# NCERT Exemplar Solutions

*Solved NCERT Exemplar Problems for Class 12th Mathematics, Chapter 5*

## Chapter 5: Continuity and Differentiability

## About this Chapter

Continuity and Differentiability extends limits into two of the cornerstone tools of calculus. The chapter builds on the $\varepsilon$-$\delta$ idea of **continuity** at a point, defines the **derivative** as the limit of a difference quotient, and gives the algebraic machinery (chain rule, implicit, logarithmic, parametric differentiation, second derivatives) that lets us differentiate every elementary function. It closes with **Rolle's** and **Mean Value** theorems, which connect the values of a function on an interval to the values of its derivative inside.

**Topics covered:** Continuity at a point • Algebra of continuous functions • Derivatives of standard functions • Chain rule • Implicit differentiation • Logarithmic differentiation • Parametric forms • Second derivative • Rolle's and Mean Value Theorems

### Quick Formula Sheet

**Continuity at $c$:**

$$\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$$

**Derivative (first principle):**

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

**Chain rule:**

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

**Parametric:**

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

**MVT:**

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

## NCERT Exemplar Exercise 5.3

**Q 5.1** **Examine the continuity of the function $f(x) = x^3 + 2x^2 - 1$ at $x = 1$.**

**SOLUTION**

**Concept used.** A function $f$ is **continuous** at a point $c$ iff three conditions hold simultaneously: (i) $f(c)$ is defined, (ii) $\lim_{x\to c} f(x)$ exists (left-hand limit equals right-hand limit), and (iii) the common limit equals the value: $\lim_{x\to c} f(x) = f(c)$. Every **polynomial** is continuous on $\mathbb{R}$ because it is built from $x$ (continuous) by finite sums and products, and continuity is preserved under those operations.

**Step 1.** Evaluate the function at $x = 1$:

$$f(1) = (1)^3 + 2(1)^2 - 1 = 1 + 2 - 1 = 2.$$

So $f(1) = 2$, which is defined.

**Step 2.** Compute the limit as $x \to 1$. Since $f$ is a polynomial, substitution gives the limit directly:

$$\lim_{x\to 1}(x^3 + 2x^2 - 1) = 1^3 + 2 \cdot 1^2 - 1 = 2.$$

**Step 3.** Compare: $\lim\limits_{x\to 1} f(x) = 2 = f(1)$. All three continuity conditions hold.

> **Final Answer:** $f$ is continuous at $x = 1$.

☞ **Polynomials and continuity**

For any polynomial $p(x)$, $\lim\limits_{x\to c} p(x) = p(c)$ for every real $c$. So polynomials never need a piecewise check; just plug in.

**EXPERT'S SOLUTION** *: Aarav Sharma, M.Sc Mathematics, IIT Bombay*

**Structural angle.** The function is an everywhere-continuous polynomial, so the answer is automatic. We still write it as a formal $\varepsilon$-style check so the student sees the structure.

**Step 1.** Identify the class: $f(x) = x^3 + 2x^2 - 1$ is a sum and product of the continuous functions $x \mapsto x$ and constants, hence is continuous on the whole of $\mathbb{R}$.

**Step 2.** Specialize to $x = 1$: $f(1) = 1 + 2 - 1 = 2$. The limit, by direct substitution legal for polynomials, is also 2.

**Step 3.** The equality $\lim_{x\to 1} f(x) = f(1) = 2$ certifies continuity at $x = 1$.

**Why this matters.** Recognising the function-class upfront (polynomial, rational with non-zero denominator, $\sin / \cos$, $e^x$, $\log x$ on its domain) lets you skip the explicit piecewise check on most problems. The piecewise machinery is reserved for the cases where a definition genuinely changes across $x = c$.

> **Final Answer:** $f$ is continuous at $x = 1$.

**Q 5.2** **Examine the continuity of** $f(x) = \begin{cases} 3x + 5, & x \geq 2 \\ x^2, & x < 2 \end{cases}$ **at** $x = 2$.

**SOLUTION**

**Concept used.** For a piecewise-defined function continuity at the break-point $c$ requires the left-hand limit (LHL, computed from the rule for $x < c$), the right-hand limit (RHL, from the rule for $x \geq c$), and the value $f(c)$ to all be equal. If any one of them disagrees with the other two, $f$ is discontinuous.

**Step 1.** Value of $f$ at $x = 2$ (using the $x \geq 2$ branch):

$$f(2) = 3(2) + 5 = 6 + 5 = 11.$$

**Step 2.** Right-hand limit, using $3x + 5$:

$$\lim_{x \to 2^+} (3x + 5) = 3(2) + 5 = 11.$$

**Step 3.** Left-hand limit, using $x^2$:

$$\lim_{x \to 2^-} x^2 = 2^2 = 4.$$

**Step 4.** Compare: LHL $= 4$ and RHL $= 11$, so LHL $\neq$ RHL. The two-sided limit does not exist.

> **Final Answer:** $f$ is **discontinuous** at $x = 2$ (jump discontinuity).

**✗ Common Mistake**

Do not use the rule $3x + 5$ to compute the LHL just because $x = 2$ falls on that branch. The LHL approaches from $x < 2$, where the rule is $x^2$. The branch label "$\geq$" only decides $f(2)$ itself.

**EXPERT'S SOLUTION** *: Vivaan Iyer, M.Sc Mathematics, ISI Kolkata*

**Picture-first.** Sketch the two branches: a parabola for $x < 2$ ending at the open point $(2, 4)$, and a line $y = 3x + 5$ from the filled point $(2, 11)$. There is a vertical gap of $11 - 4 = 7$ at $x = 2$, the signature of a jump discontinuity.

**Step 1.** LHL on the $x^2$ branch: $\lim_{x \to 2^-} x^2 = 4$.

**Step 2.** RHL on the $3x + 5$ branch: $\lim_{x \to 2^+} (3x + 5) = 11$.

**Step 3.** Since LHL $\neq$ RHL, $f$ is discontinuous at $x = 2$, and the jump is $11 - 4 = 7$.

**Why this matters.** Whenever a piecewise function changes formula at $x = c$, run the three checks (LHL, RHL, $f(c)$) mechanically. The answer follows from arithmetic, not intuition.

> **Final Answer:** Discontinuous at $x = 2$; jump of 7.

**Q 5.3** **Examine the continuity of** $f(x) = \begin{cases} \dfrac{1-\cos 2x}{x^2}, & x \neq 0 \\ 5, & x = 0 \end{cases}$ **at** $x = 0$**.**

**SOLUTION**

**Concept used.** Use the standard identity $1 - \cos 2x = 2\sin^2 x$, then apply the well-known limit $\lim_{x\to 0} \frac{\sin x}{x} = 1$. For continuity at $x = 0$ we need $\lim_{x\to 0} f(x) = f(0)$.

**Step 1.** Rewrite the numerator using $1 - \cos 2x = 2\sin^2 x$:

$$\frac{1-\cos 2x}{x^2} = \frac{2\sin^2 x}{x^2} = 2\left(\frac{\sin x}{x}\right)^2.$$

**Step 2.** Take the limit:

$$\lim_{x\to 0} \frac{1-\cos 2x}{x^2} = 2\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = 2(1)^2 = 2.$$

**Step 3.** Compare with $f(0) = 5$: limit $\neq f(0)$.

> **Final Answer:** $f$ is **discontinuous** at $x = 0$ (removable discontinuity; redefining $f(0) = 2$ would repair it).

**♥ Removable vs non-removable**

If the two-sided limit exists, the discontinuity is removable — redefining $f(c)$ to equal the limit makes the function continuous. If LHL≠RHL, no choice of $f(c)$ helps; the jump is permanent.

**EXPERT'S SOLUTION** : *Pranav Gupta, Ph.D Mathematics, IIT Delhi*

**Strategic angle.** The limit is finite; only its disagreement with $f(0)$ creates the discontinuity. Identifying this as *removable* is itself a useful classification.

**Step 1.** Identity: $1 - \cos 2x = 2\sin^2 x$. So the quotient becomes $2(\sin x/x)^2$.

**Step 2.** As $x \to 0$, $\sin x/x \to 1$, hence the quotient tends to $2 \cdot 1 = 2$.

**Step 3.** Since $f(0) = 5 \neq 2$, $f$ is discontinuous at $0$, but the limit exists — so the discontinuity is removable.

**Why this matters.** Once you spot $1 - \cos 2x$, reach for the double-angle identity automatically; it converts $\cos$-near-1 limits into $\sin$-over-$x$ limits, which are routine.

> **Final Answer:** Discontinuous (removable) at $x = 0$.

**Q 5.4** **Examine the continuity of** $f(x) = \begin{cases} \dfrac{2x^2 - 3x - 2}{x - 2}, & x \neq 2 \\ 5, & x = 2 \end{cases}$ **at** $x = 2$**.**

**SOLUTION**

**Concept used.** Factor the quadratic and cancel the common factor; the resulting linear expression gives the limit. Continuity demands $\lim_{x \to 2} f(x) = f(2)$.

**Step 1.** Factor: split $-3x$ as $-4x + x$:

$$2x^2 - 3x - 2 = 2x^2 - 4x + x - 2 = 2x(x-2) + 1(x-2) = (x-2)(2x+1).$$

**Step 2.** For $x \neq 2$, cancel:

$$f(x) = \frac{(x-2)(2x+1)}{x-2} = 2x + 1.$$

**Step 3.** Take the limit:

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (2x + 1) = 5.$$

**Step 4.** Compare with $f(2) = 5$: limit equals the value.

> **Final Answer:** $f$ is **continuous** at $x = 2$.

**✗ Branch confusion**

Always use the strict-inequality branch for the side limit and the equality branch for the value $f(c)$. Mixing the two is the most common cause of a wrong jump diagnosis.

**EXPERT'S SOLUTION** *: Aanya Mehta, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** The $0/0$ form invites factoring; once the $(x-2)$ cancels, $f$ becomes a polynomial whose value at $2$ is $5$. Since $f(2)$ was *chosen* to equal $5$, continuity is exact.

**Step 1.** $2x^2 - 3x - 2 = (2x+1)(x-2)$.

**Step 2.** Hence $f(x) = 2x + 1$ for $x \neq 2$; limit at $2$ is $5$.

**Step 3.** The piecewise definition fixes $f(2) = 5$ deliberately, so continuity holds.

**Why this matters.** Designers of piecewise definitions choose the special value $f(c)$ to fill in the hole. Always check whether the chosen value matches the limit.

> **Final Answer:** Continuous at $x = 2$.

**Q 5.5** **Examine the continuity of** $f(x) = \begin{cases} \dfrac{|x-4|}{2(x-4)}, & x \neq 4 \\ 0, & x = 4 \end{cases}$ **at** $x = 4$**.**

**SOLUTION**

**Concept used.** The sign function appears whenever $|x-c|/(x-c)$ is written. By definition $|x-4| = x-4$ if $x > 4$ and $|x-4| = -(x-4)$ if $x < 4$. So $\dfrac{|x-4|}{2(x-4)} = \dfrac{1}{2}$ for $x > 4$ and $-\dfrac{1}{2}$ for $x < 4$.

**Step 1.** Right-hand limit ($x \to 4^+$, so $x - 4 > 0$):

$$\lim_{x\to 4^+} \frac{|x-4|}{2(x-4)} = \lim_{x\to 4^+} \frac{x-4}{2(x-4)} = \frac{1}{2}.$$

**Step 2.** Left-hand limit ($x \to 4^-$, so $x - 4 < 0$):

$$\lim_{x\to 4^-} \frac{|x-4|}{2(x-4)} = \lim_{x\to 4^-} \frac{-(x-4)}{2(x-4)} = -\frac{1}{2}.$$

**Step 3.** LHL $\neq$ RHL, so $\lim_{x\to 4} f(x)$ does not exist. Continuity fails.

> **Final Answer:** $f$ is **discontinuous** at $x = 4$ (jump of 1).

### ☞ Sign-function reflex

Any expression of the form $|x-c|/(x-c)$ is the sign function shifted to $c$: it equals $+1$ for $x > c$, $-1$ for $x < c$. Two different one-sided limits make the discontinuity automatic.

**EXPERT'S SOLUTION** *: Sneha Verma, M.Tech CS, IIT Madras*

**Strategic angle.** Whenever $|x-c|$ sits over a linear factor $(x-c)$, the quotient is $\pm 1$ on the two sides of $c$. The hole at $c$ cannot be patched by any single value of $f(c)$.

**Step 1.** Right of 4: $|x-4| = x-4$, quotient $= 1/2$.

**Step 2.** Left of 4: $|x-4| = -(x-4)$, quotient $= -1/2$.

**Step 3.** LHL $\neq$ RHL $\Rightarrow$ limit does not exist; $f$ is discontinuous (jump) at 4.

**Why this matters.** $|x|/x$ is the prototypical jump discontinuity; recognise its disguised forms instantly.

> **Final Answer:** Discontinuous at $x = 4$ (jump).

**Q 5.6** **Examine the continuity of** $f(x) = \begin{cases} x\cos\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **at** $x = 0$**.**

**SOLUTION**

**Concept used.** The **Squeeze (Sandwich) Theorem**: if $g(x) \le h(x) \le k(x)$ near $c$ and $\lim g = \lim k = L$, then $\lim h = L$. Here $\cos(1/x)$ oscillates between $-1$ and $1$ but is multiplied by $x$, which tends to $0$.

**Step 1.** Bound the cosine: for all $x \neq 0$,

$$-1 \le \cos\tfrac{1}{x} \le 1.$$

**Step 2.** Multiply through by $|x| \ge 0$ (preserves inequalities for $|x|$): $-|x| \le x\cos(1/x) \le |x|$.

**Step 3.** Take limits: $\lim_{x\to 0}(-|x|) = 0$ and $\lim_{x\to 0}|x| = 0$. By the Squeeze Theorem,

$$\lim_{x\to 0} x\cos\frac{1}{x} = 0.$$

**Step 4.** Compare with $f(0) = 0$: the limit equals the value.

> **Final Answer:** $f$ is **continuous** at $x = 0$.

**☞ Exam Tip**

Pattern: whenever a factor $x^k$ $(k > 0)$ multiplies a bounded oscillation like $\sin(1/x)$ or $\cos(1/x)$, the product is squeezed to $0$ at $x = 0$. Memorise the trick.

**EXPERT'S SOLUTION** *: Aditi Singh, Ph.D Pure Mathematics, IISc Bangalore*

**Picture-first.** The graph of $x\cos(1/x)$ oscillates wildly near $0$, but the oscillations are bounded by the envelopes $y = \pm x$ which both pinch to $0$. So the graph passes through $(0, 0)$ smoothly from the squeeze.

**Step 1.** Envelope: $|x\cos(1/x)| \le |x|$.

**Step 2.** As $x \to 0$, $|x| \to 0$, hence $x\cos(1/x) \to 0$ by Squeeze.

**Step 3.** $f(0) = 0$ matches; continuity holds.

**Why this matters.** The same idea handles every "bounded times small" limit and is the foundation of many counterexamples in advanced calculus.

> **Final Answer:** Continuous at $x = 0$.

**Q 5.7** **Examine the continuity of $f(x) = (x-a)\sin\dfrac{1}{x-a}$ for $x \neq a$, with $f(a) = 0$, at $x = a$.**

## SOLUTION

**Concept used.** Squeeze theorem, exactly as in the previous problem: $(x-a)\sin\dfrac{1}{x-a}$ is "small times bounded".

**Step 1.** For every $x \neq a$, $\left|\sin\dfrac{1}{x-a}\right| \leq 1$.

**Step 2.** Multiply by $|x-a|$: $|(x-a)\sin\frac{1}{x-a}| \leq |x-a|$.

**Step 3.** As $x \to a$, $|x-a| \to 0$, so by Squeeze,

$$\lim_{x\to a}(x-a)\sin\frac{1}{x-a} = 0.$$

**Step 4.** $f(a) = 0$ matches the limit.

> **Final Answer:** $f$ is **continuous** at $x = a$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

## EXPERT'S SOLUTION : *Karan Patel, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Same template as Q6 with $x-a$ in place of $x$. The shift does not affect the squeeze.

**Step 1.** Envelope: $|(x-a)\sin\frac{1}{x-a}| \leq |x-a| \to 0$.

**Step 2.** Hence limit $= 0 = f(a)$, so $f$ continuous at $a$.

**Why this matters.** Once the template is recognised, the solution is two lines.

> **Final Answer:** Continuous at $x = a$.

**Q 5.8** **Examine the continuity of** $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **at $x = 0$.**

**SOLUTION**

**Concept used.** Behaviour of $e^{1/x}$ as $x \to 0$ depends on the side: as $x \to 0^+$, $1/x \to +\infty$, so $e^{1/x} \to \infty$; as $x \to 0^-$, $1/x \to -\infty$, so $e^{1/x} \to 0$. Compute LHL and RHL separately.

**Step 1.** Right-hand limit ($x \to 0^+$): let $t = e^{1/x} \to \infty$. Then

$$\frac{e^{1/x}}{1+e^{1/x}} = \frac{t}{1+t} = 1 - \frac{1}{1+t} \to 1 - 0 = 1.$$

So RHL $= 1$.

**Step 2.** Left-hand limit ($x \to 0^-$): $e^{1/x} \to 0$, so

$$\frac{e^{1/x}}{1+e^{1/x}} \to \frac{0}{1+0} = 0.$$

So LHL $= 0$.

**Step 3.** RHL $= 1 \neq 0 =$ LHL, so the limit does not exist; $f$ is discontinuous at $0$.

> **Final Answer:** $f$ is **discontinuous** at $x = 0$ (jump of $1$).

**✗ Common Mistake**

Do not write $\lim_{x\to 0} e^{1/x} = e^{\infty}$ as if it were a single value. The two one-sided limits of $1/x$ point in opposite directions, producing two opposite behaviours of $e^{1/x}$.

**EXPERT'S SOLUTION** : *Rahul Joshi, M.Sc Mathematics, IIT Bombay*

**Strategic angle.** Whenever $e^{1/x}$ appears, force a left/right split before doing any algebra.

**Step 1.** Right side: divide numerator and denominator by $e^{1/x}$ to get
$$\frac{1}{e^{-1/x}+1} \to \frac{1}{0+1} = 1.$$

**Step 2.** Left side: $e^{1/x} \to 0$, quotient $\to 0$.

**Step 3.** LHL $\neq$ RHL $\Rightarrow$ discontinuous.

**Why this matters.** $e^{1/x}$, $\tan^{-1}(1/x)$, $\arctan(1/x)$ all have this two-sided personality. Drilling the split prevents many algebra errors.

> **Final Answer:** Discontinuous at $x = 0$.

**Pattern memorisation.** The integrals $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(x/a) + C$, $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$, $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a) + C$ should be writable from memory. Most board problems are one substitution away from one of these standard forms.

**Generalising the trick.** The manipulation used here is part of a family: spot the algebraic disguise, apply the standard identity, reduce to a known limit or standard integral, conclude. Once you've solved three problems with the same disguise, the rest become routine.

**Q 5.9** **Examine the continuity of** $f(x) = \begin{cases} \frac{x^2}{2}, & 0 \le x \le 1 \\ 2x^2 - 3x + \frac{3}{2}, & 1 < x \le 2 \end{cases}$ **at** $x = 1$**.**

**SOLUTION**

**Concept used.** Piecewise continuity at the break-point: check $f(1)$, the left-hand limit using the first rule and the right-hand limit using the second rule.

**Step 1.** Value at $x = 1$ (first branch covers $x = 1$):

$$f(1) = \frac{1^2}{2} = \frac{1}{2}.$$

**Step 2.** Left-hand limit (use $x^2/2$):

$$\lim_{x \to 1^-} \frac{x^2}{2} = \frac{1}{2}.$$

**Step 3.** Right-hand limit (use the second branch). As $x \to 1^+$,

$$2x^2 - 3x + \frac{3}{2} \to 2 - 3 + \frac{3}{2} = \frac{1}{2}.$$

**Step 4.** LHL $=$ RHL $= f(1) = 1/2$. All three conditions match.

**Final Answer:** $f$ is **continuous** at $x = 1$.

**✗ Branch confusion**

Always use the strict-inequality branch for the side limit and the equality branch for the value $f(c)$. Mixing the two is the most common cause of a wrong jump diagnosis.

**EXPERT'S SOLUTION** *: Riya Kapoor, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** The author chose the second branch's constant $3/2$ precisely so the polynomial value matches $1/2$ at $x = 1$.

**Step 1.** LHL of $x^2/2$ at 1 is $1/2$.

**Step 2.** RHL of $2x^2 - 3x + 3/2$ at 1 is $2 - 3 + 3/2 = 1/2$.

**Step 3.** Both equal $f(1) = 1/2$; continuity holds.

**Why this matters.** Piecewise functions in textbooks are often designed so the break-point matches; verify by computation rather than trust.

**Final Answer:** Continuous at $x = 1$.

**Alternate framing.** Read the function as a single graph drawn with two coloured pens. Continuity asks: does the second pen pick up exactly where the first left off? If both pens meet the same $y$ at $x = c$ *and* the dot $f(c)$ sits on that meeting point, continuity holds; otherwise it doesn't. This visual check anticipates the algebra and exposes sign errors before they propagate.

**Q 5.10** **Examine the continuity of $f(x) = |x| + |x-1|$ at $x = 1$.**

SOLUTION

**Concept used.** $|x|$ is continuous on all of $\mathbb{R}$ (absolute value of a continuous function is continuous), and the sum of two continuous functions is continuous.

**Step 1.** $g_1(x) = |x|$ is continuous on $\mathbb{R}$.

**Step 2.** $g_2(x) = |x-1|$ is the composition of the continuous functions $x \mapsto x-1$ and $u \mapsto |u|$, hence continuous on $\mathbb{R}$.

**Step 3.** $f = g_1 + g_2$ is a sum of continuous functions, so $f$ is continuous on $\mathbb{R}$, in particular at $x = 1$.

**Step 4.** Direct verification: $f(1) = |1| + |0| = 1$; $\lim_{x\to 1}(|x| + |x-1|) = 1 + 0 = 1$.

**Final Answer:** $f$ is **continuous** at $x = 1$.

### ☞ Composition continuity

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Track both layers when locating discontinuities.

EXPERT'S SOLUTION *: Neha Reddy, M.Sc Mathematics, IIT Bombay*

**Structural angle.** The map $u \mapsto |u|$ is continuous (it satisfies $\big||a| - |b|\big| \le |a-b|$, a Lipschitz bound with constant 1. Sums and compositions of continuous functions are continuous, so any expression built from $|\cdot|$, sums and polynomial pieces is continuous.

**Step 1.** Both $|x|$ and $|x-1|$ are continuous on $\mathbb{R}$.

**Step 2.** Sum is continuous; $f(1) = 1 + 0 = 1$.

**Why this matters.** Closure of continuity under sum / product / composition is the workhorse fact that lets you skip $\varepsilon$-$\delta$ on routine examples.

> **Final Answer:** Continuous at $x = 1$.

**Q 5.11** **Find $k$ so that $f(x) = \begin{cases} 3x - 8, & x \le 5 \\ 2k, & x > 5 \end{cases}$ is continuous at $x = 5$.**

## SOLUTION

**Concept used.** For continuity at $x = 5$, set $\lim_{x\to5^-} f(x) = \lim_{x\to5^+} f(x) = f(5)$.

**Step 1.** LHL (and $f(5)$): both come from the rule $3x - 8$:

$$\lim_{x\to5^-}(3x-8) = 3(5) - 8 = 7, \quad f(5) = 7.$$

**Step 2.** RHL: $\lim_{x\to5^+}(2k) = 2k$.

**Step 3.** Force RHL $= 7$: $2k = 7 \Rightarrow k = 7/2$.

> **Final Answer:** $k = \dfrac{7}{2}$.

**☞ Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an $\varepsilon$-$\delta$ on a polynomial in an exam.

## EXPERT'S SOLUTION : *Diya Banerjee, B.Tech CSE, IIT Roorkee*

**Quick reading.** The constant branch's value $2k$ must equal the polynomial branch's value at $5$, i.e. $7$.

**Step 1.** Match: $2k = 3(5) - 8 = 7$.

**Step 2.** Hence $k = 7/2$.

**Why this matters.** "Find $k$ so that $f$ is continuous" is shorthand for "force LHL $=$ RHL"; the unknown drops out as one linear equation.

> **Final Answer:** $k = 7/2$.

**Q 5.12** **Find $k$ so that $f(x) = \begin{cases} \dfrac{2^{x+2}-16}{4^x-16}, & x \neq 2 \\ k, & x = 2 \end{cases}$ is continuous at $x = 2$.**

**SOLUTION**

**Concept used.** Factor numerator and denominator using $2^{x+2} = 4 \cdot 2^x$ and $4^x = (2^x)^2$. Then cancel.

**Step 1.** Let $t = 2^x$. As $x \to 2$, $t \to 4$. Rewrite:

$$\frac{2^{x+2}-16}{4^x-16} = \frac{4 \cdot 2^x - 16}{(2^x)^2 - 16} = \frac{4(t-4)}{(t-4)(t+4)}.$$

**Step 2.** Cancel $t-4$ (legal since $x \neq 2 \Rightarrow t \neq 4$):

$$= \frac{4}{t+4}.$$

**Step 3.** Take the limit ($t \to 4$):

$$\lim_{x \to 2} \frac{2^{x+2}-16}{4^x-16} = \frac{4}{4+4} = \frac{1}{2}.$$

**Step 4.** For continuity, $k = 1/2$.

**Final Answer:** $k = \dfrac{1}{2}$.

### ☞ L'Hospital cross-check

After finding a limit by algebra or series, you can confirm with one round of L'Hospital ($0/0$ form). Differentiating numerator and denominator independently and re-evaluating must give the same answer; if it doesn't, recheck the algebra.

**EXPERT'S SOLUTION** : *Yash Chatterjee, M.Sc Applied Mathematics, IIT Kanpur*

**Structural angle.** A $0/0$ form with exponentials becomes a $0/0$ polynomial in $t = 2^x$ once we substitute, after which it factors.

**Step 1.** Substitute $t = 2^x$: quotient becomes $\dfrac{4(t-4)}{(t-4)(t+4)} = \dfrac{4}{t+4}$.

**Step 2.** Plug $t = 4$: value is $1/2$. So $k = 1/2$.

**Why this matters.** Exponential change of variable converts many "transcendental" limits into purely algebraic ones.

> **Final Answer:** $k = 1/2$.

**Why this method is preferred.** The shortest correct route in school calculus is almost always: identify the function's category $\to$ load the standard template $\to$ substitute the specific numbers. Reinventing the technique problem-by-problem is what makes a 90-minute paper feel like 3 hours.

**Q 5.13** **Find $k$ so that $f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & 0 \le x \le 1 \end{cases}$ is continuous at $x = 0$.**

**SOLUTION**

**Concept used.** For continuity at $0$, set LHL $= f(0) =$ RHL. Use the conjugate trick to evaluate a $0/0$ surd limit.

**Step 1.** $f(0)$ comes from the $0 \le x \le 1$ branch:

$$f(0) = \frac{2(0)+1}{0-1} = -1.$$

**Step 2.** LHL (the surd branch): multiply by the conjugate $\sqrt{1+kx} + \sqrt{1-kx}$:

$$\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x} \cdot \frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}} = \frac{(1+kx)-(1-kx)}{x(\sqrt{1+kx}+\sqrt{1-kx})}.$$

Numerator simplifies: $(1+kx)-(1-kx) = 2kx$. So

$$\frac{2kx}{x(\sqrt{1+kx}+\sqrt{1-kx})} = \frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}.$$

**Step 3.** Limit as $x \to 0^-$:

$$\frac{2k}{\sqrt{1}+\sqrt{1}} = \frac{2k}{2} = k.$$

**Step 4.** Continuity: LHL $= f(0) \Rightarrow k = -1$.

> **Final Answer:** $k = -1$.

**☞ Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an $\varepsilon$-$\delta$ on a polynomial in an exam.

**EXPERT'S SOLUTION** : *Ishaan Pillai, M.Sc Mathematics, ISI Kolkata*

**Strategic angle.** The classic surd-difference-over-$x$ form yields easily to the conjugate; once rationalised, the offending $x$ cancels.

**Step 1.** Rationalise to get $\dfrac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}$.

**Step 2.** Limit at 0: $2k/2 = k$.

**Step 3.** Match RHL value $f(0) = -1$, giving $k = -1$.

**Why this matters.** Conjugate rationalisation is the go-to manoeuvre for any $\sqrt{\ } - \sqrt{\ }$ over a polynomial; commit it to muscle memory.

**Final Answer:** $k = -1$.

**Generalising the move.** Conjugate rationalisation works because $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A - B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

**Q 5.14** **Find $k$ so that** $f(x) = \begin{cases} \dfrac{1-\cos kx}{x\sin x}, & x \neq 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$ **is continuous at $x = 0$.**

**SOLUTION**

**Concept used.** Use $1 - \cos kx = 2\sin^2(kx/2)$ and the limit $\sin u/u \to 1$.

**Step 1.** Rewrite the numerator:

$$1 - \cos kx = 2\sin^2\frac{kx}{2}.$$

**Step 2.** Manipulate the quotient:

$$\frac{1-\cos kx}{x\sin x} = \frac{2\sin^2(kx/2)}{x\sin x} = 2\left(\frac{\sin(kx/2)}{kx/2}\right)^2 \cdot \frac{(kx/2)^2}{x\sin x}.$$

**Step 3.** Simplify the second factor: $\dfrac{(kx/2)^2}{x\sin x} = \dfrac{k^2x^2/4}{x\sin x} = \dfrac{k^2}{4}\cdot\dfrac{x}{\sin x}$.

**Step 4.** As $x \to 0$: $\sin(kx/2)/(kx/2) \to 1$ and $x/\sin x \to 1$, so

$$\lim_{x\to 0}\frac{1-\cos kx}{x\sin x} = 2\cdot 1\cdot\frac{k^2}{4}\cdot 1 = \frac{k^2}{2}.$$

**Step 5.** Continuity: $k^2/2 = 1/2 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$.

**Final Answer:** $k = \pm 1$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

**EXPERT'S SOLUTION** : *Krishna Nair, Ph.D Mathematics, IIT Delhi*

**Quick reading.** Both $1 - \cos$ and $\sin$ are quadratic-then-linear near $0$; the leading-order expansion gives the limit directly.

**Step 1.** Series near $0$: $1 - \cos kx \approx (kx)^2/2$ and $\sin x \approx x$.

**Step 2.** Hence $(1 - \cos kx)/(x \sin x) \approx (k^2x^2/2)/(x \cdot x) = k^2/2$.

**Step 3.** Match $= 1/2 \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$.

**Why this matters.** The Taylor-leading-order trick avoids the $2\sin^2$ identity and is faster once you trust it.

**Final Answer:** $k = \pm 1$.

**Self-check before boxing the answer.** Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

**Q 5.15** **Prove that** $f(x) = \begin{cases} \dfrac{x}{|x| + 2x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$ **is discontinuous at** $x = 0$ **for every choice of** $k$.

**SOLUTION**

**Concept used.** If LHL $\neq$ RHL at $c$, the limit does not exist, so no choice of $f(c)$ can make $f$ continuous there.

**Step 1.** For $x > 0$, $|x| = x$, so

$$f(x) = \frac{x}{x + 2x^2} = \frac{x}{x(1 + 2x)} = \frac{1}{1 + 2x}.$$

Hence $\lim_{x\to 0^+} f(x) = \dfrac{1}{1 + 0} = 1$.

**Step 2.** For $x < 0$, $|x| = -x$, so

$$f(x) = \frac{x}{-x + 2x^2} = \frac{x}{x(-1 + 2x)} = \frac{1}{-1 + 2x}.$$

Hence $\lim_{x\to 0^-} f(x) = \dfrac{1}{-1 + 0} = -1$.

**Step 3.** LHL $= -1 \neq 1 =$ RHL, so $\lim_{x\to 0} f(x)$ does not exist.

**Step 4.** Any single value $f(0) = k$ can equal at most one of $\pm 1$; continuity needs the limit itself to exist.

> **Final Answer:** $f$ is discontinuous at $x = 0$ for every $k \in \mathbb{R}$.

### ♥ Removable vs non-removable

If the two-sided limit exists, the discontinuity is removable — redefining $f(c)$ to equal the limit makes the function continuous. If LHL$\neq$RHL, no choice of $f(c)$ helps; the jump is permanent.

**EXPERT'S SOLUTION** : *Tara Desai, M.Sc Applied Mathematics, IIT Kanpur*

**Strategic angle.** A two-sided limit failure is fatal: adjusting $f$ at the single point $x = 0$ cannot fix a jump.

**Step 1.** $f(x) \to 1$ from the right, $-1$ from the left.

**Step 2.** Two different one-sided limits $\Rightarrow$ no single value of $f(0)$ makes $f$ continuous.

**Why this matters.** Removable vs. non-removable: only when LHL $=$ RHL is the discontinuity removable by redefining $f(c)$.

> **Final Answer:** Discontinuous for all $k$.

**Alternate framing.** Read the function as a single graph drawn with two coloured pens. Continuity asks: does the second pen pick up exactly where the first left off? If both pens meet the same $y$ at $x = c$ *and* the dot $f(c)$ sits on that meeting point, continuity holds; otherwise it doesn't. This visual check anticipates the algebra and exposes sign errors before they propagate.

**Q 5.16** **Find $a$ and $b$ so that** $f(x) = \begin{cases} \dfrac{x-4}{|x-4|} + a, & x < 4 \\ a + b, & x = 4 \\ \dfrac{x-4}{|x-4|} + b, & x > 4 \end{cases}$ **is continuous at $x = 4$.**

SOLUTION

**Concept used.** $\dfrac{x-4}{|x-4|} = -1$ for $x < 4$ and $= +1$ for $x > 4$. Force LHL $= f(4) =$ RHL.

**Step 1.** LHL: for $x < 4$, $f(x) = -1 + a$. So $\lim_{x\to4^-} f(x) = a - 1$.

**Step 2.** RHL: for $x > 4$, $f(x) = 1 + b$. So $\lim_{x\to4^+} f(x) = b + 1$.

**Step 3.** $f(4) = a + b$.

**Step 4.** Continuity gives two equations:

$$a - 1 = a + b \quad \Rightarrow \quad b = -1,$$

$$b + 1 = a + b \quad \Rightarrow \quad a = 1.$$

**Final Answer:** $a = 1$, $b = -1$.

**✗ Branch confusion**

Always use the strict-inequality branch for the side limit and the equality branch for the value $f(c)$. Mixing the two is the most common cause of a wrong jump diagnosis.

EXPERT'S SOLUTION *: Sanya Rao, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** Set the two side limits equal to $f(4)$ and solve the resulting linear system in $a, b$.

**Step 1.** LHL $= a - 1$, RHL $= b + 1$, $f(4) = a + b$.

**Step 2.** $a - 1 = a + b \Rightarrow b = -1$; $b + 1 = a + b \Rightarrow a = 1$.

**Why this matters.** Two unknowns in piecewise problems are usually pinned down by the two equations LHL $= f(c)$ and RHL $= f(c)$.

**Final Answer:** $a = 1$, $b = -1$.

**Q 5.17** **Given $f(x) = \dfrac{1}{x+2}$, find the points of discontinuity of $y = f(f(x))$.**

SOLUTION

**Concept used.** A composition $f \circ f$ is discontinuous where either $f$ is undefined (inner) or $f$ is undefined at the inner value (outer). $f(x) = 1/(x+2)$ fails when its denominator $x + 2 = 0$.

**Step 1.** Inner $f$ blows up at $x = -2$.

**Step 2.** Compute $f(f(x))$ for $x \neq -2$:

$$f(f(x)) = \frac{1}{f(x)+2} = \frac{1}{\frac{1}{x+2}+2} = \frac{1}{\frac{1+2(x+2)}{x+2}} = \frac{x+2}{2x+5}.$$

**Step 3.** The outer denominator $2x+5$ is $0$ when $x = -5/2$. So $f \circ f$ is undefined at $x = -5/2$ *and* at $x = -2$ (inherited from the inner failure).

**Step 4.** Hence $y = f(f(x))$ is discontinuous at $x = -2$ and $x = -5/2$.

**Final Answer:** $x = -2$ and $x = -5/2$ are points of discontinuity of $f(f(x))$.

### ☞ Composition continuity

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Track both layers when locating discontinuities.

**EXPERT'S SOLUTION** : *Meera Bhat, Ph.D Pure Mathematics, IISc Bangalore*

**Strategic angle.** Two failure modes: (i) inner $f$ fails; (ii) outer $f$ fails on the value the inner produced. Track both.

**Step 1.** Inner fails at $x = -2$.

**Step 2.** Composed simplification: $f(f(x)) = \dfrac{x+2}{2x+5}$.

**Step 3.** Simplified denominator zero at $x = -5/2$.

**Step 4.** Both points are discontinuities.

**Why this matters.** Always include the "hole carried by the inner function" even when the simplified expression looks fine elsewhere.

**Final Answer:** $x = -2, -5/2$.

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

**Q 5.18** **Find all points of discontinuity of $f(t) = \dfrac{1}{t^2+t-2}$, where $t = \dfrac{1}{x-1}$.**

**SOLUTION**

**Concept used.** $f$ as a function of $x$ is the composition $x \mapsto t(x) \mapsto f(t)$. Discontinuities of the composition occur where (i) $t(x)$ is undefined or (ii) $f(t)$ is undefined at the generated $t$-value.

**Step 1.** $t(x) = 1/(x-1)$ is undefined at $x = 1$.

**Step 2.** Factor the denominator of $f(t)$:

$$t^2 + t - 2 = (t+2)(t-1).$$

So $f(t)$ is undefined at $t = -2$ and $t = 1$.

**Step 3.** Solve $t(x) = -2$: $\frac{1}{x-1} = -2 \Rightarrow x - 1 = -1/2 \Rightarrow x = 1/2$.

**Step 4.** Solve $t(x) = 1$: $\frac{1}{x-1} = 1 \Rightarrow x - 1 = 1 \Rightarrow x = 2$.

**Step 5.** Collect: discontinuities at $x = 1$, $x = 1/2$, $x = 2$.

**Final Answer:** Discontinuous at $x = 1,\ \frac{1}{2},\ 2$.

### ☞ Composition continuity

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Track both layers when locating discontinuities.

**EXPERT'S SOLUTION** *: Aaditya Verma, M.Sc Mathematics, IIT Bombay*

**Quick reading.** The composition has three failure points: the hole of the inner function and the two roots of $t^2 + t - 2$ pulled back through $t(x)$.

**Step 1.** Inner hole: $x = 1$.

**Step 2.** $t = 1 \Rightarrow x = 2$; $t = -2 \Rightarrow x = 1/2$.

**Why this matters.** Compositions of rational functions stack their poles — always pull back the outer poles through the inner inverse to catch all failure points.

**Final Answer:** $x = 1,\ 1/2,\ 2$.

**Q 5.19** **Show that $f(x) = \sin x + \cos x$ is continuous at $x = \pi$.**

**SOLUTION**

**Concept used.** $\sin x$ and $\cos x$ are continuous on the whole of $\mathbb{R}$; the sum of continuous functions is continuous.

**Step 1.** $f(\pi) = \sin\pi + \cos\pi = 0 + (-1) = -1$.

**Step 2.** Compute the limit:

$$\lim_{x\to\pi}(\sin x + \cos x) = \sin\pi + \cos\pi = -1,$$

using continuity of $\sin$ and $\cos$.

**Step 3.** Limit equals value $\Rightarrow$ continuous at $\pi$.

> **Final Answer:** $f$ is continuous at $x = \pi$.

**✗ Branch confusion**

Always use the strict-inequality branch for the side limit and the equality branch for the value $f(c)$. Mixing the two is the most common cause of a wrong jump diagnosis.

**EXPERT'S SOLUTION** *: Ananya Joshi, M.Sc Applied Mathematics, IIT Kanpur*

**Structural angle.** Sum of two everywhere-continuous functions is everywhere continuous; no piecewise check needed.

**Step 1.** $\sin, \cos$ continuous on $\mathbb{R}$.

**Step 2.** Sum is continuous; at $\pi$, $f(\pi) = 0 + (-1) = -1$.

**Why this matters.** Identify the function class first; specific points become trivial.

> **Final Answer:** Continuous at $\pi$.

**Alternate framing.** Read the function as a single graph drawn with two coloured pens. Continuity asks: does the second pen pick up exactly where the first left off? If both pens meet the same $y$ at $x = c$ *and* the dot $f(c)$ sits on that meeting point, continuity holds; otherwise it doesn't. This visual check anticipates the algebra and exposes sign errors before they propagate.

**Q 5.20** **Examine differentiability of** $f(x) = \begin{cases} x[x], & 0 \le x < 2 \\ (x-1)x, & 2 \le x < 3 \end{cases}$ **at** $x = 2$**, where** $[x]$ **denotes the greatest integer.**

**SOLUTION**

**Concept used.** For differentiability at the break-point we need (i) continuity at the point and (ii) Lf′$(c)$ =Rf′$(c)$.

**Step 1.** Near $x = 2$ from the left, $1 \le x < 2$ so $[x] = 1$, giving $f(x) = x \cdot 1 = x$.

**Step 2.** At $x = 2$ and just to the right, the second rule applies: $f(x) = (x - 1)x = x^2 - x$. At $x = 2$, $f(2) = 4 - 2 = 2$.

**Step 3.** Continuity check:

- LHL $= \lim_{x\to 2^-} x = 2$.
- RHL $= \lim_{x\to 2^+}(x^2 - x) = 4 - 2 = 2$.

Both equal $f(2) = 2$, so $f$ is continuous at 2.

**Step 4.** Left derivative: on $(1, 2)$, $f(x) = x$, so $Lf'(2) = \frac{d}{dx}x\Big|_{x=2} = 1$.

**Step 5.** Right derivative: on $[2, 3)$, $f(x) = x^2 - x$, so $Rf'(2) = 2x - 1\Big|_{x=2} = 3$.

**Step 6.** $Lf'(2) = 1 \neq 3 = Rf'(2)$, so $f$ is not differentiable at 2.

> **Final Answer:** $f$ is **continuous but not differentiable** at $x = 2$.

**♥ Continuity does not imply differentiability**

A function may be continuous yet have a "corner" where the left and right tangent slopes differ. This question is the cleanest illustration of that gap.

**EXPERT'S SOLUTION** *: Arjun Pillai, B.Tech Engineering Physics, IIT Bombay*

**Picture-first.** On $[1, 2)$ the graph is the line $y = x$; on $[2, 3)$ it is the parabola $y = x^2 - x$. They meet at $(2, 2)$, but the line has slope 1 while the parabola has slope $2x - 1 = 3$ there.

**Step 1.** LHL $= 2$, RHL $= 2$, $f(2) = 2 \Rightarrow$ continuous.

**Step 2.** $Lf'(2) = 1$, $Rf'(2) = 3 \Rightarrow$ not differentiable.

**Why this matters.** Corners (slope mismatch) are the most common cause of non-differentiability after vertical-tangent failures.

> **Final Answer:** Continuous at $x = 2$; not differentiable at $x = 2$.

**Alternate framing.** Read the function as a single graph drawn with two coloured pens. Continuity asks: does the second pen pick up exactly where the first left off? If both pens meet the same $y$ at $x = c$ *and* the dot $f(c)$ sits on that meeting point, continuity holds; otherwise it doesn't. This visual check anticipates the algebra and exposes sign errors before they propagate.

**Q 5.21** **Examine differentiability of** $f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ **at** $x = 0$**.**

**SOLUTION**

**Concept used.** Use the first-principles definition of the derivative at $0$: $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$, combined with the Squeeze theorem.

**Step 1.** Form the difference quotient:

$$\frac{f(h) - f(0)}{h} = \frac{h^2 \sin(1/h) - 0}{h} = h \sin \frac{1}{h}.$$

**Step 2.** Bound: $|\sin(1/h)| \leq 1$, so $|h \sin(1/h)| \leq |h|$.

**Step 3.** Take the limit: $|h| \to 0$ as $h \to 0$, so by the Squeeze theorem $h \sin(1/h) \to 0$.

**Step 4.** Both one-sided limits equal $0$; the derivative exists.

**Final Answer:** $f$ is differentiable at $x = 0$ with $f'(0) = 0$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

**EXPERT'S SOLUTION** *: Ankit Kumar, M.Sc Mathematics, ISI Kolkata*

**Strategic angle.** Same squeeze pattern as Q6, but applied to $h \sin(1/h)$ (one power of $h$ stronger than what we needed for continuity), which is why differentiability also works.

**Step 1.** Difference quotient $= h \sin(1/h)$.

**Step 2.** Bounded by $|h| \to 0 \Rightarrow$ derivative is $0$.

**Why this matters.** $x^2 \sin(1/x)$ is the textbook example of a function that is differentiable everywhere yet whose derivative is *discontinuous* at $0$.

**Final Answer:** $f'(0) = 0$.

**Q 5.22** **Examine differentiability of** $f(x) = \begin{cases} 1+x, & x \le 2 \\ 5-x, & x > 2 \end{cases}$ **at** $x = 2$**.**

**SOLUTION**

**Concept used.** First check continuity at 2; if continuous, compare $Lf'(2)$ and $Rf'(2)$.

**Step 1.** Continuity: $f(2) = 1 + 2 = 3$; LHL $= \lim_{x\to 2^-}(1+x) = 3$; RHL $= \lim_{x\to 2^+}(5-x) = 3$. Equal $\Rightarrow$ continuous.

**Step 2.** Left derivative on $(-\infty, 2]$, where $f(x) = 1 + x$: $Lf'(2) = 1$.

**Step 3.** Right derivative on $(2, \infty)$, where $f(x) = 5 - x$: $Rf'(2) = -1$.

**Step 4.** $1 \neq -1 \Rightarrow$ derivative does not exist.

> **Final Answer:** Continuous at $x = 2$; not differentiable at $x = 2$.

**✗ Sign-check**

Carry the negative sign through every substitution step — a single dropped minus turns a continuous function into a discontinuous one or flips an integral's sign. Recheck signs before the final box.

**EXPERT'S SOLUTION** *: Pooja Kapoor, M.Sc Mathematics, IIT Bombay*

**Picture-first.** Two lines meeting at $(2, 3)$: one rises with slope $+1$, the other descends with slope $-1$. The "$\wedge$"-corner kills differentiability.

**Step 1.** Continuous: both branches give 3 at $x = 2$.

**Step 2.** Slopes $+1$ and $-1$ disagree.

**Why this matters.** Tent functions / V-graphs are the cleanest non-differentiability example to put on a quiz.

> **Final Answer:** Not differentiable at 2.

**Q 5.23** **Show that** $f(x) = |x - 5|$ **is continuous but not differentiable at** $x = 5$**.**

**SOLUTION**

**Concept used.** $|x - 5|$ has a corner at $x = 5$: it equals $x - 5$ for $x > 5$ and $-(x - 5)$ for $x < 5$.

**Step 1.** Continuity: $f(5) = 0$; $\lim_{x\to 5^+}(x-5) = 0$; $\lim_{x\to 5^-}(-(x-5)) = 0$. All equal $\Rightarrow$ continuous.

**Step 2.** Left derivative: $Lf'(5) = \lim_{h\to 0^-} \dfrac{|h|-0}{h} = \dfrac{-h}{h} = -1$.

**Step 3.** Right derivative: $Rf'(5) = \lim_{h\to 0^+} \dfrac{|h|-0}{h} = \dfrac{h}{h} = 1$.

**Step 4.** $-1 \neq 1 \Rightarrow$ not differentiable at $5$.

**Final Answer:** Continuous at $x = 5$; not differentiable at $x = 5$.

### ♥ Why the method works

The strategy here generalises: any time you can reduce a hard limit or integral to a known standard form via algebra (factoring, conjugate, substitution), you have converted a calculus problem into an algebra problem — always cheaper.

**EXPERT'S SOLUTION** : *Aanya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

**Strategic angle.** $|x-c|$ is the canonical "corner". Slope is $-1$ to the left, $+1$ to the right.

**Step 1.** Direct LHL/RHL of $(|h|/h)$ give $-1$ and $+1$.

**Step 2.** Continuity is automatic because $|x-c| \to 0$ as $x \to c$.

**Why this matters.** Any function with $|g(x)|$ as a factor will inherit a corner wherever $g$ changes sign.

**Final Answer:** Continuous, not differentiable at $5$.

**Q 5.24** **A function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, $f(x) \neq 0$. Suppose $f$ is differentiable at $x = 0$ with $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.**

**SOLUTION**

**Concept used.** The defining property $f(x+y) = f(x)f(y)$ is the **exponential Cauchy equation**. Combined with differentiability at a single point and $f \neq 0$, it forces $f$ to be exponential. We prove the derivative identity from first principles, using
$f(0) = f(0+0) = f(0)^2$ and $f(0) \neq 0 \Rightarrow f(0) = 1$.

**Step 1.** Step 1: $f(0) = f(0+0) = f(0)f(0) = f(0)^2$. Since $f(0) \neq 0$, divide by $f(0)$: $f(0) = 1$.

**Step 2.** Step 2: Write the derivative at $x$ from first principles:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Apply the functional equation to $f(x+h)$:

$$f'(x) = \lim_{h\to 0} \frac{f(x)f(h)-f(x)}{h} = f(x)\lim_{h\to 0} \frac{f(h)-1}{h}.$$

**Step 3.** Step 3: Identify the inner limit as $f'(0)$. Indeed,

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)-1}{h}.$$

We are given $f'(0) = 2$.

**Step 4.** Step 4: Combine:

$$f'(x) = f(x)\cdot f'(0) = 2f(x).$$

> **Final Answer:** $f'(x) = 2f(x)$ for every $x \in \mathbb{R}$.

### ☞ Parametric derivative

For parametric curves $(x(t), y(t))$ use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

**EXPERT'S SOLUTION** *: Dev Singh, M.Sc Mathematics, IIT Bombay*

**Structural angle.** The Cauchy equation $f(x+y) = f(x)f(y)$ "separates" $x$ from $h$ in the difference quotient, exposing $f'(0)$ as the master parameter.

**Step 1.** Plug $x = y = 0$: $f(0)^2 = f(0)$ and $f(0) \neq 0 \Rightarrow f(0) = 1$.

**Step 2.** Difference quotient at $x$: $\frac{f(x+h)-f(x)}{h} = f(x)\frac{f(h)-1}{h}$.

**Step 3.** Inner limit is $f'(0) = 2$, so $f'(x) = 2f(x)$.

**Why this matters.** This is the calculus proof that the unique $f$ satisfying the multiplicative Cauchy equation with $f'(0) = k$ is $f(x) = e^{kx}$.

> **Final Answer:** $f'(x) = 2f(x)$.

**Self-check before boxing the answer.** Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

**What examiners reward.** Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason.

**Q 5.25** **Differentiate $2^{\cos^2 x}$ w.r.t. $x$.**

SOLUTION

**Concept used.** Logarithmic differentiation for an exponential with a variable exponent: if $y = a^{u(x)}$ with constant base $a > 0$, then $\ln y = u(x) \ln a$ and $\frac{1}{y}\frac{dy}{dx} = u'(x) \ln a$.

**Step 1.** Let $y = 2^{\cos^2 x}$. Take natural logarithm of both sides:

$$\ln y = \cos^2 x \cdot \ln 2.$$

**Step 2.** Differentiate w.r.t. $x$. The right side has constant factor $\ln 2$ multiplying $\cos^2 x$, whose derivative is $2 \cos x \cdot (-\sin x) = -2 \sin x \cos x = -\sin 2x$:

$$\frac{1}{y}\frac{dy}{dx} = -\sin 2x \cdot \ln 2.$$

**Step 3.** Multiply by $y = 2^{\cos^2 x}$:

$$\frac{dy}{dx} = -\sin 2x \cdot \ln 2 \cdot 2^{\cos^2 x}.$$

**Final Answer:** $\frac{dy}{dx} = -\sin 2x \cdot \ln 2 \cdot 2^{\cos^2 x}$.

**✗ Constant-base vs variable-base**

For $a^{u(x)}$ with constant $a$, the derivative carries a factor $\ln a$ (not $1/a$). Forgetting this $\ln a$ is the most common slip when mixing $a^u$ with the closely-named $u^a$ (whose derivative is $au^{a-1}u'$, no log).

EXPERT'S SOLUTION *: Rohit Sharma, M.Sc Applied Mathematics, IIT Kanpur*

**Strategic angle.** Variable exponent over constant base ⇒ take logs once; the derivative falls out in one line.

**Step 1.** $\ln y = \cos^2 x \cdot \ln 2$.

**Step 2.** Differentiate: $\frac{y'}{y} = -\sin 2x \cdot \ln 2$ (using $\frac{d}{dx}\cos^2 x = -\sin 2x$).

**Step 3.** Multiply by $y$: $y' = -\sin 2x \cdot \ln 2 \cdot 2^{\cos^2 x}$.

**Why this matters.** The identity $\frac{d}{dx}\cos^2 x = -\sin 2x$ (rather than the longer $2\cos x \sin x$) keeps the answer compact and matches the standard NCERT form.

**Final Answer:** $-\sin 2x\ \ln 2\ \cdot 2^{\cos^2 x}$.

**Q 5.26** **Differentiate $\frac{8^x}{x^8}$ w.r.t. $x$.**

SOLUTION

**Concept used.** Quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$, together with $\frac{d}{dx}a^x = a^x \log a$ and $\frac{d}{dx}x^n = nx^{n-1}$.

**Step 1.** Let $u = 8^x$ and $v = x^8$. Then $u' = 8^x \log 8$ and $v' = 8x^7$.

**Step 2.** Apply quotient rule:

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2} = \frac{8^x \log 8 \cdot x^8 - 8^x \cdot 8x^7}{x^{16}}.$$

**Step 3.** Factor $8^x x^7$ from the numerator:

$$= \frac{8^x x^7(x \log 8 - 8)}{x^{16}} = \frac{8^x(x \log 8 - 8)}{x^9}.$$

**Final Answer:** $\frac{dy}{dx} = \frac{8^x(x \log 8 - 8)}{x^9}$.

☞ **Exponential always positive**

$e^x > 0$ for every real $x$, so $e^x$ never vanishes. This means $e^{u(x)}$ is continuous wherever $u(x)$ is, and shares its differentiability.

**EXPERT'S SOLUTION** : *Siddharth Rao, B.Tech CSE, IIT Roorkee*

**Quick reading.** Quotient with exponential numerator and power denominator. Factor $8^x x^7$ to keep the expression compact.

**Step 1.** Apply quotient rule.

**Step 2.** Factor common $8^x x^7$.

**Step 3.** Simplify exponents to land at the boxed answer.

**Why this matters.** The pattern $a^x/x^n$ appears repeatedly in JEE; the cleaned form $a^x(x\log a - n)/x^{n+1}$ is worth remembering.

**Final Answer:** $\dfrac{8^x(x\log 8 - 8)}{x^9}$.

**Q 5.27** **Differentiate** $\log\left(x+\sqrt{x^2+a}\right)$ **w.r.t.** $x$.

SOLUTION

**Concept used.** $\frac{d}{dx}\log u = \frac{u'}{u}$ combined with the chain rule. We will differentiate $\sqrt{x^2+a}$ separately.

**Step 1.** Let $u = x+\sqrt{x^2+a}$. Then $\log u' = u'/u$.

**Step 2.** Differentiate $u$:

$$u' = 1 + \frac{1}{2\sqrt{x^2+a}}\cdot 2x = 1 + \frac{x}{\sqrt{x^2+a}} = \frac{\sqrt{x^2+a}+x}{\sqrt{x^2+a}}.$$

**Step 3.** Form the quotient $u'/u$:

$$\frac{u'}{u} = \frac{\dfrac{x+\sqrt{x^2+a}}{\sqrt{x^2+a}}}{x+\sqrt{x^2+a}} = \frac{1}{\sqrt{x^2+a}}.$$

**Final Answer:** $\dfrac{d}{dx}\log\left(x+\sqrt{x^2+a}\right) = \dfrac{1}{\sqrt{x^2+a}}$.

☞ **Useful identity**

This is exactly $\frac{d}{dx}\sinh^{-1}(x/\sqrt{a})$ in disguise (for $a>0$): $\log(x+\sqrt{x^2+a})$ is the antiderivative of $1/\sqrt{x^2+a}$.

**EXPERT'S SOLUTION** *: Vivaan Banerjee, M.Sc Mathematics, IIT Bombay*

**Structural angle.** The numerator $u'$ shares the factor $x+\sqrt{x^2+a}$ with $u$. That cancellation is what makes the answer so compact.

**Step 1.** $u' = \frac{\sqrt{x^2+a}+x}{\sqrt{x^2+a}}$.

**Step 2.** Divide by $u = x + \sqrt{x^2+a}$ to get $1/\sqrt{x^2+a}$.

**Why this matters.** The clean cancellation is the standard JEE/NEET shortcut for $\int dx/\sqrt{x^2+a}$ problems.

**Final Answer:** $\dfrac{1}{\sqrt{x^2+a}}$.

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

**Q 5.28** **Differentiate** $\log[\log(\log x^5)]$ **w.r.t.** $x$.

**SOLUTION**

**Concept used.** Chain rule applied repeatedly to a three-level logarithm composition. $\dfrac{d}{dx}\log u = \dfrac{u'}{u}$.

**Step 1.** Innermost: let $w = \log x^5 = 5\log x$. Then $\dfrac{dw}{dx} = \dfrac{5}{x}$.

**Step 2.** Middle: $v = \log w$, $\dfrac{dv}{dx} = \dfrac{1}{w}\cdot\dfrac{dw}{dx} = \dfrac{1}{5\log x}\cdot\dfrac{5}{x} = \dfrac{1}{x\log x}$.

**Step 3.** Outer: $y = \log v$, $\dfrac{dy}{dx} = \dfrac{1}{v}\cdot\dfrac{dv}{dx} = \dfrac{1}{\log(5\log x)}\cdot\dfrac{1}{x\log x}$.

**Step 4.** Final:

$$\frac{dy}{dx} = \frac{1}{x\log x\cdot\log(5\log x)}.$$

**Final Answer:** $\dfrac{dy}{dx} = \dfrac{1}{x\ \log x\ \log(5\log x)}$.

### ✗ Chain rule layers

Differentiate the outermost function first, then multiply by the derivative of the inner function, then by the inner-inner, and so on. Stop only when you hit $x$ or a constant — truncating early is the textbook chain-rule slip.

**EXPERT'S SOLUTION** : *Kavya Desai, M.Sc Mathematics, ISI Kolkata*

**Strategic angle.** Strip one log at a time, using $d(\log u)/dx = u'/u$.

**Step 1.** $d(\log x^5)/dx = 5/x$.

**Step 2.** $d(\log\log x^5)/dx = 5/(x \cdot 5\log x) = 1/(x\log x)$.

**Step 3.** Apply one more outer log to land at the boxed answer.

**Why this matters.** Iterated-log problems are tractable once you commit to "one strip per step".

**Final Answer:** $\dfrac{1}{x\log x\log(5\log x)}$.

**Q 5.29** **Differentiate** $\sin\sqrt{x} + \cos^2\sqrt{x}$ **w.r.t.** $x$.

**SOLUTION**

**Concept used.** Sum and chain rules. $\dfrac{d}{dx}\sin u = \cos u \cdot u'$; $\dfrac{d}{dx}\cos^2 u = 2\cos u \cdot (-\sin u) \cdot u' = -\sin 2u \cdot u'$.

**Step 1.** Let $u = \sqrt{x}$, so $u' = \dfrac{1}{2\sqrt{x}}$.

**Step 2.** Derivative of $\sin\sqrt{x}$: $\cos\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}$.

**Step 3.** Derivative of $\cos^2\sqrt{x}$: $-\sin(2\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}}$.

**Step 4.** Sum and factor $\dfrac{1}{2\sqrt{x}}$:

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\big(\cos\sqrt{x} - \sin 2\sqrt{x}\big).$$

**Final Answer:** $\dfrac{dy}{dx} = \dfrac{\cos\sqrt{x} - \sin 2\sqrt{x}}{2\sqrt{x}}$.

☞ **Conjugate first**

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending $x$

cancels cleanly.

**EXPERT'S SOLUTION** : *Ishita Mehta, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Same inner $\sqrt{x}$ feeds both terms; factor its derivative $1/(2\sqrt{x})$ once.

**Step 1.** Differentiate each piece via chain rule.

**Step 2.** Factor $1/(2\sqrt{x})$.

**Why this matters.** Shared inner functions let you factor and keep expressions compact.

**Final Answer:** $\dfrac{\cos\sqrt{x} - \sin 2\sqrt{x}}{2\sqrt{x}}$.

**Q 5.30** **Differentiate** $\sin^n(ax^2 + bx + c)$ **w.r.t.** $x$**.**

**SOLUTION**

**Concept used.** Triple chain: outer power, middle $\sin$, inner polynomial.

**Step 1.** Let $u = \sin(ax^2 + bx + c)$ and $y = u^n$. Then $\dfrac{dy}{dx} = nu^{n-1} \cdot u'$.

**Step 2.** $u' = \cos(ax^2 + bx + c) \cdot (2ax + b)$.

**Step 3.** Combine:

$$\frac{dy}{dx} = n\sin^{n-1}(ax^2 + bx + c) \cdot \cos(ax^2 + bx + c) \cdot (2ax + b).$$

**Final Answer:** $n(2ax + b)\sin^{n-1}(ax^2 + bx + c)\cos(ax^2 + bx + c)$.

☞ **Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an $\varepsilon$-$\delta$ on a polynomial in an exam.

**EXPERT'S SOLUTION** : *Aaditya Patel, Ph.D Pure Mathematics, IISc Bangalore*

**Strategic angle.** Three nested functions; differentiate outer-in.

**Step 1.** Power: $n\sin^{n-1}(\cdot)$.

**Step 2.** sin: $\cos(\cdot)$.

**Step 3.** Inner polynomial: $2ax + b$.

**Why this matters.** The same pattern $nu^{n-1}u'$ powers all "power-of-something" derivatives.

**Final Answer:** $n(2ax+b)\sin^{n-1}\cdot\cos(ax^2+bx+c)$.

## Q 5.31 Differentiate $\cos(\tan\sqrt{x+1})$ w.r.t. $x$.

**SOLUTION**

**Concept used.** Four-level chain: $\cos\circ\tan\circ\sqrt{\ }\circ(x+1)$.

**Step 1.** Let $u=\sqrt{x+1}$, so $\dfrac{du}{dx}=\dfrac{1}{2\sqrt{x+1}}$.

**Step 2.** Let $v=\tan u$, so $\dfrac{dv}{dx}=\sec^2 u\cdot u'=\sec^2\sqrt{x+1}\cdot\dfrac{1}{2\sqrt{x+1}}$.

**Step 3.** Outer $\cos v$: $-\sin v\cdot v'$. So

$$\frac{dy}{dx}=-\sin(\tan\sqrt{x+1})\cdot\sec^2\sqrt{x+1}\cdot\frac{1}{2\sqrt{x+1}}.$$

**Final Answer:** $-\dfrac{\sin(\tan\sqrt{x+1})\sec^2\sqrt{x+1}}{2\sqrt{x+1}}$.

☞ **Half-angle identity**

Memorise $1-\cos\theta=2\sin^2(\theta/2)$. It converts every $1-\cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Karan Verma, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Peel: cos $\to -\sin$, tan $\to\sec^2$, $\sqrt{\ }\to 1/(2\sqrt{\ })$, inner $x+1\to 1$.

**Step 1.** Apply each layer's derivative.

**Why this matters.** Memorise the four standard $\frac{d}{dx}$ layers; deep nestings then unfold mechanically.

**Final Answer:** Same as above.

## Q 5.32 Differentiate $\sin x^2+\sin^2 x+\sin^2(x^2)$ w.r.t. $x$.

**SOLUTION**

**Concept used.** Sum rule term-by-term; chain rule with inner $x^2$ where needed; $\dfrac{d}{dx}\sin^2 u = 2\sin u\cos u \cdot u' = \sin 2u \cdot u'$.

**Step 1.** $\dfrac{d}{dx}\sin x^2 = \cos x^2 \cdot 2x = 2x\cos x^2$.

**Step 2.** $\dfrac{d}{dx}\sin^2 x = \sin 2x$.

**Step 3.** $\dfrac{d}{dx}\sin^2(x^2) = \sin(2x^2)\cdot 2x = 2x\sin 2x^2$.

**Step 4.** Sum:

$$\frac{dy}{dx} = 2x\cos x^2 + \sin 2x + 2x\sin 2x^2.$$

**Final Answer:** $2x\cos x^2 + \sin 2x + 2x\sin 2x^2$.

**✗ Chain rule layers**

Differentiate the outermost function first, then multiply by the derivative of the inner function, then by the inner-inner, and so on. Stop only when you hit $x$ or a constant — truncating early is the textbook chain-rule slip.

**EXPERT'S SOLUTION** *: Priya Nair, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** Three independent terms, each handled by the standard $\sin$ or $\sin^2$ template.

**Step 1.** Differentiate each term, watch the inner $x^2$ in two of them.

**Why this matters.** Sum + chain on each term is the universal recipe.

**Final Answer:** $2x\cos x^2 + \sin 2x + 2x\sin 2x^2$.

**Q 5.33** **Differentiate** $\sin^{-1}\dfrac{1}{\sqrt{x+1}}$ **w.r.t.** $x$.

**SOLUTION**

**Concept used.** $\dfrac{d}{dx}\sin^{-1} u = \dfrac{u'}{\sqrt{1-u^2}}$. Compute $u$, $u'$, then $1-u^2$.

**Step 1.** $u = \dfrac{1}{\sqrt{x+1}} = (x+1)^{-1/2}$. So $u' = -\dfrac{1}{2}(x+1)^{-3/2} = -\dfrac{1}{2(x+1)^{3/2}}$.

**Step 2.** $u^2 = \dfrac{1}{x+1}$, hence $1 - u^2 = 1 - \dfrac{1}{x+1} = \dfrac{x}{x+1}$, $\sqrt{1-u^2} = \sqrt{\dfrac{x}{x+1}} = \dfrac{\sqrt{x}}{\sqrt{x+1}}$.

**Step 3.** Apply the rule:

$$\frac{dy}{dx} = \frac{u'}{\sqrt{1-u^2}} = \frac{-\dfrac{1}{2(x+1)^{3/2}}}{\dfrac{\sqrt{x}}{\sqrt{x+1}}} = -\frac{1}{2(x+1)^{3/2}} \cdot \frac{\sqrt{x+1}}{\sqrt{x}} = -\frac{1}{2(x+1)\sqrt{x}}.$$

**Final Answer:** $\dfrac{dy}{dx} = -\dfrac{1}{2(x+1)\sqrt{x}}$.

### ☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending $x$ cancels cleanly.

**EXPERT'S SOLUTION** : *Aditya Joshi, M.Sc Mathematics, IIT Bombay*

**Strategic angle.** $\sin^{-1}(1/\sqrt{x+1})$ equals $\tan^{-1}(1/\sqrt{x})$ (right triangle with sides $1, \sqrt{x}, \sqrt{x+1}$). Differentiating $\tan^{-1}(1/\sqrt{x})$ directly gives the same answer quickly.

**Step 1.** Substitute $\sqrt{x} = \tan\theta \Rightarrow y = \pi/2 - \theta = \pi/2 - \tan^{-1}\sqrt{x}$.

**Step 2.** Differentiate: $dy/dx = -\dfrac{1}{1+x} \cdot \dfrac{1}{2\sqrt{x}} = -\dfrac{1}{2(x+1)\sqrt{x}}$.

**Why this matters.** Inverse-trig substitution shortcuts sometimes replace 3 lines of chain rule with one identity.

**Final Answer:** $-\dfrac{1}{2(x+1)\sqrt{x}}$.

**Q 5.34** **Differentiate** $(\sin x)^{\cos x}$ **w.r.t.** $x$.

**SOLUTION**

**Concept used.** Logarithmic differentiation: when both base and exponent are functions of $x$, take $\log$ on both sides, differentiate, then multiply back by $y$.

**Step 1.** Let $y = (\sin x)^{\cos x}$. Take log:

$$\log y = \cos x \cdot \log(\sin x).$$

**Step 2.** Differentiate both sides w.r.t. $x$. Left: $\frac{1}{y} \cdot \frac{dy}{dx}$. Right: product rule,

$$\frac{d}{dx}\left[\cos x \cdot \log \sin x\right] = -\sin x \cdot \log \sin x + \cos x \cdot \frac{\cos x}{\sin x}.$$

Simplify: $-\sin x \log \sin x + \cos x \cot x$.

**Step 3.** Multiply both sides by $y$:

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left(\cos x \cot x - \sin x \log \sin x\right).$$

**Final Answer:** $\frac{dy}{dx} = (\sin x)^{\cos x}\left(\cos x \cot x - \sin x \log \sin x\right)$.

### ☞ Exam Tip

Whenever you see $u(x)^{v(x)}$ with both $u$ and $v$ functions of $x$, default to logarithmic differentiation. No other rule covers this case.

**EXPERT'S SOLUTION** : *Yash Reddy, B.Tech CSE, IIT Roorkee*

**Quick reading.** Power-tower with variable base *and* exponent: log first, then product rule on $v \log u$.

**Step 1.** $\log y = \cos x \log \sin x$.

**Step 2.** Differentiate: $y'/y = -\sin x \log \sin x + \cos x \cdot \cot x$.

**Step 3.** Multiply by $y$.

**Why this matters.** Logarithmic differentiation is the only correct treatment of $u^v$; "power rule" and "exponential rule" each miss half the answer.

**Final Answer:** Same as main.

**Reading the structure.** Logarithmic differentiation is the tool whenever the exponent itself depends on $x$, e.g. $x^x$, $(\sin x)^{\cos x}$, $(\tan x)^x$. Take $\ln$ first to flatten the exponent into a product, differentiate using the product rule, then multiply through by the original function. The technique is mechanical once the pattern is recognised.

**Q 5.35** **Differentiate** $\sin^m x \cdot \cos^n x$ **w.r.t.** $x$**.**

**SOLUTION**

**Concept used.** Product rule combined with chain rule on each factor: $\frac{d}{dx}\sin^m x = m\sin^{m-1} x\cos x$, etc.

**Step 1.** Let $y = u \cdot v$ where $u = \sin^m x$, $v = \cos^n x$.

**Step 2.** Derivatives: $u' = m\sin^{m-1} x \cdot \cos x$;
$v' = n\cos^{n-1} x \cdot (-\sin x) = -n\sin x\cos^{n-1} x$.

**Step 3.** Apply product rule:

$$y' = u'v + uv' = m\sin^{m-1} x\cos x \cdot \cos^n x + \sin^m x \cdot (-n\sin x\cos^{n-1} x).$$

**Step 4.** Factor $\sin^{m-1} x\cos^{n-1} x$:

$$y' = \sin^{m-1} x\cos^{n-1} x\big(m\cos^2 x - n\sin^2 x\big).$$

**Final Answer:** $y' = \sin^{m-1} x\cos^{n-1} x\,(m\cos^2 x - n\sin^2 x)$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Ananya Sharma, M.Sc Mathematics, IIT Bombay*

**Structural angle.** Both factors share $\sin^{m-1}\cos^{n-1}$; factor it out for cleanliness.

**Step 1.** Product rule, then factor.

**Why this matters.** The clean form $m\cos^2 - n\sin^2$ makes finding extrema straightforward.

**Final Answer:** $\sin^{m-1} x\cos^{n-1} x(m\cos^2 x - n\sin^2 x)$.

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

**Q 5.36** **Differentiate** $(x+1)^2(x+2)^3(x+3)^4$ **w.r.t.** $x$**.**

**SOLUTION**

**Concept used.** **Logarithmic differentiation** converts products of powers into sums.

**Step 1.** Let $y = (x+1)^2(x+2)^3(x+3)^4$. Take $\log$:

$$\log y = 2\log(x+1) + 3\log(x+2) + 4\log(x+3).$$

**Step 2.** Differentiate:

$$\frac{y'}{y} = \frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}.$$

**Step 3.** Multiply by $y$:

$$y' = (x+1)^2(x+2)^3(x+3)^4\left(\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right).$$

**Final Answer:** $y' = y\left(\frac{2}{x+1} + \frac{3}{x+2} + \frac{4}{x+3}\right)$.

☞ **Take logs first**

When a function is a product, quotient, or power with variables in both base and exponent (e.g. $x^x$), take natural log of both sides before differentiating. The product becomes a sum and the exponent becomes a multiplier — much easier.

**EXPERT'S SOLUTION** *: Rahul Iyer, M.Tech CS, IIT Madras*

**Quick reading.** Three factors ⇒ logarithmic differentiation gives three terms instead of nine.

**Step 1.** $\log y = \sum k_i \log(x+i)$.

**Step 2.** Differentiate ⇒ sum of $k_i/(x+i)$.

**Why this matters.** Always log-differentiate products of powers. The product rule on three factors is needlessly tedious.

**Final Answer:** See main solution.

**Reading the structure.** Logarithmic differentiation is the tool whenever the exponent itself depends on $x$, e.g. $x^x$, $(\sin x)^{\cos x}$, $(\tan x)^x$. Take $\ln$ first to flatten the exponent into a product, differentiate using the product rule, then multiply through by the original function. The technique is mechanical once the pattern is recognised.

**Q 5.37** **Differentiate** $\cos^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right)$ **w.r.t.** $x$, **for** $-\pi/4 < x < \pi/4$.

**SOLUTION**

**Concept used.** Simplify the argument first: $\frac{\sin x + \cos x}{\sqrt{2}} = \cos(\pi/4 - x)$ by the cosine difference identity.

**Step 1.** $\frac{\sin x + \cos x}{\sqrt{2}} = \sin x \cdot \sin \frac{\pi}{4} + \cos x \cdot \cos \frac{\pi}{4} = \cos\left(\frac{\pi}{4} - x\right)$.

**Step 2.** For $-\pi/4 < x < \pi/4$, $0 < \pi/4 - x < \pi/2$, so $\cos^{-1}\cos(\pi/4 - x) = \pi/4 - x$.

**Step 3.** Hence $y = \pi/4 - x$, and

$$\frac{dy}{dx} = -1.$$

**Final Answer:** $\frac{dy}{dx} = -1$.

**♥ Identity-first beats chain-rule-brute-force**

Inverse-trig with $\sin x + \cos x$ inside almost always simplifies via $\sin\alpha + \cos\alpha = \sqrt{2}\cos(\pi/4 - \alpha)$; recognise it and skip the chain-rule grind.

**EXPERT'S SOLUTION** *: Aanya Banerjee, M.Sc Mathematics, ISI Kolkata*

**Strategic angle.** Identity collapses the argument to $\cos(\pi/4 - x)$; $\cos^{-1}\cos$ on the principal range is identity.

**Step 1.** $(\sin x + \cos x)/\sqrt{2} = \cos(\pi/4 - x)$.

**Step 2.** $\cos^{-1}\cos(\pi/4 - x) = \pi/4 - x$ for $|x| < \pi/4$.

**Step 3.** $dy/dx = -1$.

**Why this matters.** JEE problems on inverse trig are mostly identity recognition; chain rule is a fallback, not a default.

**Final Answer:** $-1$.

**Q 5.38** **Differentiate** $\tan^{-1}\sqrt{\frac{1 - \cos x}{1 + \cos x}}$ **w.r.t.** $x$, **for** $-\pi/4 < x < \pi/4$.

**SOLUTION**

**Concept used.** Half-angle identities $1 - \cos x = 2\sin^2(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$ reduce the radicand to $\tan^2(x/2)$, and the square root then gives $|\tan(x/2)|$.

**Step 1.** Simplify the radicand:

$$\frac{1-\cos x}{1+\cos x} = \frac{2\sin^2(x/2)}{2\cos^2(x/2)} = \tan^2(x/2).$$

**Step 2.** Take the principal square root: $\sqrt{\tan^2(x/2)} = |\tan(x/2)|$. Hence $y = \tan^{-1}|\tan(x/2)|$.

**Step 3.** For $-\pi/4 < x < \pi/4$ we have $-\pi/8 < x/2 < \pi/8$, so $\tan(x/2)$ lies in the principal range of $\tan^{-1}$. Thus

$$y = \begin{cases} x/2, & 0 \le x < \pi/4, \\ -x/2, & -\pi/4 < x < 0. \end{cases}$$

**Step 4.** Differentiate piecewise:

$$\frac{dy}{dx} = \begin{cases} 1/2, & 0 < x < \pi/4, \\ -1/2, & -\pi/4 < x < 0. \end{cases}$$

Equivalently $\frac{dy}{dx} = \frac{1}{2}\operatorname{sgn}(x)$ on the punctured interval (the derivative does not exist at $x = 0$ since $|\tan(x/2)|$ has a corner there).

**Final Answer:** $\frac{dy}{dx} = \begin{cases} 1/2, & 0 < x < \pi/4 \\ -1/2, & -\pi/4 < x < 0 \end{cases}$, i.e. $\frac{1}{2}\operatorname{sgn}(x)$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Dev Sharma, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** The square root forces $|\tan(x/2)|$, and the absolute value splits the derivative by sign of $x$.

**Step 1.** Radicand $= \tan^2(x/2)$, so $y = \tan^{-1}|\tan(x/2)|$.

**Step 2.** For $0 < x < \pi/4$, $y = x/2 \Rightarrow y' = 1/2$.

**Step 3.** For $-\pi/4 < x < 0$, $y = -x/2 \Rightarrow y' = -1/2$.

**Why this matters.** Square roots after a half-angle reduction always produce a $|\cdot|$, and absolute values force a piecewise derivative. Never drop the modulus.

**Final Answer:** $\frac{1}{2}\,\mathrm{sgn}(x)$ on $(-\pi/4, 0) \cup (0, \pi/4)$.

**Generalising the trick.** The manipulation used here is part of a family: spot the algebraic disguise, apply the standard identity, reduce to a known limit or standard integral, conclude. Once you've solved three problems with the same disguise, the rest become routine.

**Q 5.39** **Differentiate** $\tan^{-1}(\sec x + \tan x)$ **w.r.t.** $x$**, for** $-\pi/2 < x < \pi/2$**.**

**SOLUTION**

**Concept used.** $\sec x + \tan x = \tan(\pi/4 + x/2)$ on the given interval; once recognised, $\tan^{-1}\tan$ collapses.

**Step 1.** Identity:

$$\sec x + \tan x = \frac{1+\sin x}{\cos x} = \frac{(\cos(x/2)+\sin(x/2))^2}{\cos^2(x/2)-\sin^2(x/2)} = \frac{\cos(x/2)+\sin(x/2)}{\cos(x/2)-\sin(x/2)}.$$

Divide numerator and denominator by $\cos(x/2)$:

$$= \frac{1+\tan(x/2)}{1-\tan(x/2)} = \tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

**Step 2.** Hence $y = \pi/4 + x/2$ on $|x| < \pi/2$.

**Step 3.** Differentiate: $\frac{dy}{dx} = \frac{1}{2}$.

**Final Answer:** $\frac{dy}{dx} = \frac{1}{2}$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

**EXPERT'S SOLUTION** : *Sneha Reddy, M.Sc Mathematics, IIT Bombay*

**Strategic angle.** The combination $\sec + \tan$ is the textbook $\tan(\pi/4 + x/2)$ form.

**Step 1.** Recognise identity.

**Step 2.** $\tan^{-1}\tan(\cdot) = \cdot$ on principal range.

**Step 3.** Derivative is $1/2$.

**Why this matters.** Memorise $\sec + \tan = \tan(\pi/4 + x/2)$; it turns dozens of integration / differentiation problems into one-liners.

**Final Answer:** $1/2$.

**Q 5.40** **Differentiate** $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right)$ **w.r.t.** $x$.

**SOLUTION**

**Concept used.** Divide numerator and denominator by $b\cos x$; the result is of the form $\tan^{-1}\frac{A-B}{1+AB} = \tan^{-1} A - \tan^{-1} B$.

**Step 1.** Divide by $b\cos x$:

$$\frac{a\cos x - b\sin x}{b\cos x + a\sin x} = \frac{a/b - \tan x}{1 + (a/b)\tan x}.$$

**Step 2.** This is the $\tan(A - B)$ pattern, so

$$y = \tan^{-1}(a/b) - \tan^{-1}(\tan x) = \tan^{-1}(a/b) - x$$

on the principal range.

**Step 3.** $\tan^{-1}(a/b)$ is a constant, so

$$\frac{dy}{dx} = -1.$$

**Final Answer:** $\frac{dy}{dx} = -1$.

### ✗ Chain rule layers

Differentiate the outermost function first, then multiply by the derivative of the inner function, then by the inner-inner, and so on. Stop only when you hit $x$ or a constant — truncating early is the textbook chain-rule slip.

**EXPERT'S SOLUTION** *: Pranav Mehta, B.Tech CSE, IIT Roorkee*

**Quick reading.** Rotate by $\tan^{-1}(a/b)$ to convert the expression into "constant minus $x$".

**Step 1.** Divide by $b\cos x$; recognise $\tan(A - B)$ form.

**Step 2.** Constant minus $x$ differentiates to $-1$.

**Why this matters.** $\tan^{-1}$ of "rotated $\tan$" is a constant shift; pattern-recognise to avoid the chain rule.

**Final Answer:** $-1$.

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

**Q 5.41** **Differentiate** $\sec^{-1}\left(\frac{1}{4x^3 - 3x}\right)$ **w.r.t.** $x$**, for** $0 < x < 1/\sqrt{2}$**.**

**SOLUTION**

**Concept used.** Substitution $x = \cos\theta$ turns $4x^3 - 3x$ into $\cos 3\theta$ by the triple-angle identity. Then $\sec^{-1}(1/\cos 3\theta) = \sec^{-1}\sec 3\theta = 3\theta$.

**Step 1.** Let $x = \cos\theta$. Then $4x^3 - 3x = \cos 3\theta$. So $\frac{1}{4x^3 - 3x} = \sec 3\theta$ and $y = \sec^{-1}\sec 3\theta = 3\theta = 3\cos^{-1} x$.

**Step 2.** Differentiate: $\frac{dy}{dx} = 3 \cdot \frac{-1}{\sqrt{1 - x^2}}$.

**Final Answer:** $\frac{dy}{dx} = -\frac{3}{\sqrt{1 - x^2}}$.

☞ **Substitution checklist**

When substituting $u = g(x)$, also express $du = g'(x)\,dx$ in the integrand AND change the limits if it is a definite integral. Forgetting either step is the single most common substitution error.

**EXPERT'S SOLUTION** *: Aaditya Iyer, Ph.D Pure Mathematics, IISc Bangalore*

**Quick reading.** $4x^3 - 3x$ is the triple-angle of $\cos$; the $1/(\cdot)$ wrapper gives $\sec$, which $\sec^{-1}$ unwinds.

**Step 1.** Substitute $x = \cos\theta$.

**Step 2.** Identify $4\cos^3\theta - 3\cos\theta = \cos 3\theta$.

**Step 3.** $y = 3\theta = 3\cos^{-1} x$; derivative is $-3/\sqrt{1 - x^2}$.

**Why this matters.** Triple-angle ID for $\sin$ and $\cos$ ($3\sin\theta - 4\sin^3\theta$, $4\cos^3\theta - 3\cos\theta$) is a go-to substitution.

> **Final Answer:** $-3/\sqrt{1-x^2}$.

**Q 5.42** **Differentiate** $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$ **w.r.t.** $x$**, for** $-\dfrac{1}{\sqrt{3}} < \dfrac{x}{a} < \dfrac{1}{\sqrt{3}}$.

**SOLUTION**

**Concept used.** Triple-angle for tan: $\tan 3\phi = \dfrac{3\tan\phi - \tan^3\phi}{1 - 3\tan^2\phi}$. Substitute $x = a\tan\phi$.

**Step 1.** Let $x = a\tan\phi$, so $x/a = \tan\phi$. The fraction becomes

$$\frac{3a^2(a\tan\phi) - (a\tan\phi)^3}{a^3 - 3a(a\tan\phi)^2} = \frac{a^3(3\tan\phi - \tan^3\phi)}{a^3(1 - 3\tan^2\phi)} = \tan 3\phi.$$

**Step 2.** Hence $y = \tan^{-1}\tan 3\phi = 3\phi$ on the principal range. Solving $x = a\tan\phi$ gives $\phi = \tan^{-1}(x/a)$, so $y = 3\tan^{-1}(x/a)$.

**Step 3.** Differentiate:

$$\frac{dy}{dx} = 3 \cdot \frac{1}{1 + (x/a)^2} \cdot \frac{1}{a} = \frac{3a}{a^2 + x^2}.$$

> **Final Answer:** $\dfrac{dy}{dx} = \dfrac{3a}{a^2 + x^2}$.

### ☞ Substitution checklist

When substituting $u = g(x)$, also express $du = g'(x)\,dx$ in the integrand AND change the limits if it is a definite integral. Forgetting either step is the single most common substitution error.

**EXPERT'S SOLUTION** *: Tara Banerjee, M.Sc Mathematics, IIT Bombay*

**Structural angle.** $\tan^{-1}(\tan 3\phi) = 3\phi$ collapses the expression to a simple multiple of $\tan^{-1}(x/a)$.

**Step 1.** Substitute $x/a = \tan\phi$.

**Step 2.** Triple-angle for tan reduces to $\tan^{-1}\tan 3\phi = 3\phi$.

**Step 3.** $y = 3\tan^{-1}(x/a)$ gives derivative $3a/(a^2 + x^2)$.

**Why this matters.** The "triple-angle in disguise" pattern is a classic JEE shortcut.

**Final Answer:** $3a/(a^2 + x^2)$.

**Why this method is preferred.** The shortest correct route in school calculus is almost always: identify the function's category → load the standard template → substitute the specific numbers. Reinventing the technique problem-by-problem is what makes a 90-minute paper feel like 3 hours.

**Q 5.43** **Differentiate** $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$ **w.r.t.** $x$**, for** $-1 < x < 1$, $x \neq 0$**.**

**SOLUTION**

**Concept used.** Substitute $x^2 = \cos 2\theta$, so $\sqrt{1+x^2} = \sqrt{1+\cos 2\theta} = \sqrt{2}\cos\theta$ and $\sqrt{1-x^2} = \sqrt{2}\sin\theta$. The expression collapses to $\tan(\pi/4+\theta)$.

**Step 1.** Let $x^2 = \cos 2\theta$. Then $\sqrt{1+x^2} = \sqrt{2}\cos\theta$, $\sqrt{1-x^2} = \sqrt{2}\sin\theta$.

**Step 2.** The argument becomes

$$\frac{\sqrt{2}\cos\theta + \sqrt{2}\sin\theta}{\sqrt{2}\cos\theta - \sqrt{2}\sin\theta} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

Divide top and bottom by $\cos\theta$: $= \dfrac{1+\tan\theta}{1-\tan\theta} = \tan(\pi/4+\theta)$.

**Step 3.** Hence $y = \pi/4 + \theta = \pi/4 + \frac{1}{2}\cos^{-1}(x^2)$.

**Step 4.** Differentiate:

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{-1}{\sqrt{1-x^4}}\cdot 2x = -\frac{x}{\sqrt{1-x^4}}.$$

**Final Answer:** $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{1-x^4}}$.

### ☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending $x$ cancels cleanly.

**EXPERT'S SOLUTION** : *Krishna Verma, M.Sc Mathematics, ISI Kolkata*

**Strategic angle.** Double-angle substitution converts both surds simultaneously.

**Step 1.** $x^2 = \cos 2\theta$; surds simplify to $\sqrt{2}\cos\theta, \sqrt{2}\sin\theta$.

**Step 2.** Quotient becomes $\tan(\pi/4 + \theta)$.

**Step 3.** $y = \pi/4 + \frac{1}{2}\cos^{-1} x^2$; differentiate.

**Why this matters.** Double-angle substitution is the standard way to handle $\sqrt{1 \pm x^2}$ combinations.

> **Final Answer:** $-x/\sqrt{1-x^4}$.

**Generalising the move.** Conjugate rationalisation works because $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A - B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

**Q 5.44** **Find** $\dfrac{dy}{dx}$ **if** $x = t + \dfrac{1}{t}$, $y = t - \dfrac{1}{t}$.

**SOLUTION**

**Concept used.** Parametric form: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.

**Step 1.** $\dfrac{dx}{dt} = 1 - \dfrac{1}{t^2} = \dfrac{t^2-1}{t^2}$.

**Step 2.** $\dfrac{dy}{dt} = 1 + \dfrac{1}{t^2} = \dfrac{t^2+1}{t^2}$.

**Step 3.** Divide:

$$\frac{dy}{dx} = \frac{(t^2+1)/t^2}{(t^2-1)/t^2} = \frac{t^2+1}{t^2-1}.$$

> **Final Answer:** $\dfrac{dy}{dx} = \dfrac{t^2+1}{t^2-1}$.

### ☞ Parametric derivative

For parametric curves $(x(t), y(t))$ use $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

**EXPERT'S SOLUTION** : *Sanya Kumar, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Both $dx/dt$ and $dy/dt$ have $1/t^2$ common; ratio cancels.

**Step 1.** Derivatives w.r.t. $t$.

**Step 2.** Take the ratio.

**Why this matters.** Parametric problems usually have shared factors; cancel before plugging values.

> **Final Answer:** $(t^2 + 1)/(t^2 - 1)$.

**Q 5.45** **If** $x = 3\cos\theta - 2\cos^3\theta$ **and** $y = 3\sin\theta - 2\sin^3\theta$**, find** $\dfrac{dy}{dx}$.

**SOLUTION**

**Concept used.** Parametric differentiation: when both $x$ and $y$ are given as functions of a third variable $\theta$, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$. We also use the double-angle identity $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$ to compress the result.

**Step 1.** Differentiate $x$ w.r.t. $\theta$:

$$\frac{dx}{d\theta} = -3\sin\theta - 2\cdot 3\cos^2\theta\cdot(-\sin\theta) = -3\sin\theta + 6\cos^2\theta\sin\theta.$$

$$\text{Factor: } \frac{dx}{d\theta} = \sin\theta\,(6\cos^2\theta - 3) = 3\sin\theta\,(2\cos^2\theta - 1) = 3\sin\theta\cdot\cos 2\theta.$$

**Step 2.** Differentiate $y$ w.r.t. $\theta$:

$$\frac{dy}{d\theta} = 3\cos\theta - 2\cdot 3\sin^2\theta\cdot\cos\theta = 3\cos\theta - 6\sin^2\theta\cos\theta.$$

$$\text{Factor: } \frac{dy}{d\theta} = 3\cos\theta\,(1 - 2\sin^2\theta) = 3\cos\theta\cdot\cos 2\theta.$$

**Step 3.** Take the ratio (note $\cos 2\theta$ cancels):

$$\frac{dy}{dx} = \frac{3\cos\theta\cdot\cos 2\theta}{3\sin\theta\cdot\cos 2\theta} = \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

> **Final Answer:** $\dfrac{dy}{dx} = \cot\theta$.

☞ **Double-angle identity**

$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$. Pulling this factor out of both $dx/d\theta$ and $dy/d\theta$ is what collapses the ratio to $\cot\theta$.

**EXPERT'S SOLUTION** *: Meera Pillai, M.Sc Mathematics, IIT Bombay*

**Strategic angle.** Recognise the cubic terms as triple-angle expressions: $3\cos\theta - 2\cos^3\theta$ and $3\sin\theta - 2\sin^3\theta$ *look like* but are not the standard $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. So work directly with the chain rule; the cancellation is the moral lesson.

**Step 1.** $dx/d\theta = 3\sin\theta\cos 2\theta$ (factor $\sin\theta$, then use $2\cos^2\theta - 1 = \cos 2\theta$).

**Step 2.** $dy/d\theta = 3\cos\theta\cos 2\theta$ (factor $\cos\theta$, then use $1 - 2\sin^2\theta = \cos 2\theta$).

**Step 3.** Ratio cancels $\cos 2\theta$ and the constant 3, leaving $\cot\theta$.

**Why this matters.** The common $\cos 2\theta$ factor is the whole point of the construction: if both $dx/d\theta$ and $dy/d\theta$ share a non-trivial factor, the parametric ratio simplifies beautifully. Always factor before dividing.

**Final Answer:** $\cot\theta$.

**Q 5.46** **Find $\frac{dy}{dx}$ if $\sin(xy) + \frac{x}{y} = x^2 - y$.**

**SOLUTION**

**Concept used.** Implicit differentiation: treat $y$ as a function of $x$ and differentiate both sides; use the chain rule on every $y$ that appears.

**Step 1.** Differentiate both sides w.r.t. $x$. $\frac{d}{dx}\sin(xy) = \cos(xy)\cdot(y + xy')$ (product rule inside). $\frac{d}{dx}(x/y) = \frac{y - xy'}{y^2}$ (quotient rule). RHS: $2x - y'$.

**Step 2.** Collect:

$$\cos(xy)(y + xy') + \frac{y - xy'}{y^2} = 2x - y'.$$

**Step 3.** Group $y'$ terms on one side and constants on the other:

$$\cos(xy)\cdot xy' - \frac{xy'}{y^2} + y' = 2x - y\cos(xy) - \frac{1}{y}.$$

**Step 4.** Factor $y'$:

$$y'\left(x\cos(xy) - \frac{x}{y^2} + 1\right) = 2x - y\cos(xy) - \frac{1}{y}.$$

**Step 5.** Solve:

$$\frac{dy}{dx} = \frac{2x - y\cos(xy) - 1/y}{x\cos(xy) - x/y^2 + 1}.$$

**Final Answer:** $\dfrac{dy}{dx} = \dfrac{2xy^2 - y^3\cos(xy) - y}{xy^2\cos(xy) - x + y^2}$ (multiplying top and bottom by $y^2$).

**✗ Chain rule layers**

Differentiate the outermost function first, then multiply by the derivative of the inner function, then by the inner-inner, and so on. Stop only when you hit $x$ or a constant — truncating early is the textbook chain-rule slip.

**EXPERT'S SOLUTION** *: Aditi Kumar, Ph.D Mathematics, IIT Delhi*

**Strategic angle.** Apply $d/dx$ to every term, group all $y'$-factors, divide.

**Step 1.** Differentiate $\sin(xy)$, $x/y$, $x^2$, $y$ separately.

**Step 2.** Collect $y'$ terms; factor; divide.

**Why this matters.** Implicit problems are bookkeeping — a clean grouping at the end is more important than clever tricks.

**Final Answer:** See main solution.

**Why implicit works.** Implicit differentiation rests on the chain rule: every $y$ in the equation is secretly $y(x)$, so $\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$. Once that habit is automatic, even ugly relations like $\sin(xy) = x + y$ become routine — differentiate term-by-term, then solve linearly for $\frac{dy}{dx}$.

**Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

**Q 5.47** **If $x^m y^n = (x+y)^{m+n}$, prove that (i) $\dfrac{dy}{dx} = \dfrac{y}{x}$, and (ii) $\dfrac{d^2y}{dx^2} = 0$.**

**SOLUTION**

**Concept used.** Logarithmic differentiation of an implicit relation; then second-derivative of the resulting first-order ODE.

**Step 1.** Take $\log$ on both sides:

$$m\log x + n\log y = (m+n)\log(x+y).$$

**Step 2.** Differentiate both sides w.r.t. $x$ ($y$ depends on $x$):

$$\frac{m}{x} + \frac{n}{y}\cdot y' = (m+n)\cdot\frac{1+y'}{x+y}.$$

**Step 3.** Multiply out and collect $y'$:

$$\frac{m}{x} + \frac{ny'}{y} = \frac{(m+n)+(m+n)y'}{x+y}.$$

Combine the right side: $\dfrac{(m+n)(1+y')}{x+y}$. Bring all $y'$ to one side:

$$y'\left(\frac{n}{y} - \frac{m+n}{x+y}\right) = \frac{m+n}{x+y} - \frac{m}{x}.$$

**Step 4.** Simplify each bracket:

$$\frac{n}{y} - \frac{m+n}{x+y} = \frac{n(x+y)-(m+n)y}{y(x+y)} = \frac{nx-my}{y(x+y)}.$$

$$\frac{m+n}{x+y} - \frac{m}{x} = \frac{(m+n)x - m(x+y)}{x(x+y)} = \frac{nx-my}{x(x+y)}.$$

**Step 5.** Therefore

$$y' = \frac{(nx-my)/[x(x+y)]}{(nx-my)/[y(x+y)]} = \frac{y}{x}.$$

*Result (i)*: $\dfrac{dy}{dx} = \dfrac{y}{x}$.

**Step 6.** For (ii), differentiate $y' = y/x$ once more:

$$y'' = \frac{y'x - y\cdot 1}{x^2} = \frac{(y/x)\cdot x - y}{x^2} = \frac{y-y}{x^2} = 0.$$

**Final Answer:** (i) $\dfrac{dy}{dx} = \dfrac{y}{x}$; (ii) $\dfrac{d^2y}{dx^2} = 0$.

### ♥ Geometric meaning

$dy/dx = y/x$ means $y/x$ is constant, i.e. $y = cx$ for some $c$. So the curve $x^m y^n = (x+y)^{m+n}$ is actually a family of straight lines through the origin; that is why the second derivative is identically zero.

**EXPERT'S SOLUTION** : *Ankit Banerjee, M.Sc Mathematics, IIT Bombay*

**Strategic angle.** Logarithmic differentiation produces a clean ratio. The fact that $y/x$ is constant kills the second derivative.

**Step 1.** Log-differentiate to get $y' = y/x$.

**Step 2.** $y' = y/x$ is the ODE for $y = cx$.

**Step 3.** $y'' = 0$.

**Why this matters.** Implicit equations often hide straight-line / conic structure — always check what curve you have.

> **Final Answer:** $y' = y/x$ and $y'' = 0$.

**Reading the structure.** Logarithmic differentiation is the tool whenever the exponent itself depends on $x$, e.g. $x^x$, $(\sin x)^{\cos x}$, $(\tan x)^x$. Take $\ln$ first to flatten the exponent into a product, differentiate using the product rule, then multiply through by the original function. The technique is mechanical once the pattern is recognised.
**Why this method is preferred.** The shortest correct route in school calculus is almost always: identify the function's category $\to$ load the standard template $\to$ substitute the specific numbers. Reinventing the technique problem-by-problem is what makes a 90-minute paper feel like 3 hours.

**Q 5.48** **Find $p$ and $q$ so that** $f(x) = \begin{cases} x^2 + 3x + p, & x \le 1 \\ qx + 2, & x > 1 \end{cases}$ **is differentiable at $x = 1$.**

**SOLUTION**

**Concept used.** **Differentiable $\Rightarrow$ Continuous.** So first impose continuity at $1$ (LHL $=$ RHL $= f(1)$). Then impose $Lf'(1) = Rf'(1)$. Two equations, two unknowns.

**Step 1.** Continuity: $f(1) = 1 + 3 + p = 4 + p$; $\lim_{x \to 1^+}(qx + 2) = q + 2$. Equate: $4 + p = q + 2$, i.e. $q = p + 2$.

**Step 2.** Left derivative at $1$: differentiate the polynomial $x^2 + 3x + p$: $f'(x) = 2x + 3$, so $Lf'(1) = 5$.

**Step 3.** Right derivative at $1$: differentiate $qx + 2$: $f'(x) = q$, so $Rf'(1) = q$.

**Step 4.** Equate derivatives: $q = 5$. Substitute in the continuity equation: $5 = p + 2 \Rightarrow p = 3$.

> **Final Answer:** $p = 3$, $q = 5$.

**✗ Common Mistake**

Do not just match $Lf'$ and $Rf'$ and forget continuity. A function can have matching one-sided slopes through a jump and still be discontinuous; differentiability requires both conditions.

**EXPERT'S SOLUTION** : *Vivaan Nair, Ph.D Pure Mathematics, IISc Bangalore*

**Quick reading.** Two equations, two unknowns: one from $f(1)$ match, one from slope match.

**Step 1.** $f$ continuous at 1: $4 + p = q + 2$.

**Step 2.** $f'$ continuous at 1: $2(1) + 3 = q$, so $q = 5$.

**Step 3.** Therefore $p = 3$.

**Why this matters.** Two-piece "find the constants" problems always reduce to a $2 \times 2$ linear system.

> **Final Answer:** $p = 3$, $q = 5$.

**Generalising the trick.** The manipulation used here is part of a family: spot the algebraic disguise, apply the standard identity, reduce to a known limit or standard integral, conclude. Once you've solved three problems with the same disguise, the rest become routine.

**Q 5.49** **If $x = \sin t$ and $y = \sin pt$, prove that $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$.**

**SOLUTION**

**Concept used.** Parametric differentiation, second derivative, and the identity $1 - x^2 = \cos^2 t$.

**Step 1.** First derivatives w.r.t. $t$: $\dfrac{dx}{dt} = \cos t$, $\dfrac{dy}{dt} = p\cos pt$. Hence

$$\frac{dy}{dx} = \frac{p\cos pt}{\cos t}.$$

**Step 2.** Square both sides and use $1 - x^2 = \cos^2 t$ together with $1 - y^2 = 1 - \sin^2 pt = \cos^2 pt$. Therefore

$$\cos^2 t \cdot \left(\frac{dy}{dx}\right)^2 = p^2\cos^2 pt = p^2(1 - y^2).$$

That is, $(1 - x^2)(y')^2 = p^2(1 - y^2)$.

**Step 3.** Differentiate both sides w.r.t. $x$:

$$-2x(y')^2 + (1 - x^2)\cdot 2y'y'' = p^2 \cdot (-2y \cdot y').$$

**Step 4.** Divide by $2y'$ (assuming $y' \neq 0$ generically; the identity extends by continuity):

$$-xy' + (1-x^2)y'' = -p^2 y,$$

i.e.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2 y = 0.$$

> **Final Answer:** Proved.

☞ **Half-angle identity**

Memorise $1-\cos\theta = 2\sin^2(\theta/2)$. It converts every $1-\cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Pranav Pillai, M.Sc Mathematics, IIT Bombay*

**Structural angle.** The ODE $(1-x^2)y'' - xy' + p^2y = 0$ is the **Chebyshev equation**; its solutions are exactly $\sin(p\arcsin x)$ and $\cos(p\arcsin x)$.

**Step 1.** Parametric derivatives.

**Step 2.** Square and use $1-x^2 = \cos^2 t$, $1-y^2 = \cos^2 pt$.

**Step 3.** Differentiate, divide by $2y'$.

**Why this matters.** Recognising the answer as a standard ODE makes second-order parametric problems systematic.

> **Final Answer:** The given ODE holds.

**Common slip with parametric second derivatives.** Students often write $\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$, which is wrong. The correct formula divides $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ by $\frac{dx}{dt}$. The mistake stems from over-extending the simple parametric first-derivative rule; commit the corrected second-order formula to memory.

**Self-check before boxing the answer.** Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

**Q 5.50** **Find** $\frac{dy}{dx}$ **if** $y = x^{\tan x} + \sqrt{\frac{x^2+1}{2}}$.

**SOLUTION**

**Concept used.** Split $y$ into two parts. Differentiate $x^{\tan x}$ by logarithmic differentiation; the second term needs the chain rule on a square root of a rational function.

**Step 1.** Let $u = x^{\tan x}$. Take log: $\log u = \tan x \cdot \log x$.

**Step 2.** Differentiate:

$$\frac{u'}{u} = \sec^2 x \cdot \log x + \tan x \cdot \frac{1}{x}.$$

Therefore

$$u' = x^{\tan x}\left(\sec^2 x \log x + \frac{\tan x}{x}\right).$$

**Step 3.** Second term. Let $v = \sqrt{(x^2+1)/2}$. Write $v = \dfrac{\sqrt{x^2+1}}{\sqrt{2}}$ and apply the chain rule:

$$\frac{dv}{dx} = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{2}\sqrt{x^2+1}} = \frac{x}{\sqrt{2(x^2+1)}}.$$

**Step 4.** Sum:

$$\frac{dy}{dx} = x^{\tan x}\left(\sec^2 x \log x + \frac{\tan x}{x}\right) + \frac{x}{\sqrt{2(x^2+1)}}.$$

**Final Answer:** $\dfrac{dy}{dx} = x^{\tan x}\left(\sec^2 x \log x + \dfrac{\tan x}{x}\right) + \dfrac{x}{\sqrt{2(x^2+1)}}$.

☞ **Take logs first**

When a function is a product, quotient, or power with variables in both base and exponent (e.g. $x^x$), take natural log of both sides before differentiating. The product becomes a sum and the exponent becomes a multiplier — much easier.

**EXPERT'S SOLUTION** *: Aanya Sharma, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** $x^{\tan x}$ needs log-diff; the polynomial piece is trivial.

**Step 1.** Log-differentiate $x^{\tan x}$.

**Step 2.** Add $x$ for the polynomial part.

**Why this matters.** Splitting a sum lets you apply the appropriate technique to each piece.

**Final Answer:** See main solution.

**Reading the structure.** Logarithmic differentiation is the tool whenever the exponent itself depends on $x$, e.g. $x^x$, $(\sin x)^{\cos x}$, $(\tan x)^x$. Take $\ln$ first to flatten the exponent into a product, differentiate using the product rule, then multiply through by the original function. The technique is mechanical once the pattern is recognised.

**Q 5.51** **Verify Rolle's theorem for $f(x) = x(x-1)^2$ on $[0, 1]$.**

SOLUTION

**Concept used.** Rolle's Theorem: if $f$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $f(a) = f(b)$, then $\exists\, c \in (a, b)$ with $f'(c) = 0$.

**Step 1.** Check hypotheses. $f$ is a polynomial, so continuous on $[0, 1]$ and differentiable on $(0, 1)$. $f(0) = 0$ and $f(1) = 1 \cdot 0^2 = 0$. All three conditions hold.

**Step 2.** Compute $f'(x)$. Expand: $f(x) = x(x-1)^2 = x(x^2 - 2x + 1) = x^3 - 2x^2 + x$. So $f'(x) = 3x^2 - 4x + 1$.

**Step 3.** Solve $f'(x) = 0$: discriminant $16 - 12 = 4$, roots $x = \frac{4 \pm 2}{6} = 1$ or $\frac{1}{3}$. The root $c = 1/3 \in (0, 1)$ satisfies the conclusion.

**Final Answer:** Rolle's theorem holds; $c = 1/3$.

♥ **Differentiability $\Rightarrow$ continuity**

Differentiable at $c$ implies continuous at $c$, but not conversely. $|x|$ at $0$ is the canonical counter-example: continuous everywhere but the corner kills differentiability at the origin.

**EXPERT'S SOLUTION** *: Ananya Mehta, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Verify the three hypotheses, then find the roots of $f'$ in the open interval.

**Step 1.** Polynomial $\Rightarrow$ continuous and differentiable.

**Step 2.** $f(0) = f(1) = 0$.

**Step 3.** $f'(x) = 3x^2 - 4x + 1 = 0$ at $x = 1, 1/3$; $1/3 \in (0, 1)$ is the witness.

**Why this matters.** Rolle's hypothesis check is a 30-second mechanical drill; spend the time on solving $f'(c) = 0$.

**Final Answer:** $c = 1/3$.

**Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

**Q 5.52** **Verify mean value theorem for $f(x) = x^3 - 2x^2 - x + 3$ on $[0, 1]$.**

SOLUTION

**Concept used.** Mean Value Theorem (Lagrange): if $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, $\exists\, c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

**Step 1.** Hypotheses: $f$ is a polynomial, hence continuous and differentiable everywhere.

**Step 2.** Compute $f(0) = 3$, $f(1) = 1 - 2 - 1 + 3 = 1$. So $\dfrac{f(1) - f(0)}{1 - 0} = \dfrac{1 - 3}{1} = -2$.

**Step 3.** $f'(x) = 3x^2 - 4x - 1$. Solve $3x^2 - 4x - 1 = -2$: $3x^2 - 4x + 1 = 0$.

**Step 4.** Roots: $x = \dfrac{4 \pm \sqrt{16 - 12}}{6} = \dfrac{4 \pm 2}{6}$, i.e. $x = 1$ or $x = 1/3$. The root $c = 1/3 \in (0, 1)$ works.

**Final Answer:** MVT holds; $c = 1/3$.

☞ **Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an $\varepsilon$-$\delta$ on a polynomial in an exam.

**EXPERT'S SOLUTION** : *Aditya Banerjee, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Compute the secant slope, solve $f'(c) =$ slope.

**Step 1.** Slope $= (1 - 3)/1 = -2$.

**Step 2.** $3c^2 - 4c - 1 = -2 \Rightarrow 3c^2 - 4c + 1 = 0$.

**Step 3.** $c = 1/3$ in $(0, 1)$.

**Why this matters.** MVT problems always reduce to "secant slope $=$ derivative at $c$".

**Final Answer:** $c = 1/3$.

**Why this method is preferred.** The shortest correct route in school calculus is almost always: identify the function's category $\to$ load the standard template $\to$ substitute the specific numbers. Reinventing the technique problem-by-problem is what makes a

90-minute paper feel like 3 hours.

## Objective Type Questions

**Q 5.53** **If $f(x) = 2x$ and $g(x) = \frac{x^2}{2} + 1$, then which of the following can be a discontinuous function?**

**(A)** $f(x) + g(x)$ **(B)** $f(x) - g(x)$ **(C)** $f(x) \cdot g(x)$ **(D)** $\frac{g(x)}{f(x)}$

SOLUTION

**Correct option: (D)** $g(x)/f(x)$.

**Concept used.** Sum, difference and product of continuous functions are continuous everywhere. Quotient is continuous only where the denominator is non-zero.

**Step 1.** Both $f(x) = 2x$ and $g(x) = x^2/2 + 1$ are polynomials, continuous on $\mathbb{R}$.

**Step 2.** (A), (B), (C): closed under sums and products; continuous on $\mathbb{R}$.

**Step 3.** (D): $g(x)/f(x) = (x^2/2 + 1)/(2x)$. The denominator $2x = 0$ at $x = 0$, making the quotient undefined (and hence discontinuous) there.

**Final Answer:** Option **(D)**.

☞ **Composition continuity**

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Track both layers when locating discontinuities.

EXPERT'S SOLUTION *: Rohit Mehta, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** Polynomial / polynomial is the only operation that can introduce a discontinuity (at a zero of the denominator).

**Step 1.** Identify $f = 0$ at $x = 0$; quotient is undefined there.

**Why this matters.** Continuity is preserved by sum, product, composition; only quotient (and roots of even index) can break it.

**Final Answer:** (D).

**Q 5.54** **The function $f(x) = \dfrac{4-x^2}{4x-x^3}$ is**

**(A) discontinuous at only one point**

**(B) discontinuous at exactly two points**

**(C) discontinuous at exactly three points**

**(D) none of these.**

SOLUTION

**Correct option: (C).**

**Concept used.** A rational function is discontinuous exactly at the zeros of its denominator (within its natural domain).

**Step 1.** Factor the denominator: $4x - x^3 = x(4 - x^2) = x(2 - x)(2 + x)$.

**Step 2.** Zeros: $x = 0, 2, -2$. Three points of discontinuity.

> **Final Answer:** Option **(C)**: three points.

☞ **Save time**

Recognising the question's category (piecewise / surd / exponential / trig) lets you load the right template before reading the problem twice. Pattern-spot first, compute second.

**EXPERT'S SOLUTION** *: Karan Iyer, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Three linear factors ⇒ three poles.

**Step 1.** Factor and count.

**Why this matters.** Always factor the denominator first.

> **Final Answer:** (C).

**Self-check before boxing the answer.** Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

**Q 5.55** **The set of points where $f(x) = |2x-1|\sin x$ is differentiable is**

**(A)** $\mathbb{R}$ **(B)** $\mathbb{R} \setminus \{1/2\}$ **(C)** $(0, \infty)$ **(D) none of these.**

## SOLUTION

**Concept used.** A product $g(x)h(x)$ is differentiable at $c$ whenever both $g$ and $h$ are. If exactly one factor has a corner at $c$ but vanishes there, the product may still be differentiable because the corner is "killed" by the zero of the other factor.

**Step 1.** $|2x - 1|$ has a corner at $x = 1/2$. Elsewhere it is smooth.

**Step 2.** At $x = 1/2$, $\sin x = \sin(1/2) \approx 0.479 \neq 0$. Wait — the textbook's intended reading is $f(x) = |2x - 1| \sin x$ where $|2x - 1|$ vanishes at $x = 1/2$, not $\sin x$. Reconsider.

**Step 3.** Compute one-sided derivatives at $x = 1/2$ from first principles. Write $f(x) = |2x - 1| \sin x$. For $x > 1/2$, $|2x - 1| = 2x - 1$, so $f(x) = (2x - 1) \sin x$. For $x < 1/2$, $|2x - 1| = 1 - 2x$, so $f(x) = (1 - 2x) \sin x$. Both extensions vanish at $1/2$ since $2(1/2) - 1 = 0$, so $f(1/2) = 0$.

**Step 4.** Right derivative: $\lim_{h \to 0^+} \dfrac{(2(1/2 + h) - 1) \sin(1/2 + h) - 0}{h} = \lim_{h \to 0^+} \dfrac{2h \sin(1/2 + h)}{h} = 2 \sin(1/2)$. Left derivative: similarly $-2 \sin(1/2)$. These differ unless $\sin(1/2) = 0$, which it is not. So $f$ is *not* differentiable at $1/2$.

**Step 5.** Conclusion: $f$ is differentiable on $\mathbb{R} \setminus \{1/2\}$, matching option (B).

> **Final Answer:** Option **(B)**: $\mathbb{R} \setminus \{1/2\}$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

## EXPERT'S SOLUTION *: Sneha Iyer, Ph.D Mathematics, IIT Delhi*

**Quick reading.** $|2x - 1|$ has a corner at $1/2$. $\sin x$ does not vanish at $1/2$, so the corner survives in the product.

**Step 1.** Corner of $|2x - 1|$ at $x = 1/2$; $\sin(1/2) \neq 0$ does not cancel it.

**Step 2.** So $f$ fails differentiability only at $1/2$.

**Why this matters.** $|g| \cdot h$ is differentiable at $g = 0$ *iff* $h$ also vanishes there.

> **Final Answer:** (B).

**What examiners reward.** Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason. **Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

**Q 5.56** **The function $f(x) = \cot x$ is discontinuous on the set**
**(A)** $\{x = n\pi : n \in \mathbb{Z}\}$ **(B)** $\{x = 2n\pi : n \in \mathbb{Z}\}$ **(C)** $\{x = (2n+1)\pi/2 : n \in \mathbb{Z}\}$ **(D)** $\{x = n\pi/2 : n \in \mathbb{Z}\}$.

SOLUTION

**Correct option: (A).**
**Concept used.** $\cot x = \cos x / \sin x$ is discontinuous where the denominator $\sin x = 0$, i.e. at integer multiples of $\pi$.

**Step 1.** $\sin x = 0 \iff x = n\pi,\ n \in \mathbb{Z}$.

**Step 2.** At those points $\cot x$ is undefined; elsewhere it is continuous.

**Final Answer:** Option **(A)**.

**Trig continuity**
$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Diya Sharma, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Zeros of $\sin x$ are at $n\pi$, period $\pi$.

**Final Answer:** (A).

**Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

**Q 5.57** **The function $f(x) = e^{|x|}$ is**
**(A) continuous everywhere but not differentiable at $x = 0$**
**(B) continuous and differentiable everywhere**
**(C) not continuous at $x = 0$**

**(D) none of these.**

SOLUTION

**Correct option: (A).**
**Concept used.** $|x|$ is continuous on $\mathbb{R}$ but not differentiable at $0$; $e^u$ is continuous and differentiable everywhere. Continuity is preserved by composition; differentiability of the composition is determined by the differentiability of the *inner* function at the point (since $e^u$ is smooth).

**Step 1.** Continuity: $e^{|x|}$ is continuous on $\mathbb{R}$ (composition of continuous functions).

**Step 2.** Differentiability at $x = 0$: left derivative is $\frac{d}{dx}e^{-x}|_0 = -1$; right derivative is $\frac{d}{dx}e^{x}|_0 = 1$. These disagree, so $f$ is not differentiable at $0$.

**Final Answer:** Option **(A)**.

♥ **Differentiability $\Rightarrow$ continuity**
Differentiable at $c$ implies continuous at $c$, but not conversely. $|x|$ at $0$ is the canonical counter-example: continuous everywhere but the corner kills differentiability at the origin.

EXPERT'S SOLUTION : *Aaditya Verma, M.Sc Mathematics, IIT Bombay*

**Quick reading.** $|x|$ corner at $0$ propagates through smooth $e^u$.

**Final Answer:** (A).

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

**Q 5.58** **If $f(x) = x^2\sin(1/x)$ for $x \neq 0$, then the value of $f(0)$ that makes $f$ continuous at $0$ is**
**(A)** $0$ **(B)** $-1$ **(C)** $1$ **(D) none of these.**

SOLUTION

**Correct option:** **(A)** $0$.
**Concept used.** Squeeze theorem: $|x^2 \sin(1/x)| \le x^2 \to 0$.

**Step 1.** $-x^2 \le x^2 \sin(1/x) \le x^2$.

**Step 2.** As $x \to 0$, both bounds tend to $0$.

**Step 3.** By the Squeeze theorem, $\lim_{x\to 0} f(x) = 0$. Setting $f(0) = 0$ makes $f$ continuous at $0$.

**Final Answer:** Option **(A)**: $0$.

☞ **Pattern to memorise**

Whenever you spot *(small factor)* $\times$ *(bounded oscillation)* like $x \sin(1/x)$ or $x^k \cos(1/x)$, the squeeze theorem closes the limit at $0$ in one line. Reach for it instinctively.

EXPERT'S SOLUTION : *Kavya Verma, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** "Small times bounded" $\Rightarrow$ limit $= 0$.

**Final Answer:** (A).

**Where this fits in the toolbox.** The squeeze theorem is the controlled-oscillation tool: whenever a bounded factor is multiplied by a quantity going to zero, the product is forced to zero regardless of how wildly the oscillation behaves. Examiners often disguise this pattern by writing $\cos(1/x)$ as $\cos(\pi/x)$ or substituting $x - a$ for $x$; the template still applies once you spot it.

**Q 5.59** **If** $f(x) = \begin{cases} mx + 1, & x \le \pi/2 \\ \sin x + n, & x > \pi/2 \end{cases}$ **is continuous at** $x = \pi/2$**, then**

**(A)** $m = 1, n = 0$ **(B)** $m = \dfrac{n\pi}{2} + 1$ **(C)** $n = \dfrac{m\pi}{2}$ **(D)** $m = n = \dfrac{\pi}{2}$.

SOLUTION

**Correct option:** **(C)** $n = m\pi/2$.
**Concept used.** Continuity at $\pi/2$: LHL $=$ RHL $= f(\pi/2)$.

**Step 1.** LHL (and $f(\pi/2)$): $m\pi/2 + 1$.

**Step 2.** RHL: $\sin(\pi/2) + n = 1 + n$.

**Step 3.** Set them equal: $m\pi/2 + 1 = 1 + n \Rightarrow n = m\pi/2$.

**Final Answer:** Option **(C)**.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

**EXPERT'S SOLUTION** *: Ananya Bhat, M.Sc Mathematics, IIT Bombay*

**Quick reading.** One linear equation in $m, n$; pick the option that matches.

**Final Answer:** (C).

**Self-check before boxing the answer.** Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

**Q 5.60** **Let $f(x) = |\sin x|$. Then**

**(A) $f$ is everywhere differentiable**

**(B) $f$ is continuous everywhere but not differentiable at $x = n\pi$, $n \in \mathbb{Z}$**

**(C) $f$ is continuous everywhere but not differentiable at $x = (2n+1)\pi/2$, $n \in \mathbb{Z}$**

**(D) none of these.**

**SOLUTION**

**Correct option: (B).**

**Concept used.** $|\sin x|$ inherits corners exactly where $\sin x$ changes sign through zero, i.e. at $x = n\pi$.

**Step 1.** Sign change of $\sin x$ at $n\pi$ produces a $V$-corner of $|\sin x|$ there.

**Step 2.** Between consecutive zeros, $|\sin x|$ equals $\pm \sin x$, which is smooth.

**Final Answer:** Option **(B)**.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

**EXPERT'S SOLUTION** : *Dev Kapoor, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** $|g|$ corners are at the simple roots of $g$.

**Final Answer:** (B).

**What examiners reward.** Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason.

**Q 5.61** **If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is**

**(A)** $\frac{4x^3}{1-x^4}$ **(B)** $\frac{-4x}{1-x^4}$ **(C)** $\frac{1}{4-x^4}$ **(D)** $\frac{-4x^3}{1-x^4}$.

**SOLUTION**

**Correct option: (B)** $-4x/(1-x^4)$.

**Concept used.** $\log(u/v) = \log u - \log v$, then differentiate.

**Step 1.** $y = \log(1-x^2) - \log(1+x^2)$.

**Step 2.** Differentiate: $\frac{dy}{dx} = \frac{-2x}{1-x^2} - \frac{2x}{1+x^2}$.

**Step 3.** Common denominator:

$$= \frac{-2x(1+x^2) - 2x(1-x^2)}{(1-x^2)(1+x^2)} = \frac{-2x[(1+x^2)+(1-x^2)]}{1-x^4} = \frac{-4x}{1-x^4}.$$

**Final Answer:** Option **(B)**.

☞ **Log domain**

$\ln x$ is defined and continuous only for $x > 0$. Before using $\ln$ on an expression, confirm that the argument is strictly positive on the relevant interval.

**EXPERT'S SOLUTION** : *Aanya Joshi, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Split the log, then add fractions over $1 - x^4$.

**Final Answer:** (B).

**Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

**Q 5.62** **If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is**
**(A)** $\frac{\cos x}{2y-1}$ **(B)** $\frac{\cos x}{1-2y}$ **(C)** $\frac{\sin x}{1-2y}$ **(D)** $\frac{\sin x}{2y-1}$.

SOLUTION

**Correct option: (A)** $\cos x/(2y-1)$.
**Concept used.** Square both sides, then differentiate implicitly.

**Step 1.** Square: $y^2 = \sin x + y$.

**Step 2.** Differentiate w.r.t. $x$: $2y \cdot y' = \cos x + y'$.

**Step 3.** Collect: $y'(2y-1) = \cos x \Rightarrow y' = \cos x/(2y-1)$.

**Final Answer:** Option **(A)**.

☞ **Treat y as $y(x)$**
When differentiating implicitly, every $y$-term gets a chain-rule tail of $\frac{dy}{dx}$. After differentiating, collect all $\frac{dy}{dx}$ terms on one side and factor.

**EXPERT'S SOLUTION** *: Riya Verma, B.Tech CSE, IIT Roorkee*

**Quick reading.** Square first to clear the $\sqrt{}$; then implicit-differentiate.

**Final Answer:** (A).

**Why implicit works.** Implicit differentiation rests on the chain rule: every $y$ in the equation is secretly $y(x)$, so $\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$. Once that habit is automatic, even ugly relations like $\sin(xy) = x + y$ become routine — differentiate term-by-term, then solve linearly for $\frac{dy}{dx}$.

**Q 5.63** **The derivative of $\cos^{-1}(2x^2-1)$ w.r.t. $\cos^{-1} x$ is**
**(A)** $2$ **(B)** $-1/(2\sqrt{1-x^2})$ **(C)** $2/x$ **(D)** $1-x^2$.

**SOLUTION**

**Correct option:** **(A)** 2.

**Concept used.** $\cos 2\theta = 2\cos^2\theta - 1$. Hence $\cos^{-1}(2x^2-1) = 2\cos^{-1} x$ on the relevant range. The derivative of $f$ w.r.t. $g$ is $df/dg = (df/dx)/(dg/dx)$.

**Step 1.** $\cos^{-1}(2x^2-1) = 2\cos^{-1} x$.

**Step 2.** $\dfrac{d}{dx}\cos^{-1}(2x^2-1) = -\dfrac{2}{\sqrt{1-x^2}}$; $\dfrac{d}{dx}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$.

**Step 3.** Ratio: $\dfrac{-2/\sqrt{1-x^2}}{-1/\sqrt{1-x^2}} = 2$.

**Final Answer:** Option **(A)**: 2.

**☞ Conjugate first**

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending $x$ cancels cleanly.

**EXPERT'S SOLUTION** : *Vivaan Joshi, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** Identity $\cos^{-1}(2x^2-1) = 2\cos^{-1} x$ makes the ratio of derivatives equal 2.

**Final Answer:** (A).

**Generalising the move.** Conjugate rationalisation works because $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A - B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

**Q 5.64** **If $x = t^2$, $y = t^3$, then $\dfrac{d^2y}{dx^2}$ is**

**(A)** $3/2$ **(B)** $3/(4t)$ **(C)** $3/(2t)$ **(D)** $3/4$.

**SOLUTION**

**Correct option:** **(B)** $3/(4t)$.

**Concept used.** For parametric $x(t), y(t)$, $\dfrac{d^2y}{dx^2} = \dfrac{1}{dx/dt}\cdot\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$.

**Step 1.** $dx/dt = 2t$, $dy/dt = 3t^2$. So $\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3t}{2}$.

**Step 2.** $\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{3}{2}$.

**Step 3.** Divide by $dx/dt = 2t$: $\frac{d^2y}{dx^2} = \frac{3/2}{2t} = \frac{3}{4t}$.

**Final Answer:** Option **(B)**.

**☞ Parametric derivative**

For parametric curves $(x(t), y(t))$ use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

**EXPERT'S SOLUTION** : *Sneha Sharma, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Standard parametric $y''$ formula; do not forget the extra division by $dx/dt$.

**Final Answer:** (B).

**Common slip with parametric second derivatives.** Students often write $\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$, which is wrong. The correct formula divides $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ by $\frac{dx}{dt}$. The mistake stems from over-extending the simple parametric first-derivative rule; commit the corrected second-order formula to memory.

**Q 5.65** **The value of $c$ in Rolle's theorem for the function $f(x) = x^3 - 3x$ on $[0, \sqrt{3}]$ is** **(A)** $1$ **(B)** $-1$ **(C)** $3/2$ **(D)** $1/3$.

**SOLUTION**

**Correct option: (A)** 1.

**Concept used.** Rolle's theorem: find $c$ where $f'(c) = 0$.

**Step 1.** Check hypotheses. $f$ is a polynomial; $f(0) = 0$; $f(\sqrt{3}) = 3\sqrt{3} - 3\sqrt{3} = 0$. Yes, $f(0) = f(\sqrt{3})$.

**Step 2.** $f'(x) = 3x^2 - 3$. Solve $f'(c) = 0$: $3c^2 = 3 \Rightarrow c = \pm 1$.

**Step 3.** Only $c = 1 \in (0, \sqrt{3})$.

**Final Answer:** Option **(A)**.

### ☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending $x$ cancels cleanly.

**EXPERT'S SOLUTION** : *Pranav Iyer, M.Sc Mathematics, IIT Bombay*

**Quick reading.** $f' = 3(x^2 - 1)$ vanishes at $\pm 1$; pick the one in $(0, \sqrt{3})$.

**Final Answer:** (A).

**Generalising the move.** Conjugate rationalisation works because $(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

**Q 5.66** **For the function** $f(x) = x + \dfrac{1}{x}$ **on** $[1, 3]$**, the value of** $c$ **for the Mean Value Theorem is**

**(A)** $1$ **(B)** $\sqrt{3}$ **(C)** $2$ **(D) none of these.**

**SOLUTION**

**Correct option: (B)** $\sqrt{3}$.

**Concept used.** MVT: solve $f'(c) = (f(b) - f(a))/(b - a)$.

**Step 1.** $f(3) = 3 + 1/3 = 10/3$; $f(1) = 2$. Slope: $\dfrac{10/3 - 2}{3 - 1} = \dfrac{4/3}{2} = \dfrac{2}{3}$.

**Step 2.** $f'(x) = 1 - 1/x^2$. Set $f'(c) = 2/3$:

$$1 - 1/c^2 = 2/3 \Rightarrow 1/c^2 = 1/3 \Rightarrow c^2 = 3 \Rightarrow c = \pm\sqrt{3}.$$

**Step 3.** $c = \sqrt{3} \in (1, 3)$.

**Final Answer:** Option **(B)**.

☞ **Exam-paper habit**

Write the formula or theorem you are invoking on its own line before substituting numbers. Examiners give the method mark even if a later arithmetic slip costs the final mark.

**EXPERT'S SOLUTION** *: Aditi Verma, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Secant slope $= 2/3$ gives $c^2 = 3$.

**Final Answer:** (B).

**Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

## Fill in the Blanks

**Q 5.67** **An example of a function which is continuous everywhere but fails to be differentiable at exactly two points is ______________.**

**SOLUTION**

**Concept used.** The absolute-value functions $|x-a|$ have a corner at $x = a$. To produce exactly two corners, sum or combine two such functions at distinct centres.

**Step 1.** $f(x) = |x| + |x-1|$ has corners at $x = 0$ and $x = 1$, and is smooth (linear) elsewhere.

**Step 2.** Continuity: sum of continuous functions, so $f$ is continuous on $\mathbb{R}$.

**Step 3.** Differentiability: at $x = 0$, left slope from $-x + (1-x) = 1 - 2x$ is $-2$; right slope from $x + (1-x) = 1$ is $0$. They disagree, so not differentiable at $0$. Similarly at $x = 1$.

**Final Answer:** $f(x) = |x| + |x-1|$.

✗ **Sign-check**

Carry the negative sign through every substitution step — a single dropped minus turns a continuous function into a discontinuous one or flips an integral's sign. Recheck signs before the final box.

**EXPERT'S SOLUTION** : *Aaditya Bhat, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Two centres, two corners.

**Final Answer:** $|x| + |x - 1|$.

**Why this method is preferred.** The shortest correct route in school calculus is almost always: identify the function's category → load the standard template → substitute the specific numbers. Reinventing the technique problem-by-problem is what makes a 90-minute paper feel like 3 hours.
**Generalising the trick.** The manipulation used here is part of a family: spot the algebraic disguise, apply the standard identity, reduce to a known limit or standard integral, conclude. Once you've solved three problems with the same disguise, the rest become routine.

**Q 5.68** **Derivative of $x^2$ w.r.t. $x^3$ is __________.**

**SOLUTION**

**Concept used.** $\dfrac{d}{d(x^3)}(x^2) = \dfrac{d(x^2)/dx}{d(x^3)/dx}$.

**Step 1.** Numerator: $d(x^2)/dx = 2x$.

**Step 2.** Denominator: $d(x^3)/dx = 3x^2$.

**Step 3.** Ratio: $\dfrac{2x}{3x^2} = \dfrac{2}{3x}$.

**Final Answer:** $\dfrac{2}{3x}$.

### ♥ Why the method works

The strategy here generalises: any time you can reduce a hard limit or integral to a known standard form via algebra (factoring, conjugate, substitution), you have converted a calculus problem into an algebra problem — always cheaper.

**EXPERT'S SOLUTION** : *Yash Gupta, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** Ratio of $dx$-derivatives.

**Final Answer:** $2/(3x)$.

**Generalising the trick.** The manipulation used here is part of a family: spot the algebraic disguise, apply the standard identity, reduce to a known limit or standard integral, conclude. Once you've solved three problems with the same disguise, the rest become routine.

**Q 5.69** **If $f(x) = |\cos x|$, then $f'(\pi/4)$ is ________.**

**SOLUTION**

**Concept used.** Near $\pi/4$, $\cos x > 0$, so $|\cos x| = \cos x$. Hence $f'(\pi/4) = -\sin(\pi/4)$.

**Step 1.** Since $\cos(\pi/4) = 1/\sqrt{2} > 0$, $|\cos x| = \cos x$ in a neighbourhood of $\pi/4$.

**Step 2.** $f'(x) = -\sin x$. At $x = \pi/4$, $f'(\pi/4) = -\sin(\pi/4) = -1/\sqrt{2}$.

**Final Answer:** $-\dfrac{1}{\sqrt{2}}$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Ishita Banerjee, M.Sc Mathematics, IIT Bombay*

**Quick reading.** $|\cos x|$ is smooth wherever $\cos x \neq 0$; just differentiate the inside.

**Final Answer:** $-1/\sqrt{2}$.

**Self-check before boxing the answer.** Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

**Q 5.70** **If $f(x) = |\cos x - \sin x|$, then $f'(\pi/3)$ is ________.**

**SOLUTION**

**Concept used.** Sign of $\cos x - \sin x$ near $\pi/3$.

**Step 1.** $\cos(\pi/3) - \sin(\pi/3) = 1/2 - \sqrt{3}/2 < 0$.

**Step 2.** Hence near $\pi/3$, $|\cos x - \sin x| = -(\cos x - \sin x) = \sin x - \cos x$.

**Step 3.** $f'(x) = \cos x + \sin x$. At $x = \pi/3$: $f'(\pi/3) = 1/2 + \sqrt{3}/2 = (1 + \sqrt{3})/2$.

**Final Answer:** $\frac{1+\sqrt{3}}{2}$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Krishna Bhat, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Strip the absolute value (sign $< 0$), then differentiate.

**Final Answer:** $(1 + \sqrt{3})/2$.

**What examiners reward.** Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason.

**Q 5.71** **For the curve $\sqrt{x} + \sqrt{y} = 1$, $\frac{dy}{dx}$ at $(1/4, 1/4)$ is __________.**

**SOLUTION**

**Concept used.** Implicit differentiation.

**Step 1.** Differentiate $\sqrt{x} + \sqrt{y} = 1$ w.r.t. $x$: $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot y' = 0$.

**Step 2.** Solve: $y' = -\frac{\sqrt{y}}{\sqrt{x}}$.

**Step 3.** At $(1/4, 1/4)$: $\sqrt{y} = \sqrt{x} = 1/2$, ratio $= 1$. So $y' = -1$.

**Final Answer:** $-1$.

☞ **Treat y as $y(x)$**

When differentiating implicitly, every $y$-term gets a chain-rule tail of $\frac{dy}{dx}$. After differentiating, collect all $\frac{dy}{dx}$ terms on one side and factor.

EXPERT'S SOLUTION : *Aaditya Sharma, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** $y' = -\sqrt{y/x}$; equal coordinates give $-1$.

**Final Answer:** $-1$.

**Why implicit works.** Implicit differentiation rests on the chain rule: every $y$ in the equation is secretly $y(x)$, so $\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$. Once that habit is automatic, even ugly relations like $\sin(xy) = x + y$ become routine — differentiate term-by-term, then solve linearly for $\frac{dy}{dx}$.

## True or False

**Q 5.72** **Rolle's theorem is applicable for the function $f(x) = |x - 1|$ on $[0, 2]$.**

SOLUTION

**Concept used.** Rolle's theorem requires *differentiability* on $(a, b)$. $|x - 1|$ has a corner at $x = 1 \in (0, 2)$ and is not differentiable there.

**Step 1.** $f$ is continuous on $[0, 2]$ and $f(0) = f(2) = 1$, satisfying the endpoint condition.

**Step 2.** But $f$ is not differentiable at $x = 1 \in (0, 2)$.

**Step 3.** Therefore Rolle's theorem is **not applicable**.

**Final Answer:** **False**.

**✗ Sign-check**
Carry the negative sign through every substitution step — a single dropped minus turns a continuous function into a discontinuous one or flips an integral's sign. Recheck signs before the final box.

EXPERT'S SOLUTION : *Sneha Bhat, M.Sc Mathematics, IIT Bombay*

**Quick reading.** The non-differentiable corner inside the interval defeats Rolle.

**Final Answer:** False.

**Why this method is preferred.** The shortest correct route in school calculus is almost always: identify the function's category $\rightarrow$ load the standard template $\rightarrow$ substitute the specific numbers. Reinventing the technique problem-by-problem is what makes a

90-minute paper feel like 3 hours.

**Q 5.73** **If $f$ is continuous on its domain $D$, then $|f|$ is also continuous on $D$.**

SOLUTION

**Concept used.** The absolute-value function $u \mapsto |u|$ is continuous, and the composition of continuous functions is continuous.

**Step 1.** $|f(x)| = (|\cdot| \circ f)(x)$, composition of two continuous functions.

**Step 2.** Hence $|f|$ is continuous on $D$.

**Final Answer:** **True**.

☞ **Composition continuity**

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Track both layers when locating discontinuities.

EXPERT'S SOLUTION : *Karan Verma, M.Sc Mathematics, ISI Kolkata*

**Quick reading.** Composition of continuous functions.

**Final Answer:** True.

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

**Q 5.74** **The composition of two continuous functions is continuous.**

SOLUTION

**Concept used.** The composition theorem: if $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Applying to every point in the relevant domain gives continuity of the composition.

**Step 1.** Standard result.

**Step 2.** True whenever the composition is defined.

> **Final Answer: True.**

**☞ Composition continuity**

If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then $f \circ g$ is continuous at $a$. Track both layers when locating discontinuities.

**EXPERT'S SOLUTION** : *Tara Rao, M.Sc Mathematics, IIT Bombay*

**Quick reading.** Theorem 5.1.6 in the chapter.

> **Final Answer:** True.

**The mechanical view.** Think of the chain rule as peeling an onion: differentiate the outermost layer treating the inside as one symbol, then multiply by the derivative of that inside, and recurse. For nested radicals or compositions of trig and exponentials, write each layer on a separate line; the answer assembles itself with the right number of factors.

## Q 5.75 Trigonometric and inverse trigonometric functions are differentiable in their respective domains.

**SOLUTION**

**Concept used.** Each elementary trig function $(\sin, \cos, \tan, \cot, \sec, \csc)$ has a closed-form derivative on its domain (subject to the usual restrictions of $\tan, \sec$ at $\pi/2 + n\pi$, etc.). Each inverse trig function is differentiable on the *interior* of its domain — but *not* at the endpoints: $\sin^{-1}, \cos^{-1}$ fail at $x = \pm 1$ (vertical tangents), and $\sec^{-1}, \csc^{-1}$ fail at $x = \pm 1$.

**Step 1.** Direct trig: differentiable in their natural domain.

**Step 2.** Inverse trig: differentiable in the *interior* only.

**Step 3.** The statement claims "in their respective domains", which includes endpoints. So the statement is **false**.

> **Final Answer: False.**

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

**EXPERT'S SOLUTION** *: Meera Iyer, M.Sc Applied Mathematics, IIT Kanpur*

**Quick reading.** Endpoints $\pm 1$ of $\sin^{-1}$ are in the domain but the derivative blows up.

> **Final Answer:** False.

**What examiners reward.** Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason. **Common mistake in a different framing.** Even when the algebra is correct, students often skip naming the theorem or rule being invoked. Examiners give partial credit for stating *the right principle* (continuity laws, MVT, FTC, substitution rule) even if a later step slips. Always name the engine before turning it.

**Q 5.76** **If $f \cdot g$ is continuous at $x = a$, then $f$ and $g$ are separately continuous at $x = a$.**

**SOLUTION**

**Concept used.** The converse of "product of continuous is continuous" is false in general. A discontinuous $f$ and a $g$ that vanishes at $a$ can give a continuous product.

**Step 1.** Counterexample: $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ (discontinuous at $)and$g(x)=x$(continuous, vanishes at)$.

**Step 2.** Product $f(x) \cdot g(x) = |x|$, which is continuous on $\mathbb{R}$.

**Step 3.** Yet $f$ is discontinuous at $0$.

> **Final Answer:** **False**.

☞ **Exam-paper habit**

Write the formula or theorem you are invoking on its own line before substituting numbers. Examiners give the method mark even if a later arithmetic slip costs the final mark.

**EXPERT'S SOLUTION** : *Ananya Gupta, M.Sc Mathematics, IIT Bombay*

**Quick reading.** The zero of $g$ "hides" the jump of $f$.

**Final Answer:** False.

**Alternate framing.** Read the function as a single graph drawn with two coloured pens. Continuity asks: does the second pen pick up exactly where the first left off? If both pens meet the same $y$ at $x = c$ *and* the dot $f(c)$ sits on that meeting point, continuity holds; otherwise it doesn't. This visual check anticipates the algebra and exposes sign errors before they propagate.

### Key Takeaways

- A function is continuous at $c$ iff LHL, RHL and $f(c)$ all exist and are equal.
- Differentiable $\Rightarrow$ continuous, but the converse is false ($|x|$ at $0$).
- For piecewise functions, "find the constants for continuity / differentiability" reduces to one or two linear equations in the unknowns.
- Chain, product, quotient, logarithmic and implicit differentiation are the five workhorse algebraic techniques. Pick logarithmic for products of powers and for $u(x)^{v(x)}$; pick implicit for relations not solved for $y$.
- Inverse-trig differentiation is often shortened dramatically by an inverse-trig identity (triple angle, half angle, $\sec + \tan$, etc.).
- Rolle's theorem and MVT are routine "verify hypotheses, then solve $f'(c) = $ slope" problems; the only common pitfall is ignoring a corner inside the open interval.

End of NCERT Exemplar Exercise 5.3