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Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 5: Continuity and Differentiability

What this chapter covers: the ε-free working definition of continuity ($\lim_{x\to c} f(x) = f(c)$), algebra of continuous functions, differentiability and the derivative as a limit, the chain rule, implicit and inverse-trig differentiation, derivatives of exponential and logarithmic functions, logarithmic differentiation for x^x-type expressions, parametric derivatives, and second-order derivatives. Rolle's and Mean Value Theorems are deleted from the rationalised NCERT but tested in JEE/NEET — we include them as clearly tagged extensions.

1 Continuity

A function is **continuous** at a point if its graph passes through that point without any jump, hole, or break. Calculus rests on this idea: derivatives, integrals, and every existence theorem in the chapter quietly assume continuity wherever they act.

1.1 Definition of continuity at a point

Let f be a real function defined on an interval containing the point c. We say f is **continuous at** c if

$$\lim_{x\to c} f(x) = f(c).$$

Three things must therefore all be true, simultaneously:

1. $f(c)$ is defined (the function exists at the point).
2. $\lim_{x\to c} f(x)$ exists (the two one-sided limits agree).
3. The limit equals the value: $\lim_{x\to c} f(x) = f(c)$.

If any one fails, f is **discontinuous** at c.

Continuity at a Point — Three Conditions

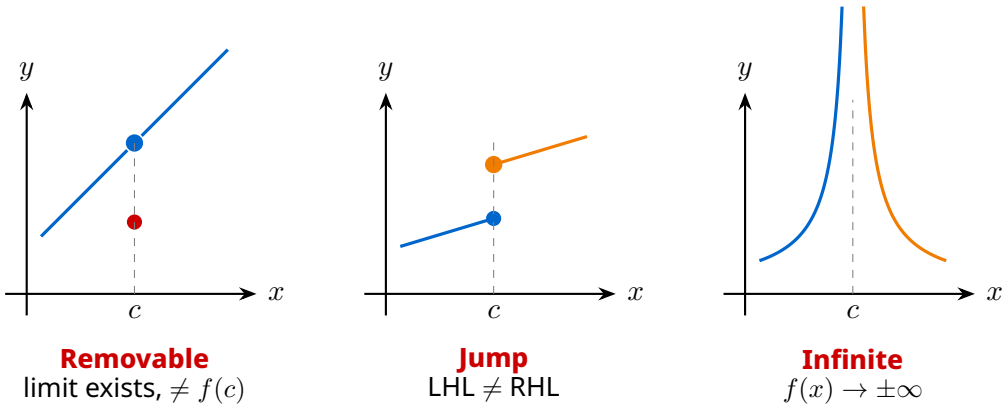
f is continuous at $x = c$ iff
(i) $f(c)$ exists (value defined).

(ii) $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$ (LHL = RHL).
(iii) $\lim_{x\to c} f(x) = f(c)$ (limit equals value).

Continuity on an interval. f is continuous on (a, b) if it is continuous at every point of (a, b). At the endpoints of a closed interval $[a, b]$, only the relevant one-sided limit is used: $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

1.2 What discontinuity looks like

Three classic ways a function can fail to be continuous at a point.



Removable
limit exists, $\neq f(c)$

Jump
LHL $\neq$ RHL

Infinite
$f(x) \to \pm\infty$

The first type (hole) is the friendly one: redefining $f(c)$ to equal the limit "repairs" the discontinuity. The other two cannot be repaired by adjusting a single value.

1.3 A worked check

Example. Decide whether

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$$

is continuous at $x = 1$.

Solution. For $x \neq 1$, $f(x) = \dfrac{(x-1)(x+1)}{x-1} = x + 1$. So

$$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x + 1) = 2, \quad \text{but} \quad f(1) = 3.$$

Since $\lim_{x\to 1} f(x) \neq f(1)$, f is *discontinuous* at $x = 1$. (A removable discontinuity — redefining $f(1) = 2$ would fix it.)

Continuity in one line

The graph of f can be drawn through the point $(c, f(c))$ without lifting the pen. Algebraically: limit = value.

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1.4 Continuous everywhere — familiar examples

Polynomials, $\sin x$, $\cos x$, e^x, $|x|$, and constant functions are continuous on all of $\mathbb{R}$. Rational functions $p(x)/q(x)$ are continuous wherever $q(x) \neq 0$. $\log x$ is continuous on $(0, \infty)$. $\tan x, \sec x$ are continuous everywhere except where $\cos x = 0$.

Standard continuous families — memorise the list

Polynomial, sin, cos, exponential, modulus, and constant: continuous on $\mathbb{R}$. Logarithm: continuous on $(0, \infty)$. tan, sec: continuous except at odd multiples of $\pi/2$. cot, csc: continuous except at multiples of π. Any combination by $+, -, \times, \div$ (denominator nonzero) and composition stays continuous.

1.5 Algebra of continuous functions

If f and g are continuous at c, then so are

$$f \pm g, \quad fg, \quad cf \ \ (c \text{ constant}), \quad \frac{f}{g} \ \ (\text{if } g(c) \neq 0).$$

Algebra of Continuity at $x = c$

If f, g continuous at c:

$$(f \pm g)(x) \text{ is continuous at } c$$
$$(f \cdot g)(x) \text{ is continuous at } c$$
$$\left(\frac{f}{g}\right)(x) \text{ is continuous at } c, \text{ provided } g(c) \neq 0$$
$$(g \circ f)(x) \text{ is continuous at } c, \text{ if } g \text{ continuous at } f(c)$$

This is why every *elementary function* — built from the standard list by arithmetic and composition — is continuous on its natural domain. Most "check the continuity" questions reduce to identifying the domain.

$\frac{f}{g}$ at a zero of g

The quotient rule for continuity requires $g(c) \neq 0$. Students sometimes declare $\tan x = \sin x / \cos x$ continuous on all of $\mathbb{R}$ — it isn't: continuity breaks at $x = \pi/2, 3\pi/2, \ldots$ where $\cos x = 0$.

1.6 Continuity of composite functions

If f is continuous at c and g is continuous at $f(c)$, then $g \circ f$ is continuous at c.

Example. $h(x) = \sin(x^2+1)$. Take $f(x) = x^2+1$ (continuous everywhere as a polynomial) and $g(u) = \sin u$ (continuous everywhere). Therefore $h = g \circ f$ is continuous on $\mathbb{R}$.

This single property handles most "prove continuous" questions in board exams — decompose the function, cite the standard list, and chain.

2 Differentiability

A function is **differentiable** at a point if its graph has a well-defined tangent line there — a slope, not a kink or a vertical line. Differentiability is a stronger condition than continuity.

2.1 Definition of the derivative

The **derivative** of f at $x = c$ is

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h},$$

when this limit exists. Equivalently,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

If the limit exists, f is **differentiable at** c; if it exists for every c in an interval, f is differentiable on that interval, and $f'(x)$ is the derivative function.

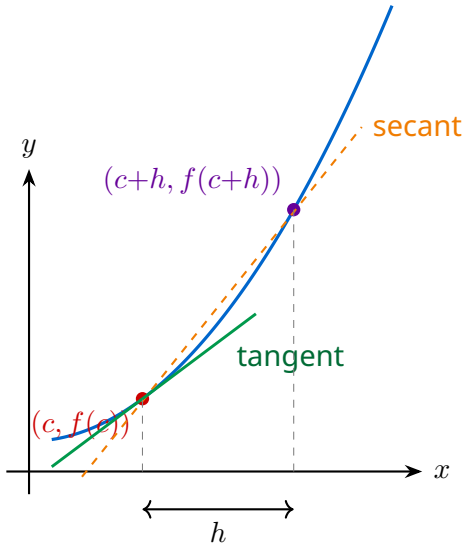
Derivative as a Limit

$$\left.\frac{df}{dx}\right|_{x=c} = f'(c) = \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}.$$

Geometric meaning: **slope of the tangent line** to $y = f(x)$ at $(c, f(c))$.

2.2 Tangent line — the geometric picture

The chord through $(c, f(c))$ and $(c+h, f(c+h))$ has slope $\dfrac{f(c+h)-f(c)}{h}$. As $h \to 0$, the chord rotates into the tangent line and its slope tends to $f'(c)$.



Slope of tangent at c = limit of slope of secant as $h \to 0 = f'(c)$.

Derivative = slope of tangent

$f'(c)$ tells you how steeply f is rising or falling at the input c. Positive $f'(c)$: increasing. Negative: decreasing. Zero: momentarily flat.

2.3 Differentiability implies continuity

If f is differentiable at c, then f is continuous at c.

Proof sketch. If $f'(c)$ exists, then

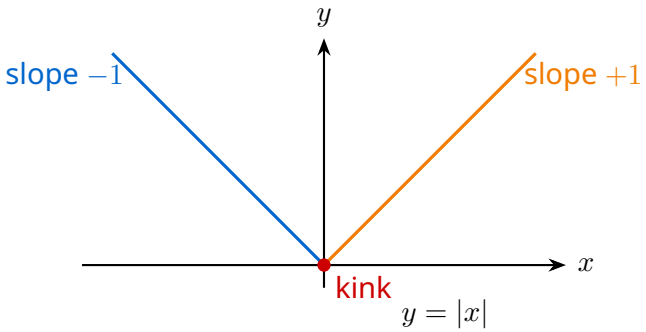
$$\lim_{x\to c}\big[f(x)-f(c)\big] = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}\cdot(x-c) = f'(c)\cdot 0 = 0,$$

so $\lim_{x\to c} f(x) = f(c)$, which is continuity.

The converse is false. A function can be continuous at a point yet not differentiable there.

2.4 The canonical counterexample: $|x|$ at 0

$f(x) = |x|$ is continuous everywhere (the graph is connected). But it is *not* differentiable at $x = 0$.



Why $|x|$ fails at 0: the left-hand and right-hand derivatives are not equal.

$$f'(0^-) = \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1,$$
$$f'(0^+) = \lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = +1.$$

Since $-1 \neq +1$, the limit doesn't exist; the tangent is not unique at the corner.

Left and Right Hand Derivatives

$$Lf'(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}, \qquad Rf'(c) = \lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h}.$$

f is differentiable at $c \iff Lf'(c) = Rf'(c)$ and both are finite.

Continuous $\Rightarrow$ Differentiable is FALSE

Differentiability is stronger. $|x|$, $x^{2/3}$, and any function with a corner, cusp, or vertical tangent are continuous but not differentiable at the offending point. Always check via LHD = RHD if a function is piecewise defined.

2.5 Algebra of derivatives

If f and g are differentiable at x, then

$$(f \pm g)'(x) = f'(x) \pm g'(x),$$
$$(cf)'(x) = c\, f'(x),$$
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x) \qquad \text{(product rule)},$$
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \text{(quotient rule, } g(x) \neq 0\text{)}.$$

Quotient rule mnemonic — "low d-high minus high d-low"

$\left(\frac{\text{high}}{\text{low}}\right)' = \frac{(\text{low})(\text{d high}) - (\text{high})(\text{d low})}{(\text{low})^2}$. The numerator is *not* symmetric — order matters.

3 The Chain Rule and Composite Functions

The chain rule is the single most important computational tool in this chapter. It tells you how to differentiate a function of a function.

3.1 The chain rule

If $y = f(u)$ and $u = g(x)$, both differentiable, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot g'(x) = f'(g(x)) \cdot g'(x).$$

Chain Rule

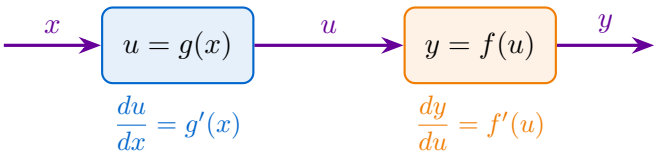
$$\frac{d}{dx}\big[f(g(x))\big] \;=\; f'(g(x)) \cdot g'(x).$$

Differentiate the outer function (leaving the inner alone), *then multiply* by the derivative of the inner function.

3.2 Visualising the chain

Think of $f \circ g$ as a two-stage pipeline: x flows in, g amplifies/distorts it to $u = g(x)$, then f acts on u. A small change dx becomes $du = g'(x)\,dx$, which becomes $dy = f'(u)\,du$. Substituting,

$$dy = f'(g(x)) \cdot g'(x) \cdot dx.$$



Multiply the rates: $\dfrac{dy}{dx} = f'(u) \cdot g'(x)$

3.3 Worked examples

Example 1. $y = \sin(x^2)$. Let $u = x^2$, $y = \sin u$.

$$\frac{dy}{dx} = \cos u \cdot 2x = 2x \cos(x^2).$$

Example 2. $y = e^{\tan x}$. Outer: $e^{(\cdot)}$. Inner: $\tan x$.

$$\frac{dy}{dx} = e^{\tan x} \cdot \sec^2 x.$$

Example 3. $y = \sqrt{1 + \cos^2 x}$. Let $u = 1 + \cos^2 x$; $y = \sqrt{u} = u^{1/2}$.

$$\frac{dy}{du} = \tfrac{1}{2}u^{-1/2}, \qquad \frac{du}{dx} = 2\cos x \cdot (-\sin x) = -\sin 2x.$$

$$\frac{dy}{dx} = \frac{-\sin 2x}{2\sqrt{1 + \cos^2 x}}.$$

"Onion" approach to nested compositions

For deeply nested functions like $y = \sin(\cos(x^3))$, peel from the outside in. Differentiate the outermost layer keeping the inside untouched, multiply by the derivative of the next layer, and continue until you've peeled all the way to x.

Outer-times-Inner

O-T-I: **O**uter derivative *evaluated at the inner function* **T**imes **I**nner derivative. Repeat for each layer. Miss the "$\times$ inner derivative" step and you lose marks on every chain-rule question.

4 Derivatives of Inverse Trigonometric, Exponential and Logarithmic Functions

This section builds the derivative library — the table you'll consult for every later problem. The derivations use the chain rule applied to identities like $\sin(\arcsin x) = x$.

4.1 Derivative of e^x and $\log x$

The exponential function is its own derivative:

$$\frac{d}{dx}(e^x) = e^x.$$

The natural logarithm has the reciprocal as its derivative:

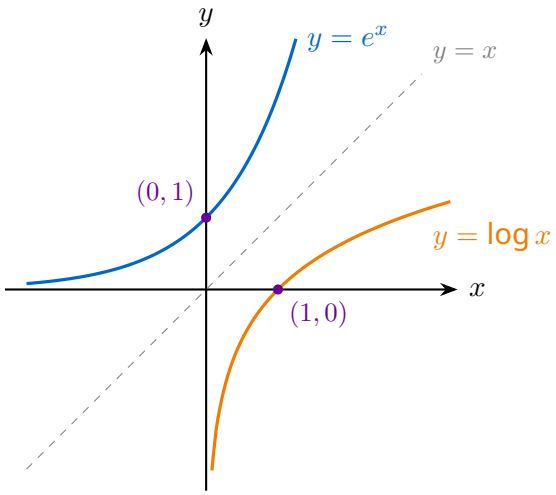
$$\frac{d}{dx}(\log x) = \frac{1}{x}, \qquad x > 0.$$

Exponential and Logarithm

$$\frac{d}{dx}(e^x) = e^x, \qquad \frac{d}{dx}(a^x) = a^x \log a,$$
$$\frac{d}{dx}(\log x) = \frac{1}{x}, \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \log a}.$$

"log" in this chapter denotes the natural logarithm $\log_e = \ln$.

The graphs of $y = e^x$ and $y = \log x$ are mirror images across $y = x$ (they are inverses).



The slope of $y = e^x$ at $(0, 1)$ is $e^0 = 1$, and the slope of $y = \log x$ at $(1, 0)$ is $1/1 = 1$. Both pass their respective points at $45°$ — a beautiful consequence of being inverses.

Why e is everywhere

Radioactive decay, compound interest, population growth, RC-circuit charging — any process whose rate of change is proportional to its current size satisfies $dN/dt = kN$, whose solution is $N(t) = N_0 e^{kt}$. The exponential's self-derivative property is exactly what makes this work.

4.2 Derivatives of inverse trigonometric functions

The standard six identities are obtained by implicit differentiation. We illustrate one in detail and tabulate the rest.

Derivation of $\frac{d}{dx}(\sin^{-1} x)$. Let $y = \sin^{-1} x$, so $\sin y = x$ with $y \in [-\pi/2, \pi/2]$. Differentiate both sides w.r.t. x:

$$\cos y \cdot \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

(We took the positive square root because $\cos y \geq 0$ on $[-\pi/2, \pi/2]$.)

Inverse Trig Derivatives — All Six

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1,$$
$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| < 1,$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$
$$\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| > 1,$$
$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| > 1.$$

Co-pairs have opposite signs

$\sin^{-1}$ and $\cos^{-1}$ differ only by a sign. Same for $\tan^{-1}$ / $\cot^{-1}$ and $\sec^{-1}$ / $\csc^{-1}$. The "co-" partner always carries the minus sign — and the domain matches.

Forgetting the absolute value in $\sec^{-1}$, $\csc^{-1}$

The $|x|$ in $\frac{1}{|x|\sqrt{x^2-1}}$ is essential — without it the derivative would carry an incorrect sign for $x < -1$. Many students drop the bars and lose marks.

4.3 Worked example with inverse trig + chain rule

Differentiate $y = \sin^{-1}(2x\sqrt{1-x^2})$.

Trick: substitute $x = \sin\theta$ where $\theta \in [-\pi/2, \pi/2]$. Then $\sqrt{1-x^2} = \cos\theta$ and

$$2x\sqrt{1-x^2} = 2\sin\theta\cos\theta = \sin 2\theta,$$

so $y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1} x$ (valid in the relevant range). Therefore

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

Substitution before differentiation

For nested inverse-trig expressions like $\sin^{-1}(2x\sqrt{1-x^2})$, $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, $\cos^{-1}(2x^2-1)$, substitute $x = \sin\theta$ or $x = \tan\theta$. The expression collapses to a multiple of θ and you skip a page of algebra.

5 Implicit, Logarithmic, and Parametric Differentiation

Not every relation between x and y comes pre-solved as $y = f(x)$. Three techniques handle the rest.

5.1 Implicit differentiation

When y is given by an equation $F(x, y) = 0$ that we cannot, or do not wish to, solve for y, we differentiate both sides of the equation w.r.t. x, treating y as a function of x and applying the chain rule everywhere y appears.

Example. Find $\dfrac{dy}{dx}$ for $x^2 + y^2 = 25$.

Differentiate both sides:

$$2x + 2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}.$$

Example. For $\sin(xy) + xy^2 = 1$:

$$\cos(xy)\left(y + x\frac{dy}{dx}\right) + y^2 + x \cdot 2y\frac{dy}{dx} = 0.$$

Collect $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left[x\cos(xy) + 2xy\right] = -\left[y\cos(xy) + y^2\right] \implies \frac{dy}{dx} = -\frac{y\cos(xy) + y^2}{x\cos(xy) + 2xy}.$$

Implicit Differentiation in One Line

Whenever you differentiate a term containing y, multiply by $\dfrac{dy}{dx}$ as a chain-rule factor; then solve algebraically for $\dfrac{dy}{dx}$.

5.2 Logarithmic differentiation

Used when $f(x)$ is (a) a product/quotient of several factors, or (b) of the form $(\text{function})^{\text{function}}$ such as x^x. The trick: take log first, then differentiate.

Example: $y = x^x$, $x > 0$.

$$\log y = x \log x.$$

Differentiate both sides w.r.t. x:

$$\frac{1}{y}\frac{dy}{dx} = \log x + x \cdot \frac{1}{x} = \log x + 1 \implies \frac{dy}{dx} = y(\log x + 1) = x^x(\log x + 1).$$

x^x-type Derivative

For $y = [f(x)]^{g(x)}$ with $f(x) > 0$:

$$\log y = g(x)\,\log f(x) \implies \frac{dy}{dx} = y\left[g'(x)\log f(x) + g(x)\,\frac{f'(x)}{f(x)}\right].$$

Example: $y = (\sin x)^{\cos x}$, $0 < x < \pi$.

$$\log y = \cos x \cdot \log \sin x.$$

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \cdot \log \sin x + \cos x \cdot \frac{\cos x}{\sin x} = -\sin x \log \sin x + \cot x \cos x.$$

$$\frac{dy}{dx} = (\sin x)^{\cos x}[\cot x \cos x - \sin x \log \sin x].$$

Spot the x^x form

If you see a variable in both base *and* exponent, reach for logarithmic differentiation **immediately**. Trying to apply $\frac{d}{dx}(x^n) = nx^{n-1}$ or $\frac{d}{dx}(a^x) = a^x \log a$ here will give the wrong answer — both rules assume one of base/exponent is constant.

x^x is neither x^n nor a^x

The wrong answer $\dfrac{d}{dx}(x^x) = x \cdot x^{x-1} = x^x$ comes from forgetting that the exponent itself varies. The correct derivative is $x^x(1+\log x)$. The two answers disagree everywhere except in your nightmares.

5.3 Parametric differentiation

Sometimes x and y are each given as functions of a third variable t (the parameter):

$$x = f(t), \quad y = g(t).$$

Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}, \quad \text{provided } f'(t) \neq 0.$$

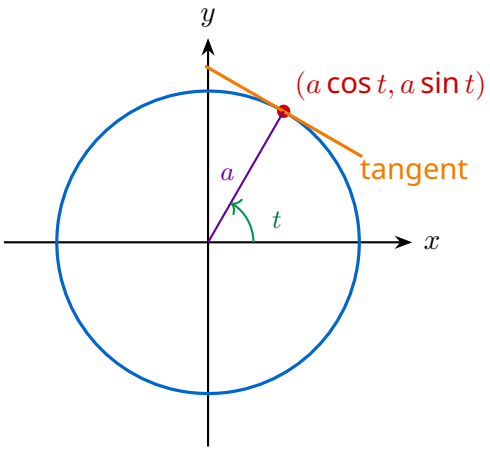
Parametric Derivative

$$\boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$$

The two derivatives w.r.t. the parameter are computed separately; their ratio is the slope in the xy-plane.

Example. Circle parametrised as $x = a\cos t$, $y = a\sin t$.

$$\frac{dx}{dt} = -a\sin t, \quad \frac{dy}{dt} = a\cos t \implies \frac{dy}{dx} = \frac{a\cos t}{-a\sin t} = -\cot t.$$



The slope $-\cot t$ matches the geometric fact that the tangent to a circle is perpendicular to the radius: the radius has slope $\tan t$, so the tangent has slope $-1/\tan t = -\cot t$.

Example. Cycloid: $x = a(t - \sin t)$, $y = a(1 - \cos t)$.

$$\frac{dx}{dt} = a(1 - \cos t), \quad \frac{dy}{dt} = a \sin t \implies \frac{dy}{dx} = \frac{\sin t}{1 - \cos t} = \cot(t/2).$$

(using the identity $\sin t = 2 \sin(t/2) \cos(t/2)$ and $1 - \cos t = 2 \sin^2(t/2)$.)

Parametric equations in physics

Projectile motion is naturally parametric: $x(t) = u \cos\theta \cdot t$, $y(t) = u \sin\theta \cdot t - \frac{1}{2} g t^2$. The slope dy/dx at any instant gives the direction of motion — horizontal at the peak (where $dy/dt = 0$), and increasingly negative on the way down.

6 Second-Order Derivatives

The derivative of $f'(x)$ is the **second derivative**, written

$$f''(x), \quad \frac{d^2y}{dx^2}, \quad y'', \quad D^2 y.$$

It measures the rate of change of the slope — i.e., **concavity**: positive f'' means the curve bends upward (smile), negative f'' means it bends downward (frown).

6.1 Computation

Example. $y = e^x \sin x$.

$$y' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x).$$

$$y'' = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x.$$

Example: implicit second derivative. For $x^2 + y^2 = 1$, we already found $y' = -x/y$. Differentiate again:

$$y'' = -\frac{y - x \cdot y'}{y^2} = -\frac{y - x \cdot (-x/y)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3}.$$

(Using $x^2 + y^2 = 1$ in the final step.)

Second Derivative Notation

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = (y')'.$$

For a parametric curve $x = f(t)$, $y = g(t)$:

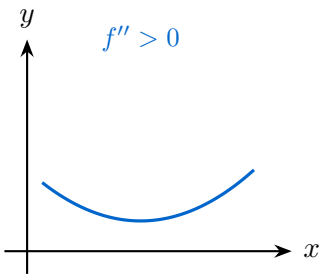
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}.$$

Do **not** write $\dfrac{d^2y/dt^2}{d^2x/dt^2}$ — that is wrong.

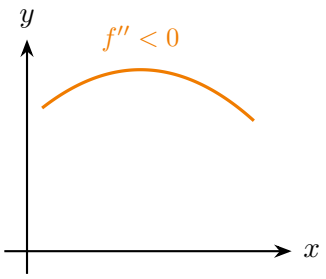
Parametric second derivative pitfall

For parametric curves, $\dfrac{d^2y}{dx^2} \neq \dfrac{d^2y/dt^2}{d^2x/dt^2}$. The correct formula differentiates $\dfrac{dy}{dx}$ w.r.t. t and then divides by $\dfrac{dx}{dt}$. Going one step at a time prevents the slip.

6.2 Geometric meaning: concavity



concave up (smile)



concave down (frown)

Reading f'' off a graph

Where $f'' > 0$, the curve looks like the bottom of a cup (smile, $\cup$). Where $f'' < 0$, the curve looks like an upside-down cup (frown, $\cap$). Points where f'' changes sign are **inflection points**.

6.3 Differential equations: a preview

Many physical laws are statements about derivatives.

- Newton's second law: $m\dfrac{d^2x}{dt^2} = F$. Position-acceleration relationship.
- Simple harmonic motion: $\dfrac{d^2x}{dt^2} + \omega^2 x = 0$. Solutions are $x = A\sin\omega t + B\cos\omega t$.
- Beam bending, heat conduction, wave propagation — all governed by second derivatives.

This is why mastering d^2y/dx^2 early pays off: it's the language of physics from Class 12 Chapter 9 onwards.

7 [JEE/NEET Extension] Rolle's and Mean Value Theorems

These two existence theorems are *deleted* from the rationalised NCERT but remain heavily tested in JEE Main, JEE Advanced, and NEET. We summarise the statements, the geometric meaning, and one worked example each.

7.1 Rolle's Theorem

Statement. If a function f is

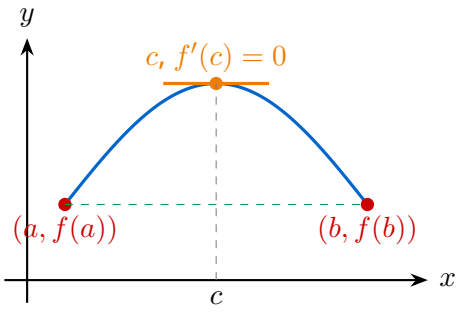
1. continuous on the closed interval $[a, b]$,
2. differentiable on the open interval (a, b), and
3. $f(a) = f(b)$,

then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Rolle's Theorem

f continuous on $[a, b]$, differentiable on (a, b), $f(a) = f(b) \implies \exists\, c \in (a, b)$ with $f'(c) = 0$.

Geometric meaning. If a smooth curve starts and ends at the same height, somewhere in between the tangent line must be horizontal — the function reaches a peak or trough.



Worked example. Verify Rolle's theorem for $f(x) = x^2 - 4x + 3$ on $[1, 3]$.

Check conditions.

- Polynomial $\Rightarrow$ continuous on $[1, 3]$ and differentiable on $(1, 3)$. ✓
- $f(1) = 1 - 4 + 3 = 0$ and $f(3) = 9 - 12 + 3 = 0$, so $f(1) = f(3)$. ✓

Find c**.** $f'(x) = 2x - 4 = 0 \Rightarrow x = 2 \in (1, 3)$. So $c = 2$ works, verifying the theorem.

7.2 Mean Value Theorem (Lagrange's MVT)

Statement. If f is

1. continuous on $[a, b]$, and
2. differentiable on (a, b),

then there exists at least one $c \in (a, b)$ such that

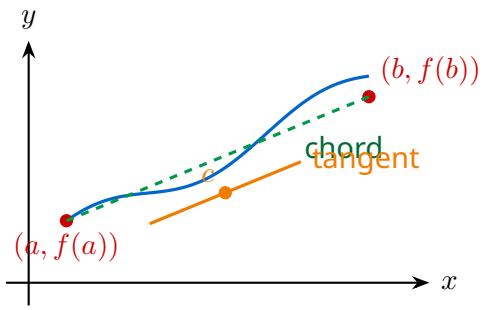
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Mean Value Theorem

$$\exists\, c \in (a, b): \quad f'(c) = \frac{f(b) - f(a)}{b - a}.$$

At some point in the interval, the instantaneous slope equals the average slope.

Geometric meaning. The chord joining $(a, f(a))$ and $(b, f(b))$ has slope $\dfrac{f(b) - f(a)}{b - a}$. The MVT says: somewhere strictly inside (a, b), the tangent to the curve is *parallel* to this chord.



Worked example. Verify MVT for $f(x) = x^2$ on $[2, 4]$.

Check conditions. f is a polynomial: continuous on $[2, 4]$, differentiable on $(2, 4)$. ✓

Find c.

$$\frac{f(4) - f(2)}{4 - 2} = \frac{16 - 4}{2} = 6. \quad f'(x) = 2x = 6 \implies x = 3 \in (2, 4).$$

So $c = 3$ verifies MVT.

Rolle vs MVT in one line

Rolle's is a special case of **MVT**: when the chord is horizontal ($f(a) = f(b)$), the parallel tangent must also be horizontal — which is exactly $f'(c) = 0$.

Existence theorems — don't try to find c explicitly

Most JEE/NEET MVT questions ask you to *prove an inequality* using MVT (e.g., $|\sin a - \sin b| \le |a - b|$), not to compute c. The trick is to apply MVT to a clever choice of f on a clever interval — and then bound f'.

8 Quick Reference Summary

A one-stop card. Every formula and rule from the chapter, condensed.

8.1 Continuity and Differentiability checklist

Concept	Definition / Condition
Continuity at c	$\lim_{x\to c} f(x) = f(c)$ (value defined, limit exists, equal)
Removable discontinuity	Limit exists, but $\neq f(c)$ (a fixable hole)
Jump discontinuity	LHL and RHL exist but are unequal
Infinite discontinuity	At least one one-sided limit is $\pm\infty$
Differentiable at c	$f'(c) = \lim_{h\to 0}(f(c+h) - f(c))/h$ exists (finite)
Differentiable $\Rightarrow$ Continuous	Always TRUE; converse fails ($\lvert x\rvert$ at 0)
Chain rule	$(d/dx)\, f(g(x)) = f'(g(x)) \cdot g'(x)$
Implicit differentiation	Differentiate both sides; chain-rule every y
Logarithmic differentiation	Use for products, quotients, $f(x)^{g(x)}$
Parametric derivative	$dy/dx = (dy/dt)/(dx/dt)$
Second derivative $f''(x)$	$(f')'$. Concavity: $+$ smile, $-$ frown

8.2 Master derivative table

Function $f(x)$	Derivative $f'(x)$	Domain note
x^n	nx^{n-1}	$n \in \mathbb{R}$
constant k	0	—
$\sin x$	$\cos x$	$\mathbb{R}$
$\cos x$	$-\sin x$	$\mathbb{R}$
$\tan x$	$\sec^2 x$	$x \neq (2k+1)\pi/2$
$\cot x$	$-\csc^2 x$	$x \neq k\pi$
$\sec x$	$\sec x \tan x$	$x \neq (2k+1)\pi/2$
$\csc x$	$-\csc x \cot x$	$x \neq k\pi$
e^x	e^x	$\mathbb{R}$
a^x	$a^x \log a$	$a > 0$
$\log x$	$1/x$	$x > 0$
$\log_a x$	$1/(x \log a)$	$x > 0$, $a > 0$, $a \neq 1$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$	$\lvert x\rvert < 1$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$	$\lvert x\rvert < 1$
$\tan^{-1} x$	$1/(1+x^2)$	$\mathbb{R}$
$\cot^{-1} x$	$-1/(1+x^2)$	$\mathbb{R}$
$\sec^{-1} x$	$1/(\lvert x\rvert\sqrt{x^2-1})$	$\lvert x\rvert > 1$
$\csc^{-1} x$	$-1/(\lvert x\rvert\sqrt{x^2-1})$	$\lvert x\rvert > 1$

8.3 Differentiation rules

Five Rules of Differentiation

Sum: $(f \pm g)' = f' \pm g'$

Constant multiple: $(cf)' = cf'$

Product: $(fg)' = f'g + fg'$

Quotient: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad g \neq 0$

Chain: $\big[f(g(x))\big]' = f'(g(x)) \cdot g'(x)$

8.4 Special techniques

Technique	When and how
Implicit differentiation	$F(x, y) = 0$. Differentiate both sides; chain-rule every y; solve for dy/dx.
Logarithmic (x^x type)	$y = f^g$. Take log, then differentiate.
Logarithmic (products)	$y = \prod f_i$. Take log to convert product to sum, differentiate.
Substitution (inverse-trig)	$\sin^{-1}, \tan^{-1}$ of compound expressions: try $x = \sin\theta$ or $x = \tan\theta$.
Parametric	$x = f(t), y = g(t)$. $dy/dx = (dy/dt)/(dx/dt)$.
Second-order parametric	$d^2y/dx^2 = \frac{d}{dt}(dy/dx) \div (dx/dt)$. NEVER second-derivatives of x and y divided.

8.5 Property-check workflow

To decide whether f is continuous at c:

1. Check $f(c)$ is defined.
2. Compute LHL: $\lim_{x\to c^-} f(x)$ and RHL: $\lim_{x\to c^+} f(x)$.
3. Check LHL $=$ RHL $= f(c)$. If yes, f is continuous; otherwise classify the discontinuity.

To decide whether f is differentiable at c:

1. Confirm f is continuous at c first (otherwise differentiability fails automatically).
2. Compute $Lf'(c) = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h}$ and $Rf'(c) = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h}$.
3. Check $Lf'(c) = Rf'(c)$ and both finite. If yes, $f'(c)$ exists.

8.6 Final word

This chapter is the entry-point to calculus. The deep idea is small: *slope of tangent $=$ limit of slope of chord*. Everything else — chain rule, implicit, log, parametric — is a computational technique built on top of that single idea.

- Memorise the derivative table cold. It's the alphabet of the rest of Class 12.
- Practise the chain rule until it's automatic; almost every JEE/NEET differentiation involves a chain.
- Recognise the x^x form on sight $\Rightarrow$ logarithmic differentiation.
- For piecewise functions, always check LHD $=$ RHD at the join.
- Rolle and MVT are deleted from boards but tested in entrance exams — know the statements and their geometric pictures.

Best of luck with your preparation!

Master the derivative table, practise the chain rule, and the rest of calculus follows.