Collegedunia NCERT Formula Sheet

Class 12 Mathematics (12th Maths) — NCERT 2026-27

Chapter 4: Determinants

Definition | Expansion | Properties | Minors & Cofactors | Adjoint & Inverse | Area of Triangle | System of Linear Equations

Chapter Snapshot. The determinant assigns a single scalar to every square matrix. This chapter defines det for orders $1, 2, 3$; lists the seven **properties** that simplify evaluation; develops **minors**, **cofactors** and the **adjoint**; gives the determinant route to the **inverse** of a non-singular matrix; uses determinants to compute the **area of a triangle**; and finally solves systems of linear equations by the **matrix method** and **Cramer's-style consistency tests**.

1 Determinant — Definition

A determinant is a function det : $M_n(\mathbb{R}) \to \mathbb{R}$. *NCERT defines it explicitly for orders* 1, 2 *and* 3.

Order 1 and order 2

$$\det[a] = a$$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

where the entries are real numbers.

The 2×2 determinant is the **cross-product rule**: main-diagonal product minus anti-diagonal product.

Order 3 — expansion along row 1

For $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$,

$$\det(A) = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

Sign pattern across the first row is $+ - +$. The same value is obtained by expansion along **any row or any column**.

Where is det defined?

The determinant is defined **only for square matrices**. For a rectangular matrix det(A) has no meaning. Notation: det$(A) = |A| = \Delta$.

2 Properties of Determinants

These seven properties cut evaluation time dramatically; together with row/column operations they reduce most 3×3 *determinants to a single product.*

P1 — Row/column interchange

$$\det(A) = \det(A^T)$$

Every property stated for rows holds verbatim for columns. "Rows can be written as columns and vice-versa without changing the determinant."

P2 — Swapping two rows

If B is obtained from A by interchanging two rows (or two columns),

$$\det(B) = -\det(A)$$

Each swap flips the **sign** but preserves the magnitude. Two swaps cancel out.

P3 — Two identical rows

If any two rows (or columns) of A are identical or proportional,

$$\det(A) = 0$$

Follows from P2: swapping the identical pair must both preserve A and flip the sign — only 0 does both.

P4 — Scalar from a row

If every element of one row of A is multiplied by k, the new determinant is $k\,\det(A)$:

$$\det\begin{bmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = k\,\det(A)$$

Consequence: $\det(kA) = k^n \det(A)$ for an $n \times n$ matrix because the scalar comes out of **each of the n rows**.

P5 — Row split

If a row is the sum of two row-vectors, the determinant splits:

$$\det\begin{bmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1+\gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} + \det\begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Linearity holds **one row at a time** — the other rows must stay identical in the two pieces.

P6 — Row-operation invariance

Adding a multiple of one row to another row does not change the determinant:

$R_i \to R_i + k\,R_j \quad (j \neq i)$ leaves $\det(A)$ unchanged.

This is the workhorse property — used to create zeros in a row/column before expansion.

P7 — Triangular matrix

If A is upper- or lower-triangular (all entries above OR below the main diagonal are zero),

$$\det(A) = a_{11} \cdot a_{22} \cdots a_{nn}$$

Determinant of a **triangular** (and hence **diagonal**) matrix is just the product of its diagonal entries.

Product rule

$\det(AB) = \det(A)\det(B)$ for square matrices A, B of the same order. Therefore $\det(A^n) = (\det A)^n$ and $\det(A^{-1}) = 1/\det(A)$ when A is non-singular.

3 Area of a Triangle

A clean geometric application of the order-3 determinant — converts three coordinate pairs into a signed area.

Area via determinant

For vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$,

$$\text{Area} = \frac{1}{2}\left|\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right|$$

The **absolute value** is essential — area must be non-negative, but the bare determinant can be negative depending on vertex order.

Collinearity test

Three points are **collinear** iff

$$\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

Zero area $\Leftrightarrow$ the three points lie on a single straight line. Useful as a one-line collinearity check.

Equation of a line through two points

The line through (x_1, y_1) and (x_2, y_2) is

$$\textbf{det}\begin{bmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{bmatrix} = 0$$

Set the area of the triangle formed by (x, y) with the two fixed points to zero — that traces out the line.

4 Minors & Cofactors

Building blocks for the expansion theorem and the adjoint. Minors are pure submatrix determinants; cofactors attach the alternating sign.

Minor M_{ij}

The minor of entry a_{ij} is the determinant of the submatrix obtained from A by **deleting row i and column j**.

For an $n \times n$ matrix the minor M_{ij} is a determinant of order $n - 1$.

Cofactor A_{ij}

$A_{ij} = (-1)^{i+j} M_{ij}$

The sign $(-1)^{i+j}$ follows the chessboard pattern starting with $+$ at $(1, 1)$ and alternating across each row and column.

Expansion of $\det(A)$ — any row, any column

Along the i-th row: $\textbf{det}(A) = \sum_{j=1}^{n} a_{ij} A_{ij}$

Along the j-th column: $\textbf{det}(A) = \sum_{i=1}^{n} a_{ij} A_{ij}$

Choose the row or column with the **most zeros** to minimise computation.

"Wrong-row" cofactor sum

For any $i \neq k$: $\sum_{j=1}^{n} a_{ij} A_{kj} = 0$

Multiplying entries of one row by the cofactors of a **different** row gives zero — the very fact that makes $A \cdot \textbf{adj}(A) = \textbf{det}(A)\, I$ work.

Don't forget the sign

M_{ij} is a pure determinant; A_{ij} has the factor $(-1)^{i+j}$. **Always use A_{ij} (cofactors), not M_{ij} (minors)** when expanding $\det(A)$ or building the adjoint. Forgetting the sign at entry $(1, 2)$ or $(2, 1)$ is the single most common slip in this chapter.

5 Adjoint & Inverse

The adjoint packages all n^2 cofactors into one matrix and unlocks a closed-form formula for A^{-1}.

Adjoint of a square matrix

$\textbf{adj}(A) = [A_{ij}]^T$ — the **transpose** of the cofactor matrix.

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $\textbf{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

Build the cofactor matrix **row-wise**, then **transpose** once. Many slips here come from skipping the transpose step.

Adjoint identity

For any square matrix A of order n,

$A \cdot \textbf{adj}(A) = \textbf{adj}(A) \cdot A = \textbf{det}(A)\, I_n$

This single identity is the foundation of the inverse formula — solving it for A^{-1} gives the next box.

Singular vs non-singular

A is **singular** if $\textbf{det}(A) = 0$.

A is **non-singular** if $\textbf{det}(A) \neq 0$.

A square matrix has an inverse **iff it is non-singular**. "Invertible = non-singular" for square matrices.

Inverse via adjoint

If $\det(A) \neq 0$,

$$A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A)$$

Compute $\det(A)$ first; if it's zero, stop — no inverse exists. Otherwise build the adjoint

and divide entry-wise.

Determinant of the adjoint

For A of order n,

$\det(\text{adj } A) = (\det A)^{n-1}$

$\text{adj}(\text{adj } A) = (\det A)^{n-2}\, A$

Useful JEE-style identities; both follow from $A \cdot \text{adj}(A) = \det(A)\, I$ by taking determinants and substituting $\text{adj}(A)$ back in.

Inverse — algebraic identities

For non-singular A, B of the same order:

$(A^{-1})^{-1} = A, \quad (A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} = B^{-1}A^{-1}, \quad \det(A^{-1}) = \frac{1}{\det A}$

The product-reversal law for inverses mirrors the one for transposes — both "flip and reverse".

2×2 inverse — closed form

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$,

$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Swap the diagonals, negate the off-diagonals, divide by the determinant.

6 System of Linear Equations

Determinants give a clean criterion for when a linear system has a unique solution, and a direct route to that solution via A^{-1}.

Matrix form of a linear system

The system

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

can be written compactly as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

A is the **coefficient matrix**, X the unknowns, B the constants. Same structure works for any number of unknowns.

Matrix method — unique solution

If $\det(A) \neq 0$, A is non-singular and

$$X = A^{-1}B = \frac{1}{\det(A)}\,\text{adj}(A)\,B$$

The system has a **unique solution**. Compute A^{-1} once and read off x, y, z from the column $A^{-1}B$.

Consistency test — singular case

If $\det(A) = 0$, examine $(\text{adj } A)\, B$:

(a) $(\text{adj } A)\, B \neq O$: **inconsistent** (no solution).

(b) $(\text{adj } A)\, B = O$: system may be **consistent** with **infinitely many** solutions or **inconsistent** — needs further inspection.

When $B = O$ (homogeneous), $X = O$ is always a solution; non-trivial solutions exist iff $\det(A) = 0$.

Decision flow

Step 1. Compute $\Delta = \det(A)$.

Step 2. If $\Delta \neq 0$: unique solution $X = A^{-1}B$.

Step 3. If $\Delta = 0$ and $(\text{adj } A)B = O$: infinitely many or inconsistent.

Step 4. If $\Delta = 0$ and $(\text{adj } A)B \neq O$:

inconsistent.

Memorise this 4-step ladder — it answers every NCERT problem on linear systems via determinants.

Three for three

For a $3{\times}3$ system: **compute** Δ**, build** adj**, divide and multiply**. Three steps — exactly enough to solve any non-singular 3×3 system in under five minutes.

Quick Reference — Chapter 4 Determinants

Compact summary of every named identity used above

Concept	Statement / Formula
2×2 determinant	$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
3×3 row-1 expansion	$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$
$\det(A^T)$	$= \det(A)$
Row interchange	flips sign: $\det(B) = -\det(A)$
Equal/proportional rows	$\det(A) = 0$
Scalar from a row	multiplies det by k; $\det(kA) = k^n \det(A)$
Row split	determinant splits when one row is a sum
$R_i \to R_i + kR_j$	det unchanged
Triangular	det $=$ product of diagonal entries
Product rule	$\det(AB) = \det(A) \det(B)$
Power rule	$\det(A^n) = (\det A)^n$
Area of triangle	$\frac{1}{2}\lvert\det \begin{smallmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{smallmatrix}\rvert$
Collinearity	det $= 0$ with $(x_i, y_i, 1)$ rows
Line through two points	det with $(x, y, 1), (x_1, y_1, 1), (x_2, y_2, 1) = 0$
Minor M_{ij}	determinant after deleting row i, column j
Cofactor A_{ij}	$(-1)^{i+j} M_{ij}$
Expansion theorem	$\det(A) = \sum_j a_{ij} A_{ij}$ (along row i)
Wrong-row sum	$\sum_j a_{ij} A_{kj} = 0$ for $i \neq k$
Adjoint	transpose of the cofactor matrix
Adjoint identity	$A \cdot \text{adj}(A) = \det(A)\, I$
Inverse	$A^{-1} = \frac{1}{\det A} \text{adj}(A)$, requires $\det A \neq 0$
$\det(\text{adj } A)$	$= (\det A)^{n-1}$
Inverse reversal	$(AB)^{-1} = B^{-1}A^{-1}$
Matrix method	$AX = B \Rightarrow X = A^{-1}B$ when $\det A \neq 0$
Singular system test	$\det A = 0$; check $(\text{adj } A)B$
Homogeneous system	$AX = O$: non-trivial soln iff $\det A = 0$