

## Determinants

### 4.1 Introduction

To every square matrix  $A = [a_{ij}]$  of order  $n$  we associate a number (real or complex) called the determinant of  $A$ , denoted by  $\det A$ ,  $|A|$  or  $\Delta$ .

For a system of linear equations :

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

Matrix form :  $AX = B$  where

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

\*

### Remarks

- (i)  $|A|$  is read as determinant of  $A$ , not modulus of  $A$ . Notation :  $|A|$ ,  $\det A$ ,  $\Delta$
- (ii) Only square matrices have determinants.
- (iii)  $\det$  is a function  $f : M \rightarrow K$  from set of square matrices to set of scalars.

In this chapter we study determinants

upto order ~~two~~ three only with real entries.

Applications : area, adjoint, inverse,  $AX = B$ .

## 4.2 Determinant

### Order 1

If  $A = [a]$  is a  $1 \times 1$  matrix, then

$$\boxed{A = a} \text{ } \leftarrow \text{single entry}$$

### Order 2

Let  $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$  . Then

$$\det A \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} a_{22} - a_{21} a_{12}$$

$$\boxed{A = ad - bc} \begin{matrix} \leftarrow \text{diagonal} \\ \leftarrow \text{anti-diagonal} \end{matrix}$$

### Example 1

Evaluate  $\begin{vmatrix} 2 & -4 \\ -1 & 2 \end{vmatrix}$

$$= 2(2) - 4(-1)$$

$$= 4 + 4 = 8$$

### Example 2

Evaluate  $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$

$$= x \cdot x - (x+1)(x-1) = x^2 - x^2 + 1 = 1$$

## Determinant of Order 3

Expansion of a  $3 \times 3$  determinant is done along a ~~diagonal~~ row or a\* column.

All six expansions give the same value.

$$\text{Let } \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} *$$

### Expansion along $R_1$

Sign rule : multiply by  $(-1)^{i+j}$

$$a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\begin{aligned} &= a_{11} (a_{22} a_{33} - a_{23} a_{32}) \\ &\quad - a_{12} (a_{21} a_{33} - a_{23} a_{31}) \\ &\quad + a_{13} (a_{21} a_{32} - a_{22} a_{31}) . \end{aligned}$$

<- pattern + - + for first row .

Each  $2 \times 2$  minor is obtained by deleting the row & column of the chosen element.

## Expansion along any Row / Column

Along C1

$$A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$+ a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} *$$

Same value as R1 expansion (verify).

Sign Chessboard

+   -   +

-   +   -   ←  $(-1)^{i+j}$  raised to it

+   -   +

Remarks

(i) Expand along the row / column with maximum zeros - faster arithmetic.

(ii) All six expansions are equal in value.

(iii) IF  $A$  is  $n \times n$  and  $B = kA$ , then

$$\boxed{B = k^n A} \quad \begin{array}{l} \leftarrow k \text{ pulled } n \text{ times} \\ \leftarrow \text{one per row} \end{array}$$

Eg.  $2A = 2^n A$ ,  $n = \text{order of } A$ .

## Worked Example

Evaluate  $\begin{vmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

Sol. Two zeros in  $C_3 \rightarrow$  expand along  $C_3$  :

$$D = 4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0 + 0$$

$$= 4 ( -1 - 12 )$$

$$= 4 ( -13 ) = -52 .$$

Eg. Evaluate  $\begin{vmatrix} 0 & \sin a & -\cos a \\ -\sin a & 0 & \sin b \\ \cos a & -\sin b & 0 \end{vmatrix}$

Expanding along  $R_1$  :

$$D = 0 - \sin a ( 0 - \sin b \cos a )$$

$$- \cos a ( \sin a \sin b - 0 )$$

$$= \sin a \sin b \cos a - \cos a \sin a \sin b$$

$$= 0 .$$

$\leftarrow$  skew-symmetric !

## Properties of Determinants

P1 .  $A' = A$

Value of det is unchanged if rows and columns are interchanged (transpose).

$\Rightarrow$  rows & columns have equal status.

P2 . Row-swap changes sign.

If any two rows (or columns) are interchanged, sign of determinant changes :

$$D' = -D$$

$\leftarrow$  one swap = one -

P3 . Two equal rows  $\rightarrow D = 0$

If any two rows (or columns) are identical or proportional, then  $D = 0$ .

Eg. 
$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = 0 \leftarrow R_1 = R_3$$

Reason : swapping the two equal rows

leaves  $D$  unchanged, but by P2 also gives  $-D$ ,

so  $D = -D \Rightarrow 2D = 0 \Rightarrow D = 0$ .

Useful for collinearity / linear dependence.

## Properties (continued)

### P4 . Scalar from a row

If each element of a row is multiplied by  $k$ , the determinant is multiplied by  $k$ .

$$\boxed{k R_i = k \cdot A} \quad \begin{array}{l} \leftarrow k \text{ taken out of} \\ \leftarrow \text{ONE row only} \end{array}$$

### P5 . Sum-row split

If a row is expressed as a sum of two rows,  $D$  splits as sum of two determinants:

$$\begin{array}{c} (a+x) \quad (b+y) \quad ; \quad c \quad d \quad ; \quad e \quad f \quad g \\ a \quad b \quad ; \quad c \quad d \quad ; \quad e \quad f \quad g \end{array} = \begin{array}{c} a \quad b \quad ; \quad c \quad d \quad ; \quad e \quad f \quad g \\ x \quad y \quad ; \quad c \quad d \quad ; \quad e \quad f \quad g \end{array}$$

### P6 . Row-operation $R_i \rightarrow R_i + k R_j$

Value of  $D$  remains unchanged if a multiple of one row is added to another row :

$$\boxed{R_i \rightarrow R_i + k R_j : D' = D} \quad \begin{array}{l} \leftarrow \text{key step for} \\ \leftarrow \text{simplification} \end{array}$$

Same operations valid for columns  $(C_i)$ .

Used to create zeros for easy expansion.

## More Properties

### P7 . Triangular determinant

IF : A is upper / lower triangular, then

$$\boxed{A = a_{11} a_{22} a_{33} \dots a_{nn}} \quad \begin{array}{l} \leftarrow \text{product of} \\ \leftarrow \text{diagonal} \end{array}$$

### P8 . Determinant of a product

$$\boxed{AB = A \cdot B} \quad \begin{array}{l} \leftarrow \text{even though} \\ \leftarrow AB \neq BA \end{array}$$

Consequences :

- $A^n = A \cdot A \cdot \dots \cdot A$  , n is a positive integer.
- $kA = k^n A$  , n = order of A .
- $I = I$  ,  $A^{-1} = I / A$

Eg.

$$\text{IF } \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^*$$

$$\text{Then } A = 2B \Rightarrow A = 2 \cdot 2 \cdot B = 4B$$

$$\text{Check : } A = 0 - 8 = -8, \quad B = 0 - 2 = -2$$

$$\text{So } A = 4(-2) = -8. \quad \text{Verified } \leftarrow n = 2$$

## Using Properties - Worked

Show that 
$$\begin{vmatrix} a & b & c \\ a+x & b+y & c+z \\ x & y & z \end{vmatrix}$$

= 0 using row operations.

### Solution

Apply  $R_2 \rightarrow R_2 - R_1 - R_3$  :

$$\begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ x & y & z \end{vmatrix} = 0 \dots \leftarrow \text{zero row}$$

Reason : a row of all zeros makes  $D = 0$

(expand <sup>\*</sup> along the zero row  $\rightarrow$  each term carries factor 0). QED.

### Useful Tricks

(i)  $R_i \rightarrow R_i + k R_j$  keeps  $D$  same.

(ii) Pull common factor out of a row.

(iii) Create two zeros in a row / column  $\rightarrow$   
only one  $2 \times 2$  minor to compute.

(iv) Symmetry  $\rightarrow$  spot equal rows quickly.

## 4.3 Area of a Triangle

Vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  :

$$\text{Area} = (1/2) \quad x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)$$

Determinant form :

$$\text{Area} = (1/2) \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \begin{array}{l} \leftarrow \text{always} \\ \leftarrow \text{take} \end{array}$$

### Remarks

- (i) Area is positive, take det.
- (ii) If area is given, use + and - signs.
- (iii) Three collinear points  $\rightarrow$  area = 0.

$\text{Collinear} \Leftrightarrow D = 0$

$\leftarrow$  test for  
 $\leftarrow$  3 pts

Eg. 6 Vertices  $(3, 8), (-4, 2), (5, 1)$

$$\text{Area} = (1/2) \begin{vmatrix} 3 & 8 & 1 \\ -4 & 2 & 1 \\ 5 & 1 & 1 \end{vmatrix}$$

## Area - calculation

Expanding along  $R_1$  :

$$\begin{aligned} & (1/2) [ 3 (2 - 1) - 8 (-4 - 5) + 1 (-4 - 10) ] \\ &= (1/2) [ 3 + 72 - 14 ] \\ &= (1/2) (61) = 61 / 2 \text{ sq. units.} \quad \leftarrow \text{answer} \end{aligned}$$

### Equation of line through $(x_1, y_1), (x_2, y_2)$

Let  $P (x, y)$  be any point on the line.

Then  $P, A, B$  are collinear  $\Rightarrow$

$$(1/2) \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

Eg. Line through  $A(1,3), B(0,0)$  :

$$(1/2) \begin{vmatrix} 0 & 0 & 1 \\ 1 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

Expand along  $R_1$  :  $(1/2)(y - 3x) = 0$

$\Rightarrow y = 3x$

$\leftarrow$  required  
 $\leftarrow$  line

## 4.4 Minors & Cofactors

### Definition - Minor $M_{ij}$

Minor of an element  $a_{ij}$  is the determinant obtained by deleting the  $i$ -th row and  $j$ -th column in which  $a_{ij}$  lies.

Order of  $M_{ij}$  is  ~~$n$~~   $n - 1$ .

### Definition - Cofactor $A_{ij}$

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

$\leftarrow$  signed  
 $\leftarrow$  minor

Sign of cofactor depends on position  $(i+j)$  :

$$\begin{array}{ccc} + & - & + \\ - & + & - \\ + & - & + \end{array} \quad \leftarrow \text{sign chart}$$

### Expansion using cofactors

$$A = \sum_j a_{ij} \cdot A_{ij}$$

$\leftarrow$  fix any row  $i$  ;  
 $\leftarrow$  sum over  $j$  .

## Minors & Cofactors - Eg.

Find minors and cofactors of

$$\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix}$$

### Minors

$$M_{11} = 3, \quad M_{12} = 4,$$

$$M_{21} = -2, \quad M_{22} = 1.$$

### Cofactors

$$A_{11} = (-1)^2 \cdot 3 = 3,$$

$$A_{12} = (-1)^3 \cdot 4 = -4,$$

$$A_{21} = (-1)^3 \cdot (-2) = 2,$$

$$A_{22} = (-1)^4 \cdot 1 = 1.$$

### Note (orthogonality)

Sum of product of a row's elements with cofactors of another row is zero :

$$a_{i1} A_{k1} + a_{i2} A_{k2} + a_{i3} A_{k3} = 0 \quad \left( \begin{array}{l} \text{key} \\ \text{lemma} \end{array} \right)$$

Reason : equivalent to det with two equal rows.

## 4.5 . Adjoint & Inverse

### Adjoint - Definition

Adjoint of a square matrix  $A = [a_{ij}]$  is the transpose of the cofactor matrix  $[A_{ij}]$ .

Denoted by  $\text{adj } A$ .

### Construction

Step 1 : compute every cofactor  $A_{ij}$ .

Step 2 : form cofactor matrix  $C = [A_{ij}]$ .

Step 3 : transpose  $\rightarrow \text{adj } A = C'$ .

For order 3 :

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{rows of} \\ \leftarrow \text{cofactor} \end{array}$$

(notice the  $i, j$  swap - that's the transpose).

### Shortcut (order 2)

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  then

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \begin{array}{l} \leftarrow \text{swap diagonal,} \\ \leftarrow \text{negate off-diag.} \end{array}$$

Eg.  $\text{adj } A$  for  $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Sol.

$$A_{11} = (-1)^2 \cdot 4 = 4, \quad A_{12} = (-1)^3 \cdot 1 = -1$$

$$A_{21} = (-1)^3 \cdot 3 = -3, \quad A_{22} = (-1)^4 \cdot 2 = 2$$

Cofactor matrix  $\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$

Transpose  $\Rightarrow$

$$\boxed{\text{adj } A} = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

\*  
 $\leftarrow$  rows  $\rightarrow$   
 $\leftarrow$  columns

Verify by shortcut

Swap diagonals  $(2, 4) \rightarrow (4, 2)$

Negate off-diagonals  $(3, 1) \rightarrow (-3, -1)$ .

Same result !

$\leftarrow$  consistent

Always check  $A \cdot \text{adj } A = A \cdot I$  as final test.

Here  $A = 2(4) - 1(3) = 5$ ; row check :

$$2(4) + 3(-1) = 5, \quad 2(-3) + 3(2) = 0. \quad \text{Good.}$$

## Adjoint Theorem

### Theorem 1.

For any square matrix  $A$  of order  $n$ ,

$$A \cdot (\text{adj } A) = (\text{adj } A) \cdot A = A \cdot I \quad \left\{ \begin{array}{l} \leftarrow \text{key} \\ \leftarrow \text{identity} \end{array} \right.$$

### Verification ( $n = 3$ )

$(i, i)$  entry of  $A \cdot (\text{adj } A) =$

$$a_{i1} A_{i1} + a_{i2} A_{i2} + a_{i3} A_{i3} = A$$

$(i, k)$  entry,  $i \neq k$  :

$$a_{i1} A_{k1} + a_{i2} A_{k2} + a_{i3} A_{k3} = 0$$

(orthogonality of cofactors).

### adj A formula

$$\text{adj } A = A^{n-1} \quad \left\{ \begin{array}{l} \leftarrow n = \text{order} \\ \leftarrow \text{of } A \end{array} \right.$$

Derivation :  $A \cdot \text{adj } A = A \cdot I = A \cdot n$

$\Rightarrow$  if  $A \neq 0$ ,  $\text{adj } A = A^{n-1}$ .

For  $n = 3$  :  $\text{adj } A = A^2$ .

## Singular & Inverse Formula

### Singular vs Non-singular

$A$  is singular if  $A = 0$ ,

$A$  is non-singular if  $A \neq 0$ .

### Theorem 4 (Inverse exists)

A square matrix  $A$  is invertible if and only if

$A$  is non-singular (i.e.  $A \neq 0$ ).

$$A^{-1} = (I \mid A) \cdot \text{adj } A \quad \leftarrow \text{valid only if } A \neq 0$$

### Proof sketch

$$A \cdot (\text{adj } A) = A \cdot I \quad (\text{Theorem 1})$$

Divide both sides by  $A$  ( $A \neq 0$ ):

$$A \cdot [(I \mid A) \text{adj } A] = I.$$

By uniqueness of inverse:

$$A^{-1} = (I \mid A) \text{adj } A. \quad \text{QED.}$$

### Quick checks

$$A^{-1} = (I \mid A)$$

$$A \cdot A^{-1} = A^{-1} \cdot A = I.$$

$$(A^{-1})^{-1} = A, \quad (A')^{-1} = (A^{-1})'$$

Eg. 13  $A^{-1}$  for a  $3 \times 3$ 

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

Step 1  $A$ Expand along  $R_1$  :

$$A = 1(16 - 9) - 3(4 - 3) + 3(3 - 4)$$

$$= 7 - 3 - 3 = 1. \quad (\neq 0, \text{ so } A^{-1} \text{ exists.})$$

Step 2 Cofactors

$$A_{11} = 7, \quad A_{12} = -1, \quad A_{13} = -1$$

$$A_{21} = -3, \quad A_{22} = 1, \quad A_{23} = 0$$

$$A_{31} = -3, \quad A_{32} = 0, \quad A_{33} = 1$$

Step 3  $\text{adj } A$  (transpose)

$$\text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Step 4  $A^{-1} = (1/A) \text{adj } A$ 

$$A^{-1} = (1/1) \cdot \text{adj } A = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

## Properties of Inverse

① Reversal law :

$$(A B)^{-1} = B^{-1} \cdot A^{-1} \quad \leftarrow \begin{array}{l} \text{note the} \\ \text{order swap} \end{array}$$

② Transpose :  $(A^{-1})' = (A')^{-1}$  .

③ Double inverse :  $(A^{-1})^{-1} = A$  .

④ Determinant :  $|A^{-1}| = \frac{1}{|A|}$  .

⑤ Identity :  $I \cdot A^{-1} = A^{-1} \cdot I = A^{-1}$  .

Quick verification  $(AB)^{-1} = B^{-1} A^{-1}$

$$A = \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$AB = \begin{pmatrix} -1 & 5 \\ 5 & -14 \end{pmatrix}, \quad AB^{-1} = \begin{pmatrix} -1 & 1 \\ 5 & -11 \end{pmatrix}$$

$$\begin{aligned} (AB)^{-1} &= \frac{1}{|AB|} \begin{pmatrix} -14 & -5 \\ -5 & -1 \end{pmatrix} \\ &= \frac{1}{11} \begin{pmatrix} 14 & 5 \\ 5 & 1 \end{pmatrix} \end{aligned}$$

$$B^{-1} A^{-1} = \frac{1}{11} \begin{pmatrix} 14 & 5 \\ 5 & 1 \end{pmatrix} \quad \text{Matches.}$$

$(A^{-1})^n = (A^n)^{-1}$  for positive  $n$  .

## 4.6 Applications - $AX = B$

### System of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

Matrix form :

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} X = B,$$

where  $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  ,  $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

### Definitions

Consistent : has at least one solution.

Inconsistent : has no solution.

Unique soln : exactly one  $(x, y, z)$ .

Infinite soln : free parameter present.

Here we study unique-solution case

(matrix method) and consistency via  $A$ .

## Matrix Method - Case I

If  $A \neq 0$  ( $A$  is non-singular), then  $A^{-1}$  exists, and

$$A X = B$$

$$\Rightarrow A^{-1} (A X) = A^{-1} B \quad (\text{pre-mult.})$$

$$\Rightarrow (A^{-1} A) X = A^{-1} B \quad (\text{assoc.})$$

$$\Rightarrow I X = A^{-1} B$$

$$X = A^{-1} \cdot B$$

$\leftarrow$  unique  
 $\leftarrow$  solution

### Steps

1. Find  $A \neq 0$ ; if 0 switch to Case II.
2. Find  $\text{adj } A$ .
3.  $A^{-1} = (1/A) \text{adj } A$ .
4.  $X = A^{-1} B \rightarrow$  read off  $x, y, z$ .

### Cramer's Rule (parallel method)

An alternative to find  $x, y, z$  using ratios of determinants - covered next page.

Both methods need  $A \neq 0$ ; if  $A = 0$  is

## Cramer's Rule - Two Variables

$$a_1 x + b_1 y = c_1$$

$$a_2 x + b_2 y = c_2$$

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \quad \leftarrow \text{coeff det.}$$

$$D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \quad \leftarrow \text{-x-col } \rightarrow c$$

$$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \quad \leftarrow \text{-y-col } \rightarrow c$$

$$\boxed{x = D_x / D, \quad y = D_y / D} \quad \leftarrow \text{if } D \neq 0$$

Eg.

$$\text{Solve } 2x + 5y = 1, \quad 3x + 2y = 7.$$

$$D = 4 - 15 = -11$$

$$D_x = 2 - 35 = -33 \Rightarrow x = 3$$

$$D_y = 14 - 3 = 11 \Rightarrow y = -1$$

Hence  $(x, y) = (3, -1)$ . Same as matrix method.

## Cramer's Rule - Three Variables

For

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad (\text{coefficient det.})$$

$$D_x = \text{replace } x\text{-column by } d\text{'s}$$

$$D_y = \text{replace } y\text{-column by } d\text{'s}$$

$$D_z = \text{replace } z\text{-column by } d\text{'s}$$

$$x = D_x/D, \quad y = D_y/D, \quad z = D_z/D$$

$\leftarrow D \neq 0$

### Three Cases

(i)  $D \neq 0 \rightarrow$  unique solution.

(ii)  $D = 0$ , all  $D_x, D_y, D_z = 0 \rightarrow$   
infinitely many solutions.

(iii)  $D = 0$ , at least one of  $D_x, D_y, D_z \neq 0 \rightarrow$   
no solution (inconsistent).

Algorithm (case  $D = 0$ ) :

compute  $(\text{adj } A) B$  - if it equals 0 then  
system may be consistent ; else inconsistent.

## Consistency - Singular Case

When  $A = 0$ ,  $A^{-1}$  does not exist.

Compute  $(\text{adj } A) B$  to decide :

①  $(\text{adj } A) B = 0$  :

system may have infinitely many solutions OR no solution. Check by substitution / dependency among rows.

②  $(\text{adj } A) B \neq 0$  :

system has no solution  $\rightarrow$  inconsistent.

### Summary

| $A$      | State    | $(\text{adj } A) B$ |
|----------|----------|---------------------|
| $\neq 0$ | unique   | -                   |
| $= 0$    | infinite | 0                   |
| $= 0$    | no soln. | $\neq 0$            |

### Tip

First compute  $A$  ; it decides the route.

Don't compute  $A^{-1}$  if  $A = 0$  (waste !).

Common ~~error~~ mistake : forgetting to compute  $(\text{adj } A) B$  in singular case before declaring.

Eg. Solve 3x3 System

Solve

$$3x - 2y + 3z = 8$$

$$2x + y - z = 1$$

$$4x - 3y + 2z = 4$$

Sol.

$$A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} A &= 3(2 \cdot 3) + 2(4 \cdot 4) + 3(-6 \cdot 4) \\ &= -3 + 16 - 30 = -17 \quad (\neq 0) \end{aligned}$$

$$A_{11} = -1, \quad A_{12} = -8, \quad A_{13} = -10$$

$$A_{21} = -5, \quad A_{22} = -6, \quad A_{23} = 1$$

$$A_{31} = -1, \quad A_{32} = 9, \quad A_{33} = 7$$

$$A^{-1} = (-1/17) \begin{bmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{bmatrix}$$

$$X = A^{-1} B = (-1/17) [-17; -34; -51]$$

$$\Rightarrow x = 1, \quad y = 2, \quad z = 3 \quad \leftarrow \text{answer}$$

## Word Problem

Sum of three numbers is 6. If the third is multiplied by 3 and the second added gives 11.

Sum of 1st and 3rd equals twice the second.

Find the numbers by matrix method.

### Setup

Let  $x, y, z$  be the numbers.

$$x + y + z = 6$$

$$y + 3z = 11$$

$$x - 2y + z = 0$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$A = 1(1+6) - 1(0-3) + 1(0-1) = 9 \neq 0$$

Compute cofactors  $\rightarrow \text{adj } A = \text{~~3x2~~ } 3 \times 3$

$$X = (1/9) (\text{adj } A), \quad B = (1/9) [9; 18; 27]$$

$x = 1, \quad y = 2, \quad z = 3$

$\leftarrow$  the three  
 $\leftarrow$  numbers

Check :  $1 + 2 + 3 = 6$ . Good.

## Key Formula Bank

① Order 2 :  $A = ad - bc$

② Order 3 : expand along R or C  
with + - + signs (chessboard).

③ Cofactor :  $A_{ij} = (-1)^{(i+j)} M_{ij}$

④ Adj A : transpose of cofactor matrix

⑤ Inverse :

$$A^{-1} = (1/\det A) \text{adj } A$$

! = central  
← formula

⑥ Area :  $\text{Area} = (1/2) \det$

⑦  $AX = B : X = A^{-1} B$

⑧ Cramer :  $x = D_x/D, y = D_y/D, z = D_z/D$

⑨  $kA : k \cdot n \cdot A$

10.  $AB = A \cdot B, A^{-1} = 1/A$

11.  $\text{adj } A = A^{(n-1)}$ . Memorise. !

## Common Mistakes & Tips

### Common Errors

① Sign chart :

forgetting the ~~+++~~ + - + pattern in expansion

② Order matters :

$(AB)^{-1} = B^{-1} A^{-1}$  NOT  $A^{-1} B^{-1}$ .

③  $k \cdot A$  :

$kA = k^n A$ , not  $k \cdot A$ .

④ Singular trap :

if  $\det A = 0 \rightarrow$  no  $A^{-1}$ . Use  $(\text{adj } A) B$ .

⑤ Area sign :

always take  $|\det|$  for area (positive).

### Exam Tips

- Use row / col with most zeros for expansion.
- Row operations preserve  $D$ , aim for triangle.
- Cross-check by  $A \cdot A^{-1} = I$  (one row enough).
- For Cramer, write  $D, D_x, D_y, D_z$  clearly.
- Collinearity :  $\det = 0$  (3 pts).
- End every solution with a final boxed answer.