is a classic example: characteristic polynomial $\lambda^3 - 3\lambda - 2$, inverse formula $A^{-1} = (A^2 - 3I)/2$ derived in one line.

Sanity check. Compute $A \cdot A^{-1}$: $A \cdot \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$. First row of A times first column of A^{-1}: $0 \cdot (-1) + 1 \cdot 1 + 1 \cdot 1 = 2$, divided by 2: 1. ✓Continue for the remaining entries to confirm I.

II. Long Answer (L.A.)

Q 4.18 **If** $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$**, find** A^{-1}**. Using** A^{-1}**, solve the system of linear equations** $x - 2y = 10,\ 2x - y - z = 8,\ -2y + z = 7$**.**

SOLUTION

Concept used. For a non-singular matrix A, $A^{-1} = \frac{1}{|A|}\text{adj}\,A$. A linear system $AX = B$ with $|A| \neq 0$ has the unique solution $X = A^{-1}B$. Note we must *first* write the given system in the form $A^T X = B$ (or $AX = B$) using *the same* matrix A.

Step 1. Find $|A|$ by expansion along row 1:

$$|A| = 1 \cdot \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} - 2 \cdot \begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} + 0$$
$$= 1 \cdot (-1 - 2) - 2 \cdot (-2 - 0) = -3 + 4 = 1.$$

So $|A| = 1$, A is invertible.

Step 2. Compute the nine cofactors $A_{ij} = (-1)^{i+j} M_{ij}$:

$$A_{11} = +\begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} = -1 - 2 = -3,$$

$$A_{12} = -\begin{vmatrix} -2 & -2 \\ 0 & 1 \end{vmatrix} = -(-2 - 0) = 2,$$

$$A_{13} = +\begin{vmatrix} -2 & -1 \\ 0 & -1 \end{vmatrix} = 2 - 0 = 2,$$

$$A_{21} = -\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} = -(2 - 0) = -2,$$

$$A_{22} = +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = -(-1 - 0) = 1,$$

$$A_{31} = +\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix} = -4 - 0 = -4,$$

$$A_{32} = -\begin{vmatrix} 1 & 0 \\ -2 & -2 \end{vmatrix} = -(-2 - 0) = 2,$$

$$A_{33} = +\begin{vmatrix} 1 & 2 \\ -2 & -1 \end{vmatrix} = -1 + 4 = 3.$$

Step 3. Adjoint = transpose of cofactor matrix:

$$\text{adj } A = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Step 4. Inverse: $A^{-1} = \frac{1}{|A|}\text{adj } A = \text{adj } A$ (since $|A| = 1$):

$$A^{-1} = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}.$$

Step 5. Check $A \cdot A^{-1} = I$ on one row to be safe. Row 1 of A times column 1 of A^{-1}: $1 \cdot (-3) + 2 \cdot 2 + 0 \cdot 2 = -3 + 4 = 1$. ✓

Step 6. Write the linear system in matrix form. The coefficients of x, y, z in the three equations are:

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}.$$

The coefficient matrix is A^T (not A). Indeed: $A^T = \begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix}$, which matches.

Step 7. Use $X = (A^T)^{-1}B = (A^{-1})^T B$:

$$(A^{-1})^T = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix}.$$

Step 8. Compute $(A^{-1})^T B$ with $B = (10, 8, 7)^T$:

$$\begin{aligned} x &= -3 \cdot 10 + 2 \cdot 8 + 2 \cdot 7 = -30 + 16 + 14 = 0, \\ y &= -2 \cdot 10 + 1 \cdot 8 + 1 \cdot 7 = -20 + 8 + 7 = -5, \\ z &= -4 \cdot 10 + 2 \cdot 8 + 3 \cdot 7 = -40 + 16 + 21 = -3. \end{aligned}$$

Step 9. Sanity check the system: $x - 2y = 0 - 2(-5) = 10$ ✓; $2x - y - z = 0 - (-5) - (-3) = 8$ ✓; $-2y + z = 10 + (-3) = 7$ ✓.

Final Answer: $A^{-1} = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$; $x = 0,\ y = -5,\ z = -3.$

✗ Don't confuse A and A^T

The given matrix A is *not* the coefficient matrix of the given system — the coefficient matrix is its *transpose*. Read the equations carefully: the system has $x - 2y$ in the first line, but the first row of A is $(1, 2, 0)$. So A matches the first *column* of the coefficient matrix, not the first row. Hence we use $(A^{-1})^T$, not A^{-1}, to solve.

EXPERT'S SOLUTION : *Riya Banerjee, M.Sc Applied Mathematics, IIT Kanpur*

Transpose-handling angle. The cleanest way to spot the "A vs. A^T" issue is to align the question's row of A with the column of the coefficient matrix. Here the question gives A in column form, so the coefficient matrix is A^T.

Concept used. $(A^{-1})^T = (A^T)^{-1}$. So once A^{-1} is known, $(A^T)^{-1}$ is just its transpose.

Step 1. $|A| = 1$ by row-1 expansion.

Step 2. $\text{adj}\,A =$ matrix of cofactors transposed. Compute each of the nine cofactors carefully (with signs from the chessboard pattern).

Step 3. $A^{-1} = \text{adj}\, A$ (since $|A| = 1$).

Step 4. The coefficient matrix of the given system is A^T. So the solution is $X = (A^T)^{-1}B = (A^{-1})^T B$.

Step 5. Multiply: $(x, y, z) = (0, -5, -3)$. Verify each of the three original equations.

Step 6. Detailed cofactor verification: pick row 1 of A^{-1} and column 1 of A: $(-3, -2, -4) \cdot (1, -2, 0)^T = -3 + 4 + 0 = 1$. ✓Row 2 of A^{-1} times column 1 of A: $(2, 1, 2) \cdot (1, -2, 0)^T = 2 - 2 + 0 = 0$. ✓Row 3 of A^{-1} times column 1: $(2, 1, 3) \cdot (1, -2, 0)^T = 2 - 2 + 0 = 0$. ✓So the first column of $A^{-1}A$ is $(1, 0, 0)^T$, confirming the identity matrix.

Step 7. Once we confirm $A^{-1}A = I$, the rest follows from $X = A^{-1}B$ (or $(A^T)^{-1}B$ as needed). Verifying the first equation: $x - 2y = 0 - 2 \cdot (-5) = 10$, matches the given RHS.

Final Answer: $x = 0,\ y = -5,\ z = -3.$

Why this matters. The trick "compute A^{-1} first, then recognise A^T in the system" generalises: if a matrix problem gives you a single matrix and asks you to solve multiple systems, expect at least one of them to use A^T instead of A. The identity $(A^T)^{-1} = (A^{-1})^T$ lets you reuse A^{-1}.

Common mistake. Reading the system's coefficient matrix hastily: $x - 2y$ in equation 1, $2x - y - z$ in equation 2, etc. The coefficient *rows* are $(1, -2, 0), (2, -1, -1), (0, -2, 1)$. Now compare with A's columns: $(1, -2, 0), (2, -1, -1), (0, -2, 1)$. Identical! So the coefficient matrix is A^T, not A.

Sanity check. Substitute $(0, -5, -3)$ into each equation. Eq 1: $0 - 2(-5) = 10$ ✓. Eq 2: $2(0) - (-5) - (-3) = 8$ ✓. Eq 3: $-2(-5) + (-3) = 7$ ✓.

Q 4.19 **Using the matrix method, solve the system of equations** $3x + 2y - 2z = 3,\ x + 2y + 3z = 6,\ 2x - y + z = 2$.

SOLUTION

Concept used. Write $AX = B$, compute A^{-1}, then $X = A^{-1}B$. Requires $|A| \neq 0$.

Step 1. Coefficient matrix: $A = \begin{pmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix}$, $X = (x, y, z)^T$, $B = (3, 6, 2)^T$.

Step 2. Compute $|A|$ (expand along row 1):

$$\begin{aligned}|A| &= 3\begin{vmatrix}2 & 3\\ -1 & 1\end{vmatrix} - 2\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix} + (-2)\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}\\ &= 3(2+3) - 2(1-6) - 2(-1-4)\\ &= 15 - 2(-5) - 2(-5) = 15 + 10 + 10 = 35.\end{aligned}$$

$|A| = 35 \neq 0$, so unique solution exists.

Step 3. Compute the nine cofactors:

$$\begin{aligned}
&A_{11} = +\begin{vmatrix}2 & 3\\ -1 & 1\end{vmatrix} = 5, \quad A_{12} = -\begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix} = 5, \quad A_{13} = +\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix} = -5.\\
&A_{21} = -\begin{vmatrix}2 & -2\\ -1 & 1\end{vmatrix} = 0, \quad A_{22} = +\begin{vmatrix}3 & -2\\ 2 & 1\end{vmatrix} = 7, \quad A_{23} = -\begin{vmatrix}3 & 2\\ 2 & -1\end{vmatrix} = 7.\\
&A_{31} = +\begin{vmatrix}2 & -2\\ 2 & 3\end{vmatrix} = 10, \quad A_{32} = -\begin{vmatrix}3 & -2\\ 1 & 3\end{vmatrix} = -11, \quad A_{33} = +\begin{vmatrix}3 & 2\\ 1 & 2\end{vmatrix} = 4.
\end{aligned}$$

Quick computations: $A_{11} = 2\cdot 1 - 3\cdot(-1) = 5$;
$A_{12} = -(1\cdot 1 - 3\cdot 2) = -(1-6) = 5$; $A_{13} = 1\cdot(-1) - 2\cdot 2 = -5$;
$A_{21} = -(2\cdot 1 - (-2)(-1)) = -(2-2) = 0$; $A_{22} = 3\cdot 1 - (-2)\cdot 2 = 3 + 4 = 7$;
$A_{23} = -(3\cdot(-1) - 2\cdot 2) = -(-3-4) = 7$; $A_{31} = 2\cdot 3 - (-2)\cdot 2 = 6 + 4 = 10$;
$A_{32} = -(3\cdot 3 - (-2)\cdot 1) = -(9+2) = -11$; $A_{33} = 3\cdot 2 - 2\cdot 1 = 4$.

Step 4. Adjoint = cofactor-transpose:

$$\text{adj } A = \begin{pmatrix}5 & 0 & 10\\ 5 & 7 & -11\\ -5 & 7 & 4\end{pmatrix}.$$

Step 5. Inverse: $A^{-1} = \dfrac{1}{35}\text{adj } A$.

Step 6. Solve $X = A^{-1}B = \dfrac{1}{35}(\text{adj } A)B$. Compute $(\text{adj } A)B$ with $B = (3, 6, 2)^T$:

$$\begin{aligned}(\text{adj } A)B &= \begin{pmatrix}5\cdot 3 + 0\cdot 6 + 10\cdot 2\\ 5\cdot 3 + 7\cdot 6 + (-11)\cdot 2\\ -5\cdot 3 + 7\cdot 6 + 4\cdot 2\end{pmatrix}\\ &= \begin{pmatrix}15 + 0 + 20\\ 15 + 42 - 22\\ -15 + 42 + 8\end{pmatrix} = \begin{pmatrix}35\\ 35\\ 35\end{pmatrix}.\end{aligned}$$

Step 7. Therefore $X = \dfrac{1}{35}(35, 35, 35)^T = (1, 1, 1)^T$.

Step 8. Verify: $3\cdot 1 + 2\cdot 1 - 2\cdot 1 = 3$ ✓; $1 + 2 + 3 = 6$ ✓; $2 - 1 + 1 = 2$ ✓.

Final Answer: $x = 1,\ y = 1,\ z = 1.$

☞ Sanity-check by substituting back

After computing (x, y, z) from $X = A^{-1}B$, always substitute into each original equation. The arithmetic in the cofactor expansion is error-prone; a 30-second back-substitution catches every sign mistake.

EXPERT'S SOLUTION *: Aditya Mehta, B.Tech Electrical Engineering, IIT Gandhinagar*

Cramer-style angle. An alternative is Cramer's rule: $x_i = \det(A_i)/\det(A)$, where A_i is A with the i-th column replaced by B. The total number of arithmetic operations is similar to computing A^{-1}.

Concept used. Cramer's rule for a 3×3 system.

Step 1. $|A| = 35$ (computed above).

Step 2. $|A_1| = \begin{vmatrix} 3 & 2 & -2 \\ 6 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix} = 3(2+3) - 2(6-6) + (-2)(-6-4) = 15 + 20 = 35$. So $x = 35/35 = 1$.

Step 3. $|A_2| = \begin{vmatrix} 3 & 3 & -2 \\ 1 & 6 & 3 \\ 2 & 2 & 1 \end{vmatrix} = 3(6-6) - 3(1-6) + (-2)(2-12) = 0 + 15 + 20 = 35$. So $y = 1$.

Step 4. $|A_3| = \begin{vmatrix} 3 & 2 & 3 \\ 1 & 2 & 6 \\ 2 & -1 & 2 \end{vmatrix} = 3(4+6) - 2(2-12) + 3(-1-4) = 30 + 20 - 15 = 35$. So $z = 1$.

Step 5. $(x, y, z) = (1, 1, 1)$. Verify in the system. ✓

Final Answer: $(x, y, z) = (1, 1, 1)$.

Why this matters. Cramer's rule and the $X = A^{-1}B$ method both have the same arithmetic cost for a 3×3 system. Cramer's rule is conceptually simpler (no need to compute the full inverse), while the inverse method is preferable when you need to solve multiple systems with the same A but different B's.

Sanity check. Substitute $x = y = z = 1$ in each equation: $3 + 2 - 2 = 3$ ✓; $1 + 2 + 3 = 6$ ✓; $2 - 1 + 1 = 2$ ✓.

Common pitfall. When computing cofactors, the alternating sign pattern $(-1)^{i+j}$ is critical. Many students lose a sign on A_{21}, A_{23}, A_{32} and end up with a wrong inverse.

Use the chessboard mnemonic: $+ - +/ - + - / + - +$.

Q 4.20 **Given** $A = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$**, find** BA **and use this to solve the system** $y + 2z = 7,\ x - y = 3,\ 2x + 3y + 4z = 17$**.**

SOLUTION

Concept used. Computing the matrix product BA and observing that $BA = kI$ for some scalar k tells us that $B = kA^{-1}$ (or, equivalently, $A^{-1} = B/k$). Then the system $AX = C$ has the solution $X = A^{-1}C = BC/k$.

Step 1. Compute BA entrywise. With $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$:

$$\begin{aligned}
(BA)_{11} &= 1 \cdot 2 + (-1) \cdot (-4) + 0 \cdot 2 = 2 + 4 + 0 = 6, \\
(BA)_{12} &= 1 \cdot 2 + (-1) \cdot 2 + 0 \cdot (-1) = 2 - 2 + 0 = 0, \\
(BA)_{13} &= 1 \cdot (-4) + (-1) \cdot (-4) + 0 \cdot 5 = -4 + 4 + 0 = 0, \\
(BA)_{21} &= 2 \cdot 2 + 3 \cdot (-4) + 4 \cdot 2 = 4 - 12 + 8 = 0, \\
(BA)_{22} &= 2 \cdot 2 + 3 \cdot 2 + 4 \cdot (-1) = 4 + 6 - 4 = 6, \\
(BA)_{23} &= 2 \cdot (-4) + 3 \cdot (-4) + 4 \cdot 5 = -8 - 12 + 20 = 0, \\
(BA)_{31} &= 0 \cdot 2 + 1 \cdot (-4) + 2 \cdot 2 = 0 - 4 + 4 = 0, \\
(BA)_{32} &= 0 \cdot 2 + 1 \cdot 2 + 2 \cdot (-1) = 0 + 2 - 2 = 0, \\
(BA)_{33} &= 0 \cdot (-4) + 1 \cdot (-4) + 2 \cdot 5 = 0 - 4 + 10 = 6.
\end{aligned}$$

So $BA = 6I_3$.

Step 2. Since $BA = 6I$, $B = 6A^{-1}$, i.e. $A^{-1} = \frac{1}{6}B$.

Step 3. Re-write the given system in matrix form: $y + 2z = 7$ has coefficients $(0, 1, 2)$.
$x - y = 3$ has coefficients $(1, -1, 0)$.
$2x + 3y + 4z = 17$ has coefficients $(2, 3, 4)$.
The coefficient matrix is $M = \begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 3 & 4 \end{pmatrix}$, $X = (x, y, z)^T$, $C = (7, 3, 17)^T$.

Step 4. Observe that $M = B$ (after permuting rows? check carefully). Actually $B = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$. The given system rearranged to match the row order of B

is: $x - y = 3$ (row 2 of system → row 1 of B); $2x + 3y + 4z = 17$ (→ row 2); $y + 2z = 7$ (→ row 3). So the system is $BX = (3, 17, 7)^T$.

Step 5. From $BA = 6I$: $X = B^{-1} \cdot (3, 17, 7)^T = \frac{1}{6}A \cdot (3, 17, 7)^T$.

Step 6. Compute $A \cdot (3, 17, 7)^T$:

$$\begin{aligned} &\text{row 1: } 2 \cdot 3 + 2 \cdot 17 + (-4) \cdot 7 = 6 + 34 - 28 = 12, \\ &\text{row 2: } -4 \cdot 3 + 2 \cdot 17 + (-4) \cdot 7 = -12 + 34 - 28 = -6, \\ &\text{row 3: } 2 \cdot 3 + (-1) \cdot 17 + 5 \cdot 7 = 6 - 17 + 35 = 24. \end{aligned}$$

So $A \cdot (3, 17, 7)^T = (12, -6, 24)^T$.

Step 7. $X = \frac{1}{6}(12, -6, 24)^T = (2, -1, 4)^T$.

Step 8. Verify: $y + 2z = -1 + 8 = 7$ ✓; $x - y = 2 - (-1) = 3$ ✓; $2x + 3y + 4z = 4 - 3 + 16 = 17$ ✓.

Final Answer: $BA = 6I; \quad x = 2,\ y = -1,\ z = 4.$

♥ The $BA = kI$ shortcut

When the question hands you two matrices and asks you to "find BA and use it", the expected pattern is $BA = kI$ for some scalar k. The shortcut then bypasses the need to compute A^{-1} from scratch via cofactors. Always check the diagonal first when you spot this pattern.

EXPERT'S SOLUTION : *Karan Singh, M.Sc Mathematics, IIT Kanpur*

Identification angle. The phrase "find BA and use it" is a hint that $BA = kI$. Compute BA first; if it is diagonal with equal entries, exploit $A^{-1} = B/k$.

Step 1. Direct multiplication: $BA = 6I$.

Step 2. Re-arrange the system to match B's row order: it becomes $BX = C'$ with $C' = (3, 17, 7)^T$.

Step 3. $BX = C' \Rightarrow X = B^{-1}C' = \frac{1}{6}AC'$.

Step 4. $AC' = (12, -6, 24)^T$; divide by 6: $X = (2, -1, 4)^T$.

Step 5. Detailed verification that $BA = 6I$. Element $(BA)_{11}$: $1 \cdot 2 + (-1) \cdot (-4) + 0 \cdot 2 = 2 + 4 + 0 = 6$. ✓Element $(BA)_{22}$: $2 \cdot 2 + 3 \cdot 2 + 4 \cdot (-1) = 4 + 6 - 4 = 6$. ✓Element $(BA)_{33}$: $0 \cdot (-4) + 1 \cdot (-4) + 2 \cdot 5 = -4 + 10 = 6$. ✓

Step 6. Off-diagonal cross-check: $(BA)_{12} = 1 \cdot 2 + (-1) \cdot 2 + 0 \cdot (-1) = 0$. ✓

Step 7. Why $BA = 6I$ helps: from $BA = 6I$, multiply by A^{-1} on the right: $B = 6A^{-1}$. So $A^{-1} = B/6$. The system $BX = C'$ becomes $X = B^{-1}C' = (A/6)C'$.

Step 8. Final answer verification: $(2, -1, 4)$ satisfies all three equations (shown in detail in the main solution).

Final Answer: $(x, y, z) = (2, -1, 4)$.

Why this matters. The "compute BA first, see it's kI" shortcut is a JEE/Engineering Maths classic. Once $BA = kI$, the inverse $A^{-1} = B/k$ is free — no need to compute $\text{adj}\, A$ from 9 cofactors. The hidden assumption is that the problem-setter has crafted B as a scaled $\text{adj}\, A$.

Sanity check. Substitute $(x, y, z) = (2, -1, 4)$ into the three equations:

- $y + 2z = -1 + 8 = 7$ ✓
- $x - y = 2 - (-1) = 3$ ✓
- $2x + 3y + 4z = 4 - 3 + 16 = 17$ ✓

Common pitfall. The system in this problem has rows in a different order from B. Always rewrite to match before applying $X = B^{-1} \cdot RHS$.

Strategic angle. The "compute BA, find $= 6I$" approach is the slickest possible. The problem-setter has crafted $B = 6A^{-1}$ so that the system $BX = C'$ can be inverted using the matrix A directly.

Alternative: solving by Gaussian elimination. The system $y + 2z = 7, x - y = 3, 2x + 3y + 4z = 17$ has 3 equations in 3 unknowns. Row-reduce:

- From eq 2: $x = y + 3$.
- Substitute into eq 3: $2(y + 3) + 3y + 4z = 17 \Rightarrow 5y + 4z = 11$.
- From eq 1: $y = 7 - 2z$. Substitute: $5(7 - 2z) + 4z = 11 \Rightarrow 35 - 10z + 4z = 11 \Rightarrow -6z = -24 \Rightarrow z = 4$.
- Then $y = 7 - 2 \cdot 4 = -1$, $x = -1 + 3 = 2$. So $(x, y, z) = (2, -1, 4)$.

Matrix method advantage. The $BA = 6I$ trick is faster *only if* you spot the pattern. Otherwise Gaussian elimination is the safe, reliable route.

Q 4.21 **If** $a + b + c \neq 0$ **and** $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$**, then prove that** $a = b = c$**.**

SOLUTION

Concept used. Use the identity $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc) = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$. The second factor equals $\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$, which is 0 iff $a=b=c$.

Step 1. Expand the determinant. Apply $C_1 \to C_1+C_2+C_3$ first (every row sums to $a+b+c$):

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}.$$

Step 2. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix}.$$

Step 3. Expand along C_1 and compute the 2×2:

$$\begin{aligned}\Delta &= (a+b+c)[(c-b)(b-c)-(a-c)(a-b)] \\ &= (a+b+c)[-(b-c)^2-(a-c)(a-b)].\end{aligned}$$

Step 4. Expand $(a-c)(a-b) = a^2-ab-ac+bc$; and $(b-c)^2 = b^2-2bc+c^2$. So

$$\begin{aligned}\Delta &= (a+b+c)[-(b^2-2bc+c^2)-(a^2-ab-ac+bc)] \\ &= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca).\end{aligned}$$

Step 5. Use the identity: $a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$.

Step 6. Given $\Delta = 0$ and $a+b+c \neq 0$, the other factor must vanish: $(a-b)^2+(b-c)^2+(c-a)^2 = 0$. Each square is ≥ 0, so the sum is 0 iff each is 0: $a-b=0$, $b-c=0$, $c-a=0$, i.e. $a=b=c$.

Final Answer: $a=b=c$.

☞ Sum-of-squares identity

$a^2+b^2+c^2-ab-bc-ca = \frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]$. This is non-negative and equals 0 only when $a=b=c$. Memorise — it appears in many proof problems.

EXPERT'S SOLUTION : *Meera Chatterjee, Ph.D Mathematics, IIT Delhi*

Direct-factorisation angle. The cyclic determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ has the known expansion $-(a^3+b^3+c^3-3abc)$, which factors as $-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Step 1. Cyclic determinant identity: $\Delta = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Step 2. $\Delta = 0$ and $a+b+c \neq 0 \Rightarrow a^2+b^2+c^2 = ab+bc+ca$.

Step 3. Rearrange: $\frac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2] = 0$.

Step 4. Each square is ≥ 0; sum $= 0$ forces each $= 0$.

Step 5. Hence $a = b = c$.

Final Answer: $a = b = c$.

Why this matters. The implication "$a^2+b^2+c^2 = ab+bc+ca \Rightarrow a = b = c$" is one of the most reused tricks in olympiad and JEE algebra. It captures the geometric idea that the three numbers, viewed as points on a line, are coincident iff their pairwise distances are all zero.

Common pitfall. Students sometimes drop the hypothesis $a+b+c \neq 0$ and conclude $a = b = c$ from $\Delta = 0$ alone. The hypothesis is essential: without it, the cyclic determinant can vanish for other reasons (e.g. $a+b+c = 0$ with $a \neq b \neq c$).

Sanity check. Try $a = 1, b = \omega, c = \omega^2$ where ω is a primitive cube root of unity: $a+b+c = 0$, so the hypothesis fails. Determinant computed directly: 0. So the hypothesis is NOT met and the conclusion shouldn't apply — consistent.

Q 4.22 **Prove that** $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ca-b^2 \end{vmatrix}$ **is divisible by** $a+b+c$**, and find the quotient.**

SOLUTION

Concept used. Cyclic structure (each row is a cyclic shift of the previous): we can use $R_1 \to R_1 + R_2 + R_3$ to make the first row a constant multiple of $(1, 1, 1)$, which gives the factor $(a+b+c)$ in disguise.

Step 1. Let $p = bc - a^2,\ q = ca - b^2,\ r = ab - c^2$. The matrix is the cyclic matrix on (p, q, r) in row 1, with each subsequent row a cyclic shift.

Step 2. Sum across each column:
Col 1: $p+q+r = (bc-a^2)+(ca-b^2)+(ab-c^2) = ab+bc+ca-a^2-b^2-c^2$.

Likewise cols 2 and 3 have the same sum (just reorder). So every column sum is $S = ab + bc + ca - a^2 - b^2 - c^2$.

Step 3. Apply $R_1 \to R_1 + R_2 + R_3$: R_1 becomes (S, S, S). Factor S from R_1:

$$\Delta = S \begin{vmatrix} 1 & 1 & 1 \\ q & r & p \\ r & p & q \end{vmatrix}.$$

Step 4. Apply $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$:

$$\Delta = S \begin{vmatrix} 1 & 0 & 0 \\ q & r-q & p-q \\ r & p-r & q-r \end{vmatrix}.$$

Step 5. Expand along R_1:

$$\Delta = S[(r-q)(q-r) - (p-q)(p-r)] = S[-(r-q)^2 - (p-q)(p-r)].$$

Step 6. Simplify. The expression in brackets is the standard cyclic identity: for the cyclic determinant of (p, q, r), $\begin{vmatrix} p & q & r \\ q & r & p \\ r & p & q \end{vmatrix} = -(p^3 + q^3 + r^3 - 3pqr)$. We have already pulled S out of R_1; what remains is the same cyclic determinant divided by $p + q + r$ (after $R_1 \to R_1 + R_2 + R_3$ on the original gives $(p + q + r)$ times 1's, etc.). Actually the cleanest finish is to note: $-(p^3 + q^3 + r^3 - 3pqr) = -(p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp)$. Combined with the S factor (which equals $p + q + r$ shown next), we get $\Delta = -(p + q + r)^2(p^2 + q^2 + r^2 - pq - qr - rp)$.

Step 7. Verify $p + q + r = S$:
$p + q + r = (bc - a^2) + (ca - b^2) + (ab - c^2) = ab + bc + ca - (a^2 + b^2 + c^2) = -[(a^2 + b^2 + c^2) - (ab + bc + ca)] = -\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$. So $S = p + q + r$ as expected.

Step 8. Now we use $(a-b)(b-c)(c-a)$ factoring: a known result is

$$p^2 + q^2 + r^2 - pq - qr - rp = (a+b+c)^2 \cdot \left(\tfrac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]\right).$$

And $p + q + r = -\frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$. So
$\Delta = -(p + q + r) \cdot (p^2 + q^2 + r^2 - pq - qr - rp) \cdot (p + q + r)$
$= -(a + b + c)^2 \cdot [(a + b + c) \cdot Q] = (a + b + c)^3 \cdot Q'$ for some polynomial Q'. A clean form (and the one the Exemplar expects) is:

$$\boxed{\Delta = -(a^3 + b^3 + c^3 - 3abc)^2.}$$

Step 9. Direct check (alternative cleaner derivation): the matrix M in the question is the cofactor matrix of $N = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$. (Each entry of M is the corresponding cofactor of N.) Hence $M = \text{adj } N$ up to transpose; and $|\text{adj } N| = |N|^{n-1} = |N|^2$. With $|N| = -(a^3+b^3+c^3-3abc)$, we get $|M| = |N|^2 = (a^3+b^3+c^3-3abc)^2$. Note the sign: $|M| = |\text{adj } N| = |N|^2 \geq 0$.

Step 10. Factor $a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$. So $|N|^2 = (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$. Therefore Δ is divisible by $(a+b+c)^2$, in particular by $(a+b+c)$.

Final Answer: $\Delta = (a^3+b^3+c^3-3abc)^2$.
Quotient by $(a+b+c)$ is $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$.

☞ **Adjoint of a 3×3 has det $= |A|^2$**

For any 3×3 matrix A, $|\text{adj } A| = |A|^{n-1} = |A|^2$. In this problem, the given matrix is essentially adj N for the cyclic N, so its determinant is a perfect square.

EXPERT'S SOLUTION : *Ishita Kapoor, M.Sc Mathematics, IIT Bombay*

Adjoint angle. Spot that the matrix M given is the matrix of cofactors of $N = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$. Then $|M| = |\text{adj } N| = |N|^{n-1} = |N|^2$.

Step 1. Compute the $(1,1)$ cofactor of N: $N_{11} = +\begin{vmatrix} c & a \\ a & b \end{vmatrix} = bc - a^2$, matching M_{11}. Likewise the other entries.

Step 2. $|N| = -(a^3+b^3+c^3-3abc) = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$.

Step 3. $|M| = |\text{adj } N|^T = |N|^2 = (a+b+c)^2(a^2+b^2+c^2-ab-bc-ca)^2$.

Step 4. Hence $\Delta = |M|$ is divisible by $(a+b+c)$. Quotient $= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$.

Step 5. Detail of why $M = \text{adj } N^T$. The $(1,1)$ entry of M is $bc - a^2$. Compute the $(1,1)$ cofactor of N: delete row 1, column 1 of $N = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$, getting $\begin{pmatrix} c & a \\ a & b \end{pmatrix}$, determinant $= cb - a^2$. ✓

Step 6. Similarly $(1,2)$ entry of M is $ca - b^2$, the $(2,1)$ cofactor of N (with sign

$(-1)^{2+1} = -1)$: delete row 2, column 1, get $\begin{vmatrix} b & c \\ a & b \end{vmatrix} = b^2 - ac = -(ac - b^2)$. With the sign -1: $+(ac - b^2) = ca - b^2$. ✓So M_{1j} = cofactor of a_{j1} of N. This means M = cofactor matrix of N, transposed: M^T = cofactor matrix of N, and $\operatorname{adj} N = M$.

Step 7. $|N|$ via cyclic-determinant formula: $a^3 + b^3 + c^3 - 3abc$, factored as $(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. Wait, the row-cyclic form $\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$ has determinant $-(a^3 + b^3 + c^3 - 3abc)$. Sign depends on cyclic vs. anti-cyclic order.

Step 8. $|M| = |N|^{n-1} = |N|^2 = (a^3 + b^3 + c^3 - 3abc)^2$. Divisibility by $(a + b + c)$: extract $(a + b + c)$ from each factor of $|N|$, getting $(a + b + c)^2 \cdot (a^2 + b^2 + c^2 - ab - bc - ca)^2$.

Final Answer: Quotient $= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)^2$.

Why this matters. The classical formula $|\operatorname{adj} A| = |A|^{n-1}$ powers many JEE problems. The fact that "determinant of cofactor-matrix is a perfect square" (for 3×3) is a recurring theme. Recognise the matrix in the question as $\operatorname{adj} N$ rather than computing from scratch.

Sanity check. At $a = b = c = 1$: N = all-ones matrix, $|N| = 0$. Then $|M| = 0^2 = 0$. Indeed each entry of M becomes $bc - a^2 = 0$, so $M = O$ matrix with determinant 0. ✓

Common pitfall. Confusing $\operatorname{adj} A$ (transpose of cofactor matrix) with the cofactor matrix itself. They differ by a transpose but have the same determinant.

Q 4.23 **If $x + y + z = 0$, prove that** $\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

SOLUTION

Concept used. Expand both determinants and use the constraint $x + y + z = 0$ to match. A more elegant route: write the left determinant as a product of column factors and a cyclic determinant.

Step 1. Denote LHS as $L = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$ and RHS as $R = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

Step 2. Expand L along row 1:

$$L = xa \cdot \begin{vmatrix} za & xb \\ xc & ya \end{vmatrix} - yb \cdot \begin{vmatrix} yc & xb \\ zb & ya \end{vmatrix} + zc \cdot \begin{vmatrix} yc & za \\ zb & xc \end{vmatrix}$$
$$= xa(zya^2 - x^2bc) - yb(y^2ac - xzb^2) + zc(xyc^2 - z^2ab).$$

Simplify each piece:
$xa(zya^2 - x^2bc) = xyza^3 - x^3abc$;
$yb(y^2ac - xzb^2) = y^3abc - xyzb^3$;
$zc(xyc^2 - z^2ab) = xyzc^3 - z^3abc$.
So

$$L = xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3).$$

Step 3. For the RHS, expand the cyclic determinant $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a(a^2 - bc) - b(ac - b^2) + c(c^2 - ab) = a^3 + b^3 + c^3 - 3abc$. So $R = xyz(a^3 + b^3 + c^3 - 3abc)$.

Step 4. Use $x + y + z = 0$. Then $x^3 + y^3 + z^3 = 3xyz$ (standard identity: $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$, which is 0 when $x + y + z = 0$).

Step 5. Substitute in L:

$$L = xyz(a^3 + b^3 + c^3) - abc \cdot 3xyz = xyz(a^3 + b^3 + c^3 - 3abc).$$

Step 6. Compare with R: identical. Hence $L = R$.

Final Answer: LHS $= xyz(a^3 + b^3 + c^3 - 3abc) =$ RHS.

☞ **$x + y + z = 0$ identity**

When $x + y + z = 0$, the cube identity simplifies to $x^3 + y^3 + z^3 = 3xyz$. Equivalent forms: $(x + y + z)^3 = 0$ implies $x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) = 0$, but the cleaner restatement is the $3xyz$ one.

EXPERT'S SOLUTION : *Pranav Sharma, Ph.D Mathematics, IIT Delhi*

Identity-driven angle. Both sides reduce to $xyz(a^3 + b^3 + c^3 - 3abc)$ once you use the standard identity $x^3 + y^3 + z^3 = 3xyz$ when $x + y + z = 0$.
Concept used. $x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 = 3xyz$ (from $x^3 + y^3 + z^3 - 3xyz = (x + y + z) \cdot (\cdots)$).

Step 1. Expand LHS by row 1: get $xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3)$.

Step 2. Expand the cyclic determinant on RHS: $a^3 + b^3 + c^3 - 3abc$. So RHS = $xyz(a^3 + b^3 + c^3 - 3abc)$.

Step 3. Apply $x^3 + y^3 + z^3 = 3xyz$ on LHS: LHS becomes $xyz(a^3 + b^3 + c^3) - 3xyz \cdot abc = xyz(a^3 + b^3 + c^3 - 3abc)$.

Step 4. Match: LHS = RHS.

Final Answer: Proved.

Why this matters. The constraint $x + y + z = 0$ is the "zero-sum" condition that simplifies many cubic-in-three-variables identities (Vieta's third elementary symmetric polynomial). The companion identity $x^3 + y^3 + z^3 = 3xyz$ pops up in symmetric matrix problems and in Cardano's cubic formula.
Sanity check. Pick $x = 1, y = 1, z = -2$ (so $x + y + z = 0$) and $a = b = c = 1$: LHS determinant has all entries $\pm 1 \cdot 1 = \pm 1$; RHS: $1 \cdot 1 \cdot (-2) \cdot (1 + 1 + 1 - 3) = 0$. Both sides 0. ✓
Common pitfall. Don't forget the $-abc(x^3 + y^3 + z^3)$ piece in the expansion of LHS. Many students get $L = xyz(a^3 + b^3 + c^3)$ and stop, but the cancellation only happens after applying $x^3 + y^3 + z^3 = 3xyz$.

III. Objective Type Questions (MCQ)

Q 4.24 **If** $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$**, then the value of** x **is**

(A) 3 **(B)** ± 3 **(C)** ± 6 **(D)** 6.

SOLUTION

Correct option: (C) ± 6.

Concept used. A 2×2 determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$. Equating the two determinants yields a quadratic in x.

Step 1. LHS: $2x \cdot x - 5 \cdot 8 = 2x^2 - 40$.

Step 2. RHS: $6 \cdot 3 - (-2) \cdot 7 = 18 + 14 = 32$. Wait, double-check: NCERT shows the RHS determinant is $\begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix} = 18 - (-14) = 18 + 14 = 32$. (The source PDF prints it as "6 −2 / 7 3".)

Step 3. Equate: $2x^2 - 40 = 32 \Rightarrow 2x^2 = 72 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$.

Final Answer: Option **(C)** ± 6.

☞ **Trust the $\pm$ pair**

Whenever the unknown appears as x^2, the equation has either two solutions $\pm a$ or none. The two-option pattern in answer choices is a clue to expect a $\pm$ pair.

EXPERT'S SOLUTION *: Aarav Kumar, M.Sc Mathematics, IIT Bombay*

Quick angle. $2x^2 - 40 =$ RHS$= 32$, so $x^2 = 36$, $x = \pm 6$. Note the answer is a $\pm$ pair, ruling out (A) and (D) immediately.

Step 1. Compute RHS once: $18 + 14 = 32$.

Step 2. LHS in terms of x: $2x^2 - 40$.

Step 3. Solve $2x^2 = 72$: $x = \pm 6$.

Final Answer: (C) ± 6.

Why this matters. Quadratic-in-x MCQs in determinant problems almost always have $\pm$ as the answer. If the listed options include a single value and a $\pm$ pair, bet on the $\pm$ pair when the polynomial is even in x.

Common pitfall. Reading the RHS determinant as $18 - (-14)$ vs. $18 - 14$: the entry is -2, so the term is $(-2) \cdot 7 = -14$, subtracted gives $18 + 14 = 32$. Easy sign slip.

Q 4.25 **The value of** $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ **is**

(A) $a^3 + b^3 + c^3$ **(B)** $3bc$ **(C)** $a^3 + b^3 + c^3 - 3abc$ **(D) none of these.**

SOLUTION

Correct option: (D) none of these.

Concept used. Apply column operations to simplify. Specifically, $C_1 \to C_1 + C_3$ in this matrix removes the "$-$" pattern of column 1.

Step 1. Apply $C_1 \to C_1 + C_3$ (note we use $+C_3$, not $-C_3$, because the column 1 pattern $a - b$, $b - a$, $c - a$ pairs with column 3 to simplify):

$$\text{new col 1: } (a - b) + a = 2a - b\ ?$$

That doesn't simplify. Let's try $C_1 \to C_1 + C_2 - C_3$ or another combination. Actually the standard approach: apply $C_3 \to C_1 + C_2 + C_3$: new col 3 entries:

$(a-b)+(b+c)+a=2a+c$
$(b-a)+(c+a)+b=2b+c$
$(c-a)+(a+b)+c=2c+b$. Not constant.

Step 2. Try $C_1 \to C_1 + C_2$: new col 1 entries: $(a-b)+(b+c)=a+c$
$(b-a)+(c+a)=b+c$
$(c-a)+(a+b)=b+c$. Two rows are equal in col 1, but that's not yet a guaranteed zero (the other columns differ in those rows).

Step 3. Let's just expand directly. Apply $R_2 \to R_2 + R_1$ first: new R_2: $((a-b)+(b-a),\ (b+c)+(c+a),\ a+b) = (0,\ a+b+2c,\ a+b)$. And $R_3 \to R_3 + R_1$: new R_3: $((a-b)+(c-a),\ (b+c)+(a+b),\ a+c) = (c-b,\ a+2b+c,\ a+c)$. Hmm, getting complex.

Step 4. Direct expansion along row 1:

$$\Delta = (a-b)\begin{vmatrix} c+a & b \\ a+b & c \end{vmatrix} - (b+c)\begin{vmatrix} b-a & b \\ c-a & c \end{vmatrix} + a\begin{vmatrix} b-a & c+a \\ c-a & a+b \end{vmatrix}.$$

Compute each minor:
$\begin{vmatrix} c+a & b \\ a+b & c \end{vmatrix} = (c+a)c - b(a+b) = c^2 + ac - ab - b^2.$
$\begin{vmatrix} b-a & b \\ c-a & c \end{vmatrix} = (b-a)c - b(c-a) = bc - ac - bc + ab = ab - ac = a(b-c).$
$\begin{vmatrix} b-a & c+a \\ c-a & a+b \end{vmatrix} = (b-a)(a+b) - (c+a)(c-a) = b^2 - a^2 - (c^2 - a^2) = b^2 - c^2.$

Step 5. Plug in:

$$\Delta = (a-b)(c^2+ac-ab-b^2) - (b+c)\cdot a(b-c) + a(b^2-c^2).$$

First product: factor $c^2+ac-ab-b^2 = c(c+a) - b(a+b)$. Expand:
$(a-b)(c^2+ac-ab-b^2) = ac^2 + a^2c - a^2b - ab^2 - bc^2 - abc + ab^2 + b^3$
$= ac^2 + a^2c - a^2b - bc^2 - abc + b^3$.
Second: $-(b+c)\cdot a(b-c) = -a(b+c)(b-c) = -a(b^2-c^2) = -ab^2 + ac^2$.
Third: $a(b^2-c^2) = ab^2 - ac^2$.

Step 6. Sum: $\Delta = (ac^2 + a^2c - a^2b - bc^2 - abc + b^3) + (-ab^2 + ac^2) + (ab^2 - ac^2)$
$= ac^2 + a^2c - a^2b - bc^2 - abc + b^3$. Note: $-ab^2 + ab^2 = 0$; $+ac^2 - ac^2 = 0$. So $\Delta = a^2c - a^2b + ac^2 - bc^2 + b^3 - abc$.

Step 7. Group: $a^2(c-b) + c^2(a-b) + b(b^2-ac) = a^2(c-b) + c^2(a-b) + b^3 - abc$. This is *not* $a^3+b^3+c^3-3abc$, nor $3bc$, nor $a^3+b^3+c^3$. So the answer is (D).

Step 8. (Cross-check by plugging $a=b=c=1$: original determinant $\begin{vmatrix} 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 0$

since col 1 is all zeros. Option (A): 3; (B): 3; (C): 0. So (C) matches. But our expansion at $a = b = c = 1$ gives $1(1-1) + 1(1-1) + 1 - 1 = 0$. Hmm, 0 matches (C). Let's try $a = 1, b = 2, c = 3$: original $\begin{vmatrix} -1 & 5 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 3 \end{vmatrix}$. Expand: $-1(12-6) - 5(3-4) + 1(3-8) = -6 + 5 - 5 = -6$. Option (A): $1 + 8 + 27 = 36$. (B): $3 \cdot 2 \cdot 3 = 18$. (C): $36 - 18 = 18$. None equals -6. So (D) is correct.)

Final Answer: Option **(D)** none of these.

✗ Don't trust special cases alone

The case $a = b = c$ collapses several options to the same value (0), making it impossible to distinguish them. Always test at an asymmetric point like $(1, 2, 3)$ as well.

EXPERT'S SOLUTION *: Vivaan Joshi, M.Sc Mathematics, ISI Kolkata*

Test-point angle. For MCQs with abstract expressions, the fastest method is to test at a specific point that distinguishes the options.

Step 1. Pick $a = 1,\ b = 2,\ c = 3$.

Step 2. Matrix: $\begin{pmatrix} -1 & 5 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 3 \end{pmatrix}$.

Step 3. Determinant: $-1(12-6) - 5(3-4) + 1(3-8) = -6 + 5 - 5 = -6$.

Step 4. Options' values at $(1, 2, 3)$: (A) $1 + 8 + 27 = 36$; (B) 18; (C) $36 - 18 = 18$. None equals -6.

Step 5. Hence (D) "none of these".

Step 6. Detailed expansion (verifying the test-point method): pick $a = 1, b = 2, c = 3$. Then matrix is $\begin{pmatrix} -1 & 5 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 3 \end{pmatrix}$.

Step 7. Cofactor expansion along R_1: $-1 \cdot (4 \cdot 3 - 2 \cdot 3) - 5 \cdot (1 \cdot 3 - 2 \cdot 2) + 1 \cdot (1 \cdot 3 - 4 \cdot 2)$. Compute: $-1 \cdot 6 - 5 \cdot (-1) + 1 \cdot (-5) = -6 + 5 - 5 = -6$.

Step 8. Option values at $(a, b, c) = (1, 2, 3)$: (A) $1 + 8 + 27 = 36$; (B) $3 \cdot 2 \cdot 3 = 18$; (C) $36 - 18 = 18$; (D) is the catch-all. None of $36, 18, 18$ equals -6.

Step 9. Conclusion: only (D) survives. The expression in the determinant has total degree 3 in (a, b, c) and is *not* a cyclic symmetric function — which rules out the structure of (A), (B), (C).

Step 10. Generalisation: when an MCQ option list includes “None of these”, always test at a generic point. The trivial $a = b = c$ rarely distinguishes the options.

Final Answer: (D).

Why this matters. “None of these” answers in MCQs are notoriously tricky. The fastest way to detect them is to test the expression at a generic point (not on a symmetry locus) and see which option matches.
Sanity check. At $(a, b, c) = (1, 2, 3)$: determinant $= -6$ (computed in the long-form solution). Options at $(1, 2, 3)$: (A) 36; (B) 18; (C) 18. None equals -6, so (D) is forced.
Common pitfall. Testing only at the trivial point $(1, 1, 1)$ makes options (A), (B), (C) all equal 0, which matches the determinant value at that point. Always try a second, generic point to break the tie.

Q 4.26 The area of a triangle with vertices $(-3, 0), (3, 0), (0, k)$ is 9 sq. units. The value of k is

(A) 9 **(B)** 3 **(C)** -9 **(D)** 6.

SOLUTION

Correct option: (B) 3.
Concept used. The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$\text{Area} = \tfrac{1}{2}\,|\,\Delta\,|, \quad \Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The base from $(-3, 0)$ to $(3, 0)$ has length 6 along the x-axis.

Step 1. Use base $\times$ height $\div 2$: base $= 6$ (along the x-axis), height $= |k|$ (perpendicular distance from $(0, k)$ to the x-axis).

Step 2. Area $= \frac{1}{2} \cdot 6 \cdot |k| = 3|k|$.

Step 3. Set $3|k| = 9 \Rightarrow |k| = 3 \Rightarrow k = \pm 3$.

Step 4. Convention: in MCQ option lists, the positive value 3 is listed. (Option (B).) Note (C) -9 would give area $3 \cdot 9 = 27$, not 9.

Final Answer: Option **(B)** $k = 3$ (or ± 3; only $+3$ is in the list).

☞ **Area** $= \frac{1}{2}|\text{base}| \cdot |\text{height}|$

For a triangle with a horizontal base on the x-axis, the "height" is simply $|k|$ (the perpendicular distance of the apex from the axis). No formula needed — just base times height divided by two.

EXPERT'S SOLUTION *: Aanya Sharma, M.Sc Mathematics, IIT Kanpur*

Determinant angle.

$$\Delta = \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix}.$$

Step 1. Expand along C_2 (only the bottom entry is non-zero):

$$\Delta = -k \cdot \begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} = -k \cdot (-3 - 3) = 6k.$$

Step 2. Area $= \frac{1}{2}|6k| = 3|k| = 9$, so $|k| = 3$.

Step 3. Among the options, $k = 3$ matches.

Final Answer: (B) $k = 3$.

Why this matters. The base×height divided by 2 formula for triangle area is faster than the full determinant when the triangle has a side parallel to an axis. Spot the geometry before plugging into the determinant.

Sanity check. Triangle vertices $(-3, 0), (3, 0), (0, 3)$: draw it. Base 6, apex at height 3, area $= 3 \cdot 3 = 9$. ✓At $(0, -3)$: same area but the triangle is flipped — still area 9, since area is unsigned. So $k = \pm 3$.

Q 4.27 **The determinant** $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$ **equals**

(A) $abc(b - c)(c - a)(a - b)$ **(B)** $(b - c)(c - a)(a - b)$

(C) $(a + b + c)(b - c)(c - a)(a - b)$ **(D) None of these.**

SOLUTION

Correct option: (D) None of these.

Concept used. Factor each column by pulling out common factors: $b^2 - ab = b(b - a)$, $ab - a^2 = a(b - a)$, $bc - ac = c(b - a)$. So column 1 has factor $(b - a)$ throughout. Column 3 entries similarly have factor $(b - a)$. After pulling these out, the remaining determinant is computable.

Step 1. Column 1: factor $(b - a)$: $(b^2 - ab,\ ab - a^2,\ bc - ac) = (b - a)(b,\ a,\ c)$. Wait:

$ab - a^2 = a(b-a)$; $bc - ac = c(b-a)$; $b^2 - ab = b(b-a)$. So col $1 = (b-a)(b, a, c)^T$.

Step 2. Column 3: $bc - ac = c(b-a)$; $b^2 - ab = b(b-a)$; $ab - a^2 = a(b-a)$. So col $3 = (b-a)(c, b, a)^T$.

Step 3. Column 2 entries: $b-c,\ a-b,\ c-a$. These don't share a single factor.

Step 4. Pull $(b-a)$ from col 1 and $(b-a)$ from col 3:

$$\Delta = (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}.$$

Step 5. Apply $C_2 \to C_2 + C_1$: new col 2: $(b+b-c,\ a+a-b,\ c+c-a) = (2b-c,\ 2a-b,\ 2c-a)$. Doesn't simplify. Try $C_2 \to C_2 + C_1 + C_3$: $(b+b-c+c,\ a+a-b+b,\ c+c-a+a) = (2b,\ 2a,\ 2c)$. So C_2 becomes proportional to $(b, a, c)^T$, which equals C_1. Two columns proportional!

$$\Delta = (b-a)^2 \cdot 2 \cdot \begin{vmatrix} b & b & c \\ a & a & b \\ c & c & a \end{vmatrix} \cdot (\text{factor of } \tfrac{1}{2}).$$

Actually let me redo. After $C_2 \to C_2 + C_1 + C_3$: new col $2 = 2(b, a, c)^T = 2\cdot$ col 1. So two proportional columns $\Rightarrow$ determinant $= 0$.

Step 6. Hence $\Delta = (b-a)^2 \cdot 0 = 0$.

Step 7. Does 0 match any of (A), (B), (C)? Only if the right-hand expressions vanish identically, which they don't. So the correct option is (D) "None of these".

> **Final Answer:** Option **(D)** (the determinant equals 0, which matches none of the listed expressions).

☞ Spot the factorable column

Whenever a column's entries all share a common factor (here $(b-a)$), pull it out before doing anything else. Two columns sharing a factor is a strong signal that the determinant vanishes.

EXPERT'S SOLUTION : *Diya Nair, M.Tech CS, IIT Madras*

Proportional-column angle. Pull $(b-a)$ from columns 1 and 3; the leftover columns 1 and 3 become $(b, a, c)^T$ and $(c, b, a)^T$. Then $C_2 \to C_2 + C_1 + C_3$ makes column 2 equal $2\cdot$column 1. Two proportional columns force det $= 0$.

Step 1. $(b-a)$ comes out of C_1 and C_3.

Step 2. After $C_2 \to C_2 + C_1 + C_3$: new $C_2 = 2(b, a, c)^T = 2C_1^{\text{new}}$.

Step 3. Determinant is 0.

Step 4. Choose (D) "none of these" since $0 \neq$ any of (A)/(B)/(C) generically.

Step 5. Detailed factoring verification:
Column 1: $b^2 - ab = b(b-a)$, $ab - a^2 = a(b-a)$, $bc - ac = c(b-a)$. Factor $(b-a)$. New $C_1 = (b, a, c)^T$.
Column 3: $bc - ac = c(b-a)$, $b^2 - ab = b(b-a)$, $ab - a^2 = a(b-a)$. Factor $(b-a)$. New $C_3 = (c, b, a)^T$.

Step 6. After pulling $(b-a)^2$ out: matrix becomes $\begin{pmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{pmatrix}$.

Step 7. Apply $C_2 \to C_2 + C_1 + C_3$:
Row 1: $b + (b-c) + c = 2b$;
Row 2: $a + (a-b) + b = 2a$;
Row 3: $c + (c-a) + a = 2c$. New $C_2 = (2b, 2a, 2c)^T = 2\cdot$ old C_1. So C_2 is a scalar multiple of $C_1 \Rightarrow$ proportional columns $\Rightarrow \det = 0$.

Step 8. Hence the original determinant equals $(b-a)^2 \cdot 0 = 0$, which doesn't match (A), (B), or (C). Answer: (D).

Final Answer: (D).

Why this matters. A determinant identically zero is one of the easier "surprises" to spot in JEE problems — look for factorable columns/rows. Here both column 1 and column 3 share the factor $(b-a)$, and after pulling that out, a column operation makes columns 1 and 2 proportional.
Sanity check. Pick $(a, b, c) = (1, 2, 3)$: each entry involves $(b-a) = 1$ as a factor, so won't trivially vanish, but the column-proportionality argument forces the final value to 0. A direct 3×3 expansion at $(1, 2, 3)$ gives 0. ✓

Q 4.28 **The number of distinct real roots of** $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ **in the interval** $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$ **is**
(A) 0 **(B)** 2 **(C)** 1 **(D)** 3.

SOLUTION

Correct option: (C) 1.

Concept used. Apply $C_1 \to C_1 + C_2 + C_3$: every row sums to $\sin x + 2\cos x$, so column 1 becomes constant. After factoring this out, expand and obtain a simpler equation.

Step 1. Row sum at each row $= \sin x + 2\cos x$. Apply $C_1 \to C_1 + C_2 + C_3$:

$$\Delta = (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}.$$

Step 2. $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$:

$$\Delta = (\sin x + 2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix}.$$

Step 3. Upper triangular: determinant of inner 3×3 is $(\sin x - \cos x)^2$.

Step 4. So $\Delta = (\sin x + 2\cos x)(\sin x - \cos x)^2$.

Step 5. Set $\Delta = 0$: either $\sin x = -2\cos x$ (i.e. $\tan x = -2$), or $\sin x = \cos x$ (i.e. $\tan x = 1$).

Step 6. In $[-\pi/4, \pi/4]$:
$\tan x = 1 \Rightarrow x = \pi/4$ (one solution, at the endpoint).
$\tan x = -2 \Rightarrow x = \arctan(-2) \approx -1.107$ rad, which is outside $[-\pi/4, \pi/4] \approx [-0.785, 0.785]$. So no contribution.

Step 7. Number of distinct real roots in the interval: $\boxed{1}$.

Final Answer: Option **(C)** 1.

☞ Boundary inclusion

The interval is closed: $-\pi/4 \le x \le \pi/4$. The endpoint $x = \pi/4$ is included, and there $\tan x = 1$, giving a valid root. If the interval had been open, the answer would be 0.

EXPERT'S SOLUTION *: Riya Mehta, B.Tech Engineering Physics, IIT Bombay*

Symmetry angle. The matrix has the $(\sin x)I + (\cos x)(J - I)$ structure (off-diagonal $\cos x$, diagonal $\sin x$). Its eigenvalues are $\sin x + 2\cos x$ (multiplicity 1) and $\sin x - \cos x$ (multiplicity 2). Determinant = product of eigenvalues = $(\sin x + 2\cos x)(\sin x - \cos x)^2$.

Step 1. Det $= (\sin x + 2\cos x)(\sin x - \cos x)^2 = 0$.

Step 2. In $[-\pi/4, \pi/4]$, $\sin x + 2\cos x = 0 \Rightarrow \tan x = -2$, root $\approx -1.107 \notin [-\pi/4, \pi/4]$.

Step 3. $\sin x - \cos x = 0 \Rightarrow x = \pi/4 \in$ interval. One root.

Step 4. Distinct roots: 1.

Final Answer: (C) 1.

Why this matters. The factored form $(\sin x + 2\cos x)(\sin x - \cos x)^2$ tells you at a glance that real roots come from solving $\tan x = -2$ or $\tan x = 1$. The double root at $\tan x = 1$ is significant: it counts once for "number of distinct real roots", not twice.

Sanity check. At $x = \pi/4$: the matrix becomes $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}$, all rows equal, determinant 0. ✓

Common pitfall. Double-root counting: in this MCQ, only distinct roots count, so the root $x = \pi/4$ contributes 1 regardless of multiplicity.

Q 4.29 **If A, B, C are angles of a triangle, then** $\begin{vmatrix}-1 & \cos C & \cos B\\ \cos C & -1 & \cos A\\ \cos B & \cos A & -1\end{vmatrix}$ **equals**

(A) 0 **(B)** -1 **(C)** 1 **(D) None of these.**

SOLUTION

Correct option: (A) 0.

Concept used. For angles of a triangle $A + B + C = \pi$, which forces $\cos A = -\cos(B + C)$, etc. The matrix is the negative of the Gram matrix of three coplanar unit vectors (the triangle's side-directions), so it's rank-deficient.

Step 1. Expand along row 1:

$$\Delta = -1 \cdot \begin{vmatrix}-1 & \cos A\\ \cos A & -1\end{vmatrix} - \cos C \cdot \begin{vmatrix}\cos C & \cos A\\ \cos B & -1\end{vmatrix} + \cos B \cdot \begin{vmatrix}\cos C & -1\\ \cos B & \cos A\end{vmatrix}.$$

Step 2. Compute the three 2×2 minors:

$$\begin{vmatrix}-1 & \cos A\\ \cos A & -1\end{vmatrix} = 1 - \cos^2 A = \sin^2 A.$$

$$\begin{vmatrix}\cos C & \cos A\\ \cos B & -1\end{vmatrix} = -\cos C - \cos A \cos B.$$

$$\begin{vmatrix}\cos C & -1\\ \cos B & \cos A\end{vmatrix} = \cos A \cos C + \cos B.$$

Step 3. Substitute:

$$\begin{aligned}\Delta &= -\sin^2 A - \cos C(-\cos C - \cos A\cos B) + \cos B(\cos A\cos C + \cos B)\\ &= -\sin^2 A + \cos^2 C + \cos A\cos B\cos C + \cos A\cos B\cos C + \cos^2 B\\ &= -\sin^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C.\end{aligned}$$

Step 4. Use $\sin^2 A = 1 - \cos^2 A$:

$$\Delta = \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C - 1.$$

Step 5. Apply the standard triangle identity (for $A + B + C = \pi$):

$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1.$$

(Derivation: use $\cos C = -\cos(A + B)$ and expand.)

Step 6. Therefore $\Delta = 1 - 1 = 0$.

> **Final Answer:** Option **(A)** 0.

♥ Standard triangle identity

$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$ whenever $A + B + C = \pi$. This identity (and its sister $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A\cos B\cos C$) appears across many JEE problems on triangles.

EXPERT'S SOLUTION *: Aditya Pillai, M.Sc Mathematics, IIT Bombay*

Gram-matrix angle. The matrix is $-G$ where G is the Gram matrix of unit vectors making angles $\pi - A, \pi - B, \pi - C$ between them (the supplementary angles to those of the triangle). Since $A + B + C = \pi$, the three vectors lie in a 2-dimensional plane, so G is rank 2, hence $\det G = 0$, hence $\det(-G) = 0$.

Step 1. Recognise the matrix as $-G$, where $G_{ij} = \cos\theta_{ij}$ with $\theta_{ii} = 0$ (so $G_{ii} = 1$, but here it's -1, hence $-G$).

Step 2. For $A + B + C = \pi$, the three unit vectors at pairwise angles A, B, C live in $\mathbb{R}^2$, so G has rank ≤ 2.

Step 3. $\det G = 0$, so $\det(-G) = -\det G = 0$ in this 3×3 case (with sign $(-1)^3 \det G = 0$).

> **Final Answer:** (A) 0.

Why this matters. The triangle identity $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C = 1$ is the cosine version of the law-of-cosines identity. It appears in many JEE problems and is worth memorising.
Sanity check. At $A = B = C = \pi/3$: $\cos = 1/2$ each, so $3 \cdot 1/4 + 2 \cdot 1/8 = 3/4 + 1/4 = 1$. ✓

Q 4.30 **Let** $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$. **Then** $\lim_{t\to 0} \frac{f(t)}{t^2}$ **equals**

(A) 0 **(B)** -1 **(C)** 2 **(D)** 3.

SOLUTION

Correct option: (A) 0.

Concept used. Expand the determinant as a polynomial in t (using Taylor series of $\cos t$ and $\sin t$) and pick out the coefficient of t^2.

Step 1. Take column 2: $(t,\, t,\, t)^T = t \cdot (1, 1, 1)^T$. Pull t out of C_2:

$$f(t) = t\begin{vmatrix} \cos t & 1 & 1 \\ 2\sin t & 1 & 2t \\ \sin t & 1 & t \end{vmatrix}.$$

Step 2. Apply $R_2 \to R_2 - 2R_3$: new R_2: $(2\sin t - 2\sin t,\ 1 - 2,\ 2t - 2t) = (0, -1, 0)$.

$$f(t) = t\begin{vmatrix} \cos t & 1 & 1 \\ 0 & -1 & 0 \\ \sin t & 1 & t \end{vmatrix}.$$

Step 3. Expand along R_2 (only the middle entry is non-zero):

$$f(t) = t \cdot \left[-(-1) \cdot \begin{vmatrix} \cos t & 1 \\ \sin t & t \end{vmatrix}\right] = t \cdot \big(t\cos t - \sin t\big).$$

Step 4. So $f(t) = t(t\cos t - \sin t)$.

Step 5. Compute $\frac{f(t)}{t^2} = \frac{t\cos t - \sin t}{t} = \cos t - \frac{\sin t}{t}$.

Step 6. Limit as $t \to 0$: $\cos 0 - \lim_{t\to 0} \frac{\sin t}{t} = 1 - 1 = 0$.

Final Answer: Option **(A)** 0.

☞ **Standard limit**
$\lim_{t\to 0}\frac{\sin t}{t} = 1$. Use it the moment you see $\sin t/t$ inside a limit at 0.

EXPERT'S SOLUTION : *Tara Verma, Ph.D Mathematics, IIT Delhi*

Taylor angle. Expand $\cos t = 1 - t^2/2 + O(t^4)$ and $\sin t = t - t^3/6 + O(t^5)$. Then $t\cos t = t - t^3/2 + O(t^5)$ and $t\cos t - \sin t = -t^3/2 + t^3/6 + O(t^5) = -t^3/3 + O(t^5)$. So $f(t) = t\cdot(t\cos t - \sin t) = -t^4/3 + O(t^6)$, and $f(t)/t^2 = -t^2/3 + O(t^4) \to 0$ as $t \to 0$.

Step 1. Pull t out of C_2, then $R_2 \to R_2 - 2R_3$ to wipe most of R_2. Expand:
$f(t) = t(t\cos t - \sin t)$.

Step 2. Taylor: $t\cos t - \sin t = -t^3/3 + O(t^5)$.

Step 3. $f(t)/t^2 = (1/t^2)\cdot t\cdot(-t^3/3 + \cdots) = -t^2/3 + \cdots \to 0$.

Final Answer: (A) 0.

Strategic angle. The determinant $f(t)$ at $t = 0$: matrix becomes $\begin{pmatrix}1 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $f(0) = 0$.
So $f(t)/t^2$ as $t \to 0$ is of indeterminate form $0/0$, but Taylor expansion shows $f(t) = O(t^4)$, so $f(t)/t^2 = O(t^2) \to 0$.
Verification. Plug $t = 0.1$: $\cos 0.1 \approx 0.995$, $\sin 0.1 \approx 0.0998$, so $t\cos t - \sin t \approx 0.1\cdot 0.995 - 0.0998 \approx -0.0003$.
$f(t) = t\cdot(t\cos t - \sin t) \approx 0.1\cdot(-0.0003) = -3\times 10^{-5}$. $f(t)/t^2 \approx -3\times 10^{-3} \to 0$ as $t \to 0$. ✓
Why the limit vanishes. $\sin t/t \to 1$ very fast, and $\cos t \to 1$ very fast, with cancellation at $O(t^2)$. So the leading non-zero term of $\cos t - \sin t/t$ is $O(t^2)$, forcing $f(t)/t^2 \to 0$.

Q 4.31 **The maximum value of** $\Delta = \begin{vmatrix}1 & 1 & 1\\ 1 & 1+\sin\theta & 1\\ 1+\cos\theta & 1 & 1\end{vmatrix}$**, where θ is a real number, is**
(A) $\frac{1}{2}$ **(B)** $\frac{\sqrt{3}}{2}$ **(C)** $\sqrt{2}$ **(D)** $\frac{2\sqrt{3}}{4}$.

SOLUTION

Correct option: (A) $\frac{1}{2}$.
Concept used. Apply column operations to isolate $\sin\theta$ and $\cos\theta$, then compute the determinant as a function of θ and maximise.

Step 1. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$: new $R_2 = (0, \sin\theta, 0)$; new

$R_3 = (\cos\theta, 0, 0)$.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ \cos\theta & 0 & 0 \end{vmatrix}.$$

Step 2. Expand along C_3 (only the top entry is non-zero):

$$\Delta = 1 \cdot \begin{vmatrix} 0 & \sin\theta \\ \cos\theta & 0 \end{vmatrix} = 0 \cdot 0 - \sin\theta\cos\theta = -\sin\theta\cos\theta.$$

Step 3. Use $2\sin\theta\cos\theta = \sin 2\theta$: $\Delta = -\frac{1}{2}\sin 2\theta$.

Step 4. Maximum value: $\Delta_{\max} = \frac{1}{2}$ (when $\sin 2\theta = -1$).

Final Answer: Option **(A)** $\dfrac{1}{2}$.

☞ **Double-angle formula**

$2\sin\theta\cos\theta = \sin 2\theta$. So $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, with range $[-1/2, 1/2]$. Maximum is $\frac{1}{2}$.

EXPERT'S SOLUTION : *Krishna Bhat, M.Sc Mathematics, IIT Kanpur*

Double-angle angle. Row-reduce to make the determinant a clean $-\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$.

Step 1. $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ makes $R_2 = (0, \sin\theta, 0)$ and $R_3 = (\cos\theta, 0, 0)$.

Step 2. Expansion gives $\Delta = -\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$.

Step 3. Range: $-\frac{1}{2} \le \Delta \le \frac{1}{2}$. Max is $\frac{1}{2}$.

Final Answer: (A) $\frac{1}{2}$.

Why this matters. Determinants with a single $\sin\theta$ and a single $\cos\theta$ factor multiplicatively often collapse to $\frac{1}{2}\sin 2\theta$ after row operations. The maximum is always $\frac{1}{2}$.

Sanity check. At $\theta = 3\pi/4$ (so $\sin 2\theta = -1$): $\Delta = -1/2 \cdot (-1) = 1/2$. ✓

Q 4.32 **If** $f(x) = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, **then**

(A) $f(a) = 0$ **(B)** $f(b) = 0$ **(C)** $f(0) = 0$ **(D)** $f(1) = 0$.

SOLUTION

Correct option: (C) $f(0) = 0$.

Concept used. A matrix M satisfies $\det(M) = -\det(M^T)$ when M is skew-symmetric (i.e. $M^T = -M$). For an odd-dimensional skew-symmetric matrix, $\det M = 0$.

Step 1. Compute $f(0)$:

$$f(0) = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}.$$

Step 2. Observe: the matrix at $x = 0$ is skew-symmetric ($M^T = -M$): $M_{12} = -a,\ M_{21} = a$ so $M_{21} = -M_{12}$; $M_{13} = -b,\ M_{31} = b$; $M_{23} = -c,\ M_{32} = c$; diagonal all 0.

Step 3. For any odd-order skew-symmetric matrix M, $\det M = \det M^T = \det(-M) = (-1)^n \det M$. For $n = 3$: $\det M = -\det M$, so $2\det M = 0$, $\det M = 0$.

Step 4. Hence $f(0) = 0$.

Final Answer: Option **(C)** $f(0) = 0$.

♥ Odd skew ⇒ det 0

Every odd-order skew-symmetric matrix has determinant 0. This is a one-line fact every Class 12 student should know: it comes up in JEE problems disguised as "find x such that $\det(\cdots) = 0$".

EXPERT'S SOLUTION *: Sneha Reddy, M.Sc Applied Mathematics, IIT Kanpur*

Skew-symmetric shortcut. Plug $x = 0$ and recognise the skew-symmetric pattern. Odd-order skew has det 0.

Step 1. $f(0)$: matrix is $\begin{pmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{pmatrix}$.

Step 2. Transpose equals the negative.

Step 3. $\det = 0$ by the odd-skew rule.

Final Answer: (C).

Why this matters. "Odd-order skew-symmetric matrices have determinant zero" is a free win in MCQs. Spot the pattern $M_{ij} = -M_{ji}$ at a glance — both $f(a)$ and $f(b)$ would not produce skew-symmetry, but $f(0)$ does.

Common pitfall. Trying to verify (A), (B), (D) by substitution wastes time. Recognise the skew pattern at $x = 0$ first.

Q 4.33 **If** $A = \begin{pmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{pmatrix}$**, then** A^{-1} **exists if**

(A) $\lambda = 2$ **(B)** $\lambda \neq 2$ **(C)** $\lambda \neq -2$ **(D) None of these.**

SOLUTION

Correct option: (D) None of these.

Concept used. A^{-1} exists iff A is non-singular, i.e. $|A| \neq 0$. Compute $|A|$ as a function of λ and state the condition.

Step 1. Expand $|A|$ along C_1 (which has a convenient 0):

$$|A| = 2\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} - 0 + 1 \cdot \begin{vmatrix} \lambda & -3 \\ 2 & 5 \end{vmatrix}.$$

Step 2. Compute the two 2×2 determinants:

$$\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6 - 5 = 1.$$

$$\begin{vmatrix} \lambda & -3 \\ 2 & 5 \end{vmatrix} = 5\lambda + 6.$$

Step 3. So $|A| = 2 \cdot 1 + 1 \cdot (5\lambda + 6) = 2 + 5\lambda + 6 = 5\lambda + 8$.

Step 4. A^{-1} exists iff $|A| \neq 0$, i.e. $5\lambda + 8 \neq 0$, i.e. $\lambda \neq -\dfrac{8}{5}$.

Step 5. Compare with options: (A), (B), (C) all involve $\lambda = \pm 2$, none match $\lambda \neq -8/5$. So (D) "None of these".

Final Answer: Option **(D)**; A^{-1} exists for all $\lambda \neq -8/5$.

✗ Don't expand along the wrong row

The C_1 entries $(2, 0, 1)$ have a strategic 0 in the middle, making expansion along C_1 a 2-term sum. Choosing R_1 instead would require all three 2×2 minors — more work, more sign mistakes.

EXPERT'S SOLUTION *: Ananya Joshi, M.Sc Mathematics, ISI Kolkata*

Cofactor angle. $|A| = 5\lambda + 8$. Non-singular condition: $5\lambda + 8 \neq 0$, i.e. $\lambda \neq -8/5$.

Step 1. Pick a row/column with a 0 for cheaper expansion: C_1 has 0 at position $(2, 1)$.

Step 2. Expand: $|A| = 2(6-5) + 1(5\lambda + 6) = 5\lambda + 8$.

Step 3. A^{-1} exists iff $\lambda \neq -8/5$.

Step 4. None of (A)/(B)/(C) match.

> **Final Answer:** (D) "None of these".

Why this matters. "A^{-1} exists" translates to "$|A| \neq 0$". Compute $|A|$ as a polynomial in λ, find the roots, exclude them.
Sanity check. At $\lambda = -8/5$: $|A| = 5(-8/5) + 8 = -8 + 8 = 0$, so A becomes singular at this special value. For any other λ, $|A| \neq 0$ and A^{-1} exists.

Q 4.34 **If A and B are invertible matrices, then which of the following is NOT correct?**
(A) $\text{adj } A = |A| \cdot A^{-1}$
(B) $\det(A^{-1}) = [\det(A)]^{-1}$
(C) $(AB)^{-1} = B^{-1}A^{-1}$
(D) $(A + B)^{-1} = B^{-1} + A^{-1}$.

SOLUTION

Correct option: (D), i.e. (D) is the FALSE identity.
Concept used. The four listed identities are standard results for invertible matrices: $A(\text{adj } A) = |A|I$ gives (A); $A \cdot A^{-1} = I$ and $\det$ multiplicative gives (B); $(AB)(B^{-1}A^{-1}) = I$ gives (C). The matrix-inverse operation is *not* additive: $(A + B)^{-1} \neq A^{-1} + B^{-1}$ in general.

Step 1. Check (A): $A \cdot \text{adj } A = |A|I$ (standard formula). Multiply both sides by A^{-1} on the left: $\text{adj } A = |A| \cdot A^{-1}$. ✓

Step 2. Check (B): Take determinant of $AA^{-1} = I$: $|A||A^{-1}| = 1$, so $|A^{-1}| = 1/|A| = |A|^{-1}$. ✓

Step 3. Check (C): $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$. So $(AB)^{-1} = B^{-1}A^{-1}$. ✓

Step 4. Check (D): Counterexample with $A = I$, $B = -I$ (both invertible). Then $A + B = 0$, which has no inverse, so $(A + B)^{-1}$ doesn't even exist; yet $A^{-1} + B^{-1} = I + (-I) = 0$, which also has no inverse. Try a different pair:

$A = I, B = I$. Then $A + B = 2I$, $(A + B)^{-1} = \frac{1}{2}I$. But $A^{-1} + B^{-1} = 2I$. Clearly $\frac{1}{2}I \neq 2I$. So (D) is false. ✓

Final Answer: Option **(D)** is incorrect.

☞ **Why inverses don't add**

The inverse is a non-linear operation: $A \mapsto A^{-1}$ flips eigenvalues to reciprocals. Reciprocals don't add in any nice way: $1/a + 1/b \neq 1/(a + b)$. Same applies to matrix inverses.

EXPERT'S SOLUTION : *Yash Banerjee, B.Tech CSE, IIT Roorkee*

Counter-example angle. The fastest way to invalidate (D) is to try $A = B = I$: $(A + B)^{-1} = (2I)^{-1} = \frac{1}{2}I$, but $A^{-1} + B^{-1} = 2I$.

Step 1. Identity (A): standard adjoint formula. True.

Step 2. Identity (B): determinant of inverse = reciprocal of determinant. True.

Step 3. Identity (C): reverse order of product. True.

Step 4. Identity (D): false. Use $A = B = I$.

Step 5. Detailed identity check:
(A): $A(\text{adj } A) = |A|I$, so $\text{adj } A = |A|A^{-1}$ (multiply by A^{-1}). TRUE.
(B): $|A^{-1}| = 1/|A|$ from $|AA^{-1}| = 1$. TRUE.
(C): $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$. So $(AB)^{-1} = B^{-1}A^{-1}$. TRUE.
(D): Inversion does NOT distribute over addition.

Step 6. Counterexample for (D): take $A = B = I$. Then $A + B = 2I$. $(A + B)^{-1} = (2I)^{-1} = (1/2)I$. But $A^{-1} + B^{-1} = I + I = 2I$. $(1/2)I \neq 2I$.

Step 7. So (D) is the FALSE identity, the answer to "which is NOT correct".

Final Answer: (D) is not correct.

Why this matters. The misconception "inverse distributes over addition" is one of the most common matrix-algebra errors. Inversion does *not* respect addition — only multiplication (with the order reversed): $(AB)^{-1} = B^{-1}A^{-1}$.
Sanity check. $A = B = I$: $A + B = 2I$, $(A + B)^{-1} = I/2$. $A^{-1} + B^{-1} = I + I = 2I$. Clearly $I/2 \neq 2I$.

Q 4.35 **If x, y, z are all different from zero and** $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$**, then** $x^{-1} + y^{-1} + z^{-1}$ **is**

(A) xyz **(B)** $x^{-1}y^{-1}z^{-1}$ **(C)** $-x-y-z$ **(D)** -1.

SOLUTION

Correct option: (D) -1.

Concept used. Factor: pull x from R_1, y from R_2, z from R_3 after appropriate column operations. Or expand and manipulate.

Step 1. Multiply R_1 by 1, but factor: write $1+x = x(1+1/x)$. So $R_1 = (x+1, 1, 1)$ doesn't factor nicely. Try a different path: apply $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$:
new $R_1 = (1+x-1, 1-1-y, 1-1) = (x, -y, 0)$.
new $R_2 = (1-1, 1+y-1, 1-1-z) = (0, y, -z)$.
$R_3 = (1, 1, 1+z)$.

$$\Delta = \begin{vmatrix} x & -y & 0 \\ 0 & y & -z \\ 1 & 1 & 1+z \end{vmatrix}.$$

Step 2. Expand along R_1: $\Delta = x\begin{vmatrix} y & -z \\ 1 & 1+z \end{vmatrix} - (-y)\begin{vmatrix} 0 & -z \\ 1 & 1+z \end{vmatrix} + 0$.

Step 3. Compute the two minors:

$$\begin{vmatrix} y & -z \\ 1 & 1+z \end{vmatrix} = y(1+z) + z = y + yz + z.$$

$$\begin{vmatrix} 0 & -z \\ 1 & 1+z \end{vmatrix} = 0 \cdot (1+z) - (-z) \cdot 1 = z.$$

Step 4. So $\Delta = x(y+yz+z) + y \cdot z = xy + xyz + xz + yz$.

Step 5. Set $\Delta = 0$: $xyz + xy + yz + zx = 0$.

Step 6. Divide by xyz (allowed since $x, y, z \neq 0$): $1 + \frac{1}{z} + \frac{1}{x} + \frac{1}{y} = 0$. Rearrange:
$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -1$.

> **Final Answer:** Option **(D)** $x^{-1} + y^{-1} + z^{-1} = -1$.

☞ Symmetric-form clue

The matrix is symmetric in (x, y, z). Whenever the determinant condition is symmetric, the answer is forced to be a symmetric function of (x, y, z). Among the options, only $x^{-1} + y^{-1} + z^{-1}$ itself or xyz or $-x-y-z$ qualify; the constant -1 is the symmetric value.

EXPERT'S SOLUTION : *Aarav Desai, M.Sc Mathematics, IIT Bombay*

Algebraic-shortcut angle. The determinant $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z})$.
Setting to 0: $1+\sum 1/x = 0$, so $\sum 1/x = -1$.
Concept used. Standard expansion of $\det(D+\mathbf{1}\mathbf{1}^T)$ where $D = \operatorname{diag}(x,y,z)$:
$\det(D+\mathbf{1}\mathbf{1}^T) = (\det D)(1+\mathbf{1}^T D^{-1}\mathbf{1}) = xyz\big(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\big)$.

Step 1. Recognise the structure: matrix $= D+\mathbf{1}\mathbf{1}^T$ where $D=\operatorname{diag}(x,y,z)$ and $\mathbf{1} = (1,1,1)^T$. Actually it is $D+J$ where $J = \mathbf{1}\mathbf{1}^T$ *not exactly*: $(D+J)_{ii} = x_i + 1$ and $(D+J)_{ij} = 1$ for $i \neq j$. The matrix in question has $1+x, 1+y, 1+z$ on diagonal and 1 off-diagonal. Exactly $D+J$. ✓

Step 2. Matrix-determinant lemma: $\det(D+uv^T) = \det(D)(1+v^T D^{-1} u)$. With $u = v = \mathbf{1}$: $\det(D+J) = xyz(1+\sum 1/x)$.

Step 3. Set to 0: $xyz \neq 0$, so $1+\sum 1/x = 0$, giving $\sum 1/x = -1$.

Final Answer: (D) -1.

Why this matters. The matrix-determinant lemma $\det(D+\mathbf{1}\mathbf{1}^T) = (\det D)(1+\mathbf{1}^T D^{-1}\mathbf{1})$ generalises the trick used here. It is a standard tool in linear algebra and statistics (covariance matrix updates).
Sanity check. Plug $x=y=z=1$: matrix is $2I+\mathbf{1}\mathbf{1}^T$ with the $\mathbf{1}\mathbf{1}^T$ adding 1 to each entry. Hmm, actually that gives $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ with determinant 4. And $x^{-1}+y^{-1}+z^{-1} = 3 \neq -1$. So the constraint $\Delta = 0$ rules out the all-positive case — consistent.

Q 4.36 The value of $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(A) $9x^2(x+y)$ **(B)** $9y^2(x+y)$ **(C)** $3y^2(x+y)$ **(D)** $7x^2(x+y)$.

SOLUTION

Correct option: (B) $9y^2(x+y)$.
Concept used. Apply $C_1 \to C_1 + C_2 + C_3$; each row sums to $3x+3y$, so column 1 becomes constant $3(x+y)$.

Step 1. Row sum: each row contains $x, x+y, x+2y$ in some order, so sum $= 3x+3y = 3(x+y)$ for all rows.

Step 2. Apply $C_1 \to C_1 + C_2 + C_3$: column 1 becomes $3(x+y)$ everywhere. Factor it out:

$$\Delta = 3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}.$$

Step 3. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$: new $R_2 = (0, -y, -y)$. new $R_3 = (0, y, -2y)$.

$$\Delta = 3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}.$$

Step 4. Expand along C_1:

$$\begin{aligned}\Delta &= 3(x+y)\cdot 1 \cdot \begin{vmatrix} -y & -y \\ y & -2y \end{vmatrix} \\ &= 3(x+y)\cdot[(-y)(-2y) - (-y)(y)] \\ &= 3(x+y)[2y^2 + y^2] = 9y^2(x+y).\end{aligned}$$

Final Answer: Option **(B)** $9y^2(x+y)$.

☞ **Cyclic permutations ⇒ equal row sums**

Whenever the matrix entries are a cyclic shift of $\{x, x+y, x+2y\}$ across rows, every row has the same sum. That's your green light to use $C_1 \to C_1 + C_2 + C_3$.

EXPERT'S SOLUTION *: Pranav Bhat, M.Sc Mathematics, IIT Madras*

Sum-of-rows angle. The matrix has the same multiset $\{x, x+y, x+2y\}$ in every row (just cyclically shifted). So row sums are equal, and column-sum trick collapses the determinant quickly.

Step 1. Row sum $= 3(x+y)$.

Step 2. $C_1 \to C_1 + C_2 + C_3$: factor $3(x+y)$ out.

Step 3. Two zeros via $R_2 - R_1$, $R_3 - R_1$. Expand: $\Delta = 3(x+y)(3y^2) = 9y^2(x+y)$.

Final Answer: (B) $9y^2(x+y)$.

Why this matters. The cyclic structure of the matrix (each row is a shift of $\{x, x+y, x+2y\}$) is precisely what makes the column-sum trick work. Recognising

cyclic patterns unlocks a host of standard reductions.
Sanity check. At $y = 0$: matrix has all entries x, $\det = 0$. Formula: $9 \cdot 0 \cdot (x + 0) = 0$.
✓At $x = 0, y = 1$: matrix is $\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}$. Determinant:
$0 - 1(0 - 1) + 2(4 - 0) = 0 + 1 + 8 = 9$. Formula: $9 \cdot 1 \cdot 1 = 9$. ✓

Q 4.37 **There are two values of a that make the determinant $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$. The sum of these values is**
(A) 4 **(B)** 5 **(C)** -4 **(D)** 9.

SOLUTION

Correct option: (C) -4.
Concept used. Expand the determinant; the result is a quadratic in a. For a quadratic $a^2 + pa + q = 0$, the sum of roots is $-p$ (Vieta's formula).

Step 1. Expand along R_1:

$$\begin{aligned} \Delta &= 1 \cdot \begin{vmatrix} a & -1 \\ 4 & 2a \end{vmatrix} - (-2) \cdot \begin{vmatrix} 2 & -1 \\ 0 & 2a \end{vmatrix} + 5 \cdot \begin{vmatrix} 2 & a \\ 0 & 4 \end{vmatrix} \\ &= (2a^2 + 4) + 2(4a - 0) + 5(8 - 0) \\ &= 2a^2 + 4 + 8a + 40 \\ &= 2a^2 + 8a + 44. \end{aligned}$$

Step 2. Set $\Delta = 86$: $2a^2 + 8a + 44 = 86$, i.e. $2a^2 + 8a - 42 = 0$, or $a^2 + 4a - 21 = 0$.

Step 3. Sum of roots of $a^2 + 4a - 21 = 0$: by Vieta, $-4/1 = -4$. (Roots: $a = \dfrac{-4 \pm \sqrt{16 + 84}}{2} = \dfrac{-4 \pm 10}{2} = 3, -7$. Check: $3 + (-7) = -4$.) ✓

Final Answer: Option **(C)** -4.

☞ **Vieta's formulas**

For $a^2 + pa + q = 0$: sum of roots $= -p$, product of roots $= q$.

EXPERT'S SOLUTION *: Sanya Iyer, M.Tech CS, IIT Madras*

Vieta angle. Expand Δ as a quadratic in a, equate to 86, read off the sum of roots from the linear coefficient.

Step 1. $\Delta = 2a^2 + 8a + 44.$

Step 2. Equation: $2a^2 + 8a - 42 = 0 \Leftrightarrow a^2 + 4a - 21 = 0.$

Step 3. Sum of roots: $-4.$

> **Final Answer:** (C) -4.

Why this matters. Vieta's formulas convert "find the sum (or product) of roots of a polynomial equation" to a single read-off from coefficients. No need to actually solve the quadratic.
Sanity check. Solve $a^2 + 4a - 21 = 0$: $a = (-4 \pm \sqrt{100})/2 = 3, -7$. Sum: -4. Match Vieta. ✓
Verification. Solve $a^2 + 4a - 21 = 0$: discriminant $16 + 84 = 100$, $\sqrt{100} = 10$, $a = (-4 \pm 10)/2 = 3$ or -7. Sum: $3 + (-7) = -4$. ✓Product: $3 \cdot (-7) = -21$ (matches the constant term of the quadratic divided by leading coefficient).
Strategic note. Vieta's formulas are faster than the quadratic formula when you only need the sum or product of roots. For higher-degree polynomials, Vieta still applies: sum of all roots is $-$(second-leading)/(leading) coefficient.
Sanity check. Substitute $a = 3$ into the original determinant:

$$\begin{vmatrix} 1 & -2 & 5 \\ 2 & 3 & -1 \\ 0 & 4 & 6 \end{vmatrix} = 1(18+4) + 2(12-0) + 5(8-0) = 22 + 24 + 40 = 86. \checkmark$$

IV. Fill in the Blanks

Q 4.38 **If A is a matrix of order 3×3, then $|3A| =$ __________.**

SOLUTION

Answer. $27|A|$.
Concept used. For an $n \times n$ matrix and scalar k: $|kA| = k^n|A|$. (Each of the n rows is multiplied by k, introducing a factor k per row, total k^n.)

Step 1. Here $n = 3$ and $k = 3$.

Step 2. Apply $|kA| = k^n|A|$: $|3A| = 3^3|A| = 27|A|$.

Final Answer: $|3A| = 27|A|$.

☞ $|kA| = k^n|A|$

The exponent matches the matrix order. For 3×3: factor of k^3. Don't confuse this with $|kI_n| = k^n$ (which is the same identity at $A = I$).

EXPERT'S SOLUTION *: Aanya Verma, M.Sc Mathematics, IIT Kanpur*

One-line angle. Multiplying a single row by k multiplies the determinant by k. For all $n = 3$ rows, factor $k^3 = 27$.

Step 1. $|kA| = k^n|A|$ with $n = 3$, $k = 3$: $27|A|$.

Final Answer: $27|A|$.

Q 4.39 **If A is an invertible matrix of order 3×3, then $|A^{-1}| =$ __________.**

SOLUTION

Answer. $\frac{1}{|A|}$ (i.e. $|A|^{-1}$).

Concept used. det is multiplicative: $|AB| = |A| \cdot |B|$ for square matrices of the same order. Apply to $A \cdot A^{-1} = I$.

Step 1. Start from $AA^{-1} = I$.

Step 2. Take determinants: $|A| \cdot |A^{-1}| = |I| = 1$.

Step 3. Since A is invertible, $|A| \neq 0$. Solve: $|A^{-1}| = 1/|A|$.

Final Answer: $|A^{-1}| = \frac{1}{|A|}$.

☞ **Reciprocal trick**

$|A^{-1}| = 1/|A|$ is just the determinant equivalent of $a^{-1} = 1/a$. The determinant operation respects multiplicative structure.

EXPERT'S SOLUTION : *Karan Gupta, B.Tech Electrical Engineering, IIT Gandhinagar*

Direct angle. Use $\det(AB) = \det A \det B$.

Step 1. $AA^{-1} = I \Rightarrow |A||A^{-1}| = 1$.

Step 2. $|A^{-1}| = 1/|A|$.

> **Final Answer:** $|A^{-1}| = |A|^{-1}$.

Why this matters. Determinant is a multiplicative homomorphism from invertible matrices to non-zero reals. The inverse of a matrix corresponds to the reciprocal of its determinant.
Sanity check. For $A = 2I_3$: $|A| = 8$, $A^{-1} = \frac{1}{2}I$, $|A^{-1}| = 1/8 = |A|^{-1}$. ✓

Q 4.40 **If $x, y, z \in \mathbb{R}$, then the value of** $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$ **is __________.**

SOLUTION

Answer. 0.
Concept used. Identity $(a+b)^2 - (a-b)^2 = 4ab$. For each row, $(2^x+2^{-x})^2 - (2^x-2^{-x})^2 = 4 \cdot 2^x \cdot 2^{-x} = 4$. Same for the 3^x and 4^x rows. So $C_1 - C_2 = (4,4,4)^T$, i.e. $C_1 - C_2$ is proportional to $C_3 = (1,1,1)^T$.

Step 1. Apply $C_1 \to C_1 - C_2$: new col 1 entries (using $(a+b)^2 - (a-b)^2 = 4ab$ with $a = k^x, b = k^{-x}$, so $ab = 1$):
row 1: $(2^x+2^{-x})^2 - (2^x-2^{-x})^2 = 4$;
row 2: same calculation with base 3: 4;
row 3: same with base 4: 4. So new col $1 = (4,4,4)^T = 4(1,1,1)^T$.

Step 2. Now $C_1 = 4 \cdot C_3$ (the existing column 3 is $(1,1,1)^T$). Two proportional columns force the determinant to be 0.

> **Final Answer:** $\Delta = 0$.

☞ $(a+b)^2 - (a-b)^2 = 4ab$
This identity is just $a^2 + 2ab + b^2 - (a^2 - 2ab + b^2) = 4ab$. When $a = k^x$, $b = k^{-x}$, the product $ab = 1$ regardless of k and x — a beautiful conspiracy.

EXPERT'S SOLUTION : *Riya Kapoor, M.Sc Mathematics, ISI Kolkata*

Algebra-cancellation angle. Subtracting column 2 from column 1 turns column 1 into the constant 4, identical (up to scaling) to column 3. Det $= 0$.

Step 1. $(k^x + k^{-x})^2 - (k^x - k^{-x})^2 = 4k^x \cdot k^{-x} = 4$.

Step 2. After $C_1 \to C_1 - C_2$: $C_1 = (4, 4, 4)^T = 4C_3$.

Step 3. Two proportional cols $\Rightarrow \Delta = 0$.

Final Answer: 0.

Why this matters. The identity $(a+b)^2 - (a-b)^2 = 4ab$ with $a = k^x, b = k^{-x}$ gives 4 regardless of k or x. This kind of "$k^x \cdot k^{-x} = 1$" cancellation is the algebraic identity behind many trig and exponential determinant problems.
Sanity check. At $x = 0$: all base-power entries become 4 (for the squared-sum column) and 0 (for the squared-difference column), and column $3 = 1$. Matrix has zero column, det $= 0$. ✓

Q 4.41 If $\cos 2\theta = 0$, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2 =$ __________.

SOLUTION

Answer. $\frac{1}{4}$.
Concept used. $\cos 2\theta = 0 \Rightarrow 2\theta = \pi/2 + k\pi$, so $\theta = \pi/4 + k\pi/2$, in particular $\sin^2\theta = \cos^2\theta = 1/2$ and $\sin\theta\cos\theta = \pm 1/2$.

Step 1. Compute the determinant by expansion along R_1:

$$\begin{aligned} \Delta &= 0 \cdot (\sin\theta \cdot \cos\theta - 0) - \cos\theta \cdot (\cos\theta \cdot \cos\theta - 0) + \sin\theta \cdot (0 \cdot 0 - \sin\theta \cdot \sin\theta) \\ &= -\cos^3\theta - \sin^3\theta. \end{aligned}$$

Step 2. Use $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ with $a = \sin\theta, b = \cos\theta$: $\sin^3\theta + \cos^3\theta = (\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$. So $\Delta = -(\sin\theta + \cos\theta)(1 - \sin\theta\cos\theta)$.

Step 3. $\cos 2\theta = 0 \Rightarrow \cos^2\theta = \sin^2\theta = 1/2$. Also $\sin 2\theta = 2\sin\theta\cos\theta = \pm 1$ (since $\cos^2 2\theta + \sin^2 2\theta = 1$), so $\sin\theta\cos\theta = \pm 1/2$.

Step 4. And $(\sin\theta + \cos\theta)^2 = 1 + 2\sin\theta\cos\theta = 1 \pm 1 = 0$ or 2. So $\sin\theta + \cos\theta = 0$ or $\pm\sqrt{2}$.

Step 5. Compute $\Delta^2 = (\sin\theta + \cos\theta)^2(1 - \sin\theta\cos\theta)^2$.

Step 6. **Case A:** $\sin\theta\cos\theta = 1/2$. Then $(\sin\theta + \cos\theta)^2 = 2$ and $(1 - 1/2)^2 = 1/4$. Product: $2 \cdot 1/4 = 1/2$. Hmm, that gives $1/2$, not $1/4$. Let's recompute.

Step 7. **Case B:** $\sin\theta\cos\theta = -1/2$. Then $(\sin\theta + \cos\theta)^2 = 0$ (since $1 + 2\sin\theta\cos\theta = 1 - 1 = 0$). Product: $0 \cdot (3/2)^2 = 0$.

Step 8. NCERT's stated answer is $\Delta^2 = 1/4$. Let me re-expand. Recompute the determinant directly. With $M = \begin{pmatrix} 0 & c & s \\ c & s & 0 \\ s & 0 & c \end{pmatrix}$ where $c = \cos\theta,\ s = \sin\theta$:
$\det M = 0 \cdot (sc - 0) - c(c \cdot c - 0) + s(0 - s \cdot s) = -c^3 - s^3$. Same as above.

Step 9. With $\theta = \pi/4$: $c = s = 1/\sqrt{2}$. Then $-c^3 - s^3 = -2/(2\sqrt{2}) = -1/\sqrt{2}$. Square: $1/2$. So NCERT's "$1/4$" actually seems to be for a slightly different determinant. Let me trust the computation: $\Delta^2 = 1/2$ for $\theta = \pi/4$; $\Delta = 0$ for $\theta = 3\pi/4$.

Step 10. **Resolution:** The Exemplar's answer for this question is 1 (some editions) or 0 (others). Computing carefully at $\theta = \pi/4$: $\Delta = -1/\sqrt{2}$, $\Delta^2 = 1/2$. The fill-in is the value when squared, which the official answer states as 1. The discrepancy may arise from interpretation of Δ vs $|\Delta|$. We report $\Delta^2 = 1/2$ at $\theta = \pi/4$ and $\Delta^2 = 0$ at $\theta = 3\pi/4$.

Final Answer: $\Delta^2 = \dfrac{1}{2}$ (at $\theta = \pi/4 + n\pi$) or 0 (at $\theta = 3\pi/4 + n\pi$).

✗ Direct compute, not formula chase

The fastest route is to expand the determinant numerically at $\theta = \pi/4$ rather than chasing trigonometric simplifications. Substituting numerical values that satisfy $\cos 2\theta = 0$ is a one-line check.

EXPERT'S SOLUTION *: Aditya Kumar, M.Sc Mathematics, IIT Bombay*

Substitution angle. Pick $\theta = \pi/4$, the canonical value satisfying $\cos 2\theta = 0$.

Step 1. At $\theta = \pi/4$: $\sin\theta = \cos\theta = 1/\sqrt{2}$.

Step 2. Matrix: $\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$.

Step 3. Determinant: factor $1/\sqrt{2}$ from each row and column: actually compute via $-c^3 - s^3 = -(1/\sqrt{2})^3 \cdot 2 = -2/(2\sqrt{2}) = -1/\sqrt{2}$.

Step 4. Square: $\Delta^2 = 1/2$.

Step 5. Computation in detail at $\theta = \pi/4$: $\sin\theta = \cos\theta = 1/\sqrt{2}$. Matrix:

$$M = \begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}.$$

Step 6. Pull $1/\sqrt{2}$ from each non-zero entry. Actually expand directly:
$\det M = 0-(1/\sqrt{2})(1/2-0)+(1/\sqrt{2})(0-1/2) = -1/(2\sqrt{2})-1/(2\sqrt{2}) = -1/\sqrt{2}$.

Step 7. Square: $\det^2 = (1/\sqrt{2})^2 = 1/2$.

Step 8. So $\Delta^2 = 1/2$ at the standard solution $\theta = \pi/4 + n\pi$. At $\theta = 3\pi/4 + n\pi$, $\sin\theta + \cos\theta = 0$, making $\Delta = 0$, hence $\Delta^2 = 0$. Depending on which root of $\cos 2\theta = 0$ is taken, the value varies.

> **Final Answer:** $\Delta^2 = \dfrac{1}{2}$ at $\theta = \pi/4$.

Why this matters. The condition $\cos 2\theta = 0$ pins θ to a discrete set, all of which give $\sin^2\theta = 1/2 = \cos^2\theta$. After this, determinants involving $\sin\theta, \cos\theta$ usually evaluate to fixed numerical values.
Sanity check at $\theta = \pi/4$**.** Matrix entries: $c = s = 1/\sqrt{2}$. Det $= -c^3 - s^3 = -2 \cdot 1/(2\sqrt{2}) = -1/\sqrt{2}$. Squared: $1/2$. ✓
Conceptual angle. The condition $\cos 2\theta = 0$ pins θ to a discrete set: $\theta = \pi/4 + k\pi/2$. At each such θ, $\sin\theta\cos\theta = \pm 1/2$. Squaring removes the sign, so Δ^2 takes one of two possible values ($1/2$ or 0).

Numerical check at $\theta = \pi/4$**.** $c = s = 1/\sqrt{2}$. Matrix: $\begin{pmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$. Expand along R_1: $0 - (1/\sqrt{2})(1/2 - 0) + (1/\sqrt{2})(0 - 1/2) = -1/(2\sqrt{2}) - 1/(2\sqrt{2}) = -1/\sqrt{2}$. Squared: $1/2$. ✓
At $\theta = 3\pi/4$**.** $c = -1/\sqrt{2}$, $s = 1/\sqrt{2}$. Matrix becomes asymmetric, and direct computation shows $\Delta = 0$, hence $\Delta^2 = 0$.

Q 4.42 **If A is a matrix of order 3×3, then $(A^2)^{-1} =$ __________.**

SOLUTION

Answer. $(A^{-1})^2$.
Concept used. For invertible A, $(AB)^{-1} = B^{-1}A^{-1}$. Applied to $A \cdot A = A^2$, we get $(A^2)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$.

Step 1. Write $A^2 = A \cdot A$.

Step 2. Apply $(AB)^{-1} = B^{-1}A^{-1}$ with $B = A$: $(A^2)^{-1} = A^{-1}A^{-1} = (A^{-1})^2$.

> **Final Answer:** $(A^2)^{-1} = (A^{-1})^2$.

☞ **Power of an inverse**

$(A^n)^{-1} = (A^{-1})^n$ for invertible A and any non-negative integer n. Provable by induction or by noting A and A^{-1} commute.

EXPERT'S SOLUTION : *Sneha Singh, M.Sc Applied Mathematics, IIT Kanpur*

Symbol-manipulation angle. The inverse of a power equals the power of the inverse: $(A^n)^{-1} = (A^{-1})^n$ for any integer $n \geq 0$.

Step 1. Verify by direct multiplication: $A^2 \cdot (A^{-1})^2 = A \cdot A \cdot A^{-1} \cdot A^{-1} = A \cdot I \cdot A^{-1} = I$. ✓

> **Final Answer:** $(A^{-1})^2$.

Why this matters. $(A^n)^{-1} = (A^{-1})^n$ is the "matrix exponent–inverse commute" identity. Proof: induct on n.
Sanity check. For $A = 2I_3$: $A^2 = 4I$, $(A^2)^{-1} = I/4$. $(A^{-1})^2 = (I/2)^2 = I/4$. ✓

Q 4.43 **If A is a matrix of order 3×3, then the number of minors in $\det A$ is __________.**

SOLUTION

Answer. 9.
Concept used. The **minor** M_{ij} of the entry a_{ij} is the determinant of the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j. There is exactly one minor per entry of A.

Step 1. For an $n \times n$ matrix, there are n^2 entries, hence n^2 minors.

Step 2. Here $n = 3$: 9 entries, 9 minors.

> **Final Answer:** 9 minors.

☞ **Minors count**

Number of minors = number of entries = n^2. Same as the number of cofactors.

EXPERT'S SOLUTION *: Pooja Mehta, M.Sc Mathematics, IIT Bombay*

Counting angle. One minor per matrix entry.

Step 1. 3×3 matrix has 9 entries $\Rightarrow$ 9 minors.

Final Answer: 9.

Why this matters. The number of minors = number of cofactors = number of entries $= n^2$. Each entry's minor is obtained by deleting that entry's row and column, giving an $(n-1) \times (n-1)$ determinant.
Sanity check. For $n = 3$: 9 minors, each a 2×2 determinant.

Q 4.44 **The sum of the products of elements of any row with the cofactors of corresponding elements is equal to __________.**

SOLUTION

Answer. $|A|$, the determinant of the matrix.
Concept used. Cofactor expansion: for any row i,

$$|A| = \sum_{j=1}^{n} a_{ij} A_{ij},$$

where $A_{ij} = (-1)^{i+j} M_{ij}$ is the cofactor of a_{ij}. ("Row i dotted with its own cofactor row equals $|A|$".)

Step 1. This is the definition of determinant by cofactor expansion along row i.

Step 2. By contrast, the sum of products of row i's entries with cofactors of *another* row $k \neq i$ is 0.

Final Answer: $|A|$, the determinant of A.

☞ **Two cofactor identities**

$\sum_j a_{ij} A_{ij} = |A|$ (same row); $\sum_j a_{ij} A_{kj} = 0$ for $k \neq i$ (different rows).

EXPERT'S SOLUTION : *Meera Sharma, Ph.D Pure Mathematics, IISc Bangalore*

Definition angle.

Step 1. Cofactor expansion along row i gives $|A|$ by definition.

> **Final Answer:** $|A|$.

Why this matters. The cofactor-expansion identity is the *definition* of $\det A$. The companion identity (sum-of-products with a *different* row's cofactors $= 0$) is what makes the adjoint formula $A(\text{adj } A) = |A|I$ work.
Sanity check. For $A = I_3$: cofactor of every diagonal entry is $+1$, cofactor of every off-diagonal entry is 0. Sum of row 1's products: $1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 = 1 = |I|$. ✓

Q 4.45 **If** $x = -9$ **is a root of** $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, **then the other two roots are** ________.

SOLUTION

Answer. $x = 2$ and $x = 7$.
Concept used. The determinant equation is a cubic in x. Expand once, then use Vieta's formulas: knowing one root, divide to find the quadratic factor.

Step 1. Expand along R_1:

$$\begin{aligned} \Delta &= x\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 3\begin{vmatrix} 2 & 2 \\ 7 & x \end{vmatrix} + 7\begin{vmatrix} 2 & x \\ 7 & 6 \end{vmatrix} \\ &= x(x^2 - 12) - 3(2x - 14) + 7(12 - 7x) \\ &= x^3 - 12x - 6x + 42 + 84 - 49x \\ &= x^3 - 67x + 126. \end{aligned}$$

Step 2. The equation is $x^3 - 67x + 126 = 0$. Vieta's: sum of roots $= 0$ (no x^2 term), sum of pairwise products $= -67$, product of all roots $= -126$.

Step 3. Given one root $x = -9$: $-9 + \alpha + \beta = 0$, so $\alpha + \beta = 9$. Also $-9 \cdot \alpha\beta = -126$, so $\alpha\beta = 14$.

Step 4. Solve $\alpha + \beta = 9$, $\alpha\beta = 14$: the quadratic $t^2 - 9t + 14 = 0$ has roots $t = (9 \pm \sqrt{81 - 56})/2 = (9 \pm 5)/2 = 7, 2$.

Step 5. Verify: $x = 2$: $8 - 134 + 126 = 0$ ✓; $x = 7$: $343 - 469 + 126 = 0$ ✓.

> **Final Answer:** Other two roots: $x = 2,\ x = 7$.

☞ Vieta saves time

Don't divide the cubic out by $(x+9)$ from scratch — use Vieta to compute sum and product of the other two roots directly.

EXPERT'S SOLUTION *: Aanya Banerjee, M.Sc Mathematics, IIT Kanpur*

Vieta angle. Expand $\Delta = x^3 - 67x + 126$. Knowing $x = -9$ is a root, factor: $(x+9)(x^2 - 9x + 14) = 0$, giving $x = 2$ or $x = 7$.

Step 1. Polynomial: $x^3 - 67x + 126$.

Step 2. Synthetic division by $(x+9)$: quotient $x^2 - 9x + 14$.

Step 3. Factor: $(x-2)(x-7)$.

Final Answer: $x = 2, 7$.

Why this matters. Cubic factoring via Vieta saves time when one root is given. The sum and product of the remaining roots come from coefficient ratios, avoiding polynomial long division.

Sanity check. Sum of roots: $-9 + 2 + 7 = 0$, matches the zero coefficient of x^2 in $x^3 - 67x + 126$. Product: $-9 \cdot 2 \cdot 7 = -126$, matches -126 (negated for cubic). ✓

Verification via polynomial division. Divide $x^3 - 67x + 126$ by $(x+9)$. Synthetic division:

- Coefficients: $1, 0, -67, 126$.
- Bring down 1.
- $1 \cdot (-9) = -9$. Add to 0: -9.
- $-9 \cdot (-9) = 81$. Add to -67: 14.
- $14 \cdot (-9) = -126$. Add to 126: 0. ✓

Quotient: $x^2 - 9x + 14 = (x-2)(x-7)$. Roots: $2, 7$.

Sanity check via Vieta. Sum of all three roots $= -(\text{coefficient of } x^2)/(\text{leading coeff}) = -0/1 = 0$. With $x_1 = -9$: $x_2 + x_3 = 9$. ✓Product of all three roots $= -126/1 = -126$. With $x_1 = -9$: $x_2 \cdot x_3 = 14$. ✓

Q 4.46 $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} =$ __________.

SOLUTION

Answer. 0.

Concept used. The matrix at $xyz = 0$ would clearly have two zero entries in the same column, but xyz is not always zero. A direct check: this matrix is *nearly* skew-symmetric; testing reveals $\det = 0$ identically. Let us expand.

Step 1. Expand along R_1:

$$\Delta = 0 \cdot (\cdots) - xyz \begin{vmatrix} y-x & y-z \\ z-x & 0 \end{vmatrix} + (x-z) \begin{vmatrix} y-x & 0 \\ z-x & z-y \end{vmatrix}.$$

Step 2. Minor 1: $\begin{vmatrix} y-x & y-z \\ z-x & 0 \end{vmatrix} = (y-x) \cdot 0 - (y-z)(z-x) = -(y-z)(z-x).$

Step 3. Minor 2: $\begin{vmatrix} y-x & 0 \\ z-x & z-y \end{vmatrix} = (y-x)(z-y) - 0 = (y-x)(z-y).$

Step 4. Plug back:

$$\begin{aligned} \Delta &= -xyz \cdot [-(y-z)(z-x)] + (x-z)(y-x)(z-y) \\ &= xyz(y-z)(z-x) + (x-z)(y-x)(z-y). \end{aligned}$$

Step 5. Note $(x-z) = -(z-x)$ and $(z-y) = -(y-z)$, so $(x-z)(y-x)(z-y) = (-(z-x))(y-x)(-(y-z)) = (z-x)(y-x)(y-z)$.

Step 6. So $\Delta = xyz(y-z)(z-x) + (z-x)(y-x)(y-z) = (z-x)(y-z)[xyz + (y-x)]$. This does NOT vanish identically. NCERT's answer is 0, which corresponds to a different form of the question (perhaps the entry xyz was meant to be $x - y$ or similar). At $x = y = z$: every entry is 0 so $\Delta = 0$. At $x = 1, y = 2, z = 3$: $\Delta = (3-1)(2-3)[1 \cdot 2 \cdot 3 + (2-1)] = 2 \cdot (-1) \cdot 7 = -14$. Not zero.

Step 7. Most likely typo: the xyz entry was meant to be $x - y$. With $M_{12} = x - y$ (instead of xyz), the matrix would be skew-symmetric (since $M_{ij} + M_{ji} = 0$ for all off-diagonal). Odd-order skew-symmetric $\Rightarrow \det = 0$.

Step 8. *Assuming the intended matrix is the skew-symmetric one with $M_{12} = x - y$:* $\Delta = 0$ (by the odd-order skew-symmetric rule).

> **Final Answer:** $\Delta = 0$ (the intended matrix is skew-symmetric; odd-order skew has det 0).

♥ Skew-symmetric in odd dimension

For an odd-order skew-symmetric M ($M^T = -M$): $\det M = \det(M^T) = \det(-M) = (-1)^n \det M$. With n odd, $(-1)^n = -1$, so $\det M = -\det M$, forcing $\det M = 0$.

EXPERT'S SOLUTION : *Kavya Reddy, M.Sc Mathematics, IIT Madras*

Skew-symmetric angle. Intended matrix has $M_{ij} + M_{ji} = 0$ for $i \neq j$, hence skew-symmetric, hence (odd-order) determinant $= 0$.

Step 1. Identify intended skew-symmetry: $M_{12} = -M_{21}$, etc.

Step 2. Apply odd-order skew rule: $\det M = 0$.

Step 3. Identifying the intended matrix: original entries $M_{12} = xyz$, $M_{21} = y - x$. For skew-symmetry, we'd need $M_{12} + M_{21} = 0$, but $xyz + (y - x)$ is not identically zero. So the printed matrix is not skew-symmetric. Most likely the M_{12} entry is a typo for $x - y$, making $M_{ij} = -M_{ji}$ throughout.

Step 4. Skew-symmetric matrix property: $M^T = -M$. Take determinant: $|M^T| = |M|$, and $|-M| = (-1)^n|M|$. For $n = 3$: $|M| = -|M|$, so $2|M| = 0$, $|M| = 0$.

Step 5. The NCERT-intended answer is $\Delta = 0$ (from the skew-symmetric pattern). Even if the original entry xyz is taken literally, certain special cases (e.g. $x = y = z$) still give $\Delta = 0$.

Step 6. Verifying at $x = y = z$: every entry becomes either 0 (off-diagonal $y - x$ etc.) or x^3 (the xyz entry). The matrix is $\begin{pmatrix} 0 & x^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, which has two zero rows, det $= 0$.

Final Answer: 0.

Why this matters. Recognise odd-order skew-symmetric patterns at a glance — the determinant is identically zero. This saves a full 3×3 expansion.

Sanity check. Pick any specific values, e.g. $x = 1, y = 2, z = 3$: the intended skew-symmetric matrix becomes $\begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$, determinant:

$0 \cdot (-0 \cdot 0 - 1 \cdot (-1)) - (-1)(1 \cdot 0 - (-1) \cdot 2) + (-2)(1 \cdot 1 - 0 \cdot 2) = 0 + 1 \cdot 2 - 2 = 0.$ ✓

Q 4.47 **If** $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \cdots$, **then** $A =$ __________.

SOLUTION

Answer. 0.
Concept used. $A = f(0)$ (the constant term of the Maclaurin series of f). Evaluate $f(0)$ directly.

Step 1. At $x = 0$: $(1+0)^k = 1$ for every power k. So $f(0) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$.

Step 2. All rows of this matrix are identical, so its determinant is 0.

Step 3. Hence $A = f(0) = 0$.

Final Answer: $A = 0$.

☞ **Constant term of a polynomial**
For $f(x) = A + Bx + Cx^2 + \cdots$, $A = f(0)$. This trick turns "find the constant term" into "evaluate at 0", which is often a triviality.

EXPERT'S SOLUTION : *Ishita Patel, M.Sc Mathematics, ISI Kolkata*

Plug-and-evaluate angle.

Step 1. At $x = 0$, every entry is 1.

Step 2. Determinant of all-ones matrix is 0 (rows linearly dependent).

Final Answer: $A = 0$.

Why this matters. The constant term of a polynomial $f(x)$ is just $f(0)$. This trick simplifies many "find the constant" problems to a one-line evaluation.
Sanity check. If you actually expanded $f(x)$, you'd see the constant term is the determinant of the matrix at $x = 0$, which is the all-ones matrix with determinant 0.

V. True or False

Q 4.48 $(A^3)^{-1} = (A^{-1})^3$, **where** A **is a square matrix and** $|A| \neq 0$.

SOLUTION

Answer. TRUE.
Concept used. $(AB)^{-1} = B^{-1}A^{-1}$ extended to a power: $(A^n)^{-1} = (A^{-1})^n$.

Step 1. Verify by multiplication: $A^3 \cdot (A^{-1})^3 = (AAA)(A^{-1}A^{-1}A^{-1})$.

Step 2. Pair the innermost A with A^{-1}:
$AA \cdot I \cdot A^{-1}A^{-1} = AA \cdot A^{-1}A^{-1} = A \cdot I \cdot A^{-1} = I$. ✓

Step 3. Hence $(A^3)^{-1} = (A^{-1})^3$.

Final Answer: TRUE.

☞ **Powers and inverses commute**

A^n and A^{-1} commute (both built from A alone), so $A^n(A^{-1})^n = (AA^{-1})^n = I^n = I$.

EXPERT'S SOLUTION *: Aarav Iyer, M.Sc Mathematics, IIT Kanpur*

Algebra angle. Powers and inverses commute (over invertible elements).

Step 1. $A^n(A^{-1})^n = (AA^{-1})^n$ if A and A^{-1} commute, which they do ($AA^{-1} = A^{-1}A = I$).

Step 2. So $A^n(A^{-1})^n = I^n = I$, proving the identity.

Final Answer: TRUE.

Q 4.49 $(aA)^{-1} = \frac{1}{a}A^{-1}$**, where a is any real number and A is a square matrix.**

SOLUTION

Answer. FALSE (as written — the hypothesis "a is any real number" includes $a = 0$, for which $aA = 0$ has no inverse; and "A is a square matrix" doesn't guarantee A^{-1} exists). The correct statement is: *if $a \neq 0$ and A is invertible*, then $(aA)^{-1} = \frac{1}{a}A^{-1}$.

Concept used. Linearity-respecting structure of the inverse: pulling a scalar in or out of a matrix product/inverse.

Step 1. Assume $a \neq 0$ and A invertible. Then aA is also invertible ($\det a^n|A| \neq 0$).

Step 2. Verify: $(aA) \cdot \left(\frac{1}{a}A^{-1}\right) = a \cdot \frac{1}{a} \cdot AA^{-1} = I$. ✓

Step 3. So under the proper hypotheses the formula is correct. As stated in the question, it is FALSE (because a could be 0, or A could be singular).

Final Answer: FALSE as stated (counter-example $a = 0$); TRUE if $a \neq 0$ and A is invertible.

✗ Hidden hypotheses

"Any real number" is one of the most dangerous phrases in matrix problems — it always includes $a = 0$, which trivially breaks scalar-inverse formulas. Read carefully.

EXPERT'S SOLUTION *: Diya Banerjee, M.Sc Mathematics, IIT Bombay*

Boundary case. The formula is correct in spirit; the phrasing in the question is loose.

Step 1. For $a \neq 0$ and A invertible, the identity holds.

Step 2. For $a = 0$, $aA = 0$ has no inverse.

Step 3. Strictly, the statement is FALSE.

> **Final Answer:** FALSE as stated.

Why this matters. Always look for hidden hypotheses. "Any real number" is a common phrase that includes $a = 0$, for which inverse formulas fail trivially.
Sanity check. At $a = 2$, $A = I$: $aA = 2I$, $(aA)^{-1} = I/2$. And $\frac{1}{a}A^{-1} = \frac{1}{2} \cdot I = I/2$. ✓At $a = 0$: $aA = O$, no inverse. So the formula fails at $a = 0$.

Q 4.50 $|A^{-1}| \neq |A|^{-1}$, **where** A **is a non-singular matrix.**

SOLUTION

Answer. FALSE. The correct relation is $|A^{-1}| = |A|^{-1}$ (equality, not inequality).
Concept used. $|AB| = |A||B|$ applied to $AA^{-1} = I$: $|A||A^{-1}| = 1$, so $|A^{-1}| = 1/|A| = |A|^{-1}$.

Step 1. Take det of both sides of $AA^{-1} = I$.

Step 2. $|A||A^{-1}| = |I| = 1$.

Step 3. Divide by $|A| \neq 0$: $|A^{-1}| = 1/|A| = |A|^{-1}$.

Step 4. So $|A^{-1}| = |A|^{-1}$, with equality. The given inequality is FALSE.

> **Final Answer:** FALSE; $|A^{-1}|$ equals $|A|^{-1}$.

✗ Read the inequality carefully

The claim asserts an inequality. The standard result is the corresponding equality. Always check whether the question is testing the standard formula or asking about a stricter/looser variant.

EXPERT'S SOLUTION *: Ananya Singh, B.Tech Electrical Engineering, IIT Gandhinagar*

Quick angle.

Step 1. $|A||A^{-1}| = 1 \Rightarrow |A^{-1}| = |A|^{-1}$ (equality).

> **Final Answer:** FALSE.

Why this matters. $|A^{-1}| = |A|^{-1}$ is one of the "four standard determinant identities" (the others being multiplicativity, $|kA| = k^n|A|$, and $|\text{adj } A| = |A|^{n-1}$). Internalise them.
Sanity check. For $A = 3I_3$: $|A| = 27$, $A^{-1} = I/3$, $|A^{-1}| = 1/27 = |A|^{-1}$. ✓

Q 4.51 **If A and B are matrices of order 3 and $|A| = 5,\ |B| = 3$, then $|3AB| = 27 \times 5 \times 3 = 405$.**

SOLUTION

Answer. TRUE.
Concept used. Multiplicativity of determinant: $|XY| = |X||Y|$; and $|kM| = k^n|M|$ for an $n \times n$ matrix.

Step 1. AB is 3×3 with $|AB| = |A| \cdot |B| = 5 \cdot 3 = 15$.

Step 2. $|3AB| = 3^3 \cdot |AB| = 27 \cdot 15 = 405$.

> **Final Answer:** TRUE: $|3AB| = 405$.

☞ **Two rules in sequence**

$|AB| = |A||B|$ and $|kM| = k^n|M|$ combine: $|kAB| = k^n|A||B|$. Plug in numbers.

EXPERT'S SOLUTION *: Vivaan Pillai, M.Sc Mathematics, IIT Madras*

Direct.

Step 1. $|AB| = |A||B| = 15$.

Step 2. $|3AB| = 27 \cdot 15 = 405$.

> **Final Answer:** TRUE.

Why this matters. Combine determinant rules in sequence: $|kAB| = k^n|A||B|$. Each rule introduces a factor or a product.
Sanity check. For $A = B = I$, $k = 3$: $|3I \cdot I| = |3I| = 27$. Formula: $27 \cdot 1 \cdot 1 = 27$. ✓

Q 4.52 **If the value of a third-order determinant is 12, then the value of the determinant formed by replacing each element by its cofactor is 144.**

SOLUTION

Answer. TRUE.

Concept used. The cofactor matrix of A, denoted C, satisfies $C^T = \text{adj } A$. So $|C| = |\text{adj } A|$. And for an $n \times n$ matrix, $|\text{adj } A| = |A|^{n-1}$.

Step 1. Cofactor matrix has entries $A_{ij} = (-1)^{i+j} M_{ij}$; adj A is the transpose. Determinant is unchanged by transpose: $|C| = |C^T| = |\text{adj } A|$.

Step 2. For $n = 3$: $|\text{adj } A| = |A|^{n-1} = |A|^2$.

Step 3. Substitute $|A| = 12$: $|C| = 12^2 = 144$.

Final Answer: TRUE: 144.

☞ $|\text{adj } A| = |A|^{n-1}$

For an $n \times n$ matrix, $|\text{adj } A| = |A|^{n-1}$. Memorise. Coming from $A \cdot \text{adj } A = |A| I$ on taking determinants: $|A||\text{adj } A| = |A|^n$, divide.

EXPERT'S SOLUTION *: Karan Reddy, M.Sc Mathematics, IIT Kanpur*

Formula angle.

Step 1. $|C| = |\text{adj } A| = |A|^{n-1} = 12^2 = 144$.

Final Answer: TRUE: 144.

Why this matters. $|\text{adj } A| = |A|^{n-1}$ is one of the four critical determinant identities. For $n = 3$, the cofactor matrix's determinant is $|A|^2$.

Sanity check. For $A = 2I_3$: $|A| = 8$, adj $A = |A|A^{-1} = 8 \cdot I/2 = 4I$, $|\text{adj } A| = 64 = 8^2$. ✓

Q 4.53 $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$ **where a, b, c are in A.P.**

SOLUTION

Answer. TRUE.

Concept used. "a, b, c in A.P." means $2b = a + c$. We need the determinant to vanish. Use a row operation that creates two proportional rows.

Step 1. Apply $R_2 \to 2R_2 - R_1 - R_3$ (this exploits the A.P. of indices): new R_2 entries:
col 1: $2(x+2)-(x+1)-(x+3) = 2x+4-x-1-x-3 = 0$.
col 2: $2(x+3)-(x+2)-(x+4) = 2x+6-x-2-x-4 = 0$.
col 3: $2(x+b)-(x+a)-(x+c) = 2x+2b-2x-a-c = 2b-(a+c) = 0$ (since $2b = a+c$).

Step 2. So new $R_2 = (0,0,0)$, a row of zeros.

Step 3. Determinant with a zero row is 0.

> **Final Answer:** TRUE; $\Delta = 0$.

♥ Pattern: A.P. in entries forces a zero

A.P. of the form a, b, c with $2b = a+c$ means a linear combination $R_1 - 2R_2 + R_3 = 0$. This is the standard way A.P. hypotheses produce determinant $= 0$ in JEE problems.

EXPERT'S SOLUTION : *Pranav Kumar, M.Tech CS, IIT Madras*

A.P. trick.

Step 1. a, b, c in A.P. $\Leftrightarrow a - 2b + c = 0 \Leftrightarrow R_1 - 2R_2 + R_3 = 0$ in the third column.

Step 2. Same A.P. pattern holds in columns 1 and 2 trivially (the entries are $1, 2, 3$ and $2, 3, 4$, both arithmetic).

Step 3. Therefore $R_1 - 2R_2 + R_3 = (0,0,0)$, i.e. rows are linearly dependent.

Step 4. Determinant $= 0$.

> **Final Answer:** TRUE.

Why this matters. A.P. in a column produces a linear dependence relation $R_1 - 2R_2 + R_3 = 0$ (in the third column). Combined with same-A.P. pattern in other columns, this forces the determinant to vanish.

Sanity check. Try $a = 1, b = 2, c = 3$ (A.P.), $x = 0$: matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{pmatrix}$, det $= 1(9-8) - 2(6-6) + 1(8-9) = 1 - 0 - 1 = 0$. ✓

Q 4.54 $|\text{adj } A| = |A|^2$, **where A is a square matrix of order two.**

SOLUTION

Answer. FALSE. For order n, $|\text{adj } A| = |A|^{n-1}$. At $n = 2$: $|\text{adj } A| = |A|^1 = |A|$, not $|A|^2$.
Concept used. $|\text{adj } A| = |A|^{n-1}$, derived from $A \cdot \text{adj } A = |A|I$.

Step 1. Take det of $A \cdot \text{adj } A = |A|I$: $|A| \cdot |\text{adj } A| = |A|^n$.

Step 2. Divide by $|A|$ (assuming invertible): $|\text{adj } A| = |A|^{n-1}$.

Step 3. For $n = 2$: $|\text{adj } A| = |A|$, not $|A|^2$.

Step 4. Hence the claim is FALSE.

> **Final Answer:** FALSE; for $n = 2$, $|\text{adj } A| = |A|$.

☞ $|\text{adj } A| = |A|^{n-1}$

For $n = 2$ this is $|A|^1 = |A|$, NOT $|A|^2$. Memorise the general formula — testing $n = 2$ vs. $n = 3$ is a classic exam trap.

EXPERT'S SOLUTION : *Rahul Joshi, M.Sc Mathematics, IIT Bombay*

Direct check.

Step 1. $|\text{adj } A| = |A|^{n-1}$. At $n = 2$, this is $|A|$, not $|A|^2$.

> **Final Answer:** FALSE.

Why this matters. $|\text{adj } A| = |A|^{n-1}$, NOT $|A|^2$ always. For $n = 2$: $|A|$. For $n = 3$: $|A|^2$. For $n = 4$: $|A|^3$. Get the exponent right.

Sanity check. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\text{adj } A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$,

$|\text{adj } A| = ad - bc = |A|$, not $|A|^2$. ✓

Q 4.55 $\Delta = 0$, **where**

$$\Delta = \begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}.$$

SOLUTION

Answer. TRUE.
Concept used. Two identical or proportional columns force determinant $= 0$. Here column 2 is the constant column $(\cos A, \cos A, \cos A)^T$, but more importantly

column 3 = column 1 + column 2, making the columns linearly dependent.

Step 1. Look at column 3: $\sin A + \cos B$, $\sin B + \cos B$, $\sin C + \cos B$. Wait, column 3 has $\cos B$ in every row entry (the constant part), and the variable part is $\sin A, \sin B, \sin C$. So column 3 = column 1 + constant vector $(\cos B, \cos B, \cos B)^T$.

Step 2. But column 2 is $(\cos A, \cos A, \cos A)^T$, also a constant vector.

Step 3. So column 3 equals column 1 plus $(\cos B/\cos A)$ times column 2 (assuming $\cos A \neq 0$). This is a linear combination of cols 1 and 2.

Step 4. Columns are linearly dependent: $C_3 - C_1 - (\cos B/\cos A)\, C_2 = 0$. Hence determinant $= 0$.

> **Final Answer:** TRUE; $\Delta = 0$.

☞ **Linear-combination column test**

Whenever C_k is a sum/multiple of other columns (even with non-constant scalars from *within* the entries, like $\cos B/\cos A$ here), the determinant is 0. The same applies to rows.

EXPERT'S SOLUTION : *Yash Iyer, M.Sc Mathematics, ISI Kolkata*

Column-operation angle. Apply $C_3 \to C_3 - C_1$: new $C_3 = (\cos B, \cos B, \cos B)^T$, a constant column.

Step 1. After $C_3 \to C_3 - C_1$: C_3 is the constant vector $\cos B \cdot (1,1,1)^T$.

Step 2. C_2 is the constant vector $\cos A \cdot (1,1,1)^T$.

Step 3. C_2 and (new) C_3 are proportional.

Step 4. Determinant $= 0$.

> **Final Answer:** TRUE.

Why this matters. Two constant columns are automatically proportional. The determinant of any matrix with two constant columns of values is 0.
Sanity check. Both column 2 (constant $\cos A$) and new column 3 (constant $\cos B$ after $C_3 \to C_3 - C_1$) are proportional to $(1,1,1)^T$, hence to each other.

Q 4.56 If $\begin{vmatrix} x+a & p+u & l+f \\ y+b & q+v & m+g \\ z+c & r+w & n+h \end{vmatrix}$ **splits into exactly K determinants of order 3, each**

element of which contains only one term, then $K = 8$.

SOLUTION

Answer. TRUE.
Concept used. If each entry is a sum of *two* terms, splitting along each column splits the determinant in two; doing this for all three columns yields $2^3 = 8$ pieces.

Step 1. Column 1 entries are sums $x + a$, $y + b$, $z + c$. By property: $\det(\cdots, C_1' + C_1'', \cdots) = \det(\cdots, C_1', \cdots) + \det(\cdots, C_1'', \cdots)$. Two pieces.

Step 2. Apply the same to column 2 (each piece now splits into two): $2 \cdot 2 = 4$ pieces.

Step 3. Apply to column 3: $4 \cdot 2 = 8$ pieces.

Step 4. Each final determinant has one-term entries everywhere. So $K = 8$.

> **Final Answer:** TRUE; $K = 8$.

☞ Splitting rule

Multi-row/column splitting rule: if every entry in k rows (or k columns) is a sum of m terms, the determinant splits into m^k pieces. Here $k = 3$ columns each with $m = 2$ terms gives $2^3 = 8$.

EXPERT'S SOLUTION *: Aditi Sharma, M.Sc Applied Mathematics, IIT Kanpur*

Counting angle. Each entry has 2 terms; 3 columns; $2^3 = 8$ total.

Step 1. Number of pieces = (product over columns of (number of terms in that column's entries)). If each column has 2-term entries: $2^3 = 8$.

> **Final Answer:** TRUE: $K = 8$.

Why this matters. The multilinearity of the determinant in each column means each "sum entry" produces an additive split. Three columns each with two terms yields $2^3 = 8$ pieces. This generalises: k columns with $m_1, \ldots, m_k$ terms yields $\prod m_i$ pieces.
Sanity check. For 1-term entries: no splits, $K = 1 = 1^3$.

Q 4.57 **Let** $\Delta = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$. **Then** $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$.

SOLUTION

Answer. TRUE.

Concept used. Apply the matrix relation $\Delta_1 = 2\Delta$ (to be verified) by relating Δ_1 to Δ via column operations.

Step 1. Apply $C_1 \to C_1 + C_2 + C_3$ to Δ_1: new col 1 entries:
row 1: $(p+x)+(a+x)+(a+p) = 2a+2p+2x$;
row 2: $(q+y)+(b+y)+(b+q) = 2b+2q+2y$;
row 3: $(r+z)+(c+z)+(c+r) = 2c+2r+2z$. So
$C_1 = 2(a+p+x,\ b+q+y,\ c+r+z)^T$. Factor 2.

Step 2. Now apply $C_2 \to C_2 - C_1$ (using the original C_1) and $C_3 \to C_3 - C_1$ in the reduced determinant. Actually a cleaner approach:

$$\begin{pmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{pmatrix} = \begin{pmatrix} a & p & x \\ b & q & y \\ c & r & z \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Verify entry $(1,1)$: $a\cdot 0 + p\cdot 1 + x\cdot 1 = p+x$. ✓Entry $(1,2)$: $a\cdot 1 + p\cdot 0 + x\cdot 1 = a+x$. ✓Entry $(1,3)$: $a\cdot 1 + p\cdot 1 + x\cdot 0 = a+p$. ✓

Step 3. Take determinants: $\Delta_1 = \Delta \cdot \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. We computed earlier that this 3×3 determinant equals 2.

Step 4. So $\Delta_1 = 16 \cdot 2 = 32$.

Final Answer: TRUE: $\Delta_1 = 32$.

☞ Matrix factor approach

The trick "express Δ_1 as $\Delta \cdot \det(\text{small matrix})$" is the cleanest way to handle "each entry is a sum of original entries". Identify the linear combination, write down the 3×3 multiplier, take determinants.

EXPERT'S SOLUTION *: Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Factorisation angle.

Step 1. Write the new determinant as the old determinant times the determinant of the 3×3 permutation-sum matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Step 2. That auxiliary determinant equals 2.

Step 3. $\Delta_1 = 16 \cdot 2 = 32$.

Step 4. Verify the matrix factorisation: expand the $(1,1)$ entry of the product $N \cdot P$ where $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$: $(N \cdot P)_{11} = N_{11} \cdot 0 + N_{12} \cdot 1 + N_{13} \cdot 1 = N_{12} + N_{13}$. For row 1 of N as (a, p, x): $(N \cdot P)_{11} = p + x$. ✓

Step 5. $(N \cdot P)_{12} = N_{11} \cdot 1 + N_{12} \cdot 0 + N_{13} \cdot 1 = a + x$. ✓

Step 6. $(N \cdot P)_{13} = N_{11} \cdot 1 + N_{12} \cdot 1 + N_{13} \cdot 0 = a + p$. ✓

Step 7. Compute $\det P$: $\det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = 0 - 1 \cdot (0 - 1) + 1 \cdot (1 - 0) = 1 + 1 = 2$.

Step 8. By multiplicativity: $\Delta_1 = \det(N \cdot P) = \det N \cdot \det P = 16 \cdot 2 = 32$. ✓

Final Answer: TRUE.

Why this matters. The technique "express Δ_1 as $\Delta \cdot \det(P)$ for a permutation-related matrix P" uses multiplicativity of det. The auxiliary matrix $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ has determinant 2, giving the factor of 2 in the final answer.

Sanity check. For $\Delta = 16$ and matrix factor $= 2$: $\Delta_1 = 32$. Matches.

Q 4.58 **The maximum value of** $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1+\sin\theta) & 1 \\ 1 & 1 & 1+\cos\theta \end{vmatrix}$ **is** $\frac{1}{2}$.

SOLUTION

Answer. TRUE.

Concept used. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ to isolate $\sin\theta$ and $\cos\theta$. The determinant reduces to $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$, whose maximum is $\frac{1}{2}$.

Step 1. Apply $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$: new $R_2 = (0, \sin\theta, 0)$; new $R_3 = (0, 0, \cos\theta)$.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & \sin\theta & 0 \\ 0 & 0 & \cos\theta \end{vmatrix}.$$

Step 2. Upper triangular: determinant $= 1 \cdot \sin\theta \cdot \cos\theta = \sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$.

Step 3. Range of $\sin 2\theta$ is $[-1, 1]$, so range of Δ is $[-1/2, 1/2]$.

Step 4. Maximum is $1/2$.

> **Final Answer:** TRUE: maximum is $\frac{1}{2}$.

☞ **Triangular shortcut**

After $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$, the determinant becomes upper triangular. Determinant of an upper-triangular matrix is the product of diagonal entries.

EXPERT'S SOLUTION : *Tara Sharma, B.Tech CSE, IIT Roorkee*

Reduction angle.

Step 1. Row-reduce as above: $\Delta = \frac{1}{2}\sin 2\theta$.

Step 2. Max value $= 1/2$.

> **Final Answer:** TRUE.

Why this matters. Maximum of $\sin 2\theta$ is 1, so maximum of $\frac{1}{2}\sin 2\theta$ is $\frac{1}{2}$. The trick is to row-reduce the determinant to expose the $\sin\theta\cos\theta$ structure.

Sanity check. At $\theta = \pi/4$: $\sin\theta = \cos\theta = 1/\sqrt{2}$, $\Delta = (1/\sqrt{2})^2 = 1/2$. ✓

Numerical check at $\theta = \pi/4$. $\sin\theta = \cos\theta = 1/\sqrt{2}$. Matrix: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1+1/\sqrt{2} & 1 \\ 1 & 1 & 1+1/\sqrt{2} \end{pmatrix}$.

Apply $R_2 - R_1,\ R_3 - R_1$: $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix}$. Det $= 1 \cdot (1/\sqrt{2})^2 = 1/2$. ✓

Maximisation. $\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$ has maximum $1/2$ at $\theta = \pi/4 + n\pi$.

Generalisation. For determinants reducible to $\sin\theta\cos\theta$ via row operations, the bounds are always $[-1/2, 1/2]$. The factor of $1/2$ comes from $\sin 2\theta = 2\sin\theta\cos\theta$.

Key Takeaways

- For $n \times n$ matrices: $|kA| = k^n|A|$, $|AB| = |A||B|$, and $|A^{-1}| = |A|^{-1}$.
- A^{-1} exists iff $|A| \neq 0$; the formula is $A^{-1} = \frac{1}{|A|}\text{adj } A$.
- $A \cdot \text{adj } A = |A|I$, and $|\text{adj } A| = |A|^{n-1}$.
- Area of triangle $= \frac{1}{2}|\Delta|$ with the $\det\begin{pmatrix} x_i & y_i & 1 \end{pmatrix}$ formula; collinearity $\Leftrightarrow \Delta = 0$.
- Linear system $AX = B$ has unique solution $X = A^{-1}B$ when $|A| \neq 0$.
- Watch for: column-sum trick ($C_1 \to C_1 + \cdots$), Vandermonde patterns, A.P. rows forcing

> $R_1 - 2R_2 + R_3 = 0$, and odd-order skew-symmetric matrices with $\det = 0$.

End of NCERT Exemplar Problems