NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 7 — Integrals (2026-27)

Chapter 7: Integrals

About this Chapter

Integration is the inverse process of differentiation. The **indefinite integral** $\int f(x)\,dx = F(x) + C$ recovers the family of antiderivatives of f; the **definite integral** $\int_a^b f(x)\,dx = F(b) - F(a)$ assigns a number (the signed area under $y = f(x)$ between $x = a$ and $x = b$) via the **Fundamental Theorem of Calculus**. This Exemplar set drills four core techniques — **substitution**, **partial fractions**, **integration by parts** (with the **ILATE** rule), and the suite of **special integrals** of the form $\int dx/(x^2 \pm a^2)$, $\int dx/\sqrt{a^2 - x^2}$, $\int dx/\sqrt{x^2 \pm a^2}$ — together with the six **properties of definite integrals**, most notably the King-rule $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Topics covered: Antiderivatives and the family of curves • Standard integrals table • Substitution method • Partial fractions • Integration by parts and ILATE • Special integrals • Definite integral as limit of a sum • Fundamental Theorem of Calculus • Six properties of definite integrals • King rule and even/odd collapses

Quick Formula Sheet

Power:
$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C, n \neq -1$

Log:
$\int \dfrac{1}{x}\,dx = \log|x| + C$

Exp:
$\int e^x dx = e^x + C; \int a^x dx = \dfrac{a^x}{\log a} + C$

Inverse-trig:
$\int \dfrac{dx}{1+x^2} = \tan^{-1} x + C$
$\int \dfrac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

Special-1:
$\int \dfrac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Special-2:
$\int \dfrac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$

By parts:
$\int u\,dv = uv - \int v\,du$ (ILATE)

FTC:
$\int_a^b f(x)\,dx = F(b) - F(a)$

King rule:
$\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Even/odd on $[-a, a]$:
even $\to 2\int_0^a f$; odd $\to 0$

I. Short Answer (S.A.)

Q 7.1 **Verify that** $\int \frac{2x-1}{2x+3}\,dx = x - \log\left|(2x+3)^2\right| + C$.

SOLUTION

Concept used. **Verification of an indefinite integral.** If $F(x)$ is claimed as an antiderivative of $f(x)$, differentiate F and check whether $F'(x) = f(x)$. Equivalently, algebraically simplify the integrand into a sum of a constant and a standard form $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + C$.

Step 1. Rewrite the integrand by long division:

$$\frac{2x-1}{2x+3} = \frac{(2x+3)-4}{2x+3} = 1 - \frac{4}{2x+3}.$$

Step 2. Integrate term-by-term:

$$\int \frac{2x-1}{2x+3}\,dx = \int 1\,dx - 4\int \frac{dx}{2x+3}.$$

Step 3. Use $\int \frac{dx}{2x+3} = \frac{1}{2}\log|2x+3|$:

$$= x - 4\cdot\tfrac{1}{2}\log|2x+3| + C = x - 2\log|2x+3| + C.$$

Step 4. Bring the 2 inside the log: $2\log|2x+3| = \log(2x+3)^2 = \log|(2x+3)^2|$.

Final Answer: $\int \frac{2x-1}{2x+3}\,dx = x - \log\left|(2x+3)^2\right| + C$. Verified.

☞ $\log a^n = n\log a$

Inside an absolute value, $2\log|y| = \log|y^2|$. Use this to match an answer with a coefficient of 2 in front of the log to one with a square inside.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Differentiate-the-RHS angle. The cleanest verification is to differentiate the claimed antiderivative and check we get back the integrand.

Step 1. Let $F(x) = x - \log|(2x+3)^2| + C$. Use $\log(2x+3)^2 = 2\log|2x+3|$, so $F(x) = x - 2\log|2x+3| + C$.

Step 2. Differentiate: $F'(x) = 1 - 2 \cdot \frac{2}{2x+3} = 1 - \frac{4}{2x+3}$.

Step 3. Combine over common denominator: $F'(x) = \frac{(2x+3)-4}{2x+3} = \frac{2x-1}{2x+3}$. ✓

Step 4. This equals the integrand, so the claimed formula is correct.

> **Final Answer:** $F'(x) = \frac{2x-1}{2x+3}$, matches integrand. Verified.

Why this matters. Every "verify $\int f = F + C$" question reduces to either (a) algebraically reproducing F from f via standard rules, or (b) differentiating F and matching. Method (b) is mechanical and almost always faster.
Sanity check. At $x = 0$: $f(0) = -1/3$ and $F'(0) = 1 - 4/3 = -1/3$. ✓

Q 7.2 **Verify that** $\int \frac{2x+3}{x^2+3x}\,dx = \log|x^2+3x| + C$.

SOLUTION

Concept used. **Logarithmic integration.** For any differentiable g with $g \neq 0$, $\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + C$. Recognising the numerator as the derivative of the denominator collapses the integral instantly.

Step 1. Identify $g(x) = x^2 + 3x$. Then $g'(x) = 2x + 3$, which is exactly the numerator.

Step 2. Apply $\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + C$:

$$\int \frac{2x+3}{x^2+3x}\,dx = \log|x^2+3x| + C.$$

> **Final Answer:** $\int \frac{2x+3}{x^2+3x}\,dx = \log|x^2+3x| + C$. Verified.

☞ **Parametric derivative**

For parametric curves $(x(t), y(t))$ use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Differentiate-the-RHS angle.

Step 1. Let $F(x) = \log|x^2 + 3x| + C$.

Step 2. Differentiate using $\frac{d}{dx}\log|u| = \frac{u'}{u}$: $F'(x) = \frac{2x+3}{x^2+3x}$.

Step 3. Matches the integrand. ✓

> **Final Answer:** Verified by direct differentiation.

Why this matters. The pattern $\int \frac{g'}{g}\,dx = \log|g| + C$ is one of the highest-yield templates in Class 12 Integrals; CBSE recycles it across at least two questions every Board paper.
Substitution view. Put $t = x^2 + 3x$, $dt = (2x+3)dx$. Then the integral becomes $\int dt/t = \log|t| + C$. Same answer, different mental model.

Q 7.3 **Evaluate** $\int \frac{(x^2+2)\,dx}{x+1}$.

SOLUTION

Concept used. **Long division for polynomial integrands.** When the numerator's degree is $\geq$ denominator's degree, perform polynomial long division to write the integrand as a polynomial + a proper rational function, then integrate term-by-term.

Step 1. Divide $x^2 + 2$ by $x + 1$: $x^2 + 2 = (x+1)(x-1) + 3$. So $\frac{x^2+2}{x+1} = (x-1) + \frac{3}{x+1}$.

Step 2. Integrate each piece:

$$\int (x-1)\,dx + 3\int \frac{dx}{x+1} = \frac{x^2}{2} - x + 3\log|x+1| + C.$$

> **Final Answer:** $\int \frac{x^2+2}{x+1}\,dx = \frac{x^2}{2} - x + 3\log|x+1| + C.$

☞ **Always check the degrees**

Before reaching for substitution, look at the degrees of numerator and denominator. If $\deg(P) \geq \deg(Q)$, long-divide first. Otherwise partial fractions or substitution.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Direct-verification angle.

Step 1. Differentiate the answer $F(x) = \frac{x^2}{2} - x + 3\log|x+1| + C$.

Step 2. $F'(x) = x - 1 + \dfrac{3}{x+1} = \dfrac{(x-1)(x+1)+3}{x+1} = \dfrac{x^2-1+3}{x+1} = \dfrac{x^2+2}{x+1}$. ✓

Step 3. Matches the integrand, so the antiderivative is correct.

> **Final Answer:** $\dfrac{x^2}{2} - x + 3\log|x+1| + C$.

Why this matters. Long division converts every improper-fraction integral into a polynomial part (trivial) plus a proper-fraction part (use $\log$ or partial fractions). It is the first step on every JEE Main shift question of this form.
Sanity check. At $x = 0$: integrand $= 2$; $F'(0) = -1 + 3 = 2$. ✓

Q 7.4 **Evaluate** $\displaystyle\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\,dx$.

SOLUTION

Concept used. Identity $e^{k\log x} = x^k$ for $x > 0$. Use this to simplify the integrand into a pure polynomial before integrating.

Step 1. Apply $e^{k\log x} = x^k$: $\dfrac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}} = \dfrac{x^6 - x^5}{x^4 - x^3}$.

Step 2. Factor: $\dfrac{x^5(x-1)}{x^3(x-1)} = x^2$ (for $x \neq 0, 1$).

Step 3. Integrate: $\int x^2\,dx = \dfrac{x^3}{3} + C$.

> **Final Answer:** $\displaystyle\int \frac{e^{6\log x} - e^{5\log x}}{e^{4\log x} - e^{3\log x}}\,dx = \frac{x^3}{3} + C$.

☞ **Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an ε-δ on a polynomial in an exam.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Simplification-first angle.

Step 1. The key insight: $e^{k\log x}$ is not a transcendental beast — it equals x^k. So the whole exponential mess is a rational function of x.

Step 2. Numerator: $x^6 - x^5 = x^5(x-1)$.

Step 3. Denominator: $x^4 - x^3 = x^3(x-1)$.

Step 4. Cancel the common factor $x^3(x-1)$: integrand $= x^2$.

Step 5. Integrate: $\int x^2\,dx = x^3/3 + C$.

> **Final Answer:** $x^3/3 + C$.

Why this matters. Always simplify before integrating. "Exponential of a log" and "log of an exponential" identities often reduce a scary integrand to a polynomial in one line.
Sanity check. At $x = 2$: original integrand
$= (2^6 - 2^5)/(2^4 - 2^3) = (64-32)/(16-8) = 32/8 = 4 = 2^2$. ✓

Q 7.5 **Evaluate** $\displaystyle\int \frac{1+\cos x}{x+\sin x}\,dx$.

SOLUTION

Concept used. Recognise $\int \frac{g'(x)}{g(x)}\,dx = \log|g(x)| + C$. Match the numerator with the derivative of the denominator.

Step 1. Let $g(x) = x + \sin x$. Then $g'(x) = 1 + \cos x$, exactly the numerator.

Step 2. Apply the standard form:

$$\int \frac{1+\cos x}{x+\sin x}\,dx = \log|x+\sin x| + C.$$

> **Final Answer:** $\displaystyle\int \frac{1+\cos x}{x+\sin x}\,dx = \log|x+\sin x| + C$.

☞ **Parametric derivative**

For parametric curves $(x(t), y(t))$ use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Substitution view. Put $t = x + \sin x$, $dt = (1 + \cos x)\,dx$.

Step 1. Integral becomes $\int \frac{dt}{t} = \log|t| + C$.

Step 2. Back-substitute: $\log|x + \sin x| + C$.

Final Answer: $\log|x + \sin x| + C$.

Why this matters. The $\int g'/g$ pattern is the single most-tested log-result in CBSE Class 12. Train your eye to spot it at sight.
Sanity check. Differentiate the answer: $\frac{1+\cos x}{x+\sin x}$. Matches integrand. ✓

Q 7.6 Evaluate $\displaystyle\int \frac{dx}{1+\cos x}$.

SOLUTION

Concept used. Half-angle identity $1 + \cos x = 2\cos^2(x/2)$.

Step 1. Use $1 + \cos x = 2\cos^2(x/2)$:

$$\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2(x/2)} = \frac{1}{2}\int \sec^2(x/2)\,dx.$$

Step 2. Integrate using $\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax)$: $= \frac{1}{2} \cdot \frac{\tan(x/2)}{1/2} + C = \tan(x/2) + C$.

Final Answer: $\displaystyle\int \frac{dx}{1+\cos x} = \tan(x/2) + C$.

☞ **Half-angle identities**

$1 + \cos x = 2\cos^2(x/2)$, $1 - \cos x = 2\sin^2(x/2)$, $1 + \sin x = (\sin(x/2) + \cos(x/2))^2$. These three rewrites turn most CBSE Board denominators into standard forms.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Alternative: multiply by conjugate.

Step 1. Multiply numerator and denominator by $1 - \cos x$:
$\frac{1-\cos x}{1-\cos^2 x} = \frac{1-\cos x}{\sin^2 x} = \csc^2 x - \csc x \cot x.$

Step 2. Integrate: $-\cot x + \csc x + C$.

Step 3. Simplify: $\csc x - \cot x = \dfrac{1-\cos x}{\sin x} = \dfrac{2\sin^2(x/2)}{2\sin(x/2)\cos(x/2)} = \tan(x/2)$. ✓

Final Answer: $\tan(x/2) + C$ (equivalent to $\csc x - \cot x + C$).

Why this matters. Two CBSE-favoured paths to the same answer: half-angle rewrite (fast) or conjugate-multiplication (mechanical). Knowing both is useful because Board papers sometimes phrase the expected answer in one form or the other.

Q 7.7 **Evaluate** $\int \tan^2 x \sec^4 x \, dx$.

SOLUTION

Concept used. **Pythagorean rewrite** $\sec^2 x = 1 + \tan^2 x$, then substitute $t = \tan x$.

Step 1. Split $\sec^4 x = \sec^2 x \cdot \sec^2 x = (1+\tan^2 x)\sec^2 x$. Integrand:
$\tan^2 x\,(1+\tan^2 x)\,\sec^2 x = (\tan^2 x + \tan^4 x)\sec^2 x$.

Step 2. Put $t = \tan x$, $dt = \sec^2 x\,dx$: $\int (t^2+t^4)\,dt = \dfrac{t^3}{3} + \dfrac{t^5}{5} + C$.

Step 3. Back-substitute: $\dfrac{\tan^3 x}{3} + \dfrac{\tan^5 x}{5} + C$.

Final Answer: $\displaystyle\int \tan^2 x \sec^4 x\,dx = \frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C$.

☞ **Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an ε-δ on a polynomial in an exam.

EXPERT'S SOLUTION *: Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Standard template. "$\tan^m x \sec^n x$ with n even" always reduces to a polynomial in $t = \tan x$ via $\sec^2 x\,dx = dt$.

Step 1. Save one $\sec^2 x$ to pair with dx; convert the rest of the $\sec^{n-2} x$ into $(1+\tan^2 x)^{(n-2)/2}$.

Step 2. Here $n = 4$, leftover $\sec^2 x = 1 + \tan^2 x$.

Step 3. Integral becomes $\int t^2(1+t^2)\,dt = \int (t^2+t^4)\,dt = t^3/3 + t^5/5 + C$.

Step 4. Back-substitute $t = \tan x$.

Final Answer: $\tan^3 x/3 + \tan^5 x/5 + C$.

Why this matters. This rewrite template appears in JEE Main every alternate shift. Memorise the cue: "$\sec^n x$ with n even $\Rightarrow$ peel off $\sec^2 x$, substitute $t = \tan x$".
Sanity check. Differentiate:
$t^2 \sec^2 x + t^4 \sec^2 x = (\tan^2 x + \tan^4 x)\sec^2 x = \tan^2 x(1 + \tan^2 x)\sec^2 x = \tan^2 x \sec^4 x$. ✓

Q 7.8 **Evaluate** $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}}\, dx$.

SOLUTION

Concept used. **Perfect-square identity** $1 + \sin 2x = (\sin x + \cos x)^2$.

Step 1. Use $\sin 2x = 2\sin x \cos x$ and $\sin^2 x + \cos^2 x = 1$:

$$1 + \sin 2x = \sin^2 x + \cos^2 x + 2\sin x \cos x = (\sin x + \cos x)^2.$$

Step 2. So $\sqrt{1 + \sin 2x} = |\sin x + \cos x|$.

Step 3. Integrand becomes $\frac{\sin x + \cos x}{|\sin x + \cos x|} = \pm 1$ (sign depends on interval). Taking the principal-value branch where $\sin x + \cos x > 0$: $\int 1\, dx = x + C$.

Final Answer: $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}}\, dx = x + C$ (on intervals where $\sin x + \cos x > 0$).

☞ **Half-angle identity**
Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Two perfect-square companions worth memorising:

Step 1. $1 + \sin 2x = (\sin x + \cos x)^2$.

Step 2. $1 - \sin 2x = (\sin x - \cos x)^2$.

Step 3. Both turn "$\sqrt{1 \pm \sin 2x}$" into a clean linear-trig sum.

Final Answer: $x + C$ (modulo sign branch).

Why this matters. CBSE places at least one "$\sqrt{1 \pm \sin 2x}$" question every alternate

Board paper. The whole challenge is spotting the perfect square.
Branch comment. Strictly the answer is $\pm x + C$; CBSE accepts $x + C$ as the principal antiderivative because the question implicitly assumes $\sin x + \cos x > 0$.

Q 7.9 **Evaluate** $\int \sqrt{1 + \sin x}\, dx$.

SOLUTION

Concept used. Perfect-square identity $1 + \sin x = (\sin(x/2) + \cos(x/2))^2$ (derived from the half-angle identities together with $\sin^2(x/2) + \cos^2(x/2) = 1$).

Step 1. Use $\sin x = 2\sin(x/2)\cos(x/2)$ and $1 = \sin^2(x/2) + \cos^2(x/2)$:
$1 + \sin x = (\sin(x/2) + \cos(x/2))^2$.

Step 2. So $\sqrt{1 + \sin x} = |\sin(x/2) + \cos(x/2)|$. Take principal branch:
$\int(\sin(x/2) + \cos(x/2))\, dx$.

Step 3. Integrate term-by-term: $\int \sin(x/2)\, dx = -2\cos(x/2)$;
$\int \cos(x/2)\, dx = 2\sin(x/2)$.

Step 4. Sum: $-2\cos(x/2) + 2\sin(x/2) + C = 2(\sin(x/2) - \cos(x/2)) + C$.

Final Answer: $\int \sqrt{1 + \sin x}\, dx = 2(\sin(x/2) - \cos(x/2)) + C$.

☞ Differentiate to verify

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Half-angle path.

Step 1. Rewrite the integrand as $\sin(x/2) + \cos(x/2)$.

Step 2. Each piece integrates via $\int \sin(ax)\, dx = -\cos(ax)/a$.

Step 3. Result: $-2\cos(x/2) + 2\sin(x/2) + C$.

Final Answer: $2(\sin(x/2) - \cos(x/2)) + C$.

Why this matters. The companion $\sqrt{1 - \sin x}$ similarly becomes $\sin(x/2) - \cos(x/2)$

(modulo branch). Both half-angle identities live on page 1 of every Class 12 Integrals cheat sheet.
Sanity check. Differentiate: $\cos(x/2) + \sin(x/2) = \sqrt{1 + \sin x}$ on the principal branch. ✓

Q 7.10 **Evaluate $\int \frac{x}{\sqrt{x}+1}\, dx$. (Hint: put $\sqrt{x} = z$.)**

SOLUTION

Concept used. **Substitution** to eliminate the square root: $z = \sqrt{x}$ gives $x = z^2$, $dx = 2z\, dz$.

Step 1. Substitute: integral becomes $\int \frac{z^2}{z+1} \cdot 2z\, dz = 2\int \frac{z^3}{z+1}\, dz$.

Step 2. Long-divide z^3 by $z+1$: $z^3 = (z+1)(z^2 - z + 1) - 1$. So $\frac{z^3}{z+1} = z^2 - z + 1 - \frac{1}{z+1}$.

Step 3. Integrate: $2\int \left(z^2 - z + 1 - \frac{1}{z+1}\right) dz = 2\left(\frac{z^3}{3} - \frac{z^2}{2} + z - \log|z+1|\right) + C$.

Step 4. Back-substitute $z = \sqrt{x}$:

$$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\log|\sqrt{x}+1| + C.$$

Final Answer: $\int \frac{x}{\sqrt{x}+1}\, dx = \frac{2x\sqrt{x}}{3} - x + 2\sqrt{x} - 2\log(\sqrt{x}+1) + C.$

☞ **Conjugate first**

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Why $z = \sqrt{x}$. Substitutions that turn a "square root in denominator" into a pure polynomial denominator always pay off.

Step 1. $z = \sqrt{x} \Rightarrow x = z^2,\ dx = 2z\, dz$.

Step 2. Integrand transforms to $\frac{2z^3}{z+1}$.

Step 3. Polynomial long division produces $z^2 - z + 1 - 1/(z+1)$.

Step 4. Integrate and back-substitute.

Final Answer: $\frac{2x\sqrt{x}}{3} - x + 2\sqrt{x} - 2\log(\sqrt{x}+1) + C$.

Why this matters. The "set $\sqrt{x} = z$" substitution is one of the four core S.A.-level substitutions you must own (along with $\tan x = t$, $e^x = t$, and $\sin x = t$ / $\cos x = t$).

Q 7.11 **Evaluate** $\int \sqrt{\dfrac{a+x}{a-x}}\, dx$.

SOLUTION

Concept used. Rationalising the radical by multiplying numerator and denominator inside the square root by $\sqrt{a+x}$, plus the standard inverse-sine/$\sqrt{a^2-x^2}$ integrals.

Step 1. Multiply inside the radical: $\sqrt{\dfrac{a+x}{a-x}} = \dfrac{a+x}{\sqrt{a^2-x^2}}$.

Step 2. Split the integral: $\int \dfrac{a+x}{\sqrt{a^2-x^2}}\, dx = a\int \dfrac{dx}{\sqrt{a^2-x^2}} + \int \dfrac{x\, dx}{\sqrt{a^2-x^2}}$.

Step 3. Standard results: $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$. For the second, put $u = a^2 - x^2$, $du = -2x\, dx$: $\int \dfrac{x\, dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$.

Step 4. Combine: $a\sin^{-1}(x/a) - \sqrt{a^2-x^2} + C$.

Final Answer: $\int \sqrt{\dfrac{a+x}{a-x}}\, dx = a\sin^{-1}\dfrac{x}{a} - \sqrt{a^2-x^2} + C$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Rationalisation pattern. Whenever you see $\sqrt{(a+x)/(a-x)}$ or $\sqrt{(a-x)/(a+x)}$, multiply numerator and denominator (inside) by $\sqrt{a+x}$ or $\sqrt{a-x}$ respectively; you always land in $\sqrt{a^2-x^2}$ territory.

Step 1. Rationalise to $\dfrac{a+x}{\sqrt{a^2-x^2}}$.

Step 2. Split, use $\int dx/\sqrt{a^2-x^2} = \sin^{-1}(x/a)$ and the substitution $u = a^2 - x^2$.

> **Final Answer:** $a\sin^{-1}(x/a) - \sqrt{a^2-x^2} + C$.

Why this matters. The companion formula $\int \sqrt{(a-x)/(a+x)}\,dx = a\sin^{-1}(x/a) + \sqrt{a^2-x^2} + C$ uses the same rationalisation. Both appear in NCERT examples and CBSE Board.

Q 7.12 **Evaluate** $\displaystyle\int \frac{x^{1/2}}{1+x^{3/4}}\,dx$. **(Hint: put** $x = z^4$.**)**

SOLUTION

Concept used. **Substitution** to clear all fractional powers simultaneously. Choose the exponent equal to the LCM of the fractional powers: $\operatorname{lcm}(2,4) = 4$, so $x = z^4$.

Step 1. Substitute: $x = z^4$, $dx = 4z^3\,dz$, $x^{1/2} = z^2$, $x^{3/4} = z^3$. Integrand becomes $\dfrac{z^2}{1+z^3}\cdot 4z^3\,dz = \dfrac{4z^5}{1+z^3}\,dz$.

Step 2. Long-divide $4z^5$ by $1+z^3$: $4z^5 = (1+z^3)\cdot 4z^2 - 4z^2$. So $\dfrac{4z^5}{1+z^3} = 4z^2 - \dfrac{4z^2}{1+z^3}$.

Step 3. For $\int \dfrac{4z^2\,dz}{1+z^3}$: numerator $= \frac{4}{3}\cdot 3z^2 = \frac{4}{3}\cdot\dfrac{d}{dz}(1+z^3)$, so this integrates to $\frac{4}{3}\log|1+z^3|$.

Step 4. Combine: $\int \dfrac{4z^5\,dz}{1+z^3} = \dfrac{4z^3}{3} - \dfrac{4}{3}\log|1+z^3| + C$.

Step 5. Back-substitute $z = x^{1/4}$: $\dfrac{4x^{3/4}}{3} - \dfrac{4}{3}\log|1+x^{3/4}| + C$.

> **Final Answer:** $\displaystyle\int \frac{x^{1/2}}{1+x^{3/4}}\,dx = \frac{4}{3}x^{3/4} - \frac{4}{3}\log|1+x^{3/4}| + C$.

☞ **Log domain**

$\ln x$ is defined and continuous only for $x > 0$. Before using $\ln$ on an expression, confirm that the argument is strictly positive on the relevant interval.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

LCM-of-denominators rule. For $\int R(x^{p/q}, x^{r/s})\,dx$ with R rational, put $x = z^{\text{lcm}(q,s)}$ to clear all roots in one stroke.

Step 1. Here $p/q = 1/2, r/s = 3/4 \Rightarrow \text{lcm} = 4 \Rightarrow x = z^4$.

Step 2. Integrand becomes a rational function of z, integrable by long division + log.

> **Final Answer:** $\frac{4}{3}x^{3/4} - \frac{4}{3}\log(1 + x^{3/4}) + C$.

Why this matters. JEE Main loves the LCM-substitution trick. Without it, you cannot integrate any rational function with mixed fractional powers.

Q 7.13 **Evaluate** $\int \frac{\sqrt{1+x^2}}{x^4}\,dx$.

SOLUTION

Concept used. **Substitution** $x = 1/t$, classic trick for integrands with high inverse powers of x.

Step 1. Put $x = 1/t$, $dx = -dt/t^2$. Then $x^4 = 1/t^4$ and
$\sqrt{1+x^2} = \sqrt{1+1/t^2} = \sqrt{(t^2+1)/t^2} = \sqrt{t^2+1}/|t|$.

Step 2. Integrand $\cdot dx$:
$\frac{\sqrt{t^2+1}/|t|}{1/t^4} \cdot \left(-\frac{dt}{t^2}\right) = -\frac{t^4\sqrt{t^2+1}}{|t|} \cdot \frac{dt}{t^2} = -t^2\sqrt{t^2+1} \cdot \text{sgn}(t)/t \cdot dt$. Taking $t > 0$ for the principal branch: $= -t\sqrt{1+t^2}\,dt$.

Step 3. Put $u = 1 + t^2$, $du = 2t\,dt$:
$\int -t\sqrt{1+t^2}\,dt = -\frac{1}{2}\int \sqrt{u}\,du = -\frac{1}{2} \cdot \frac{2}{3}u^{3/2} = -\frac{(1+t^2)^{3/2}}{3}$.

Step 4. Back-substitute $t = 1/x$: $1 + t^2 = 1 + 1/x^2 = (x^2+1)/x^2$, so
$(1+t^2)^{3/2} = (x^2+1)^{3/2}/|x|^3$. Hence integral
$= -(x^2+1)^{3/2}/(3|x|^3) + C = -(1+x^2)^{3/2}/(3x^3) + C$ (taking $x > 0$).

> **Final Answer:** $\int \frac{\sqrt{1+x^2}}{x^4}\,dx = -\frac{(1+x^2)^{3/2}}{3x^3} + C$.

✗ Chain rule layers

Differentiate the outermost function first, then multiply by the derivative of the inner function, then by the inner-inner, and so on. Stop only when you hit x or a constant —

truncating early is the textbook chain-rule slip.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Reciprocal substitution. Whenever you see a high inverse power of x in the denominator (like x^4 here) and a square root of a quadratic in the numerator, put $x = 1/t$ to flip the relative powers.

Step 1. $x = 1/t \Rightarrow dx = -dt/t^2$, $\sqrt{1+x^2} = \sqrt{1+t^2}/t$.

Step 2. Integrand condenses to $-t\sqrt{1+t^2}\,dt$.

Step 3. Easy substitution $u = 1 + t^2$ finishes the job.

Final Answer: $-\dfrac{(1+x^2)^{3/2}}{3x^3} + C$.

Why this matters. The reciprocal substitution is a top-3 trick in any Class 12 + JEE Maths toolbox. The cue is "x^n in the denominator with $n \geq 3$".
Sanity check. Differentiate $F(x) = -(1+x^2)^{3/2}/(3x^3)$. Quotient + chain rule gives $F'(x) = \sqrt{1+x^2}/x^4$ after simplification. ✓

Q 7.14 **Evaluate** $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}}$.

SOLUTION

Concept used. Standard integral $\int \dfrac{dx}{\sqrt{a^2-u^2}} = \sin^{-1}(u/a) + C$ with a linear substitution $u = 3x$.

Step 1. Rewrite: $16 - 9x^2 = 4^2 - (3x)^2$.

Step 2. Put $u = 3x$, $du = 3\,dx$: $\int \dfrac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\int \dfrac{du}{\sqrt{4^2-u^2}} = \frac{1}{3}\sin^{-1}(u/4) + C$.

Step 3. Back-substitute: $\frac{1}{3}\sin^{-1}(3x/4) + C$.

Final Answer: $\displaystyle\int \frac{dx}{\sqrt{16-9x^2}} = \frac{1}{3}\sin^{-1}\frac{3x}{4} + C$.

☞ **Conjugate first**

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x

cancels cleanly.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Pattern match. $\int dx/\sqrt{a^2 - b^2x^2} = \frac{1}{b}\sin^{-1}(bx/a) + C$.

Step 1. $a = 4, b = 3 \Rightarrow$ answer $= \frac{1}{3}\sin^{-1}(3x/4) + C$.

> **Final Answer:** $\frac{1}{3}\sin^{-1}(3x/4) + C$.

Why this matters. The whole family of special integrals reduces to template-matching once you spot a and b.

Sanity check. Differentiate:

$$\frac{1}{3} \cdot \frac{3/4}{\sqrt{1-(3x/4)^2}} = \frac{1/4}{\sqrt{1-9x^2/16}} = \frac{1/4}{\frac{1}{4}\sqrt{16-9x^2}} = \frac{1}{\sqrt{16-9x^2}}. \checkmark$$

Q 7.15 Evaluate $\displaystyle\int \frac{dt}{\sqrt{3t-2t^2}}$.

SOLUTION

Concept used. **Completing the square** inside the radical, then match to $\int du/\sqrt{a^2-u^2} = \sin^{-1}(u/a)$.

Step 1. Complete the square:

$$3t - 2t^2 = -2(t^2 - \tfrac{3}{2}t) = -2\left[(t-\tfrac{3}{4})^2 - \tfrac{9}{16}\right] = \tfrac{9}{8} - 2(t-\tfrac{3}{4})^2.$$

Step 2. So $\sqrt{3t-2t^2} = \sqrt{\frac{9}{8} - 2(t-\frac{3}{4})^2} = \sqrt{2}\sqrt{\frac{9}{16} - (t-\frac{3}{4})^2}$.

Step 3. Integral: $\frac{1}{\sqrt{2}}\int \dfrac{dt}{\sqrt{(3/4)^2 - (t-3/4)^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\dfrac{t-3/4}{3/4} + C$.

Step 4. Simplify: $\frac{1}{\sqrt{2}}\sin^{-1}\dfrac{4t-3}{3} + C$.

> **Final Answer:** $\displaystyle\int \frac{dt}{\sqrt{3t-2t^2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{4t-3}{3} + C$.

☞ **Completing the square in quadratics under a root**

For $\sqrt{ax^2+bx+c}$ with $a < 0$ (so the radicand opens downward), always complete the square to write it as $\sqrt{R^2-(x-h)^2}$ (up to a constant factor), then use $\sin^{-1}$.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Mechanical recipe.

Step 1. Factor out the coefficient of t^2: $3t - 2t^2 = -2(t^2 - \frac{3}{2}t)$.

Step 2. Complete inside the bracket: $t^2 - \frac{3}{2}t = (t - \frac{3}{4})^2 - \frac{9}{16}$.

Step 3. Distribute: $3t - 2t^2 = \frac{9}{8} - 2(t - \frac{3}{4})^2$.

Step 4. Match $\int du/\sqrt{a^2 - u^2}$ template.

> **Final Answer:** $\frac{1}{\sqrt{2}} \sin^{-1}((4t-3)/3) + C$.

Why this matters. Every "$\sqrt{\text{quadratic}}$" in the denominator on a CBSE/JEE question reduces to one of the three special-integral templates via completion of square. Get fluent at it.

Q 7.16 **Evaluate** $\displaystyle\int \frac{3x-1}{x^2+9}\,dx$.

SOLUTION

Concept used. Split the numerator into a part proportional to the derivative of the denominator plus a constant, then use $\int g'/g$ and $\int dx/(x^2+a^2)$.

Step 1. $\frac{d}{dx}(x^2+9) = 2x$. Write $3x = \frac{3}{2} \cdot 2x$. So $\displaystyle\frac{3x-1}{x^2+9} = \frac{3}{2} \cdot \frac{2x}{x^2+9} - \frac{1}{x^2+9}$.

Step 2. Integrate: $\frac{3}{2} \log|x^2+9| - \frac{1}{3}\tan^{-1}(x/3) + C$.

> **Final Answer:** $\displaystyle\int \frac{3x-1}{x^2+9}\,dx = \frac{3}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}\frac{x}{3} + C.$

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Numerator-splitting template. For $\int (Ax+B)/(x^2+a^2)\,dx$: split numerator as $A/2$ times $2x$ (the derivative of x^2+a^2) plus B.

Step 1. Here $A = 3, B = -1, a^2 = 9$.

Step 2. Log-piece: $\frac{A}{2}\log(x^2+a^2) = \frac{3}{2}\log(x^2+9)$.

Step 3. Arctan-piece: $\frac{B}{a}\tan^{-1}(x/a) = -\frac{1}{3}\tan^{-1}(x/3)$.

> **Final Answer:** $\frac{3}{2}\log(x^2+9) - \frac{1}{3}\tan^{-1}(x/3) + C$.

Why this matters. This split is a 3-mark CBSE template, recycled every 2 years.

Sanity check. Differentiate: $\frac{3}{2}\cdot\dfrac{2x}{x^2+9} - \frac{1}{3}\cdot\dfrac{1/3}{1+x^2/9} = \dfrac{3x}{x^2+9} - \dfrac{1}{x^2+9} = \dfrac{3x-1}{x^2+9}$. ✓

Q 7.17 **Evaluate** $\displaystyle\int \sqrt{5-2x+x^2}\,dx$.

SOLUTION

Concept used. Completing the square inside the radical, then the standard formula

$$\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2} + \frac{a^2}{2}\log\left|u+\sqrt{u^2+a^2}\right| + C.$$

Step 1. Complete the square: $x^2-2x+5 = (x-1)^2+4 = (x-1)^2+2^2$.

Step 2. Substitute $u = x-1$, $du = dx$, $a = 2$: $\displaystyle\int \sqrt{u^2+4}\,du$.

Step 3. Apply the standard formula:

$$\int \sqrt{u^2+4}\,du = \frac{u}{2}\sqrt{u^2+4} + \frac{4}{2}\log\left|u+\sqrt{u^2+4}\right| + C$$
$$= \frac{u}{2}\sqrt{u^2+4} + 2\log\left|u+\sqrt{u^2+4}\right| + C.$$

Step 4. Back-substitute $u = x-1$ and $u^2+4 = x^2-2x+5$:

$$\frac{x-1}{2}\sqrt{x^2-2x+5} + 2\log\left|x-1+\sqrt{x^2-2x+5}\right| + C.$$

> **Final Answer:** $\displaystyle\int \sqrt{5-2x+x^2}\,dx = \frac{x-1}{2}\sqrt{x^2-2x+5} + 2\log\left|x-1+\sqrt{x^2-2x+5}\right| + C.$

☞ Differentiate to verify

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Mechanical recipe.

Step 1. Complete the square inside the radical to obtain $(x-1)^2+4$.

Step 2. Match the form $\sqrt{u^2+a^2}$ with $u=x-1$, $a=2$.

Step 3. Use $\int \sqrt{u^2+a^2}\,du = \frac{u}{2}\sqrt{u^2+a^2}+\frac{a^2}{2}\log\left|u+\sqrt{u^2+a^2}\right|+C$.

Step 4. Back-substitute $u=x-1$.

Final Answer: $\frac{x-1}{2}\sqrt{x^2-2x+5}+2\log\left|x-1+\sqrt{x^2-2x+5}\right|+C.$

Why this matters. The three radical templates $\sqrt{a^2-u^2}$, $\sqrt{u^2-a^2}$, $\sqrt{u^2+a^2}$ are all NCERT-listed standard integrals. Recognise which one falls out after completing the square, and the answer is one substitution away.

Q 7.18 Evaluate $\int \frac{x}{x^4-1}\,dx$.

SOLUTION

Concept used. Substitution $t=x^2$ to convert x^4-1 to t^2-1 and use $\int dt/(t^2-a^2)=\frac{1}{2a}\log|(t-a)/(t+a)|$.

Step 1. Put $t=x^2$, $dt=2x\,dx$, so $x\,dx=dt/2$. Integral: $\int \frac{x\,dx}{x^4-1}=\frac{1}{2}\int\frac{dt}{t^2-1}$.

Step 2. Standard: $\int\frac{dt}{t^2-1}=\frac{1}{2}\log\left|\frac{t-1}{t+1}\right|$.

Step 3. So integral $=\frac{1}{4}\log\left|\frac{t-1}{t+1}\right|=\frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right|+C$.

Final Answer: $\int\frac{x}{x^4-1}\,dx=\frac{1}{4}\log\left|\frac{x^2-1}{x^2+1}\right|+C.$

☞ **Parametric derivative**

For parametric curves $(x(t),y(t))$ use $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Quartic-to-quadratic trick. Any $\int (x \cdot \text{stuff}(x^2))\,dx$ becomes a t-integral via $t = x^2$.

Step 1. $t = x^2 \Rightarrow x\,dx = dt/2$.

Step 2. Convert: $\int dt/(2(t^2-1))$.

Step 3. Use $1/(t^2-1) = \frac{1}{2}[1/(t-1) - 1/(t+1)]$.

> **Final Answer:** $\frac{1}{4}\log|(x^2-1)/(x^2+1)| + C$.

Why this matters. Hidden $t = x^2$ substitutions appear in roughly one shift question per JEE Main session. The cue is "odd power of x outside, even power inside".

Q 7.19 **Evaluate $\int \frac{x^2}{1-x^4}\,dx$. (Put $x^2 = t$.)**

SOLUTION

Concept used. The hint $x^2 = t$ converts the integrand to a rational function of t. Combined with partial fractions of $t/(1-t^2)$, the integral reduces to standard logs.

Step 1. NB: the hint $x^2 = t$ would require $2x\,dx = dt$, so $dx = dt/(2x) = dt/(2\sqrt{t})$. Replace: $\int \frac{t}{1-t^2}\cdot\frac{dt}{2\sqrt{t}} = \frac{1}{2}\int \frac{\sqrt{t}}{1-t^2}\,dt$. That's still hard. The cleaner path is partial fractions in x.

Step 2. Factor: $1-x^4 = (1-x^2)(1+x^2) = (1-x)(1+x)(1+x^2)$. Write $\frac{x^2}{1-x^4} = \frac{1}{2}\left[\frac{1}{1-x^2} - \frac{1}{1+x^2}\right]$ (verify: $\frac{1}{2}[(1+x^2)-(1-x^2)]/[(1-x^2)(1+x^2)] = x^2/(1-x^4)$. ✓)

Step 3. Integrate each piece: $\int \frac{dx}{1-x^2} = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|$; $\int \frac{dx}{1+x^2} = \tan^{-1}x$.

Step 4. Combine: $\frac{1}{2}\cdot\frac{1}{2}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C$.

> **Final Answer:** $\int \frac{x^2}{1-x^4}\,dx = \frac{1}{4}\log\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\tan^{-1}x + C.$

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Partial-fraction approach.

Step 1. $x^2/(1-x^4) = \frac{1}{2}\cdot 1/(1-x^2) - \frac{1}{2}\cdot 1/(1+x^2)$.

Step 2. Each piece is a standard log / arctan.

Step 3. Combine and simplify.

> **Final Answer:** $\frac{1}{4}\log|(1+x)/(1-x)| - \frac{1}{2}\tan^{-1}x + C$.

Why this matters. Mixed real-pole + complex-pole partial fractions (here: $1-x^2$ has real roots, $1+x^2$ has complex roots) produce a log + arctan answer. Memorise the mix.
Generalising the move. Conjugate rationalisation works because $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A-B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

Q 7.20 **Evaluate** $\displaystyle\int \sqrt{2ax - x^2}\,dx$.

SOLUTION

Concept used. Completing the square inside the radical, then standard $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}(u/a) + C$.

Step 1. $2ax - x^2 = -(x^2-2ax) = -((x-a)^2 - a^2) = a^2 - (x-a)^2$.

Step 2. Put $u = x-a$: integral $= \int \sqrt{a^2-u^2}\,du$.

Step 3. Apply standard: $= \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}(u/a) + C$.

Step 4. Back-substitute: $= \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\dfrac{x-a}{a} + C$.

> **Final Answer:** $\displaystyle\int \sqrt{2ax-x^2}\,dx = \frac{x-a}{2}\sqrt{2ax-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x-a}{a} + C$.

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A}\pm\sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A}\mp\sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Standard library. The integral $\int \sqrt{a^2 - u^2}\, du$ is one of three NCERT-listed templates (the other two are $\int \sqrt{u^2 + a^2}$ and $\int \sqrt{u^2 - a^2}$). All three have the same shape $\frac{u}{2}\sqrt{\cdot} + \frac{a^2}{2} \cdot$ (inverse-trig or log).

Step 1. Complete square in $2ax - x^2$.

Step 2. Apply standard.

Final Answer: $\frac{x-a}{2}\sqrt{2ax - x^2} + \frac{a^2}{2}\sin^{-1}((x-a)/a) + C$.

Why this matters. The three "$\sqrt{\text{quadratic}}\, dx$" templates appear in every CBSE Board paper as a 5-marker. Memorise them along with their completing-square setups.

Q 7.21 Evaluate $\displaystyle\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx$.

SOLUTION

Concept used. Trigonometric substitution $x = \sin\theta$. Then $\sin^{-1} x = \theta$, $\sqrt{1-x^2} = \cos\theta$, $dx = \cos\theta\, d\theta$.

Step 1. Sub: integrand becomes $\dfrac{\theta}{\cos^3\theta} \cdot \cos\theta\, d\theta = \dfrac{\theta}{\cos^2\theta}\, d\theta = \theta \sec^2\theta\, d\theta$.

Step 2. Integration by parts: $u = \theta$, $dv = \sec^2\theta\, d\theta \Rightarrow du = d\theta$, $v = \tan\theta$.
$\int \theta \sec^2\theta\, d\theta = \theta\tan\theta - \int \tan\theta\, d\theta = \theta\tan\theta + \log|\cos\theta|$.

Step 3. Back-substitute: $\theta = \sin^{-1} x$, $\tan\theta = x/\sqrt{1-x^2}$, $\cos\theta = \sqrt{1-x^2}$.
$= \dfrac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2}\log(1-x^2) + C$.

Final Answer: $\displaystyle\int \frac{\sin^{-1} x}{(1-x^2)^{3/2}}\, dx = \frac{x\sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2}\log(1-x^2) + C$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Why $x = \sin\theta$ here. The $\sin^{-1} x$ becomes a clean θ; the $(1-x^2)^{3/2}$ becomes $\cos^3\theta$. The leftover is $\theta\sec^2\theta$, which is a textbook by-parts.

Step 1. Sub $x = \sin\theta$.

Step 2. By parts on $\theta\sec^2\theta$.

Step 3. Back-substitute.

Final Answer: $\dfrac{x\sin^{-1}x}{\sqrt{1-x^2}} + \frac{1}{2}\log(1-x^2) + C$.

Why this matters. Trig substitution + by parts is a top-3 combo for "inverse-trig in numerator, $(1-x^2)^k$ in denominator" problems.

Q 7.22 **Evaluate** $\displaystyle\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx$.

SOLUTION

Concept used. **Sum-to-product** formulas and the identity $\sin 3A - \sin A = 2\cos 2A \sin A$ etc. The key manipulation: multiply numerator and denominator by $\sin(3x/2)$ or use the symmetric sum-to-product.

Step 1. Use $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$: $\cos 5x + \cos 4x = 2\cos(9x/2)\cos(x/2)$.

Step 2. For the denominator, use $1 - 2\cos 3x = -[2\cos 3x - 1]$. Recall $2\cos A - 1 = -2(2\sin^2(A/2) - \sin A)/(\text{stuff})$. A cleaner path is: rewrite $1 - 2\cos 3x = -\dfrac{2\cos 3x \cdot \sin(3x/2) - \sin(3x/2)}{\sin(3x/2)}$.
Cleanest path: factor numerator and denominator using $\cos kx = \frac{1}{2}\sin(3x/2)/\sin(3x/2) \cdot 2\cos kx$. Use the product expansion: $\dfrac{\cos 5x + \cos 4x}{1 - 2\cos 3x}$ simplifies to $-(\cos 2x + \cos x)$, which can be verified by multiplying both sides by $1 - 2\cos 3x$ and applying product-to-sum.

Step 3. Integrate the simplified form: $\int -(\cos 2x + \cos x)\,dx = -\frac{\sin 2x}{2} - \sin x + C$.

Final Answer: $\displaystyle\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}\,dx = -\frac{\sin 2x}{2} - \sin x + C$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Verification of the algebraic simplification. Use the identity $(1-2\cos 3x)(\cos 2x+\cos x) = -(\cos 5x+\cos 4x)$, which follows from product-to-sum: $2\cos 3x\cdot\cos 2x = \cos 5x+\cos x$; $2\cos 3x\cdot\cos x = \cos 4x+\cos 2x$. Summing: $2\cos 3x(\cos 2x+\cos x) = \cos 5x+\cos 4x+\cos 2x+\cos x$. Hence $(1-2\cos 3x)(\cos 2x+\cos x) = (\cos 2x+\cos x)-2\cos 3x(\cos 2x+\cos x) =$ $(\cos 2x+\cos x)-(\cos 5x+\cos 4x+\cos 2x+\cos x) = -(\cos 5x+\cos 4x)$. ✓

Step 1. So $\dfrac{\cos 5x+\cos 4x}{1-2\cos 3x} = -(\cos 2x+\cos x)$.

Step 2. Integrate: $-\frac{1}{2}\sin 2x-\sin x+C$.

> **Final Answer:** $-\frac{1}{2}\sin 2x-\sin x+C$.

Why this matters. Product-to-sum + sum-to-product identities turn "mess of cosines" integrands into clean polynomials in $\sin kx, \cos kx$.

Always test the integrand. Before integrating over a symmetric interval $[-a,a]$, compute $f(-x)$ and compare with $f(x)$. If f is odd, the answer is 0 with no further work; if even, fold the integral to $2\int_0^a f$. Missing this short-cut is the most common waste of exam time on definite integrals.

Q 7.23 **Evaluate** $\displaystyle\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}\,dx$.

SOLUTION

Concept used. **Algebraic identity** $\sin^6 x+\cos^6 x = 1-3\sin^2 x\cos^2 x$, then split.

Step 1. $\sin^6 x+\cos^6 x = (\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^2 x(\sin^2 x+\cos^2 x) =$ $1-3\sin^2 x\cos^2 x$.

Step 2. Integrand: $\dfrac{1-3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x} = \dfrac{1}{\sin^2 x\cos^2 x}-3$.

Step 3. Simplify $\dfrac{1}{\sin^2 x\cos^2 x} = \dfrac{4}{\sin^2 2x} = 4\csc^2 2x$.

Step 4. Integrate: $\int 4\csc^2 2x\,dx-\int 3\,dx = -2\cot 2x-3x+C$.

> **Final Answer:** $\displaystyle\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cos^2 x}\,dx = -2\cot 2x-3x+C$.

☞ **Differentiate to verify**

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Identity cue. Whenever you see $\sin^6 + \cos^6$, recall the cube-sum identity $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$ applied with $a = \sin^2 x, b = \cos^2 x$.

Step 1. Reduce numerator: $1 - 3\sin^2 x \cos^2 x$.

Step 2. Split into $\csc^2 2x - 3$ form.

Step 3. Integrate.

Final Answer: $-2\cot 2x - 3x + C$.

Why this matters. The companion identity $\sin^4 x + \cos^4 x = 1 - 2\sin^2 x \cos^2 x$ is its quadratic sibling. Both appear in JEE Main multiple times per session.

Q 7.24 **Evaluate $\int \frac{x}{a^3 - x^3}\,dx$. (No hint, but a substitution $u = a^3 - x^3$ does not directly fit because of the x in the numerator vs $3x^2$ needed.)**

SOLUTION

Concept used. Partial fractions on $\frac{x}{(a-x)(a^2+ax+x^2)}$.

Step 1. Factor: $a^3 - x^3 = (a-x)(a^2+ax+x^2)$. Set

$$\frac{x}{(a-x)(a^2+ax+x^2)} = \frac{A}{a-x} + \frac{Bx+C}{a^2+ax+x^2}.$$

Step 2. Multiply out: $x = A(a^2+ax+x^2) + (Bx+C)(a-x)$. At $x = a$: $a = A \cdot 3a^2 \Rightarrow A = 1/(3a)$. Match x^2 coefficient: $0 = A - B \Rightarrow B = A = 1/(3a)$. Match constant: $0 = Aa^2 + Ca \Rightarrow C = -Aa = -1/3$.

Step 3. Integrate piece 1: $\frac{1}{3a}\int \frac{dx}{a-x} = -\frac{1}{3a}\log|a-x|$.

Step 4. Piece 2: $\int \frac{(x/(3a)) - 1/3}{a^2+ax+x^2}\,dx = \frac{1}{3a}\int \frac{x-a}{a^2+ax+x^2}\,dx$. Write $x - a = \frac{1}{2}(2x+a) - \frac{3a}{2}$. The piece $(2x+a)$ is the derivative of a^2+ax+x^2. So this splits into a log and an arctan after completing the square: $a^2+ax+x^2 = (x+a/2)^2 + 3a^2/4$.

Step 5. After algebra: $\frac{1}{6a}\log(a^2+ax+x^2) - \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x+a}{a\sqrt{3}}$.

Step 6. Combine: $-\frac{1}{3a}\log|a-x| + \frac{1}{6a}\log(a^2+ax+x^2) - \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x+a}{a\sqrt{3}} + C.$

Final Answer: $\int \frac{x}{a^3-x^3}\,dx = \frac{1}{6a}\log\frac{a^2+ax+x^2}{(a-x)^2} - \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x+a}{a\sqrt{3}} + C.$

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A}\pm\sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A}\mp\sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Partial-fraction recipe for cubic denominators. Factor the cubic; if it has one real linear factor and one irreducible quadratic, use $A/(linear) + (Bx+C)/(quadratic)$.

Step 1. Factor $a^3 - x^3$.

Step 2. Solve for A, B, C.

Step 3. Integrate the linear piece (log) and the quadratic piece (log + arctan after completing the square).

Final Answer: $\frac{1}{6a}\log\frac{a^2+ax+x^2}{(a-x)^2} - \frac{1}{a\sqrt{3}}\tan^{-1}\frac{2x+a}{a\sqrt{3}} + C.$

Why this matters. Cubic-denominator partial fractions are a 5-mark CBSE staple. The work splits cleanly into linear + quadratic pieces.

Q 7.25 Evaluate $\int \frac{\cos x - \cos 2x}{1-\cos x}\,dx.$

SOLUTION

Concept used. **Double-angle identity** $\cos 2x = 2\cos^2 x - 1$, plus algebraic simplification.

Step 1. $\cos x - \cos 2x = \cos x - (2\cos^2 x - 1) = \cos x - 2\cos^2 x + 1 = -(2\cos^2 x - \cos x - 1) = -(2\cos x + 1)(\cos x - 1).$

Step 2. So $\frac{\cos x - \cos 2x}{1-\cos x} = \frac{-(2\cos x+1)(\cos x - 1)}{1-\cos x} = \frac{(2\cos x+1)(1-\cos x)}{1-\cos x} = 2\cos x + 1.$

Step 3. Integrate: $\int(2\cos x+1)\,dx = 2\sin x + x + C$.

Final Answer: $\displaystyle\int \frac{\cos x-\cos 2x}{1-\cos x}\,dx = 2\sin x + x + C.$

☞ Half-angle identity

Memorise $1-\cos\theta = 2\sin^2(\theta/2)$. It converts every $1-\cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Algebraic simplification first. Always check whether the integrand factors before applying heavy machinery.

Step 1. Use $\cos 2x = 2\cos^2 x - 1$ to write the numerator as a quadratic in $\cos x$.

Step 2. Factor and cancel.

Step 3. Integrate term-by-term.

Final Answer: $2\sin x + x + C$.

Why this matters. Catching the factorable form saves ~ 5 minutes versus blindly attacking with half-angle or by-parts.
Sanity check. Differentiate: $2\cos x + 1$. Recover the simplified integrand. ✓

Q 7.26 **Evaluate** $\displaystyle\int \frac{dx}{x\sqrt{x^4-1}}$. **(Hint: put** $x^2=\sec\theta$.**)**

SOLUTION

Concept used. Trig substitution $x^2=\sec\theta$, so $2x\,dx=\sec\theta\tan\theta\,d\theta$, hence $x\,dx=\frac{1}{2}\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^4-1}=\sqrt{\sec^2\theta-1}=\tan\theta$.

Step 1. Rewrite integrand: $\dfrac{dx}{x\sqrt{x^4-1}} = \dfrac{1}{x^2\sqrt{x^4-1}}\cdot x\,dx$.

Step 2. Substitute: $\dfrac{1}{\sec\theta\cdot\tan\theta}\cdot\frac{1}{2}\sec\theta\tan\theta\,d\theta = \frac{1}{2}\,d\theta$.

Step 3. Integrate: $\frac{1}{2}\theta + C$.

Step 4. Back-substitute: $\theta=\sec^{-1}(x^2)$, so $\displaystyle\int\frac{dx}{x\sqrt{x^4-1}} = \frac{1}{2}\sec^{-1}(x^2)+C$.

Final Answer: $\int \frac{dx}{x\sqrt{x^4-1}} = \frac{1}{2}\sec^{-1}(x^2) + C.$

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Trig-sub mechanics.

Step 1. $x^2 = \sec\theta \Rightarrow 2x\,dx = \sec\theta\tan\theta\,d\theta.$

Step 2. $\sqrt{x^4-1} = \tan\theta.$

Step 3. Integrand simplifies to $d\theta/2$.

Final Answer: $\frac{1}{2}\sec^{-1}(x^2) + C.$

Why this matters. The cue "$\sqrt{x^4-1}$ or $\sqrt{x^{2n}-1}$" demands a $\sec\theta$ substitution on the inner variable x^n.

Sanity check. Differentiate: $\frac{1}{2}\cdot\frac{2x}{x^2\sqrt{x^4-1}} = \frac{1}{x\sqrt{x^4-1}}$. ✓

Q 7.27 **Evaluate $\int_0^2 (x^2+3)\,dx$ as a limit of sums.**

SOLUTION

Concept used. **Definite integral as limit of a sum.** For a continuous f on $[a,b]$: $\int_a^b f(x)\,dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$, where $h=(b-a)/n$.

Step 1. Here $a=0, b=2, h=2/n, x_r = 0+rh = 2r/n$.
$f(x_r) = (2r/n)^2 + 3 = 4r^2/n^2 + 3.$

Step 2. Sum: $h\sum_{r=0}^{n-1} f(x_r) = \frac{2}{n}\sum_{r=0}^{n-1}\left[\frac{4r^2}{n^2}+3\right] = \frac{8}{n^3}\sum_{r=0}^{n-1} r^2 + \frac{6}{n}\cdot n = \frac{8}{n^3}\cdot\frac{(n-1)n(2n-1)}{6} + 6.$

Step 3. Simplify: $\frac{8(n-1)(2n-1)}{6n^2} + 6 = \frac{4(n-1)(2n-1)}{3n^2} + 6.$

Step 4. As $n \to \infty$: $\dfrac{(n-1)(2n-1)}{n^2} \to 2$, so $\dfrac{4 \cdot 2}{3} + 6 = \dfrac{8}{3} + 6 = \dfrac{26}{3}$.

Step 5. Cross-check by FTC: $\int_0^2 (x^2+3)\,dx = [x^3/3 + 3x]_0^2 = 8/3 + 6 = 26/3$. ✓

> **Final Answer:** $\displaystyle\int_0^2 (x^2+3)\,dx = \frac{26}{3}$.

✗ Limits get substituted

When you substitute $u = g(x)$ inside a definite integral, the new limits are $u = g(a)$ and $u = g(b)$, NOT a and b. Forgetting to change the limits is the most-marked error in board scripts.

EXPERT'S SOLUTION *: Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Mechanical recipe for limit-of-sums.

Step 1. Set $h = (b-a)/n$, $x_r = a + rh$.

Step 2. Plug into f, sum from $r = 0$ to $n-1$, multiply by h.

Step 3. Use $\sum r = n(n-1)/2$, $\sum r^2 = n(n-1)(2n-1)/6$, $\sum r^3 = [n(n-1)/2]^2$.

Step 4. Take limit $n \to \infty$.

> **Final Answer:** $26/3$.

Why this matters. CBSE places a "limit of sum" question every Board paper. The setup is mechanical; only the algebra varies.

Q 7.28 Evaluate $\displaystyle\int_0^2 e^x\,dx$ as a limit of sums.

SOLUTION

Concept used. Limit of sum with a geometric progression: when f is exponential, the sum $\sum e^{rh}$ is a GP whose closed form lets us take $n \to \infty$.

Step 1. $h = 2/n, x_r = rh$. $f(x_r) = e^{rh}$.

Step 2. Sum: $h\sum_{r=0}^{n-1} e^{rh} = h \cdot \dfrac{e^{nh}-1}{e^h-1} = h \cdot \dfrac{e^2-1}{e^h-1}$ (since $nh = 2$).

Step 3. As $n \to \infty$, $h \to 0$ and $\dfrac{h}{e^h-1} \to 1$ (standard limit).

Step 4. Hence limit $= (e^2 - 1) \cdot 1 = e^2 - 1$.

Step 5. Cross-check: $\int_0^2 e^x\,dx = [e^x]_0^2 = e^2 - 1$. ✓

Final Answer: $\displaystyle\int_0^2 e^x\,dx = e^2 - 1$.

☞ **Standard limit** $\lim_{h\to 0}(e^h - 1)/h = 1$

This limit is the bridge between limit-of-sums for exponentials and the FTC value. Memorise it.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

GP collapse.

Step 1. Sum is a finite GP with ratio e^h.

Step 2. Closed form: $h(e^2 - 1)/(e^h - 1)$.

Step 3. Limit: $e^2 - 1$.

Final Answer: $e^2 - 1$.

Why this matters. Whenever the integrand is e^{kx} or a^x, expect a GP closed-form. Polynomial integrands lead to power-sum closed forms.

Q 7.29 **Evaluate** $\displaystyle\int_0^1 \frac{dx}{e^x + e^{-x}}$.

SOLUTION

Concept used. **Multiply numerator and denominator by e^x**, then standard arctan substitution.

Step 1. $\dfrac{1}{e^x + e^{-x}} = \dfrac{e^x}{e^{2x} + 1}$.

Step 2. Put $t = e^x$, $dt = e^x dx$: $\int \dfrac{dt}{t^2 + 1} = \tan^{-1} t$.

Step 3. Limits: $x = 0 \to t = 1$; $x = 1 \to t = e$. So definite integral
$= \tan^{-1}(e) - \tan^{-1}(1) = \tan^{-1}(e) - \pi/4$.

Final Answer: $\int_0^1 \frac{dx}{e^x + e^{-x}} = \tan^{-1}(e) - \frac{\pi}{4}$.

✗ Limits get substituted

When you substitute $u = g(x)$ inside a definite integral, the new limits are $u = g(a)$ and $u = g(b)$, NOT a and b. Forgetting to change the limits is the most-marked error in board scripts.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Clear-the-negative-exponent trick.

Step 1. Multiply by e^x/e^x to remove e^{-x}.

Step 2. Substitute $t = e^x$.

Step 3. Arctan integral.

Final Answer: $\tan^{-1}(e) - \pi/4$.

Why this matters. "Integrand with both e^x and e^{-x}" is a fixed CBSE setup; always multiply by e^x first.

Q 7.30 Evaluate $\int_0^{\pi/2} \frac{\tan x\, dx}{1 + m^2 \tan^2 x}$.

SOLUTION

Concept used. Sub $u = \tan x$ (so $du = \sec^2 x\, dx = (1 + \tan^2 x)dx$). Need to rewrite the integral first.

Step 1. Multiply numerator and denominator by $\cos^2 x$:
$$\frac{\tan x}{1 + m^2 \tan^2 x} = \frac{\sin x \cos x}{\cos^2 x + m^2 \sin^2 x}.$$

Step 2. Use $\cos^2 x = 1 - \sin^2 x$: denominator $= 1 + (m^2 - 1)\sin^2 x$.

Step 3. Put $u = \sin^2 x$, $du = 2 \sin x \cos x\, dx$. So $\sin x \cos x\, dx = du/2$. Integral
$$= \frac{1}{2} \int_0^1 \frac{du}{1 + (m^2 - 1)u}.$$

Step 4. This is $\frac{1}{2} \cdot \frac{1}{m^2 - 1} \log |1 + (m^2 - 1)u|\Big|_0^1 = \frac{1}{2(m^2 - 1)} \log m^2 = \frac{\log m}{m^2 - 1}$ (for $m^2 \neq 1$).

> **Final Answer:** $\displaystyle\int_0^{\pi/2} \frac{\tan x\, dx}{1+m^2\tan^2 x} = \frac{\log m}{m^2-1}$ (for $m \neq \pm 1$).

☞ Half-angle identity

Memorise $1-\cos\theta = 2\sin^2(\theta/2)$. It converts every $1-\cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Rewrite $\tan x$ first.

Step 1. Multiply top and bottom by $\cos^2 x$ to clear the $\tan^2$.

Step 2. Substitute $u = \sin^2 x$.

Step 3. Standard log integral over u.

> **Final Answer:** $\dfrac{\log m}{m^2-1}$.

Why this matters. The trick "multiply by $\cos^2 x/\cos^2 x$" is the standard cure for $\tan^n x$ in the denominator. Memorise it.

Sanity check. At $m=1$, the integrand is $\tan x/(1+\tan^2 x) = \sin x\cos x = \frac{1}{2}\sin 2x$ and $\int_0^{\pi/2}\frac{1}{2}\sin 2x\,dx = 1/2$. The closed-form $\log m/(m^2-1) \to 1/2$ as $m\to 1$ by L'Hôpital. ✓

Q 7.31 Evaluate $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}}$.

SOLUTION

Concept used. Completing the square inside the radical $(x-1)(2-x) = -(x^2-3x+2)$ and standard $\sin^{-1}$ integral.

Step 1. Expand: $(x-1)(2-x) = -x^2+3x-2 = -(x^2-3x+2)$. Complete square: $x^2-3x+2 = (x-3/2)^2 - 1/4$, so $(x-1)(2-x) = 1/4-(x-3/2)^2$.

Step 2. Put $u = x-3/2$: integral $= \int_{-1/2}^{1/2}\dfrac{du}{\sqrt{1/4-u^2}}$.

Step 3. Standard: $\int du/\sqrt{a^2-u^2} = \sin^{-1}(u/a)$ with $a=1/2$.
$= \sin^{-1}(2u)\big|_{-1/2}^{1/2} = \sin^{-1}(1)-\sin^{-1}(-1) = \pi/2-(-\pi/2) = \pi$.

Final Answer: $\displaystyle\int_1^2 \frac{dx}{\sqrt{(x-1)(2-x)}} = \pi.$

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Completing-the-square + symmetry.

Step 1. Rewrite radicand as $(1/2)^2 - (x - 3/2)^2$.

Step 2. Sub $u = x - 3/2$.

Step 3. Arcsin evaluates to π on $[-1/2, 1/2]$.

Final Answer: π.

Why this matters. The integral $\int_a^b dx/\sqrt{(x-a)(b-x)} = \pi$ for any $a < b$. This is a fixed result; memorise it as a one-line theorem.

Q 7.32 Evaluate $\displaystyle\int_0^1 \frac{x\,dx}{1+x^2}$.

SOLUTION

Concept used. **Sub** $u = 1 + x^2$, $du = 2x\,dx$.

Step 1. Integral $= \frac{1}{2}\int_1^2 \dfrac{du}{u} = \frac{1}{2}\log 2$.

Final Answer: $\displaystyle\int_0^1 \frac{x\,dx}{1+x^2} = \frac{1}{2}\log 2.$

☞ **Differentiate to verify**

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Pure g'/g template.

Step 1. $g = 1 + x^2$, $g' = 2x$. Numerator $x = g'/2$.

Step 2. Integral $= \frac{1}{2}\log|g|\big|_0^1$.

Final Answer: $\frac{1}{2}\log 2$.

Why this matters. 1-mark template; appears every CBSE paper.

Q 7.33 **Evaluate** $\displaystyle\int_0^{\pi} x\sin x\cos^2 x\,dx$.

SOLUTION

Concept used. King-rule property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ with $a = \pi$.

Step 1. Let $I = \int_0^{\pi} x\sin x\cos^2 x\,dx$. Replace x by $\pi - x$: $\sin(\pi - x) = \sin x$, $\cos(\pi - x) = -\cos x$, $\cos^2(\pi - x) = \cos^2 x$. So $I = \int_0^{\pi}(\pi - x)\sin x\cos^2 x\,dx$.

Step 2. Add: $2I = \int_0^{\pi}\pi\sin x\cos^2 x\,dx = \pi\int_0^{\pi}\sin x\cos^2 x\,dx$.

Step 3. Compute $\int_0^{\pi}\sin x\cos^2 x\,dx$ via $u = \cos x$, $du = -\sin x\,dx$:
$= -\int_1^{-1} u^2\,du = \int_{-1}^{1} u^2\,du = 2/3$.

Step 4. So $2I = 2\pi/3 \Rightarrow I = \pi/3$.

Final Answer: $\displaystyle\int_0^{\pi} x\sin x\cos^2 x\,dx = \frac{\pi}{3}$.

☞ **King rule on $[0, \pi]$**

$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} f(\pi - x)\,dx$ when the integrand involves $\sin x$ (invariant), $\cos x$ (changes sign), and a polynomial in x.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

King-rule template. Whenever you see "$x \cdot$ (symmetric trig) on $[0, \pi]$", use the King rule.

Step 1. $I + I_{\text{flip}} = \pi\int_0^{\pi}(\text{symmetric trig})\,dx$.

Step 2. Compute the symmetric piece via u-sub.

Step 3. Divide by 2.

Final Answer: $\pi/3$.

Why this matters. King-rule + by-parts together solve 60% of CBSE 5-mark definite-integral questions.

Q 7.34 **Evaluate** $\displaystyle\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$**. (Hint:** $x = \sin\theta$.**)**

SOLUTION

Concept used. Trig substitution $x = \sin\theta$.

Step 1. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $\sqrt{1-x^2} = \cos\theta$, $1 + x^2 = 1 + \sin^2\theta$. Integrand: $\dfrac{\cos\theta\, d\theta}{(1+\sin^2\theta)\cos\theta} = \dfrac{d\theta}{1+\sin^2\theta}$.

Step 2. Divide numerator and denominator by $\cos^2\theta$: $\dfrac{\sec^2\theta\, d\theta}{\sec^2\theta + \tan^2\theta} = \dfrac{\sec^2\theta\, d\theta}{1 + 2\tan^2\theta}$.

Step 3. Put $t = \tan\theta$, $dt = \sec^2\theta\, d\theta$: $\int \dfrac{dt}{1+2t^2} = \frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}\, t)$.

Step 4. Limits: $x = 0 \to \theta = 0 \to t = 0$; $x = 1/2 \to \theta = \pi/6 \to t = 1/\sqrt{3}$. Value: $\frac{1}{\sqrt{2}} \tan^{-1}(\sqrt{2}/\sqrt{3}) = \frac{1}{\sqrt{2}} \tan^{-1}\sqrt{2/3}$.

Final Answer: $\displaystyle\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \frac{1}{\sqrt{2}} \tan^{-1}\sqrt{\frac{2}{3}}$.

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Trig-sub recipe.

Step 1. $x = \sin\theta$ removes both the $\sqrt{1-x^2}$ and brings $1 + x^2$ into $1 + \sin^2\theta$.

Step 2. Divide by $\cos^2\theta$ to convert to $\sec^2\theta, \tan^2\theta$.

Step 3. Sub $t = \tan\theta$.

Final Answer: $\frac{1}{\sqrt{2}}\tan^{-1}\sqrt{2/3}$.

Why this matters. When $\sqrt{1-x^2}$ and $1+x^2$ both appear, $x=\sin\theta$ converts everything into trig.

II. Long Answer (L.A.)

Q 7.35 **Evaluate** $\displaystyle\int \frac{x^2\,dx}{x^4-x^2-12}$.

SOLUTION

Concept used. **Substitution** $t=x^2$ converts the quartic into a quadratic in t, then partial fractions.

Step 1. Factor $x^4-x^2-12=(x^2-4)(x^2+3)$, since $(-4)\cdot 3=-12$ and $-4+3=-1$.

Step 2. Partial fractions in $t=x^2$:

$$\frac{t}{(t-4)(t+3)}=\frac{A}{t-4}+\frac{B}{t+3}.$$

Solve $t=A(t+3)+B(t-4)$. At $t=4$: $4=7A\Rightarrow A=4/7$. At $t=-3$: $-3=-7B\Rightarrow B=3/7$.

Step 3. So $\displaystyle\frac{x^2}{(x^2-4)(x^2+3)}=\frac{4/7}{x^2-4}+\frac{3/7}{x^2+3}$.

Step 4. Integrate: $\frac{4}{7}\displaystyle\int\frac{dx}{x^2-4}+\frac{3}{7}\int\frac{dx}{x^2+3}=\frac{4}{7}\cdot\frac{1}{4}\log\left|\frac{x-2}{x+2}\right|+\frac{3}{7}\cdot\frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}$.

Final Answer: $\displaystyle\int\frac{x^2\,dx}{x^4-x^2-12}=\frac{1}{7}\log\left|\frac{x-2}{x+2}\right|+\frac{\sqrt{3}}{7}\tan^{-1}\frac{x}{\sqrt{3}}+C.$

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x=\frac{\pi}{2}+k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Quartic-to-quadratic partial fraction.

Step 1. Spot that x^4-x^2-12 is a quadratic in x^2.

Step 2. Factor in x^2, then split into two quadratic denominators.

Step 3. Each piece is one of the special integrals: $1/(x^2-a^2)$ or $1/(x^2+a^2)$.

> **Final Answer:** $\frac{1}{7}\log|(x-2)/(x+2)| + \frac{\sqrt{3}}{7}\tan^{-1}(x/\sqrt{3}) + C$.

Why this matters. Disguised quadratics in x^2 are the bread-and-butter of 5-mark CBSE partial-fraction questions.

Q 7.36 **Evaluate** $\displaystyle\int \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)}$ **(with** $a \neq b$**).**

SOLUTION

Concept used. Partial fractions in $t = x^2$ identity:
$$\frac{t}{(t+a^2)(t+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{t+a^2} - \frac{b^2}{t+b^2}\right].$$

Step 1. Set $\dfrac{x^2}{(x^2+a^2)(x^2+b^2)} = \dfrac{A}{x^2+a^2} + \dfrac{B}{x^2+b^2}$. Cross-multiply: $x^2 = A(x^2+b^2) + B(x^2+a^2)$. Match: $1 = A+B$; $0 = Ab^2 + Ba^2$. Solve: $A = a^2/(a^2-b^2)$, $B = -b^2/(a^2-b^2) = b^2/(b^2-a^2)$.

Step 2. Integrate each piece: $\int \dfrac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$; same for b.

Step 3. Combine: $\dfrac{a^2}{a^2-b^2}\cdot\dfrac{1}{a}\tan^{-1}(x/a) - \dfrac{b^2}{a^2-b^2}\cdot\dfrac{1}{b}\tan^{-1}(x/b) + C = \dfrac{1}{a^2-b^2}\left[a\tan^{-1}(x/a) - b\tan^{-1}(x/b)\right] + C$.

> **Final Answer:** $\displaystyle\int \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right] + C.$

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Symmetric partial fraction.

Step 1. Sub $t = x^2$ mentally; partial-fraction the resulting linear-in-t form.

Step 2. Each piece is an arctan in x.

Step 3. Final answer is anti-symmetric in a, b (sign flip when $a \leftrightarrow b$).

Final Answer: $\frac{1}{a^2-b^2}[a\tan^{-1}(x/a) - b\tan^{-1}(x/b)] + C.$

Why this matters. A 5-mark CBSE template; the symmetric/antisymmetric structure is a sanity-check.

Q 7.37 Evaluate $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+\sin x}$.

SOLUTION

Concept used. King rule on $[0, \pi]$ with $\sin(\pi - x) = \sin x$.

Step 1. Let $I = \int_0^{\pi} \dfrac{x\,dx}{1+\sin x}$. By King rule, $I = \int_0^{\pi} \dfrac{(\pi - x)\,dx}{1+\sin(\pi - x)} = \int_0^{\pi} \dfrac{(\pi - x)\,dx}{1+\sin x}$.

Step 2. Add: $2I = \pi \int_0^{\pi} \dfrac{dx}{1+\sin x}$.

Step 3. Compute $\int_0^{\pi} \dfrac{dx}{1+\sin x}$: multiply numerator and denominator by $1 - \sin x$: $\dfrac{1-\sin x}{1-\sin^2 x} = \dfrac{1-\sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x$.

Step 4. Integrate: $\tan x - \sec x$ from 0 to π. At π and 0, both $\tan$ and $\sec$ have issues at $\pi/2$. The improper integral evaluates (using half-angle: $1 + \sin x = (\sin(x/2) + \cos(x/2))^2$, so $1/(1 + \sin x) = 1/(\sin(x/2) + \cos(x/2))^2$): the antiderivative is $-2/(\tan(x/2) + 1)$ (or equivalently $\tan(x/2 - \pi/4) - 1$ form), evaluating from 0 to π gives 2.

Step 5. So $2I = 2\pi \Rightarrow I = \pi$.

Final Answer: $\displaystyle\int_0^{\pi} \frac{x\,dx}{1+\sin x} = \pi.$

✗ Improper integrals need limits

If the integrand is unbounded at an endpoint or the interval is infinite, replace the offending endpoint by a parameter, integrate, then take the limit. Skipping this and substituting ∞ directly is invalid.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

King + half-angle.

Step 1. King rule converts the integral into π times a symmetric one.

Step 2. $1 + \sin x = (\sin(x/2) + \cos(x/2))^2$ gives a clean antiderivative.

Step 3. Evaluate at the endpoints carefully.

> **Final Answer:** π.

Why this matters. The half-angle perfect-square + King-rule combo gives elegant closed forms for $\int x/(\text{symmetric trig})$.
Always test the integrand. Before integrating over a symmetric interval $[-a, a]$, compute $f(-x)$ and compare with $f(x)$. If f is odd, the answer is 0 with no further work; if even, fold the integral to $2\int_0^a f$. Missing this short-cut is the most common waste of exam time on definite integrals.

Q 7.38 **Evaluate** $\displaystyle\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx$.

SOLUTION

Concept used. Partial fractions with three distinct linear factors.

Step 1. Set $\dfrac{2x-1}{(x-1)(x+2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{x-3}$.

Step 2. Cover-up rule: At $x = 1$: $A = \dfrac{1}{(3)(-2)} = -\dfrac{1}{6}$. At $x = -2$: $B = \dfrac{-5}{(-3)(-5)} = -\dfrac{1}{3}$.
At $x = 3$: $C = \dfrac{5}{(2)(5)} = \dfrac{1}{2}$.

Step 3. Integrate each: $-\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C$.

> **Final Answer:** $\displaystyle\int \frac{2x-1}{(x-1)(x+2)(x-3)}\,dx = -\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C$.

☞ Cover-up rule

For distinct linear factors $(x - a_i)$, the coefficient over $(x - a_i)$ is the numerator evaluated at $x = a_i$, divided by the product of $(a_i - a_j)$ over $j \neq i$.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Cover-up mechanics.

Step 1. Coefficient of $1/(x-a)$: plug $x = a$ into the remaining factors in the denominator.

Step 2. Each integral is a log.

> **Final Answer:** $-\frac{1}{6}\log|x-1| - \frac{1}{3}\log|x+2| + \frac{1}{2}\log|x-3| + C$.

Why this matters. Cover-up saves 60% of the algebra on distinct-linear partial fractions.

Q 7.39 **Evaluate** $\displaystyle\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$.

SOLUTION

Concept used. Form $\int e^{f(x)}(f'(x)\cdot g(x) + g'(x))\,dx = e^{f(x)}g(x) + C$ with $f(x) = \tan^{-1}x$.

Step 1. $f(x) = \tan^{-1}x \Rightarrow f'(x) = 1/(1+x^2)$.

Step 2. Write integrand: $e^{\tan^{-1}x}\cdot\dfrac{1+x+x^2}{1+x^2} = e^{\tan^{-1}x}\left[1 + \dfrac{x}{1+x^2}\right]$. Split:
$= e^{\tan^{-1}x}\cdot 1 + e^{\tan^{-1}x}\cdot\dfrac{x}{1+x^2}$.

Step 3. Try $g(x) = x$: $g'(x) = 1$. Then
$f'(x)\cdot g(x) + g'(x) = x/(1+x^2) + 1 = (1+x+x^2)/(1+x^2)$. ✓

Step 4. So integral $= e^{\tan^{-1}x}\cdot x + C = x\,e^{\tan^{-1}x} + C$.

> **Final Answer:** $\displaystyle\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx = x\,e^{\tan^{-1}x} + C$.

☞ Differentiate to verify

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION *: Sneha Pillai, M.Sc Mathematics, IIT Bombay*

The $e^f\cdot$ (stuff) pattern. Whenever the integrand has $e^{f(x)}$ times a rational function, look for $g(x)$ such that the rational piece equals $f'(x)g(x) + g'(x)$.

Step 1. Split the rational piece: $1 + x/(1+x^2)$.

Step 2. Match $g = x$.

Final Answer: $xe^{\tan^{-1} x} + C$.

Why this matters. The companion $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$ is the JEE Main one-line shortcut. The $\tan^{-1}$ version generalises it.

Q 7.40 **Evaluate** $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\,dx$. **(Hint:** $x = a\tan^2\theta$.**)**

SOLUTION

Concept used. Trig substitution $x = a\tan^2\theta$ makes $\sqrt{x/(a+x)} = \sin\theta$, so the inverse-sine becomes θ, then by parts.

Step 1. Sub: $x = a\tan^2\theta$, $dx = 2a\tan\theta\sec^2\theta\,d\theta$.

$$\frac{x}{a+x} = \frac{a\tan^2\theta}{a + a\tan^2\theta} = \frac{\tan^2\theta}{\sec^2\theta} = \sin^2\theta.$$

Step 2. So $\sin^{-1}\sqrt{x/(a+x)} = \theta$. Integral: $\int \theta \cdot 2a\tan\theta\sec^2\theta\,d\theta = 2a\int \theta\tan\theta\sec^2\theta\,d\theta$.

Step 3. By parts: $u = \theta$, $dv = \tan\theta\sec^2\theta\,d\theta = d(\tan^2\theta/2)$. $\int \theta\tan\theta\sec^2\theta\,d\theta =$

$$\theta\cdot\frac{\tan^2\theta}{2} - \int \frac{\tan^2\theta}{2}\,d\theta = \frac{\theta\tan^2\theta}{2} - \tfrac{1}{2}\int(\sec^2\theta - 1)\,d\theta = \frac{\theta\tan^2\theta}{2} - \tfrac{1}{2}(\tan\theta - \theta).$$

Step 4. Multiply by $2a$:

$$a\theta\tan^2\theta - a(\tan\theta - \theta) + C = a\theta(\tan^2\theta + 1) - a\tan\theta + C = a\theta\sec^2\theta - a\tan\theta + C.$$

But $\sec^2\theta = 1 + \tan^2\theta = (a+x)/a$, $\tan\theta = \sqrt{x/a}$. Back-substitute:

$$= (a+x)\theta - a\sqrt{x/a} + C = (a+x)\sin^{-1}\sqrt{x/(a+x)} - \sqrt{ax} + C.$$

Final Answer: $\int \sin^{-1}\sqrt{\dfrac{x}{a+x}}\,dx = (a+x)\sin^{-1}\sqrt{\dfrac{x}{a+x}} - \sqrt{ax} + C.$

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Trig-sub motivation. $x/(a+x) = \sin^2\theta$ for $x = a\tan^2\theta$ is exactly the right cue.

Step 1. Sub $x = a\tan^2\theta$.

Step 2. By parts on $\theta\tan\theta\sec^2\theta$.

Step 3. Back-substitute.

> **Final Answer:** $(a+x)\sin^{-1}\sqrt{x/(a+x)} - \sqrt{ax} + C$.

Why this matters. "Inverse-trig of $\sqrt{x/(a+x)}$" is a top JEE Advanced template; the $\tan^2$ substitution is the key.
Why each pairing works. $\sqrt{a^2-x^2} \to a\cos\theta$ via $x = a\sin\theta$ uses $1-\sin^2 = \cos^2$; $\sqrt{a^2+x^2} \to a\sec\theta$ via $x = a\tan\theta$ uses $1+\tan^2 = \sec^2$; $\sqrt{x^2-a^2} \to a\tan\theta$ via $x = a\sec\theta$ uses $\sec^2 -1 = \tan^2$. The Pythagorean identity always kills the radical.
What examiners reward. Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason.

Q 7.41 Evaluate $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx$.

SOLUTION

Concept used. Half-angle identities $1+\cos x = 2\cos^2(x/2)$, $1-\cos x = 2\sin^2(x/2)$.

Step 1. Use half-angle: $\sqrt{1+\cos x} = \sqrt{2}\,|\cos(x/2)|$,
$(1-\cos x)^{5/2} = (2\sin^2(x/2))^{5/2} = 4\sqrt{2}\,\sin^5(x/2)$ (taking positive branch).

Step 2. Integrand: $\dfrac{\sqrt{2}\cos(x/2)}{4\sqrt{2}\sin^5(x/2)} = \dfrac{\cos(x/2)}{4\sin^5(x/2)}$.

Step 3. Substitute $u = \sin(x/2)$, $du = \frac{1}{2}\cos(x/2)\,dx$. So $\cos(x/2)\,dx = 2\,du$. Integral:
$\dfrac{1}{4}\displaystyle\int \frac{2\,du}{u^5} = \frac{1}{2}\int u^{-5}\,du = -\frac{1}{8u^4}$.

Step 4. Limits: $x = \pi/3 \to u = \sin(\pi/6) = 1/2$, so $u^4 = 1/16$, $1/u^4 = 16$, contribution -2; $x = \pi/2 \to u = \sin(\pi/4) = 1/\sqrt{2}$, $u^4 = 1/4$, $1/u^4 = 4$, contribution $-1/2$.

Step 5. Evaluate: $-1/2 - (-2) = -1/2 + 2 = 3/2$.

> **Final Answer:** $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\sqrt{1+\cos x}}{(1-\cos x)^{5/2}}\,dx = \frac{3}{2}$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1-\cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Half-angle reduction.

Step 1. Replace both $1 \pm \cos x$ via half-angle.

Step 2. Substitute $u = \sin(x/2)$.

Step 3. Power-rule integration.

Final Answer: $3/2$.

Why this matters. "$\sqrt{1+\cos x}/(1-\cos x)^k$" integrals reduce to power-rule integrals via half-angle.

Generalising the move. Conjugate rationalisation works because $(\sqrt{A}-\sqrt{B})(\sqrt{A}+\sqrt{B}) = A - B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

Q 7.42 **Evaluate** $\int e^{-3x}\cos^3 x\,dx$.

SOLUTION

Concept used. **Reduction via triple-angle formula** $\cos^3 x = \frac{3\cos x + \cos 3x}{4}$, then standard $\int e^{ax}\cos bx\,dx$.

Step 1. $\cos^3 x = \frac{1}{4}(3\cos x + \cos 3x)$. Integral $= \frac{3}{4}\int e^{-3x}\cos x\,dx + \frac{1}{4}\int e^{-3x}\cos 3x\,dx$.

Step 2. Standard: $\int e^{ax}\cos bx\,dx = \dfrac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$. With $a = -3$:

$$\int e^{-3x}\cos x\,dx = \frac{e^{-3x}(-3\cos x + \sin x)}{9+1} = \frac{e^{-3x}(\sin x - 3\cos x)}{10};$$

$$\int e^{-3x}\cos 3x\,dx = \frac{e^{-3x}(-3\cos 3x + 3\sin 3x)}{9+9} = \frac{e^{-3x}(\sin 3x - \cos 3x)}{6}.$$

Step 3. Combine the two pieces with the $\frac{3}{4}$ and $\frac{1}{4}$ factors:

$$= \frac{3}{4}\cdot\frac{e^{-3x}(\sin x - 3\cos x)}{10} + \frac{1}{4}\cdot\frac{e^{-3x}(\sin 3x - \cos 3x)}{6} + C$$

$$= \frac{e^{-3x}(3\sin x - 9\cos x)}{40} + \frac{e^{-3x}(\sin 3x - \cos 3x)}{24} + C.$$

Final Answer: $\displaystyle\int e^{-3x}\cos^3 x\,dx = \frac{e^{-3x}(3\sin x - 9\cos x)}{40} + \frac{e^{-3x}(\sin 3x - \cos 3x)}{24} + C.$

☞ **Half-angle identity**

Memorise $1-\cos\theta = 2\sin^2(\theta/2)$. It converts every $1-\cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Triple-angle reduction. The identity $\cos^3 x = (3\cos x + \cos 3x)/4$ turns $\int e^{ax}\cos^3 x\,dx$ into two standard $\int e^{ax}\cos(bx)\,dx$ integrals.

Step 1. Replace $\cos^3 x$.

Step 2. Apply $\int e^{ax}\cos bx\,dx = e^{ax}(a\cos bx + b\sin bx)/(a^2+b^2)$ twice.

Final Answer: Sum of the two standard pieces with $a=-3$, $b=1,3$.

Why this matters. JEE Main and CBSE both place "$\int e^{ax}\cos^n x$" and "$\int e^{ax}\sin^n x$" as 5-mark questions. The triple-angle reduction is the cleanest path.

Q 7.43 **Evaluate** $\displaystyle\int \sqrt{\tan x}\,dx$. **(Hint:** $\tan x = t^2$.**)**

SOLUTION

Concept used. **Substitution** $\tan x = t^2$ converts the radical to t.

Step 1. $\tan x = t^2 \Rightarrow \sec^2 x\,dx = 2t\,dt \Rightarrow dx = \dfrac{2t\,dt}{1+t^4}$. Integrand:

$$\sqrt{\tan x}\,dx = t\cdot\frac{2t\,dt}{1+t^4} = \frac{2t^2\,dt}{1+t^4}.$$

Step 2. Split: $\dfrac{2t^2}{1+t^4} = \dfrac{(t^2+1)+(t^2-1)}{1+t^4} = \dfrac{t^2+1}{1+t^4} + \dfrac{t^2-1}{1+t^4}$.

Step 3. Divide by t^2 in each: $\dfrac{t^2+1}{1+t^4} = \dfrac{1+1/t^2}{t^2+1/t^2}$, sub $u = t-1/t$, $du = (1+1/t^2)\,dt$, $u^2 = t^2-2+1/t^2$. $\dfrac{t^2-1}{1+t^4} = \dfrac{1-1/t^2}{t^2+1/t^2}$, sub $v = t+1/t$, $dv = (1-1/t^2)\,dt$, $v^2 = t^2+2+1/t^2$.

Step 4. First: $\int du/(u^2+2) = \frac{1}{\sqrt2}\tan^{-1}(u/\sqrt2)$. Second:

$$\int dv/(v^2-2) = \frac{1}{2\sqrt2}\log\left|\frac{v-\sqrt2}{v+\sqrt2}\right|.$$

Step 5. Combine and back-substitute $t = \sqrt{\tan x}$, so $u = \sqrt{\tan x} - 1/\sqrt{\tan x}$ and $v = \sqrt{\tan x} + 1/\sqrt{\tan x}$. With $T = \sqrt{2\tan x}$, the answer becomes

$$\int \sqrt{\tan x}\, dx = \tfrac{1}{\sqrt{2}} \tan^{-1}\left(\tfrac{\tan x - 1}{T}\right) + \tfrac{1}{2\sqrt{2}} \log\left|\tfrac{\tan x + 1 - T}{\tan x + 1 + T}\right| + C.$$

Final Answer: Letting $T = \sqrt{2\tan x}$:

$$\int \sqrt{\tan x}\, dx = \tfrac{1}{\sqrt{2}} \tan^{-1}\left(\tfrac{\tan x - 1}{T}\right) + \tfrac{1}{2\sqrt{2}} \log\left|\tfrac{\tan x - T + 1}{\tan x + T + 1}\right| + C.$$

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Symmetric-split trick. Whenever you see $\int 2t^2/(1+t^4)\, dt$ or $\int 1/(1+t^4)\, dt$, split the numerator into $(t^2+1) + (t^2-1)$ and use the conjugate substitutions $u = t \pm 1/t$.

Step 1. Split the numerator symmetrically.

Step 2. Divide by t^2 inside each fraction.

Step 3. Substitute u, v separately.

Step 4. Each piece is a $\tan^{-1}$ or partial-fraction log.

Final Answer: Standard answer above.

Why this matters. The $1/(1+t^4)$ integral is a top-3 trick in JEE Main; expect at least one shift per session to hide it.

Generalising the move. Conjugate rationalisation works because $(\sqrt{A} - \sqrt{B})(\sqrt{A} + \sqrt{B}) = A - B$ — a polynomial difference that we can simplify. The same identity in disguise drives the limit $\lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} = \frac{1}{2}$ and the standard derivative $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$. Recognise the family.

Q 7.44 Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2}$.

SOLUTION

Concept used. **Divide by** $\cos^4 x$ (per the hint) to convert to a rational function of $\tan x$.

Step 1. Divide top and bottom by $\cos^4 x$: $\dfrac{\sec^4 x}{(a^2 + b^2 \tan^2 x)^2}$.

Step 2. $\sec^4 x = \sec^2 x \cdot \sec^2 x = (1 + \tan^2 x)\sec^2 x$. Sub $t = \tan x$, $dt = \sec^2 x\, dx$. Limits: $x : 0 \to \pi/2$ gives $t : 0 \to \infty$. Integral: $\int_0^\infty \dfrac{(1+t^2)\, dt}{(a^2 + b^2 t^2)^2}$.

Step 3. Split: $\dfrac{1+t^2}{(a^2+b^2t^2)^2} = \dfrac{1}{(a^2+b^2t^2)^2} + \dfrac{t^2}{(a^2+b^2t^2)^2}$. Use $\int_0^\infty dt/(a^2+b^2t^2)^2 = \dfrac{\pi}{4a^3b}$, and $\int_0^\infty t^2\, dt/(a^2+b^2t^2)^2 = \dfrac{\pi}{4ab^3}$ (standard contour / by-parts results).

Step 4. Sum: $\dfrac{\pi}{4ab}\left[\dfrac{1}{a^2} + \dfrac{1}{b^2}\right] = \dfrac{\pi(a^2+b^2)}{4a^3b^3}$.

Final Answer: $\displaystyle\int_0^{\pi/2} \frac{dx}{(a^2\cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi(a^2+b^2)}{4a^3b^3}$.

☞ **Parametric derivative**

For parametric curves $(x(t), y(t))$ use $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$. Never combine into a single derivative until the very last step — treat the two derivatives separately and divide.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Divide-by-$\cos^4 x$ + split.

Step 1. Convert to a rational function of $t = \tan x$.

Step 2. Split the $1 + t^2$ numerator into 1 and t^2.

Step 3. Use the two standard $\int_0^\infty \cdot/(a^2+b^2t^2)^2$ results.

Final Answer: $\pi(a^2+b^2)/(4a^3b^3)$.

Why this matters. The two "$1/(a^2+b^2t^2)^2$" standard integrals are NCERT-listed and worth memorising.

Self-check before boxing the answer. Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.

Q 7.45 **Evaluate** $\int_0^1 x\log(1+2x)\,dx$.

SOLUTION

Concept used. **Integration by parts** with $u = \log(1+2x)$, $dv = x\,dx$.

Step 1. $u = \log(1+2x), du = 2\,dx/(1+2x)$; $dv = x\,dx, v = x^2/2$. By parts:

$$\int x\log(1+2x)\,dx = \frac{x^2}{2}\log(1+2x) - \int \frac{x^2}{2}\cdot\frac{2}{1+2x}\,dx = \frac{x^2}{2}\log(1+2x) - \int \frac{x^2}{1+2x}\,dx.$$

Step 2. Long-divide x^2 by $1+2x$: $4x^2 = (1+2x)(2x-1)+1$, so

$$\frac{x^2}{1+2x} = \frac{2x-1}{4} + \frac{1}{4(1+2x)}.\ \int = \frac{x^2}{4} - \frac{x}{4} + \frac{1}{8}\log|1+2x|.$$

Step 3. Combine: $\frac{x^2}{2}\log(1+2x) - \frac{x^2}{4} + \frac{x}{4} - \frac{1}{8}\log|1+2x| + C$.

Step 4. Evaluate 0 to 1: at $x = 1$: $\frac{1}{2}\log 3 - 1/4 + 1/4 - \frac{1}{8}\log 3 = (\frac{1}{2} - \frac{1}{8})\log 3 = \frac{3}{8}\log 3$. At $x = 0$: all terms 0.

Final Answer: $\int_0^1 x\log(1+2x)\,dx = \frac{3\log 3}{8}$.

☞ **Log domain**

$\ln x$ is defined and continuous only for $x > 0$. Before using $\ln$ on an expression, confirm that the argument is strictly positive on the relevant interval.

EXPERT'S SOLUTION *: Sneha Pillai, M.Sc Mathematics, IIT Bombay*

ILATE on $x\log(\cdot)$. Logarithmic beats Algebraic, so $u = \log(1+2x)$, $dv = x\,dx$.

Step 1. Apply by parts.

Step 2. Long-divide the residual rational integrand.

Step 3. Evaluate.

Final Answer: $3\log 3/8$.

Why this matters. "$x\log$(linear)" is one of the most-repeated CBSE 5-markers. The setup is mechanical once you remember ILATE.

Q 7.46 **Evaluate** $\int_0^\pi x\log\sin x\,dx$.

SOLUTION

Concept used. **King rule** on $[0, \pi]$ then reduce to the classic $\int_0^{\pi/2} \log \sin x \, dx = -(\pi \log 2)/2$.

Step 1. $I = \int_0^{\pi} x \log \sin x \, dx$. King:
$I = \int_0^{\pi} (\pi - x) \log \sin(\pi - x) \, dx = \int_0^{\pi} (\pi - x) \log \sin x \, dx$.

Step 2. Add: $2I = \pi \int_0^{\pi} \log \sin x \, dx$.

Step 3. By symmetry $\log \sin x$ on $[0, \pi]$ is symmetric about $\pi/2$:
$\int_0^{\pi} \log \sin x \, dx = 2 \int_0^{\pi/2} \log \sin x \, dx = 2 \cdot (-\pi \log 2/2) = -\pi \log 2$.

Step 4. So $2I = \pi \cdot (-\pi \log 2) = -\pi^2 \log 2$, hence $I = -\pi^2 \log 2/2$.

Final Answer: $\displaystyle\int_0^{\pi} x \log \sin x \, dx = -\frac{\pi^2 \log 2}{2}$.

☞ $\int_0^{\pi/2} \log \sin x \, dx = -(\pi \log 2)/2$
This is one of the most-tested "classical" definite integrals. Memorise the value and the King-rule derivation.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

King + classical $\log \sin$ **value.**

Step 1. King rule converts $x \log \sin x$ on $[0, \pi]$ into $\pi/2$ times $\int \log \sin x$.

Step 2. Use symmetry to halve the interval.

Step 3. Quote the classical value $-\pi \log 2/2$ on $[0, \pi/2]$.

Final Answer: $-\pi^2 \log 2/2$.

Why this matters. CBSE 5-mark template that uses both King rule AND the classical $\log \sin$ value.

Q 7.47 **Evaluate** $\displaystyle\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x) \, dx$.

SOLUTION

Concept used. **Even/odd test** on the integrand on the symmetric interval $[-\pi/4, \pi/4]$.

Step 1. Note $\sin(-x) + \cos(-x) = -\sin x + \cos x = \cos x - \sin x$. So

$f(-x) = \log(\cos x - \sin x)$. Then
$f(x)+f(-x) = \log[(\sin x+\cos x)(\cos x-\sin x)] = \log(\cos^2 x-\sin^2 x) = \log\cos 2x$.

Step 2. Hence $\int_{-\pi/4}^{\pi/4}[f(x) + f(-x)]\,dx = \int_{-\pi/4}^{\pi/4} \log\cos 2x\,dx$. But also $\int_{-\pi/4}^{\pi/4} f(-x)\,dx = \int_{-\pi/4}^{\pi/4} f(u)\,du$ (sub $u = -x$), so $2I = \int_{-\pi/4}^{\pi/4} \log\cos 2x\,dx$.

Step 3. Sub $t = 2x$, $dt = 2\,dx$. Limits: $-\pi/2$ to $\pi/2$.
$\int_{-\pi/2}^{\pi/2} \log\cos t\,dt/2 = \frac{1}{2}\cdot 2\int_0^{\pi/2} \log\cos t\,dt = \int_0^{\pi/2} \log\cos t\,dt = -(\pi\log 2)/2$.

Step 4. So $2I = -\pi\log 2/2$, $I = -\pi\log 2/4$.

Final Answer: $\displaystyle\int_{-\pi/4}^{\pi/4} \log(\sin x + \cos x)\,dx = -\frac{\pi\log 2}{4}$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

$f(x) + f(-x)$ **trick.** On a symmetric interval, $\int_{-a}^{a} f = \frac{1}{2}\int_{-a}^{a}(f(x) + f(-x))\,dx$. Used when $f(x) + f(-x)$ simplifies dramatically.

Step 1. Compute $f(x) + f(-x) = \log\cos 2x$.

Step 2. Substitute $t = 2x$.

Step 3. Use the classical $\int_0^{\pi/2} \log\cos t\,dt = -\pi\log 2/2$.

Final Answer: $-\pi\log 2/4$.

Why this matters. The classical $\int\log\sin$ / $\int\log\cos$ values are heavily reused in JEE Advanced and CBSE 5-markers.

III. Objective Type Questions (MCQ)

Q 7.48 $\displaystyle\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta}\,dx$ **equals**

(A) $2(\sin x + x\cos\theta) + C$ **(B)** $2(\sin x - x\cos\theta) + C$
(C) $2(\sin x + 2x\cos\theta) + C$ **(D)** $2(\sin x - 2x\cos\theta) + C$

SOLUTION

Answer. (A).
Concept used. **Double-angle identity** $\cos 2x = 2\cos^2 x - 1$.

Step 1. $\cos 2x - \cos 2\theta = 2(\cos^2 x - \cos^2\theta) = 2(\cos x - \cos\theta)(\cos x + \cos\theta)$.

Step 2. Cancel $\cos x - \cos\theta$: integrand $= 2(\cos x + \cos\theta)$.

Step 3. Integrate: $2\sin x + 2x\cos\theta + C$. Option (A).

> **Final Answer:** (A) $2(\sin x + x\cos\theta) + C$.

☞ **Half-angle identity**
Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Algebraic simplification first. Identity $\cos 2\alpha - \cos 2\beta = 2(\cos^2\alpha - \cos^2\beta)$ factors as difference of squares.

Step 1. Apply identity.

Step 2. Cancel.

Step 3. Integrate.

> **Final Answer:** (A).

Why this matters. Whenever you see "$\cos 2x - \cos 2\theta$", factor it via difference-of-squares; the $\cos x - \cos\theta$ in the denominator cancels.

Q 7.49 $\displaystyle\int \frac{dx}{\sin(x-a)\sin(x-b)}$ **equals**

(A) $\sin(b-a)\log\left|\dfrac{\sin(x-b)}{\sin(x-a)}\right| + C$ **(B)** $\operatorname{cosec}(b-a)\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right| + C$

(C) $\operatorname{cosec}(b-a)\log\left|\dfrac{\sin(x-b)}{\sin(x-a)}\right| + C$ **(D)** $\sin(b-a)\log\left|\dfrac{\sin(x-a)}{\sin(x-b)}\right| + C$

SOLUTION

Answer. (C).
Concept used. **Add-zero trick**: $(x-a) - (x-b) = b - a$, a constant. So $\sin(b-a) = \sin((x-a) - (x-b))$ is a constant that we can pull into the numerator.

Step 1. Multiply and divide the integrand by $\sin(b-a)$:

$$\frac{1}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(b-a)} \cdot \frac{\sin(b-a)}{\sin(x-a)\sin(x-b)}.$$

Step 2. Expand $\sin(b-a) = \sin((x-a)-(x-b))$ using $\sin(P-Q) = \sin P\cos Q - \cos P\sin Q$ with $P = x-a$, $Q = x-b$:

$$\sin(b-a) = \sin(x-a)\cos(x-b) - \cos(x-a)\sin(x-b).$$

Step 3. Divide both terms by $\sin(x-a)\sin(x-b)$:

$$\frac{\sin(b-a)}{\sin(x-a)\sin(x-b)} = \cot(x-b) - \cot(x-a).$$

Step 4. Therefore the integrand equals $\operatorname{cosec}(b-a)\,[\cot(x-b) - \cot(x-a)]$.

Step 5. Integrate term by term using $\int \cot u\,du = \log|\sin u|$:

$$\int \frac{dx}{\sin(x-a)\sin(x-b)} = \operatorname{cosec}(b-a)\,[\log|\sin(x-b)| - \log|\sin(x-a)|] + C.$$

Step 6. Combine the logs: $\operatorname{cosec}(b-a)\ \log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$. This matches option (C).

Final Answer: (C) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$.

☞ Differentiate to verify

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Trick: $\sin((x-a)-(x-b))$ **is constant.**

Step 1. Multiply and divide by $\sin(b-a)$.

Step 2. Expand $\sin(b-a) = \sin((x-a)-(x-b))$ with $\sin(P-Q) = \sin P\cos Q - \cos P\sin Q$.

Step 3. The integrand becomes $\operatorname{cosec}(b-a)\,[\cot(x-b) - \cot(x-a)]$, integrable as a log.

Final Answer: (C): $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right| + C$.

Why this matters. The "$1/(\sin(x-a)\sin(x-b))$" integral is a top JEE Main template; the trick of writing $1 = \sin(b-a)/\sin(b-a)$ and then expanding the constant numerator into a difference of cotangents is the key move.
Self-check before boxing the answer. Re-substitute the answer into the original problem (continuity check, derivative check, or definite-integral re-derivation). 30 seconds spent on this catches almost every sign-flip and missing-factor error before the marker sees them.
What examiners reward. Step-by-step justification, the right theorem name written out, and a clearly boxed final answer collectively earn more marks than a terse correct numerical answer. The expert solution above is laid out exactly this way for that reason.

Q 7.50 $\int \tan^{-1}\sqrt{x}\,dx$ **equals**

(A) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$ **(B)** $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$
(C) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$ **(D)** $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

SOLUTION

Answer. (A).
Concept used. Substitution $\sqrt{x}=t$, then by parts.

Step 1. $\sqrt{x}=t \Rightarrow x=t^2, dx=2t\,dt$. Integral $=\int \tan^{-1} t\cdot 2t\,dt$.

Step 2. By parts: $u=\tan^{-1} t, du=dt/(1+t^2)$; $dv=2t\,dt, v=t^2$.
$\int 2t\tan^{-1} t\,dt = t^2\tan^{-1} t-\int \frac{t^2}{1+t^2}\,dt = t^2\tan^{-1} t-\int\left(1-\frac{1}{1+t^2}\right)dt =$
$t^2\tan^{-1} t - t+\tan^{-1} t + C$.

Step 3. Back-substitute $t=\sqrt{x}$:
$x\tan^{-1}\sqrt{x}-\sqrt{x}+\tan^{-1}\sqrt{x}+C=(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$. Option (A).

Final Answer: (A) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$.

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A}\pm\sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A}\mp\sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Sub + by-parts.

Step 1. $\sqrt{x} = t \Rightarrow dx = 2t\,dt$.

Step 2. By parts on $\int 2t \tan^{-1} t\,dt$.

Step 3. Combine $x\tan^{-1}\sqrt{x}$ and $\tan^{-1}\sqrt{x}$ into $(x+1)\tan^{-1}\sqrt{x}$.

Final Answer: (A).

Why this matters. Inverse-trig of $\sqrt{x}$ is always handled by "substitute $\sqrt{x} = t$ first, then by-parts."

Q 7.51 $\int e^x \left(\frac{1-x}{1+x^2}\right)^2 dx$ **equals**

(A) $\frac{e^x}{1+x^2} + C$ **(B)** $-\frac{e^x}{1+x^2} + C$

(C) $\frac{e^x}{(1+x^2)^2} + C$ **(D)** $-\frac{e^x}{(1+x^2)^2} + C$

SOLUTION

Answer. (A).

Concept used. Pattern $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.

Step 1. Expand $(1-x)^2 = 1 - 2x + x^2$. Write

$$\frac{(1-x)^2}{(1+x^2)^2} = \frac{1+x^2-2x}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}.$$

Step 2. Try $f(x) = 1/(1+x^2)$. Then $f'(x) = -2x/(1+x^2)^2$. So

$$f(x) + f'(x) = \frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2} = \text{exactly our integrand!}$$

Step 3. Hence $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C = \frac{e^x}{1+x^2} + C$. Option (A).

Final Answer: (A) $\frac{e^x}{1+x^2} + C$.

☞ Differentiate to verify

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

$e^x[f + f']$ **recognition.** Always check whether an e^x-times-rational integrand splits as $f + f'$.

Step 1. Expand the rational piece.

Step 2. Try $f(x) = 1/(1 + x^2)$.

Step 3. Confirm $f' = -2x/(1 + x^2)^2$.

> **Final Answer:** (A).

Why this matters. The $e^x[f + f']$ shortcut is a top-5 JEE Main MCQ trick. Train your eye to recognise it.

Q 7.52 $\displaystyle\int \frac{x^9\,dx}{(4x^2+1)^6}$ **equals**

(A) $\dfrac{1}{5x}\left(4+\dfrac{1}{x^2}\right)^{-5} + C$ **(B)** $\dfrac{1}{5}\left(4+\dfrac{1}{x^2}\right)^{-5} + C$

(C) $\dfrac{1}{10x}(1+4)^{-5} + C$ **(D)** $\dfrac{1}{10}\left(\dfrac{1}{x^2}+4\right)^{-5} + C$

SOLUTION

Answer. (D).

Concept used. **Sub** $1/x^2 + 4 = u$ (or equivalently, factor x^2 out of $4x^2 + 1$).

Step 1. Write $4x^2 + 1 = x^2(4 + 1/x^2)$. So $(4x^2 + 1)^6 = x^{12}(4 + 1/x^2)^6$. Integrand:

$$\frac{x^9}{x^{12}(4+1/x^2)^6} = \frac{1}{x^3(4+1/x^2)^6}.$$

Step 2. Put $u = 4 + 1/x^2$, $du = -2/x^3\,dx$, so $dx/x^3 = -du/2$. Integral:

$$-\frac{1}{2}\int\frac{du}{u^6} = -\frac{1}{2}\cdot\frac{u^{-5}}{-5} = \frac{1}{10}u^{-5} = \frac{1}{10}\left(4+\frac{1}{x^2}\right)^{-5}.$$

Step 3. Option (D).

> **Final Answer:** (D) $\dfrac{1}{10}\left(\dfrac{1}{x^2}+4\right)^{-5} + C.$

☞ Differentiate to verify

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Factor-out trick. For high-power integrands $\int x^m/(ax^2+b)^n\,dx$, factor x^2 out of the bracket to get an integrand in $1/x^2$, then substitute.

Step 1. Factor $4x^2+1 = x^2(4+1/x^2)$.

Step 2. Sub $u = 4+1/x^2$.

Step 3. Power-rule.

Final Answer: (D).

Why this matters. The factor-out trick converts seemingly hopeless high-power rational integrals into power-rule problems. Top JEE Main + Adv trick.

Q 7.53 **If** $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$, **then**

(A) $a=-1/10, b=-2/5$ **(B)** $a=1/10, b=-2/5$

(C) $a=-1/10, b=2/5$ **(D)** $a=1/10, b=2/5$

SOLUTION

Answer. (C).

Concept used. Partial fractions for $\frac{1}{(x+2)(x^2+1)}$: one term for the linear factor and one $(Bx+C)/(x^2+1)$ for the irreducible quadratic.

Step 1. Write $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$. Multiply both sides by $(x+2)(x^2+1)$: $1 = A(x^2+1)+(Bx+C)(x+2)$.

Step 2. Put $x=-2$ (cover-up): $1 = A((-2)^2+1) = 5A \Rightarrow A = 1/5$.

Step 3. Expand the right side:
$A(x^2+1)+(Bx+C)(x+2) = Ax^2+A+Bx^2+2Bx+Cx+2C$
$=(A+B)x^2+(2B+C)x+(A+2C)$. Matching with $1 = 0\cdot x^2+0\cdot x+1$:
$A+B=0 \Rightarrow B=-A=-1/5;$
$A+2C=1 \Rightarrow 2C = 1-1/5 = 4/5 \Rightarrow C = 2/5;$ (check:
$2B+C = -2/5+2/5 = 0.$ ✓)

Step 4. Integrate term by term: $\int \frac{A}{x+2}\,dx = \frac{1}{5}\log|x+2|;$
$\int \frac{Bx}{x^2+1}\,dx = \frac{B}{2}\log(1+x^2) = -\frac{1}{10}\log|1+x^2|;$
$\int \frac{C}{x^2+1}\,dx = C\tan^{-1}x = \frac{2}{5}\tan^{-1}x.$

Step 5. Sum: $\int \frac{dx}{(x+2)(x^2+1)} = -\frac{1}{10}\log|1+x^2| + \frac{2}{5}\tan^{-1}x + \frac{1}{5}\log|x+2| + C$.
Compare with the given template $a\log|1+x^2| + b\tan^{-1}x + \frac{1}{5}\log|x+2| + C$: $a = -1/10$, $b = 2/5$. Option (C).

Final Answer: (C) $a = -1/10,\ b = 2/5$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Differentiate-and-match (the shortcut). Instead of doing partial fractions, differentiate the RHS template and match against the integrand $1/[(x+2)(x^2+1)]$.

Step 1. Differentiate:
$$\frac{d}{dx}\left[a\log(1+x^2) + b\tan^{-1}x + \tfrac{1}{5}\log|x+2|\right] = \frac{2ax}{1+x^2} + \frac{b}{1+x^2} + \frac{1}{5(x+2)}.$$

Step 2. Place over the common denominator $(1+x^2)(x+2)$ and set the numerator equal to 1: $2ax(x+2) + b(x+2) + \frac{1}{5}(1+x^2) = 1$. Expand:
$\left(2a + \frac{1}{5}\right)x^2 + (4a+b)x + \left(2b + \frac{1}{5}\right) = 1$.

Step 3. Match coefficients. x^2: $2a + 1/5 = 0 \Rightarrow a = -1/10$. x^0: $2b + 1/5 = 1 \Rightarrow b = 2/5$. x^1 check: $4a + b = -2/5 + 2/5 = 0$. ✓

Final Answer: (C) $a = -1/10,\ b = 2/5$.

Why this matters. "Coefficient-matching" questions are pure algebra disguised as integrals. Speed them up by skipping the integration entirely — just differentiate the RHS.

Setup discipline. List the factors of the denominator first; assign one term per factor (linear, repeated linear, irreducible quadratic) with unknown numerators. The cover-up method (substitute the root) handles linear factors instantly; comparing coefficients handles the rest. The integration step itself is then a sum of $\ln$ and $\arctan$ pieces.

Q 7.54 $\int \frac{x^3\,dx}{x+1}$ **equals**

(A) $x + \frac{x^2}{2} + \frac{x^3}{3} - \log|1-x| + C$

(B) $x + \frac{x^2}{2} - \frac{x^3}{3} - \log|1-x| + C$

(C) $x - \frac{x^2}{2} - \frac{x^3}{3} - \log|1+x| + C$

(D) $x - \frac{x^2}{2} + \frac{x^3}{3} - \log|1+x| + C$

SOLUTION

Answer. (D).

Concept used. Polynomial long division for $x^3/(x+1)$.

Step 1. Divide x^3 by $x+1$: $x^3 = (x+1)(x^2-x+1)-1$. So $\dfrac{x^3}{x+1} = x^2 - x + 1 - \dfrac{1}{x+1}$.

Step 2. Integrate: $\dfrac{x^3}{3} - \dfrac{x^2}{2} + x - \log|x+1| + C$. Option (D).

Final Answer: (D) $x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \log|1+x| + C$.

☞ **Polynomial reflex**

Polynomials are continuous and differentiable on all of $\mathbb{R}$. Any limit of a polynomial is just substitution; never run an ε-δ on a polynomial in an exam.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Long division.

Step 1. Divide x^3 by $x+1$.

Step 2. Integrate each term.

Final Answer: (D).

Why this matters. 1-mark template; whenever numerator's degree exceeds denominator's, long-divide first.

Q 7.55 $\displaystyle\int \frac{x+\sin x}{1+\cos x}\,dx$ **equals**

(A) $\log|1+\cos x| + C$ **(B)** $\log|x+\sin x| + C$

(C) $x - \tan(x/2) + C$ **(D)** $x\cdot\tan(x/2) + C$

SOLUTION

Answer. (D).

Concept used. Half-angle identities $1+\cos x = 2\cos^2(x/2)$, $\sin x = 2\sin(x/2)\cos(x/2)$.

Step 1. Numerator: $x + 2\sin(x/2)\cos(x/2)$. Denominator: $2\cos^2(x/2)$.

Step 2. Split: $\dfrac{x}{2\cos^2(x/2)} + \dfrac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} = \frac{x}{2}\sec^2(x/2) + \tan(x/2)$.

Step 3. By parts on $\frac{x}{2}\sec^2(x/2)$: $u = x/2, dv = \sec^2(x/2)\,dx, du = dx/2, v = 2\tan(x/2)$.

Integral $= \frac{x}{2} \cdot 2\tan(x/2) - \int 2\tan(x/2) \cdot dx/2 = x\tan(x/2) - \int \tan(x/2)\, dx$.

Step 4. $\int \tan(x/2)\, dx = -2\log|\cos(x/2)| = \log\sec^2(x/2)$ (etc.). Combine with the leftover $\int \tan(x/2)\, dx$ from step 2: Total $= x\tan(x/2) - \int \tan(x/2)\, dx + \int \tan(x/2)\, dx = x\tan(x/2) + C$. Option (D).

> **Final Answer:** (D) $x\tan(x/2) + C$.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Half-angle reduction + cancellation.

Step 1. Half-angle rewrite turns the integrand into $(x/2)\sec^2(x/2) + \tan(x/2)$.

Step 2. By parts on the first piece produces $x\tan(x/2) - \int \tan(x/2)\, dx$.

Step 3. The $-\int \tan(x/2)$ cancels the $+\int \tan(x/2)$ from the original split.

> **Final Answer:** (D).

Why this matters. Elegant cancellation; CBSE recycles this exact integrand every 2 years.

Sanity check. Differentiate $x\tan(x/2)$: $\tan(x/2) + (x/2)\sec^2(x/2) = \tan(x/2) + (x/2)(1 + \tan^2(x/2))$. Need $(x + \sin x)/(1 + \cos x)$. Check at $x = 0$: derivative $= 0 + 0 = 0$; integrand $= 0/2 = 0$. ✓

Q 7.56 If $\int \frac{x^3\, dx}{\sqrt{1+x^2}} = a(1+x^2)^{3/2} + b\sqrt{1+x^2} + C$, then

(A) $a = 1/3, b = 1$ **(B)** $a = -1/3, b = 1$

(C) $a = -1/3, b = -1$ **(D)** $a = 1/3, b = -1$

SOLUTION

Answer. (D).

Concept used. Sub $u = 1 + x^2$, $du = 2x\, dx$. $x^2 = u - 1$.

Step 1. $\int \frac{x^3}{\sqrt{1+x^2}}\, dx = \int \frac{x^2 \cdot x}{\sqrt{1+x^2}}\, dx = \int \frac{(u-1)}{\sqrt{u}} \cdot \frac{du}{2} = \frac{1}{2}\int (u^{1/2} - u^{-1/2})\, du$.

Step 2. Integrate: $\frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right] = \frac{1}{3}u^{3/2} - u^{1/2}$.

Step 3. Back-substitute: $\frac{1}{3}(1+x^2)^{3/2} - \sqrt{1+x^2} + C$. So $a = 1/3, b = -1$. Option (D).

Final Answer: (D) $a = 1/3,\ b = -1$.

☞ **Conjugate first**

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x cancels cleanly.

EXPERT'S SOLUTION *: Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Sub $u = 1 + x^2$.

Step 1. Use $x^2 = u - 1$.

Step 2. Integrate $u^{1/2} - u^{-1/2}$.

Step 3. Read off a, b.

Final Answer: (D).

Why this matters. The substitution $u = 1 + x^2$ converts $x^3\,dx/\sqrt{1+x^2}$ into a clean polynomial-in-$u^{1/2}$. Master template.

Q 7.57 Evaluate $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

SOLUTION

Answer. (A).

Concept used. Double-angle $1 + \cos 2x = 2\cos^2 x$.

Step 1. $\dfrac{1}{1+\cos 2x} = \dfrac{1}{2\cos^2 x} = \frac{1}{2}\sec^2 x$.

Step 2. Integral: $\frac{1}{2}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2}[1 - (-1)] = 1$.

Final Answer: (A) 1.

☞ **Half-angle identity**

Memorise $1 - \cos\theta = 2\sin^2(\theta/2)$. It converts every $1 - \cos$ limit into a $\sin u/u$ limit, which is routine.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Half-angle of $\cos 2x$.

Step 1. $1 + \cos 2x = 2\cos^2 x$.

Step 2. $\sec^2 x$ integrates to $\tan x$.

> **Final Answer:** (A).

Why this matters. 1-mark template; recognise $1 + \cos 2x = 2\cos^2 x$ at sight.

Q 7.58 **Evaluate** $\int_0^{\pi/2} \sqrt{1 - \sin 2x}\, dx$.

(A) $2\sqrt{2}$ **(B)** $2(\sqrt{2} + 1)$ **(C)** 2 **(D)** $2(\sqrt{2} - 1)$

SOLUTION

Answer. (D).

Concept used. Perfect square $1 - \sin 2x = (\sin x - \cos x)^2$.

Step 1. $\sqrt{1 - \sin 2x} = |\sin x - \cos x|$.

Step 2. On $[0, \pi/4]$, $\cos x \geq \sin x$, so $|\sin x - \cos x| = \cos x - \sin x$. On $[\pi/4, \pi/2]$, $\sin x \geq \cos x$, so $|\sin x - \cos x| = \sin x - \cos x$.

Step 3. $\int_0^{\pi/4} (\cos x - \sin x)\, dx = [\sin x + \cos x]_0^{\pi/4} = \sqrt{2} - 1$.
$\int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx = [-\cos x - \sin x]_{\pi/4}^{\pi/2} = -1 - (-\sqrt{2}) = \sqrt{2} - 1$.

Step 4. Sum: $2(\sqrt{2} - 1)$. Option (D).

> **Final Answer:** (D) $2(\sqrt{2} - 1)$.

☞ **Differentiate to verify**

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Branch-aware integration.

Step 1. Perfect-square the radicand.

Step 2. Split the interval at $x = \pi/4$ where $\sin x = \cos x$.

Step 3. Add the two pieces.

Final Answer: (D).

Why this matters. "$\sqrt{1 \pm \sin 2x}$" on intervals crossing $\pi/4$ or $3\pi/4$ requires interval-splitting. Don't drop the absolute-value bars.

IV. Fill in the Blanks

Q 7.59 $\displaystyle\int_0^{\pi/2} \cos x\, e^{\sin x}\, dx =$ ________.

SOLUTION

Concept used. Sub $u = \sin x$, $du = \cos x\, dx$.

Step 1. Integral $= \int_0^1 e^u\, du = e - 1$.

Final Answer: $e - 1$.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Pure $u = \sin x$ template.

Step 1. $u = \sin x$ kills the $\cos x\, dx$.

Step 2. $\int e^u\, du = e^u$.

Final Answer: $e - 1$.

Why this matters. "$\cos x\, e^{\sin x}$" is one of the cleanest substitutions; recognise on sight.

Q 7.60 $\displaystyle\int \frac{x+3}{(x+4)^2}\, e^x\, dx =$ ________.

SOLUTION

Concept used. Pattern $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$.

Step 1. Try $f(x) = 1/(x+4)$. Then $f'(x) = -1/(x+4)^2$.

$$f(x) + f'(x) = \frac{1}{x+4} - \frac{1}{(x+4)^2} = \frac{(x+4)-1}{(x+4)^2} = \frac{x+3}{(x+4)^2}. \checkmark$$

Step 2. Hence integral $= e^x f(x) + C = \dfrac{e^x}{x+4} + C$.

Final Answer: $\dfrac{e^x}{x+4} + C$.

☞ **Differentiate to verify**

After integrating, differentiate the answer mentally and check it equals the original integrand. This 5-second check catches almost every algebra slip.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

$e^x[f + f']$ **recognition.**

Step 1. Try $f = 1/(x+4)$.

Step 2. Check $f + f' = (x+3)/(x+4)^2$.

Step 3. Read off the answer.

Final Answer: $e^x/(x+4) + C$.

Why this matters. "$(x+k)/(x+k+1)^2 \cdot e^x$" is the canonical $f + f'$ setup. JEE Main reuses it every 2 sessions.

Q 7.61 If $\displaystyle\int_0^a \frac{dx}{1+4x^2} = \frac{\pi}{8}$, then $a =$ __________.

SOLUTION

Concept used. Standard $\int dx/(1+(bx)^2) = (1/b)\tan^{-1}(bx)$.

Step 1. $\int_0^a \dfrac{dx}{1+4x^2} = \frac{1}{2}\tan^{-1}(2x)\big|_0^a = \frac{1}{2}\tan^{-1}(2a)$.

Step 2. Set $\frac{1}{2}\tan^{-1}(2a) = \pi/8 \Rightarrow \tan^{-1}(2a) = \pi/4 \Rightarrow 2a = 1 \Rightarrow a = 1/2$.

Final Answer: $a = 1/2$.

✗ Limits get substituted

When you substitute $u = g(x)$ inside a definite integral, the new limits are $u = g(a)$ and $u = g(b)$, NOT a and b. Forgetting to change the limits is the most-marked error in board scripts.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Solve-for-the-limit.

Step 1. Use the standard $\tan^{-1}$ result.

Step 2. Solve the resulting trig equation.

Final Answer: $a = 1/2$.

Why this matters. "Find a given a definite integral equals a target" is a 1-mark template; mechanical.

Q 7.62 $\int \frac{\sin x}{3 + 4\cos^2 x}\, dx =$ __________.

SOLUTION

Concept used. **Sub** $u = \cos x$, $du = -\sin x\, dx$, then arctan.

Step 1. Integral

$$= -\int \frac{du}{3 + 4u^2} = -\int \frac{du}{4(u^2 + 3/4)} = -\frac{1}{4} \cdot \frac{1}{\sqrt{3}/2} \tan^{-1} \frac{u}{\sqrt{3}/2} = -\frac{1}{2\sqrt{3}} \tan^{-1} \frac{2u}{\sqrt{3}}.$$

Step 2. Back-substitute: $-\frac{1}{2\sqrt{3}} \tan^{-1} \frac{2\cos x}{\sqrt{3}} + C$.

Final Answer: $-\frac{1}{2\sqrt{3}} \tan^{-1} \frac{2\cos x}{\sqrt{3}} + C$.

☞ Conjugate first

For any $0/0$ limit of the form $(\sqrt{A} \pm \sqrt{B})/x$, multiply numerator and denominator by the conjugate $\sqrt{A} \mp \sqrt{B}$. The radicals become a polynomial difference and the offending x

cancels cleanly.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Mathematics, IIT Kanpur*

Standard $u = \cos x$ + arctan.

Step 1. Sub.

Step 2. Arctan template.

Step 3. Back-substitute.

Final Answer: $-\frac{1}{2\sqrt{3}} \tan^{-1}(2\cos x/\sqrt{3}) + C$.

Why this matters. 1-mark template; the $\sin x\, dx$ in the numerator cues $u = \cos x$ immediately.

Q 7.63 $\int_{-\pi}^{\pi} \sin^3 x \cos^2 x\, dx =$ ________.

SOLUTION

Concept used. Odd function on symmetric interval integrates to 0.

Step 1. $\sin^3(-x)\cos^2(-x) = -\sin^3 x \cdot \cos^2 x$. So integrand is odd.

Step 2. Integral over $[-\pi, \pi]$ (symmetric about 0) equals 0.

Final Answer: 0.

☞ **Trig continuity**

$\sin x$ and $\cos x$ are continuous and differentiable on all of $\mathbb{R}$. $\tan x$ is continuous on its domain (excluding $x = \frac{\pi}{2} + k\pi$); list the exclusions explicitly before using continuity.

EXPERT'S SOLUTION : *Sneha Pillai, M.Sc Mathematics, IIT Bombay*

Even/odd test first. Always test even/odd on symmetric intervals before computing.

Step 1. Replace $x \to -x$.

Step 2. $\sin^3 x$ flips sign (odd power); $\cos^2 x$ unchanged (even power).

Step 3. Product is odd; integral over $[-a, a]$ is 0.

Final Answer: 0.

Why this matters. The 1-second answer to "odd on $[-a, a]$". Don't waste time computing.

Key Takeaways

- **Antiderivative family:** $\int f(x)\,dx = F(x) + C$; always carry the $+C$.
- g'/g **template:** $\int g'(x)/g(x)\,dx = \log|g(x)| + C$. Spot at sight.
- **Standard table:** memorise power, $1/x$, e^x, a^x, six trig, two inverse-trig; together they handle 30% of every paper.
- **Substitution patterns:** $\sqrt{x} = t$; $\tan x = t$; $\sin x = t$ or $\cos x = t$; $x^2 = t$; $x = 1/t$ (reciprocal); $x = a\tan^2\theta$ for $\sqrt{x/(a+x)}$.
- **Partial fractions:** distinct linear factors → cover-up rule; quadratic factors → $(Ax + B)/(\text{quadratic})$, then split into log + arctan.
- **Integration by parts (ILATE):** $I > L > A > T > E$; pick u as the earliest in this list.
- **Special integrals:** $\int dx/(x^2 \pm a^2)$, $\int dx/\sqrt{a^2 - x^2}$, $\int dx/\sqrt{x^2 \pm a^2}$, $\int \sqrt{a^2 \pm x^2}\,dx$, $\int \sqrt{x^2 - a^2}\,dx$.
- $e^x[f + f']$ **shortcut:** $\int e^x[f(x) + f'(x)]\,dx = e^x f(x) + C$. Top JEE Main trick.
- **Definite-integral properties:** swap limits flips sign; equal limits give 0; split at c; King rule $\int_0^a f(x)dx = \int_0^a f(a-x)dx$; even/odd collapse on $[-a, a]$; doubling rule on $[0, 2a]$.
- **Classical values:** $\int_0^{\pi/2} \log\sin x\,dx = \int_0^{\pi/2} \log\cos x\,dx = -(\pi\log 2)/2$.
- **Watch for:** dropped $+C$; missing absolute-value bars in $\log|g|$; wrong choice of u in by-parts; partial-fraction template mismatch.

End of NCERT Exemplar Problems