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Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 4: Determinants

What this chapter covers: the determinant of a square matrix (orders 1, 2, 3), expansion by cofactors along any row/column, the geometric meaning as a signed area, minors and cofactors, the adjoint and inverse via adj(A), and using A^{-1} to solve systems of linear equations. We also flag classic properties of determinants and Cramer's rule as *[JEE/NEET Extension]* since they are tested heavily in entrances although de-emphasised in the rationalised NCERT.

1 Determinant of a Square Matrix

A matrix is an array of numbers; a **determinant** is a single number we extract from a *square* matrix that captures, in one scalar, several geometric and algebraic facts about it — whether the matrix is invertible, how it scales areas/volumes, and whether the linear system it represents has a unique solution.

Throughout this chapter, $A = [a_{ij}]$ is a square matrix of order n (with $n = 1, 2,$ or 3), and we write the determinant as $|A|$, $\det(A)$, or Δ. Crucially: **only square matrices have determinants.** The notation $|A|$ stands for "determinant of A", *not* the absolute value of A.

1.1 Determinant of order 1 and 2

For a 1×1 matrix $A = [a]$, the determinant is simply the entry: $|A| = a$.

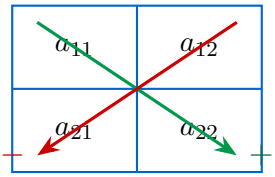
For a 2×2 matrix, the rule is a cross-product of diagonals:

Determinant of a 2×2 Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\,a_{22} - a_{21}\,a_{12}.$$

(Main diagonal product minus anti-diagonal product.)



Down-right diagonal: positive. Down-left: negative.

Example. For $A = \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix}$, we have $|A| = (2)(2) - (-1)(4) = 4 + 4 = 8$.

Determinant as a Function

We can think of det as a function $\det : M_n \to \mathbb{R}$ that takes any $n \times n$ matrix and returns a single real number. Square matrix in, scalar out.

Contents

1 Determinant of a Square Matrix **1**
1.1 Determinant of order 1 and 2 . . . 1
1.2 Determinant of order 3 by cofactor expansion . . . 3
1.3 Cofactor expansion works along any row or column . . . 4
1.4 Row-2 and column expansions in detail . . . 5
1.5 Geometric meaning of the determinant . . . 5

2 Properties of Determinants *[JEE/NEET Extension]* **6**
2.1 The seven core properties . . . 6
2.2 Useful corollaries . . . 7
2.3 When the row operation introduces a scalar . . . 8
2.4 Worked example using properties . . . 8

3 Area of a Triangle **8**
3.1 The area formula . . . 9
3.2 Collinearity test . . . 9

4 Minors, Cofactors, Adjoint and Inverse **10**
4.1 Minor and Cofactor . . . 10
4.2 Adjoint of a matrix . . . 11
4.3 Singular and non-singular matrices . . . 12

4.4 Inverse via the adjoint 12
4.5 Properties of inverse and adjoint 13

5 Solving Linear Systems with A^{-1} 13
5.1 Matrix form of a linear system 14
5.2 Solution by inverse 14
5.3 Consistency of a linear system 15
5.4 Cramer's Rule *[JEE/NEET Extension]* 16

6 Quick Reference Summary 17
6.1 Determinant formulas 17
6.2 Identities at a glance 18
6.3 Solution checklist for a linear system 18
6.4 Determinant property cheat-card 18
6.5 Final word 18

1.2 Determinant of order 3 by cofactor expansion

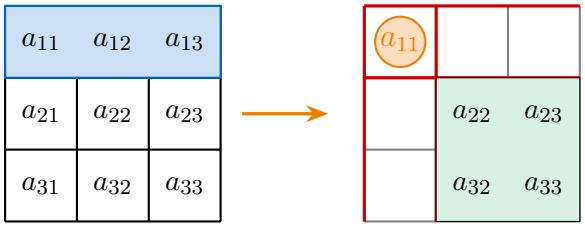
For a 3×3 matrix, the determinant is built from **three** 2×2 **sub-determinants**, weighted by entries of one chosen row (or column), with alternating signs.

Determinant of a 3×3 Matrix — Expansion along R_1

For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$,

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

Each 2×2 block is the determinant of the matrix you get by *deleting the row and column* of the multiplying entry. The signs follow a fixed checkerboard pattern, $+, -, +, -, \ldots$, starting from the top-left corner.



Expand along Row 1 **Minor M_{11}**

The figure shows the first step: we strike out the row and column of a_{11}, leaving a

2×2 **minor** whose determinant multiplies a_{11}. Repeat for a_{12} (with a minus sign) and a_{13} (with a plus sign again).

Sign Pattern $+ \; - \; + \; - \; \ldots$

The cofactor sign at position (i, j) is $(-1)^{i+j}$. For a 3×3 this gives the chessboard:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

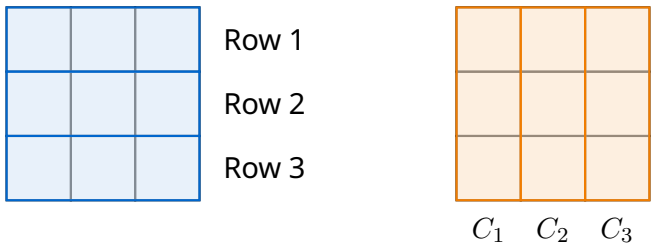
Memory hook: *top-left is plus; flip each step*.

1.3 Cofactor expansion works along any row or column

A central, non-obvious fact: you can expand along any of the three rows or any of the three columns, and you always get the same number.

Six Ways to Expand — One Answer

For a 3×3 matrix, there are 3 rows $+3$ columns $= 6$ valid expansion lines. All six give the identical determinant. **Choose the row or column with the most zeros** — it minimises arithmetic.



All six expansion lines (3 rows in blue, 3 columns in orange) yield the same $|A|$.

Example. Take $A = \begin{bmatrix} 1 & 2 & 4 \\ -1 & 3 & 0 \\ 4 & 1 & 0 \end{bmatrix}$. Column 3 has two zeros, so expand along C_3:

$$|A| = 4 \begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} - 0 + 0 = 4(-1 - 12) = 4(-13) = -52.$$

Comparison example. Verify that row-1 and row-3 expansion agree for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix}.$$

Along R_1**:** $|A| = 1(0 - 24) - 2(0 - 20) + 3(0 - 5) = -24 + 40 - 15 = 1$.
Along R_3**:** $|A| = 5(8 - 3) - 6(4 - 0) + 0 = 25 - 24 + 0 = 1$.
Both routes give $|A| = 1$, as the theorem promises.

1.4 Row-2 and column expansions in detail

For completeness, here are the formulas for the remaining five expansion lines. They differ only in the entry-weights and the sign pattern; the underlying minors come from the same 3×3 matrix.

Expansion along R_2 and C_2

Along R_2: $|A| = -a_{21}\,M_{21} + a_{22}\,M_{22} - a_{23}\,M_{23}$.

Along C_2: $|A| = -a_{12}\,M_{12} + a_{22}\,M_{22} - a_{32}\,M_{32}$.

The signs come straight from $(-1)^{i+j}$: row-2 entries carry signs $-, +, -$; column-2 entries also $-, +, -$.

Hunt for Zeros First

Before plugging into the row-1 expansion blindly, scan the matrix. If any row or column has two zeros, expand along it and the calculation collapses to a single 2×2 determinant.

Sign Slip on a_{12}

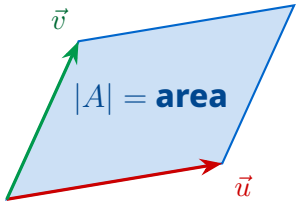
The middle entry of any row (or column) carries a *minus* sign because $(-1)^{1+2} = -1$. Students routinely forget this and get the wrong sign for the whole determinant. Write the signs $+, -, +$ *first*, then the entries.

1.5 Geometric meaning of the determinant

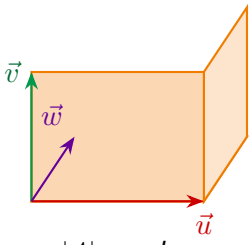
A determinant is more than an arithmetic gadget — it has a vivid geometric interpretation that explains every algebraic property in one picture.

Determinant as a Signed Area / Volume

For a 2×2 matrix whose columns are the vectors $\vec{u}$ and $\vec{v}$ in the plane, $|A|$ is the *signed area* of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.
For a 3×3 matrix with columns $\vec{u}, \vec{v}, \vec{w}$ in 3-D, $|A|$ is the *signed volume* of the parallelepiped spanned by them.
Sign is positive if the basis is right-handed, negative if left-handed. Zero if the vectors are linearly dependent (parallelogram squashed flat, parallelepiped flattened).



Columns of A span this parallelogram.



$|A| =$ *volume*

This picture makes seven of the standard properties (next section) obvious at sight. For instance, two equal columns flatten the parallelogram to a line $\Rightarrow$ zero area; scaling a column by k stretches the parallelogram by k in one direction $\Rightarrow$ area scales by k; swapping two columns reverses orientation $\Rightarrow$ sign flips.

Determinants in Physics and Engineering

The Jacobian determinant in multivariable calculus measures how a coordinate change distorts volumes — it's exactly this signed-volume idea applied locally to curved coordinates. Robotics uses determinants to detect when a robot arm reaches a *singular configuration* (the Jacobian goes to zero), at which point the arm temporarily loses a degree of freedom.

2 Properties of Determinants *[JEE/NEET Extension]*

The rationalised NCERT 2026–27 syllabus for Class 12 no longer derives the seven classical properties as a standalone section, but they remain non-negotiable for JEE/NEET and frequently shorten a board-level computation as well. Learn them in this exact order — they build on each other.

2.1 The seven core properties

The Seven Properties of $|A|$

P1. Row–Column Symmetry. The determinant of a matrix equals that of its transpose:

$$|A| = |A^T|.$$

P2. Row/Column Interchange. Swapping two rows (or two columns) multiplies the determinant by -1.

$$R_i \leftrightarrow R_j \implies |A| \to -|A|.$$

P3. Two Identical Rows/Columns. If any two rows (or columns) are identical, then $|A| = 0$.

$$R_i = R_j \implies |A| = 0.$$

P4. Scalar Multiple of a Row/Column. If every entry of a row (or column) is multiplied by k, the determinant scales by k:

$$R_i \to k\,R_i \implies |A| \to k\,|A|.$$

Corollary: $|kA| = k^n|A|$ for an $n \times n$ matrix. **P5. Sum-Splitting in One Row/Column.** If a single row is a sum of two row-vectors, the determinant splits:

$$\begin{vmatrix} a_1+p_1 & a_2+p_2 & a_3+p_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & p_2 & p_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

P6. Row Operation $R_i \to R_i + k\,R_j$. Adding a multiple of one row to another *does not change* the determinant:

$$R_i \to R_i + k\,R_j \implies |A| \text{ unchanged.}$$

P7. Determinant of a Product. For square matrices of the same order,

$$|AB| = |A|\,|B|.$$

Propert	Statement	Effect on $\lvert A\rvert$
P1	$\lvert A\rvert = \lvert A^T\rvert$	Unchanged
P2	Swap two rows (or columns)	Multiplied by -1
P3	Two identical rows/columns	Equals 0
P4	Multiply a row by k	Multiplied by k
P5	Row is a sum of two row-vectors	Determinant splits as a sum
P6	$R_i \to R_i + k\,R_j$	Unchanged
P7	Product rule	$\lvert AB\rvert = \lvert A\rvert\,\lvert B\rvert$

2.2 Useful corollaries

A few derived facts follow straight from the seven properties; together with them, they form the working toolkit for simplifying any determinant before evaluating.

- If a row (or column) is entirely zero, $|A| = 0$. (Apply P4 with $k = 0$.)
- If one row is a scalar multiple of another, $|A| = 0$. (Apply P4 to reduce to two identical rows, then P3.)
- For an upper-triangular (or lower-triangular) matrix, $|A|$ = product of diagonal entries.
- $|A^{-1}| = 1/|A|$, provided $|A| \neq 0$. (Apply P7 to $AA^{-1} = I$, with $|I| = 1$.)

Triangular Trick

If you can reduce a matrix to upper-triangular form using P6 (which leaves the determinant unchanged), the answer is just the product of the diagonal. This is faster than full cofactor expansion for 4×4 and bigger matrices — and

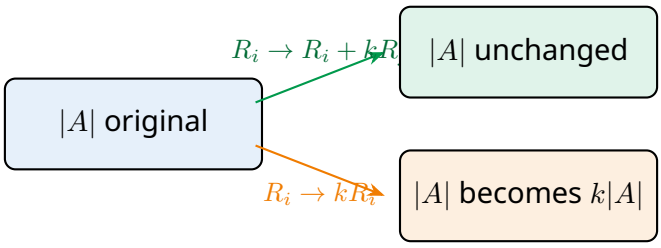
shows up as a JEE shortcut.

$|A+B| \neq |A|+|B|$

The determinant is multiplicative ($|AB| = |A||B|$) but *not* additive. Writing $|A+B| = |A|+|B|$ is the single most common error in this chapter. There is no general formula for $|A+B|$.

2.3 When the row operation introduces a scalar

Be careful about pulling scalars when applying row operations. The rule $R_i \to R_i + kR_j$ leaves the determinant unchanged. But a single *scalar multiplication* $R_i \to kR_i$ multiplies the determinant by k — you must compensate by dividing by k outside the determinant.



2.4 Worked example using properties

Evaluate $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ (both leave Δ unchanged by P6):

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & (b-a)(b+a) \\ c-a & (c-a)(c+a) \end{vmatrix}.$$

Factor out $(b-a)$ from R_1 of the 2×2 and $(c-a)$ from R_2 (P4 twice):

$$\Delta = (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} = (b-a)(c-a)(c-b).$$

This is the classical **Vandermonde determinant**, ubiquitous in JEE algebra problems.

3 Area of a Triangle

Determinants give a clean coordinate-free formula for the area of a triangle in the plane, with one beautiful consequence: **three points are collinear iff the corresponding determinant vanishes**.

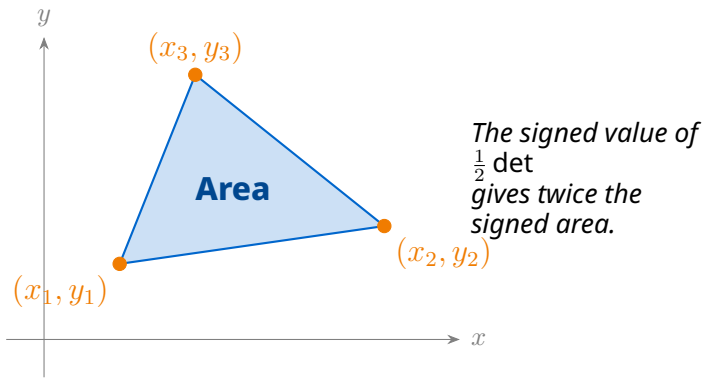
3.1 The area formula

Area of a Triangle via Determinant

The area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|.$$

The outer $|\cdot|$ is the absolute value — area is non-negative. The inner $|\cdot|$ is the determinant.



Why the formula? Expanding along C_3 (the column of 1's) recovers the familiar shoelace identity:

$$\text{Area} = \tfrac{1}{2}\left|\, x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \,\right|.$$

So the determinant view and the shoelace formula are literally the same thing, with the determinant being easier to remember.

3.2 Collinearity test

Three Points are Collinear $\iff$ Determinant is Zero

If $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ lie on a single straight line, the area of the "triangle" they form is zero — so

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

Conversely, if this determinant equals zero, the three points are collinear.

Example. Are $(1, 2), (3, 6), (5, 10)$ collinear?

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 5 & 10 & 1 \end{vmatrix} = 1(6 - 10) - 2(3 - 5) + 1(30 - 30) = -4 + 4 + 0 = 0.$$

Yes — the determinant vanishes, so the three points lie on the line $y = 2x$.

Example. Find the area of the triangle with vertices $A(2,7), B(1,1), C(10,8)$.

$$\text{Area} = \tfrac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix} = \tfrac{1}{2}\big|\,2(1-8) - 7(1-10) + 1(8-10)\,\big|.$$

Computing: $2(-7) - 7(-9) + 1(-2) = -14 + 63 - 2 = 47$. So the area is $47/2 = 23.5$ square units.

Equation of a Line through Two Points

The line through (x_1, y_1) and (x_2, y_2) has equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

This is just the collinearity test applied to a generic third point (x, y). A single determinant gives the line in one shot.

Determinants in Computer Graphics

Game engines and rendering pipelines use this exact formula thousands of times per frame to decide which side of a triangle's edge a pixel lies on — and therefore which triangles overlap. A sign change in the determinant means the pixel jumped from inside to outside the triangle.

4 Minors, Cofactors, Adjoint and Inverse

We've used minors implicitly during cofactor expansion; this section formalises them, and uses them to build the **adjoint** of a matrix and ultimately its **inverse** — two of the most exam-critical objects in Class 12.

4.1 Minor and Cofactor

Minor and Cofactor

The **minor** M_{ij} of an entry a_{ij} is the determinant of the sub-matrix obtained by deleting the i-th row and the j-th column from A.
The **cofactor** of a_{ij} is

$$A_{ij} = (-1)^{i+j}\, M_{ij}.$$

So the cofactor equals the minor with a sign flip whenever $i + j$ is odd.

a_{11}	a_{12}	a_{13}
a_{21}	a_{22}	a_{23}
a_{31}	a_{32}	a_{33}

$$\Rightarrow M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

Cofactor: $A_{22} = (-1)^{2+2} M_{22} = M_{22}$

Determinant via Cofactors — Compact Form

Expanding along row i:

$$|A| = \sum_{j=1}^{n} a_{ij}\, A_{ij}.$$

Expanding along column j:

$$|A| = \sum_{i=1}^{n} a_{ij}\, A_{ij}.$$

Six choices of i or j in a 3×3, one answer.

Cross-row check (a powerful zero-check). If you multiply the entries of row i by the cofactors of a *different* row k (with $i \neq k$), the sum is zero:

$$\sum_{j=1}^{n} a_{ij}\, A_{kj} = 0 \quad (i \neq k).$$

This is why the adjoint construction (next subsection) works.

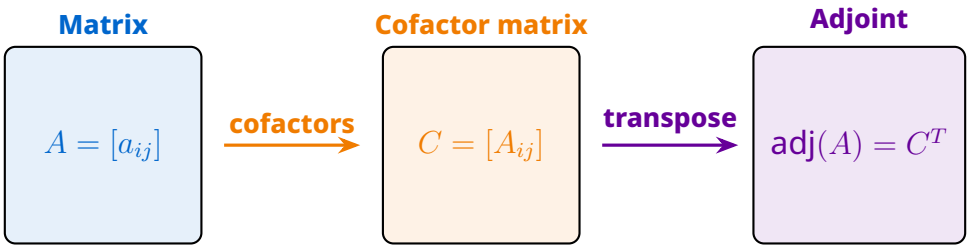
4.2 Adjoint of a matrix

Adjoint of a Matrix

The **adjoint** of $A = [a_{ij}]$, written $\text{adj}(A)$, is the *transpose of the cofactor matrix*:

$$\text{adj}(A) = [A_{ij}]^T.$$

So in $\text{adj}(A)$, the (i, j)-entry is the cofactor A_{ji} of the original matrix — indices swap.



The key identity. The adjoint satisfies

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = |A|\, I_n,$$

where I_n is the $n \times n$ identity matrix. This single identity is the gateway to the inverse formula.

Why $A \cdot \text{adj}(A) = |A|I$

The diagonal entries of $A \cdot \text{adj}(A)$ are sums of the form $\sum_j a_{ij}A_{ij}$ — which equal $|A|$ by cofactor expansion. The off-diagonal entries are the cross-row sums $\sum_j a_{ij}A_{kj}$ with $i \neq k$, which equal zero. So the product is $|A|$ times the identity.

Worked 2×2. For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, the cofactors are $A_{11} = 4,\ A_{12} = -1,\ A_{21} = -3,\ A_{22} = 2$. So

$$\text{adj}(A) = \begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}.$$

Check: $A \cdot \text{adj}(A) = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 5I_2$, and indeed $|A| = 8 - 3 = 5$. ✓

2×2 Adjoint — Swap and Negate

For a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the adjoint is simply $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ — *swap the main diagonal, negate the anti-diagonal*. Memorise this; it saves time on every 2×2 inverse.

4.3 Singular and non-singular matrices

Singular vs Non-Singular

A square matrix A is

- **non-singular** if $|A| \neq 0$ — it has an inverse,
- **singular** if $|A| = 0$ — it does *not* have an inverse.

The geometric picture: a singular matrix *collapses dimensions* (it sends a 3-D parallelepiped to a flat sheet or a line, killing volume), and you can't reverse that collapse. A non-singular matrix preserves dimension, so it is reversible.

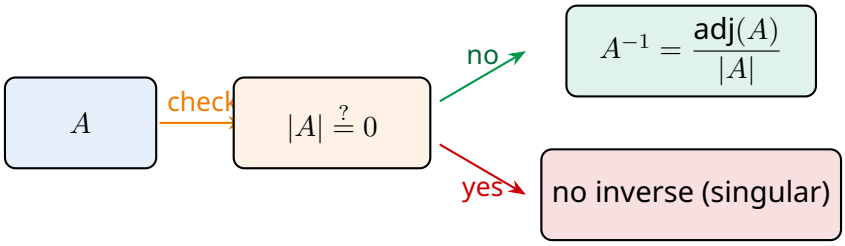
4.4 Inverse via the adjoint

Inverse Formula

If $|A| \neq 0$, then

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A).$$

This is the unique inverse: $A^{-1}A = AA^{-1} = I_n$.



Continuing the example. For $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, since $|A| = 5$ and $\text{adj}(A) = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$,

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}.$$

Check: $AA^{-1} = \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = I_2$. ✓

4.5 Properties of inverse and adjoint

Inverse & Adjoint Identities

For non-singular square matrices A and B of the same order n:

$$\begin{aligned} (AB)^{-1} &= B^{-1}A^{-1} \\ (A^T)^{-1} &= (A^{-1})^T \\ |A^{-1}| &= \frac{1}{|A|} \\ |\text{adj}(A)| &= |A|^{n-1} \\ \text{adj}(AB) &= \text{adj}(B)\,\text{adj}(A) \\ \text{adj}(A^T) &= [\text{adj}(A)]^T \end{aligned}$$

Reversal Rule

$(AB)^{-1} = B^{-1}A^{-1}$, *not* $A^{-1}B^{-1}$. Order *reverses* when you invert a product. The same holds for transposes: $(AB)^T = B^TA^T$. Easy to mix up under time pressure.

Socks-and-Shoes

To dress: **socks first, then shoes**. To undress (invert): **shoes first, then socks**. Reverses the order. Same idea for $(AB)^{-1} = B^{-1}A^{-1}$.

5 Solving Linear Systems with A^{-1}

The whole machinery — determinant, cofactors, adjoint, inverse — pays off in this one application: solving systems of linear equations cleanly, and deciding when a

solution even exists.

5.1 Matrix form of a linear system

A system of three linear equations in three unknowns

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

can be written compactly as $AX = B$, where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

A is the **coefficient matrix**, X is the **variable matrix**, B is the **constant matrix**.

5.2 Solution by inverse

If $|A| \neq 0$, we can pre-multiply $AX = B$ by A^{-1}:

$$AX = B \implies A^{-1}(AX) = A^{-1}B \implies (A^{-1}A)X = A^{-1}B \implies X = A^{-1}B.$$

Matrix Method for Linear Systems

If $|A| \neq 0$, the unique solution of $AX = B$ is

$$X = A^{-1}\,B = \frac{1}{|A|}\,\text{adj}(A)\,B.$$

Worked example (2 variables). Solve

$$\begin{aligned} 2x + 3y &= 8 \\ x + 4y &= 9. \end{aligned}$$

Here $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$. We computed $|A| = 5$ and $A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$ in Section 4.4. So

$$X = A^{-1}B = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 8 \\ 9 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 32 - 27 \\ -8 + 18 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

So $x = 1$, $y = 2$.

Worked example (3 variables). Solve

$$\begin{aligned} x + y + z &= 6 \\ x - y + z &= 2 \\ 2x + y - z &= 1. \end{aligned}$$

Step 1. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Expanding along R_1,

$$|A| = 1(1-1) - 1(-1-2) + 1(1+2) = 0 + 3 + 3 = 6 \neq 0.$$

So a unique solution exists.

Step 2. Compute the cofactor matrix and then the adjoint:

$$\text{adj}(A) = \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}.$$

Step 3.

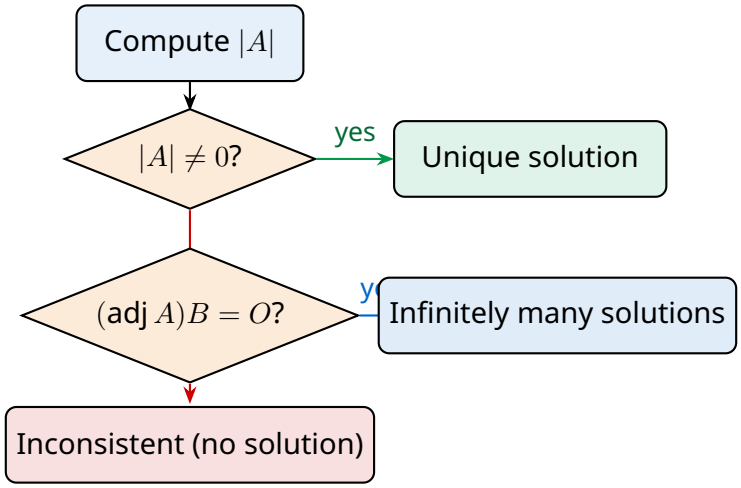
$$X = \frac{1}{6}\,\text{adj}(A)\,B = \frac{1}{6}\begin{bmatrix} 0+4+2 \\ 18-6+0 \\ 18+2-2 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

So $x = 1$, $y = 2$, $z = 3$.

5.3 Consistency of a linear system

A system is **consistent** if it has at least one solution and **inconsistent** if it has none. Three cases arise based on $|A|$ and $\text{adj}(A)\,B$:

Condition	Status of system	Nature of solution
$\lvert A\rvert \neq 0$	Consistent	Unique solution $X = A^{-1}B$
$\lvert A\rvert = 0$ and $(\text{adj}\,A)\,B = O$	Consistent	Infinitely many solutions
$\lvert A\rvert = 0$ and $(\text{adj}\,A)\,B \neq O$	Inconsistent	No solution



Homogeneous systems ($B = O$). The equation becomes $AX = O$ and always has the **trivial solution** $X = O$.

- If $|A| \neq 0$: the trivial solution is the *only* solution.
- If $|A| = 0$: non-trivial solutions exist — infinitely many of them.

Geometric Picture of Consistency

Each linear equation in three unknowns represents a plane in $\mathbb{R}^3$.

- **Unique solution**: the three planes meet at a single point.
- **Infinitely many**: the three planes share a common line (or are the same plane).
- **No solution**: at least two planes are parallel but distinct, so no common point exists.

Always Compute $|A|$ First

For any "solve the system" question, the first move is to compute $|A|$. If non-zero, jump straight to $X = A^{-1}B$. If zero, you must check $(\text{adj}\, A)\, B$ before declaring the system inconsistent.

5.4 Cramer's Rule *[JEE/NEET Extension]*

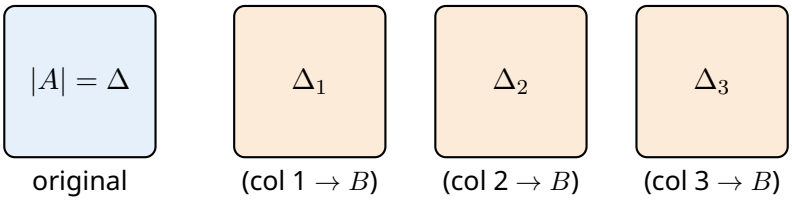
An alternative to the matrix method, especially handy when only one or two variables are asked.

Cramer's Rule for 3×3

Let $\Delta = |A|$ (with $|A| \neq 0$). Define $\Delta_1, \Delta_2, \Delta_3$ as the determinants obtained from A by replacing the 1st, 2nd, 3rd column respectively with B. Then

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}.$$

If $\Delta = 0$ and any $\Delta_i \neq 0$, the system is inconsistent. If $\Delta = 0$ and all $\Delta_i = 0$, it has infinitely many solutions.



Cramer's rule and the matrix method always agree, since $X = A^{-1}B = \frac{1}{|A|}\text{adj}(A)B$ expands to exactly these ratios.

Linear Systems in Engineering

Electrical circuits with multiple loops (Kirchhoff's laws), chemical reaction balance, traffic flow at road intersections, and economic input-output models all reduce to systems of linear equations. The matrix method scales to hundreds of variables on a computer; the determinant check tells the engineer immediately whether the system is well-posed.

Don't Confuse $\Delta = 0$ with No Solution

$\Delta = 0$ does *not* automatically mean "no solution". It means the system is *either* inconsistent *or* has infinitely many solutions. You must check the Δ_i (Cramer) or $(\text{adj}\, A)\, B$ (matrix method) to distinguish.

6 Quick Reference Summary

A one-stop checklist for the chapter — everything that goes onto a formula sheet.

6.1 Determinant formulas

Core Formulas — Memorise

$$\begin{aligned}
2\times2: &\quad |A| = a_{11}a_{22} - a_{12}a_{21} \\
3\times3 \text{ (row 1)}: &\quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \\
\text{Cofactor}: &\quad A_{ij} = (-1)^{i+j} M_{ij} \\
\text{Area of triangle}: &\quad \text{Area} = \tfrac{1}{2}\left|\det \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}\right| \\
\text{Adjoint}: &\quad \text{adj}(A) = [A_{ij}]^T \\
\text{Inverse}: &\quad A^{-1} = \tfrac{1}{|A|}\,\text{adj}(A) \\
\text{Linear system}: &\quad X = A^{-1}B \\
\text{Cramer}: &\quad x_i = \Delta_i/\Delta
\end{aligned}$$

6.2 Identities at a glance

Identity	Notes
$\lvert A^T\rvert = \lvert A\rvert$	Rows and columns play symmetric roles
$\lvert kA\rvert = k^n\lvert A\rvert$	For $n \times n$; scalar pulls out per row
$\lvert AB\rvert = \lvert A\rvert\,\lvert B\rvert$	Multiplicativity
$\lvert A^{-1}\rvert = 1/\lvert A\rvert$	Provided $\lvert A\rvert \neq 0$
$\lvert\text{adj}(A)\rvert = \lvert A\rvert^{n-1}$	In particular $\lvert A\rvert^2$ for 3×3
$A\,\text{adj}(A) = \text{adj}(A)\,A = \lvert A\rvert I$	The defining property
$(AB)^{-1} = B^{-1}A^{-1}$	Reversal rule
$\text{adj}(AB) = \text{adj}(B)\,\text{adj}(A)$	Adjoint reversal

6.3 Solution checklist for a linear system

1. Write the system in the form $AX = B$.
2. Compute $\lvert A\rvert$.
3. If $\lvert A\rvert \neq 0$: solve $X = A^{-1}B$. Done.
4. If $\lvert A\rvert = 0$: compute $(\text{adj}\,A)\,B$.
   - If $(\text{adj}\,A)\,B = O$: infinitely many solutions.
   - If $(\text{adj}\,A)\,B \neq O$: no solution (inconsistent).

6.4 Determinant property cheat-card

1. $\lvert A\rvert = \lvert A^T\rvert$ (transpose invariance)
2. Swap two rows/columns $\Rightarrow$ sign flips.
3. Two equal rows/columns $\Rightarrow \lvert A\rvert = 0$.
4. Multiply a row by $k \Rightarrow$ multiplies $\lvert A\rvert$ by k.
5. Row $=$ sum $\Rightarrow$ determinant splits.
6. $R_i \to R_i + kR_j \Rightarrow \lvert A\rvert$ unchanged.
7. $\lvert AB\rvert = \lvert A\rvert\,\lvert B\rvert$.
8. Triangular matrix $\Rightarrow \lvert A\rvert =$ product of diagonal.

6.5 Final word

Determinants tie three threads together. *Algebraically*, they collapse a square matrix into a single number that decides invertibility. *Geometrically*, they measure signed area/volume and detect when a transformation flattens space. *Computationally*, they unlock the inverse formula and so let us solve every well-posed system of linear equations in a uniform way.

If you remember three things from this chapter:

- $\lvert A\rvert \neq 0 \iff A$ is invertible $\iff$ the system $AX = B$ has a unique solution.

- $A^{-1} = \dfrac{1}{|A|}\,\text{adj}(A)$ — the workhorse formula.
- Cofactor expansion gives the same answer along every row and every column — exploit zeros.

Best of luck with your preparation!

Revise, practise, and trust your reasoning.