Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 7: Integrals

What this chapter covers: integration as the inverse of differentiation, standard integral forms, the three big techniques (substitution, partial fractions, integration by parts), special integrals of $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$, the definite integral, the Fundamental Theorem of Calculus, and the seven properties used to simplify definite integrals. Sections marked *[JEE/NEET Extension]* cover definite integral as a limit of a sum — removed from the rationalised NCERT but still tested in entrance exams.

1 Integration: The Inverse of Differentiation

Differential calculus answers: "given a function, find its rate of change." Integral calculus reverses the question: "given the rate of change, can we recover the original function?" This reverse process is called **integration** or **anti-differentiation**, and the family of functions it produces is the **indefinite integral**.

A second, geometrically motivated question — "what is the area under a curve?" — turns out, remarkably, to be solved by the same machinery. The bridge between the two is the Fundamental Theorem of Calculus. This chapter develops both viewpoints and the techniques needed to compute integrals that do not yield to a one-line guess.

1.1 Anti-derivative and indefinite integral

If $F'(x) = f(x)$ on an interval I, then F is called an **anti-derivative** (or **primitive**) of f. Anti-derivatives are not unique: for any real constant C,

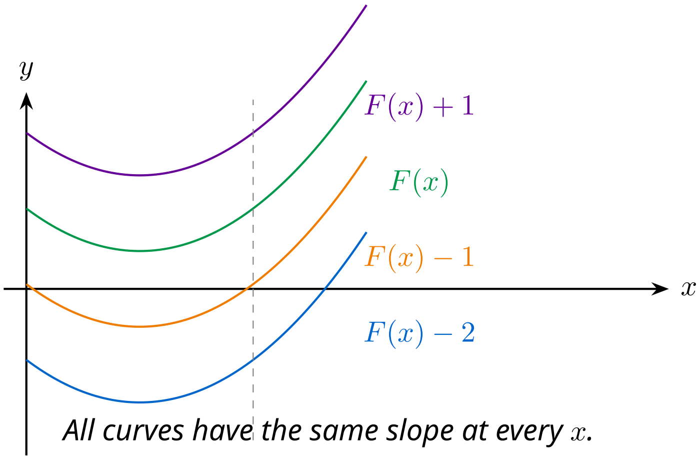
$$\frac{d}{dx}\big[F(x) + C\big] = F'(x) = f(x).$$

So $\{F + C : C \in \mathbb{R}\}$ is an entire family of anti-derivatives. The symbol $\int f(x)\,dx$ denotes this whole family and is read "the indefinite integral of f with respect to x."

Indefinite Integral — Definition

$$\int f(x)\,dx = F(x) + C, \qquad \text{where } F'(x) = f(x).$$

Here $f(x)$ is the *integrand*, x is the *variable of integration*, and C is the *constant of integration*.



All curves have the same slope at every x.

Geometrically, the family $F(x) + C$ is an infinite stack of vertically shifted copies of the same curve. Every copy has the *same slope* at every x, which is exactly what $F'(x) = f(x)$ records.

Why the "$+C$" Matters

Functions with the same derivative on an interval differ by a constant. So whenever we write $\int f(x)\,dx$, we are naming a *family*, not a single function. Forgetting $+C$ is the most common mark-loser in this chapter.

Contents

1.2 Standard integrals: the look-up table

Every derivative rule from Class 11 gives a corresponding integral. The thirteen results below are the foundation of the entire chapter — you must recognise them on sight.

Standard Indefinite Integrals

Integrand	Integral (add $+C$)
$x^n \quad (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
1	x
$\frac{1}{x}$	$\log \lvert x \rvert$
e^x	e^x
a^x	$\frac{a^x}{\log a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\text{cosec}^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\text{cosec}\, x \cot x$	$-\text{cosec}\, x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Common Mistake

$\int \frac{dx}{x} = \log |x| + C$, *not* $\log x$. The absolute value matters because log is undefined for negative reals. Examiners deduct marks for missing the modulus.

1.3 Properties of indefinite integrals

The integral is a linear operator. Three rules cover almost every algebraic simplification in this chapter.

Linearity & Inverse Properties

$$\text{(I)} \quad \frac{d}{dx}\int f(x)\,dx = f(x)$$

$$\int f'(x)\,dx = f(x) + C$$

$$\text{(II)} \quad \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

$$\text{(III)} \quad \int k\,f(x)\,dx = k\int f(x)\,dx \quad (k \in \mathbb{R})$$

Quick Tip

> Always split a sum or pull out constants *before* reaching for substitution or by-parts. Many "hard" integrals collapse to a row of the standard table after one line of algebra.

1.4 Method of inspection: worked examples

The fastest way to integrate is to *guess* an anti-derivative and verify by differentiation. The skill is pattern recognition.

Example. An anti-derivative of $\cos 2x$ is $\frac{1}{2}\sin 2x$, because $\dfrac{d}{dx}\left(\frac{1}{2}\sin 2x\right) = \cos 2x$.

Example. Find $\displaystyle\int \left(x^{3/2} + 2e^x - \tfrac{1}{x}\right) dx$.

Split by linearity:

$$\int x^{3/2}\,dx + 2\int e^x\,dx - \int \tfrac{1}{x}\,dx = \tfrac{2}{5}x^{5/2} + 2e^x - \log|x| + C.$$

Example. $\displaystyle\int (\sin x + \cos x)\,dx = -\cos x + \sin x + C$. Two table look-ups, added.

Workflow for a New Integrand

1. Algebraic simplification first — expand brackets, split fractions, use identities.
2. Compare each piece against the standard table.
3. Only if no piece matches, move to substitution (Section 2.1) or by-parts (Section 5).

1.5 Differentiation vs. Integration: a side-by-side comparison

Differentiation	**Integration**
Always exists for differentiable f.	May not be expressible in elementary form (e.g. e^{-x^2}).
Sum / product / quotient rules cover almost everything.	Substitution, by-parts, partial fractions — choice of method matters.
Output of $f(x)$ is a *single* function $f'(x)$.	Output is a *family* of functions $F(x) + C$.
Mechanical: differentiate term by term.	Often requires creative manipulation or a non-obvious substitution.
Useful for slopes, rates, and optimisation.	Useful for areas, totals, displacements, probabilities.

2 Methods of Integration

For functions outside the standard table, the integral is rarely guessable by inspection. NCERT develops three general-purpose techniques:

1. Integration by substitution (Section 2.1)
2. Integration using trigonometric identities (Section 2.2)
3. Integration of some particular functions (Section 3)

The first turns one variable into another that simplifies the integrand. The second uses identities to rewrite trig powers and products. The third applies a small list of memorised results to quadratic-in-denominator forms.

2.1 Integration by substitution

Substitution is the chain rule run in reverse. If we set $x = g(t)$, then $dx = g'(t)\,dt$ and

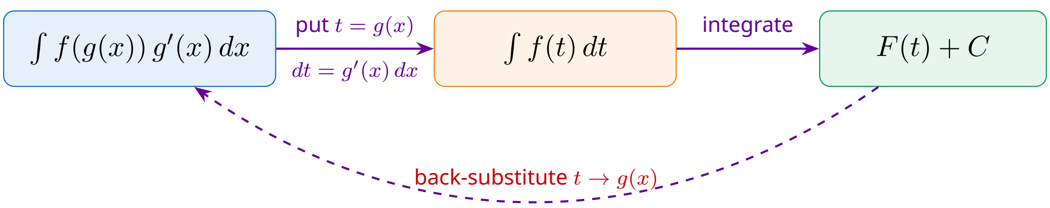
$$\int f(x)\,dx = \int f\big(g(t)\big)\,g'(t)\,dt.$$

The art is choosing g so that the new integrand is one of the standard forms. The cue is usually: *the derivative of an inner function already sits in the integrand*.

Substitution Rule

If $\int f(x)\,dx = F(x) + C$, then

$$\int f\big(g(t)\big)\,g'(t)\,dt = F\big(g(t)\big) + C.$$



Worked example. Compute $\int 2x\,\sin(x^2+1)\,dx$.

Notice that $\dfrac{d}{dx}(x^2+1) = 2x$, which already appears multiplicatively. Put $t = x^2 + 1$, so $dt = 2x\,dx$:

$$\int 2x\,\sin(x^2+1)\,dx = \int \sin t\,dt = -\cos t + C = -\cos(x^2+1) + C.$$

Quick Tip

Scan the integrand for an inner function whose derivative is also present (perhaps up to a constant). That inner function is your t. If no such inner-derivative pair exists, substitution is probably not the right move.

Standard integrals via substitution

Four trig integrals are obtained by substitution and then used as look-ups everywhere:

Trig Standard Forms

$$\int \tan x\,dx = \log|\sec x| + C$$
$$\int \cot x\,dx = \log|\sin x| + C$$
$$\int \sec x\,dx = \log|\sec x + \tan x| + C$$
$$\int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

Memory Aid

TCSC (**T**an gives se**C**, co**T** gives **S**in, se**C** integrates with se**C**+tan, **C**osec integrates with cosec − cot). The first two integrals carry a positive log; the last two never need a minus inside the modulus on the right side.

2.2 Integration using trigonometric identities

When the integrand is a power or product of trig functions, a clever identity often turns it into a sum of standard pieces.

Identities You Will Use Constantly

$$\sin^2 x = \tfrac{1-\cos 2x}{2}, \quad \cos^2 x = \tfrac{1+\cos 2x}{2}$$
$$\sin^3 x = \tfrac{3\sin x - \sin 3x}{4}, \quad \cos^3 x = \tfrac{3\cos x + \cos 3x}{4}$$
$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\cos B = \cos(A-B) + \cos(A+B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

For example,

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{x}{2} + \tfrac{1}{4}\sin 2x + C.$$

Pattern Recognition for Trig Integrals

- Even powers of sin/cos $\to$ double-angle reduction.
- Odd powers of sin/cos $\to$ peel one off and substitute the other (e.g. for $\sin^3 x$, write $\sin^2 x \cdot \sin x = (1-\cos^2 x)\sin x$, put $t = \cos x$).
- Products $\sin mx \cos nx$ etc. $\to$ product-to-sum identities.

Worked example. Find $\displaystyle\int \sin 2x \cos 3x \, dx$.

Using $2\sin A \cos B = \sin(A+B) + \sin(A-B)$:

$$\sin 2x \cos 3x = \tfrac{1}{2}\big[\sin 5x + \sin(-x)\big] = \tfrac{1}{2}\sin 5x - \tfrac{1}{2}\sin x.$$

Therefore

$$\int \sin 2x \cos 3x \, dx = -\tfrac{1}{10}\cos 5x + \tfrac{1}{2}\cos x + C.$$

Worked example. Find $\int \sin^3 x \, dx$.

Use the identity $\sin 3x = 3\sin x - 4\sin^3 x$, so $\sin^3 x = \frac{3\sin x - \sin 3x}{4}$. Then

$$\int \sin^3 x \, dx = \tfrac{3}{4}\int \sin x \, dx - \tfrac{1}{4}\int \sin 3x \, dx = -\tfrac{3}{4}\cos x + \tfrac{1}{12}\cos 3x + C.$$

3 Integrals of Some Particular Functions

Six quadratic-denominator forms occur so often that NCERT lists them as standard results. Memorise the table; you will use it on almost every problem involving completing the square.

Six Standard Integrals

(1) $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$$

(2) $$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$$

(3) $$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

(4) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left|x + \sqrt{x^2 - a^2}\right| + C$$

(5) $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

(6) $$\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left|x + \sqrt{x^2 + a^2}\right| + C$$

Quick Tip

Note the sign pattern: $a^2 - x^2$ inside the radical gives $\sin^{-1}$, $x^2 + a^2$ in the denominator gives $\tan^{-1}/a$, and anything with $\sqrt{x^2 \pm a^2}$ gives a logarithm. Picking the wrong row is the most common error.

Common Mistake

Confusing (1) and (2): when the quadratic is $x^2 - a^2$, the inside of the log is $\frac{x-a}{x+a}$. When it is $a^2 - x^2$, the inside is $\frac{a+x}{a-x}$. Both look similar, but the constants and signs differ.

3.1 Reducing general quadratic integrals

Integrals of the type

$$\int \frac{dx}{ax^2 + bx + c}, \qquad \int \frac{dx}{\sqrt{ax^2 + bx + c}}$$

are handled by *completing the square*: write

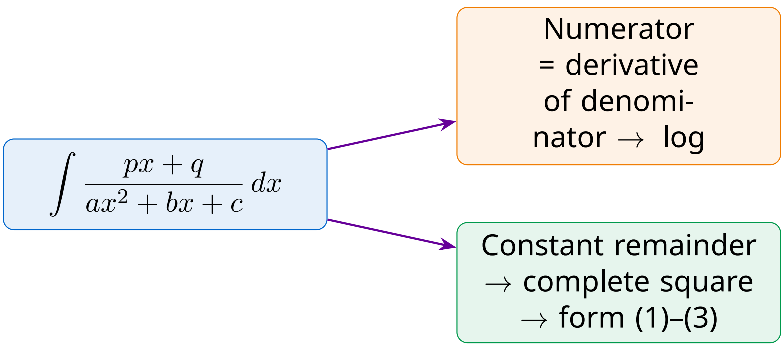
$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 + \left(\frac{c}{a} - \frac{b^2}{4a^2}\right)\right].$$

Setting $t = x + b/(2a)$ reduces the integral to one of forms (1)–(6).

For

$$\int \frac{px + q}{ax^2 + bx + c}\,dx \quad \text{or} \quad \int \frac{px + q}{\sqrt{ax^2 + bx + c}}\,dx,$$

write $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A(2ax + b) + B$. Find A, B by comparing coefficients. The first part integrates as a log (or square root); the second reduces to a known form.



Three-Step Recipe

Step 1. If the numerator is linear, split it into "derivative of denominator" + "constant." **Step 2.** Integrate the first piece directly (log or surd). **Step 3.** Complete the square in what remains and apply the matching standard integral.

Worked example. $\int \frac{dx}{x^2 - 6x + 13}$.

Complete the square: $x^2 - 6x + 13 = (x-3)^2 + 4 = (x-3)^2 + 2^2$. Put $t = x - 3$:

$$\int \frac{dt}{t^2 + 2^2} = \tfrac{1}{2}\tan^{-1}\tfrac{t}{2} + C = \tfrac{1}{2}\tan^{-1}\left(\tfrac{x-3}{2}\right) + C.$$

Worked example. $\int \frac{x+2}{2x^2 + 6x + 5}\,dx$.

The derivative of the denominator is $4x + 6$. Write

$$x + 2 = A(4x + 6) + B = \tfrac{1}{4}(4x + 6) + \tfrac{1}{2}.$$

So

$$\int \frac{x+2}{2x^2+6x+5}\,dx = \tfrac{1}{4}\int \frac{4x+6}{2x^2+6x+5}\,dx + \tfrac{1}{2}\int \frac{dx}{2x^2+6x+5}.$$

The first piece equals $\frac{1}{4}\log\lvert 2x^2 + 6x + 5\rvert$; the second is a standard $\tan^{-1}$ after completing the square. The final answer:

$$\tfrac{1}{4}\log\lvert 2x^2 + 6x + 5\rvert + \tfrac{1}{2}\tan^{-1}(2x + 3) + C.$$

4 Integration by Partial Fractions

A rational function $\frac{P(x)}{Q(x)}$ is called **proper** if $\deg P < \deg Q$ and **improper** otherwise. Improper rationals reduce to proper ones by long division; the polynomial part integrates term-by-term.

A proper rational function with a factorable denominator can be split into a sum of simpler fractions whose integrals are known. The five standard decompositions are catalogued below.

Partial Fraction Decompositions

Rational function form	Partial fraction form
$\dfrac{px+q}{(x-a)(x-b)},\ a \neq b$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}$
$\dfrac{px+q}{(x-a)^2}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}$
$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

(x^2+bx+c in the last row is assumed irreducible over the reals.)

After decomposition, each piece is a standard integral:

- $\dfrac{A}{x-a} \to A\log|x-a|$
- $\dfrac{B}{(x-a)^2} \to -\dfrac{B}{x-a}$
- $\dfrac{Bx+C}{x^2+bx+c} \to$ split (as in Section 3.1) into a log piece and a $\tan^{-1}$ piece.

Quick Tip

Cover-up rule for distinct linear factors. For $\dfrac{P(x)}{(x-a)(x-b)\cdots}$, the coefficient A over $(x-a)$ equals $\dfrac{P(a)}{(a-b)\cdots}$. Just "cover up" the $(x-a)$ factor and plug $x=a$ into what's left. Saves an entire system of equations.

Worked example. Find $\int \dfrac{dx}{(x+1)(x+2)}$.

Write $\dfrac{1}{(x+1)(x+2)} = \dfrac{A}{x+1}+\dfrac{B}{x+2}$. Cover-up gives $A=1$, $B=-1$. So

$$\int \frac{dx}{(x+1)(x+2)} = \log|x+1| - \log|x+2| + C = \log\left|\frac{x+1}{x+2}\right| + C.$$

Common Mistake

For an improper fraction (degree of numerator $\geq$ degree of denominator), *always* divide first. Trying to write $\dfrac{x^2+1}{x-1}$ directly as partial fractions is impossible and wastes time.

Worked example (improper fraction). $\displaystyle\int \frac{x^2+1}{x^2-5x+6}\,dx.$

Since degrees are equal, divide first:

$$\frac{x^2+1}{x^2-5x+6} = 1 + \frac{5x-5}{(x-2)(x-3)}.$$

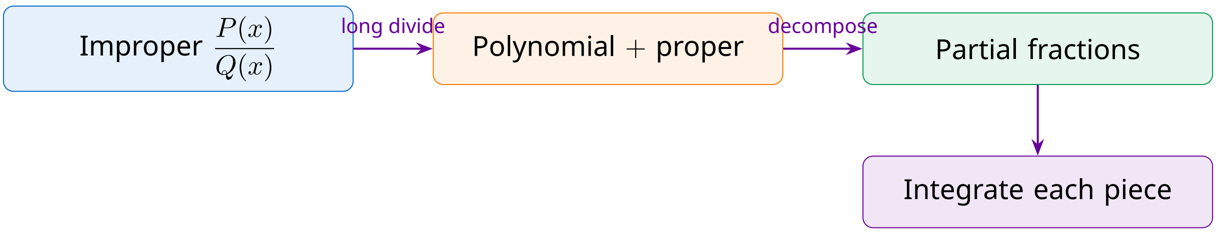
Decompose: $\dfrac{5x-5}{(x-2)(x-3)} = \dfrac{-5}{x-2} + \dfrac{10}{x-3}$. Hence

$$\int \frac{x^2+1}{x^2-5x+6}\,dx = x - 5\log|x-2| + 10\log|x-3| + C.$$

Worked example (repeated linear factor). $\displaystyle\int \frac{3x-2}{(x+1)^2(x+3)}\,dx.$

By Table row 4: $\dfrac{3x-2}{(x+1)^2(x+3)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+3}$. Equating coefficients gives $A = \frac{11}{4}$, $B = -\frac{5}{2}$, $C = -\frac{11}{4}$. Therefore

$$\int \frac{3x-2}{(x+1)^2(x+3)}\,dx = \tfrac{11}{4}\log\left|\frac{x+1}{x+3}\right| + \frac{5}{2(x+1)} + C.$$



5 Integration by Parts

The product rule $(uv)' = u'v + uv'$, integrated, gives the most powerful general technique in the chapter:

Integration by Parts

$$\int u\,v\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx.$$

In words: "the integral of a product equals (first function) × (integral of second) − integral of [(derivative of first) × (integral of second)]."

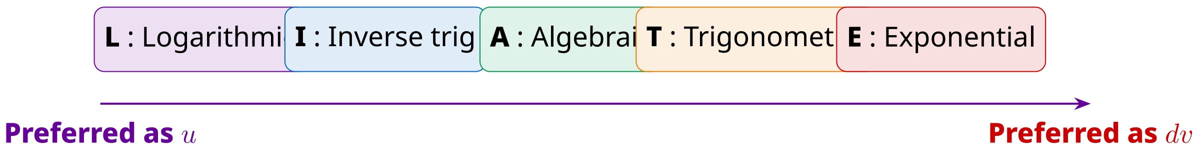
The result is only useful if the new integral on the right is simpler than the original. So the choice of which factor is "first" (u) and which is "second" (v) matters enormously.

5.1 LIATE: choosing the first function

The LIATE rule orders five function families by how desirable each is as the *first* function (the one we differentiate).

Memory Aid

LIATE:
L – Logarithmic ($\log x$)
I – Inverse trigonometric ($\sin^{-1} x$, $\tan^{-1} x, \ldots$)
A – Algebraic (x^n, polynomials)
T – Trigonometric ($\sin x$, $\cos x$)
E – Exponential (e^x, a^x)
Whichever class is *higher on the list* becomes u. So for $\int x \log x \, dx$, log beats algebra: $u = \log x$.



Worked example. Compute $\int x \cos x \, dx$.

Algebra beats trig in LIATE, so $u = x$, $dv = \cos x \, dx$. Then $du = dx$, $v = \sin x$, and

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Common Mistake

Swapping the choice ($u = \cos x$, $dv = x \, dx$) yields $\int x \cos x \, dx = \frac{x^2}{2} \cos x + \int \frac{x^2}{2} \sin x \, dx$, which is *worse* than the original. The integral has not been reduced — it has been promoted.

5.2 The $\int e^x [f(x) + f'(x)] \, dx$ formula

A by-parts result that is asked in some form almost every year:

A Useful Special Form

$$\int e^x \left[f(x) + f'(x) \right] dx = e^x f(x) + C.$$

The proof is one line: differentiate the right side and watch the product rule give back the integrand.

Worked example. $\int e^x \left(\tan^{-1} x + \dfrac{1}{1 + x^2} \right) dx$. With $f(x) = \tan^{-1} x$, we have $f'(x) = \dfrac{1}{1 + x^2}$, so the integral is $e^x \tan^{-1} x + C$.

Quick Tip

Whenever you see e^x multiplying a sum where one term is the derivative of the other, jump straight to this formula. Do not actually do by-parts.

5.3 Worked by-parts examples

Example 1. $\int \log x\, dx$. There is no obvious second factor — introduce 1.

Take $u = \log x$, $dv = 1\, dx$, so $du = \frac{1}{x} dx$, $v = x$:

$$\int \log x\, dx = x \log x - \int x \cdot \tfrac{1}{x}\, dx = x \log x - x + C.$$

Example 2. $\int xe^x\, dx$. By LIATE, $u = x$, $dv = e^x dx$:

$$\int xe^x\, dx = xe^x - \int e^x\, dx = xe^x - e^x + C = (x-1)e^x + C.$$

Example 3 (repeated by-parts). $I = \int e^x \sin x\, dx$.

Take $u = e^x$, $dv = \sin x\, dx$. Then

$$I = -e^x \cos x + \int e^x \cos x\, dx.$$

Apply by-parts again on the right side ($u = e^x$, $dv = \cos x\, dx$):

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx = e^x \sin x - I.$$

Substitute: $I = -e^x \cos x + e^x \sin x - I$, so $2I = e^x(\sin x - \cos x)$, giving

$$\int e^x \sin x\, dx = \tfrac{e^x}{2}(\sin x - \cos x) + C.$$

Recursive By-Parts

When by-parts produces an integral that looks like the original, do *not* despair. Apply by-parts again; the original integral usually re-appears with a different coefficient, and you solve algebraically. This trick handles every $\int e^{ax} \sin bx\, dx$ and $\int e^{ax} \cos bx\, dx$.

5.4 Integrals of $\sqrt{x^2 \pm a^2}$ and $\sqrt{a^2 - x^2}$

Applying by-parts with the constant function 1 as the second factor produces three further standard results — handle them as look-ups.

Square-Root Standard Forms

$$\int \sqrt{x^2-a^2}\,dx = \tfrac{x}{2}\sqrt{x^2-a^2} - \tfrac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C$$
$$\int \sqrt{x^2+a^2}\,dx = \tfrac{x}{2}\sqrt{x^2+a^2} + \tfrac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right| + C$$
$$\int \sqrt{a^2-x^2}\,dx = \tfrac{x}{2}\sqrt{a^2-x^2} + \tfrac{a^2}{2}\sin^{-1}\tfrac{x}{a} + C$$

Pattern

Each result is "$\frac{x}{2}$ times the original surd, plus $\frac{a^2}{2}$ times a familiar inverse function (log or $\sin^{-1}$)." The sign in front of the second piece matches the sign inside the original radical.

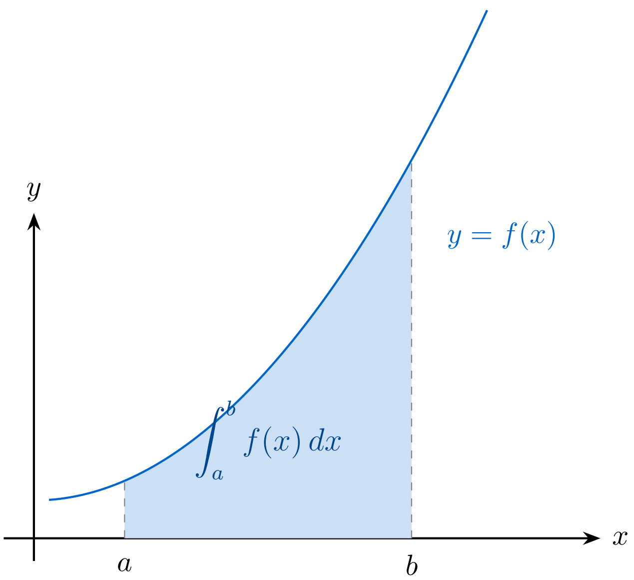
6 The Definite Integral

A **definite integral** pairs the indefinite integral with two real limits a (lower) and b (upper):

$$\int_a^b f(x)\,dx.$$

Unlike the indefinite integral, this is a single real number, not a family of functions. The constant of integration C cancels, so it never appears in the answer.

Geometrically, for a positive integrand $f(x) > 0$ on $[a, b]$, the definite integral equals the area of the region bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$, $x = b$.



Two Equivalent Definitions

The definite integral $\int_a^b f(x)\,dx$ can be introduced either

- as the **limit of a Riemann sum** (originally Section 7.7 of NCERT; see Section 9 for the JEE/NEET extension), or
- as the **difference of anti-derivative values** $F(b) - F(a)$, via the Fundamental Theorem.

Both routes give the same number for continuous integrands.

6.1 Fundamental Theorem of Calculus

Two theorems make the link between the area question and the anti-derivative question precise.

First Fundamental Theorem. Let f be continuous on $[a, b]$, and define the *area function*

$$A(x) = \int_a^x f(t)\,dt, \qquad x \in [a, b].$$

Then A is differentiable on (a, b) and $A'(x) = f(x)$ for every $x \in [a, b]$.

Second Fundamental Theorem. Let f be continuous on $[a, b]$ and let F be *any* anti-derivative of f. Then

$$\int_a^b f(x)\,dx = \big[F(x)\big]_a^b = F(b) - F(a).$$

Fundamental Theorem of Calculus

For continuous f on $[a, b]$:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x), \qquad \int_a^b f(x)\,dx = F(b) - F(a).$$

The first theorem says integration "un-does" differentiation; the second turns area into an arithmetic problem.

Real-World Application

Speed–distance and area-under-curve are the same calculation. If $v(t)$ is the velocity of a body at time t, then $\int_a^b v(t)\,dt$ equals the displacement between times a and b. This is exactly why definite integration is unavoidable in physics, economics, and probability.

6.2 Three-step procedure

1. Find an anti-derivative $F(x)$ of f. The constant C is unnecessary.
2. Evaluate F at the upper and lower limits.

3. Subtract: $F(b) - F(a)$.

Example. $\int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$.

Common Mistake

The integrand must be continuous on $[a, b]$. Writing $\int_{-1}^{1} \frac{1}{x^2}\,dx$ as $[-1/x]_{-1}^{1} = -2$ is *wrong* — the function is unbounded at $x = 0$. Always check the domain before invoking the second fundamental theorem.

7 Substitution in Definite Integrals & Properties

When a definite integral demands a substitution $t = g(x)$, you have two options:

1. Integrate as an indefinite integral, back-substitute to x, then evaluate at the original limits a and b.
2. *Change the limits along with the variable*: as x runs from a to b, t runs from $g(a)$ to $g(b)$. Evaluate the new integral in t at those new limits, with no back-substitution.

Method 2 is usually faster.

Example. Compute $\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx$. Put $t = x^5 + 1$, $dt = 5x^4\,dx$. When $x = -1$, $t = 0$; when $x = 1$, $t = 2$. So

$$\int_{-1}^{1} 5x^4\sqrt{x^5+1}\,dx = \int_0^2 \sqrt{t}\,dt = \left[\tfrac{2}{3}t^{3/2}\right]_0^2 = \tfrac{4\sqrt{2}}{3}.$$

Quick Tip

Change-of-limits is required if the substitution is not invertible in elementary form over $[a, b]$ (e.g. $t = \sin x$ on $[0, \pi]$ folds back on itself). Otherwise either method works.

7.1 Seven properties of definite integrals

The properties below are the engine of *every* cleverly simplified board/JEE definite-integral problem. Learn them by name.

Properties of Definite Integrals

$$P_0:\ \int_a^b f(x)\,dx=\int_a^b f(t)\,dt \quad \text{(dummy variable)}$$
$$P_1:\ \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx,\ \int_a^a f(x)\,dx=0 \quad \text{(swap)}$$
$$P_2:\ \int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx \quad \text{(split)}$$
$$P_3:\ \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx \quad \text{(reflect)}$$
$$P_4:\ \int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx \quad \text{(key board property)}$$
$$P_5:\ \int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx+\int_0^a f(2a-x)\,dx$$
$$P_6:\ \int_0^{2a} f(x)\,dx=2\int_0^a f(x)\,dx \text{ if } f(2a-x)=f(x)$$
$$=0 \text{ if } f(2a-x)=-f(x)$$
$$P_7:\ \int_{-a}^a f(x)\,dx=2\int_0^a f(x)\,dx\ (f\text{ even});\ \ =0\ (f\text{ odd})$$

Quick Tip

P_4 **is the most-tested property**. Whenever you see $\int_0^a$ with a denominator involving $\sin x$ and $\cos x$ (or $\tan x$, or $\sec x$, etc.), apply P_4: replace x with $a-x$, add the two integrals, and the trig usually telescopes.

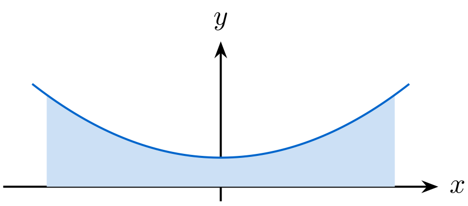
Classic example. Show that $I=\displaystyle\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx=\frac{\pi}{4}$.

By P_4,

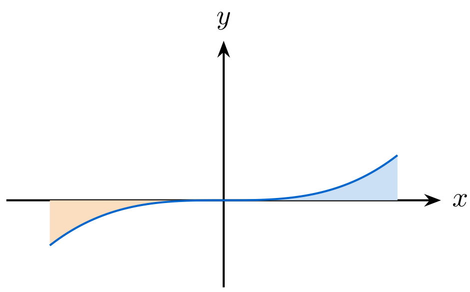
$$I=\int_0^{\pi/2}\frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}\,dx=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx.$$

Adding the two expressions for I:

$$2I=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}\,dx=\int_0^{\pi/2}1\,dx=\tfrac{\pi}{2}\implies I=\tfrac{\pi}{4}.$$



Even function: area on $[-a,0]$ equals area on $[0,a]$



Odd function: areas above and below the axis cancel

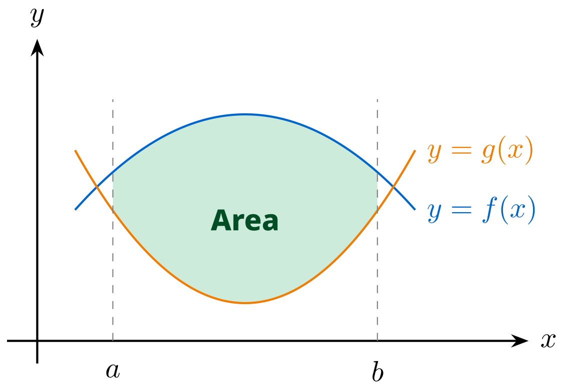
Memory Aid

For property P_7: *"Even doubles, odd vanishes."* When the limits are symmetric about zero, check parity *first*. If $f(-x) = f(x)$, fold the integral over $[0, a]$ and double. If $f(-x) = -f(x)$, the answer is 0 before you compute anything.

7.2 Area between two curves [JEE/NEET]

A geometric application that appears often in entrance exams (full treatment is in Chapter 8 "Application of Integrals," but the formula appears here too):

$$\text{Area} = \int_a^b \left[\, f(x) - g(x) \,\right] dx, \qquad f(x) \geq g(x) \text{ on } [a, b].$$



8 Definite Integral as Limit of a Sum [JEE/NEET Extension]

This was Section 7.7 in pre-rationalisation NCERT and remains a standard JEE Main / Advanced topic — the rigorous definition that gives the second fundamental theorem its meaning.

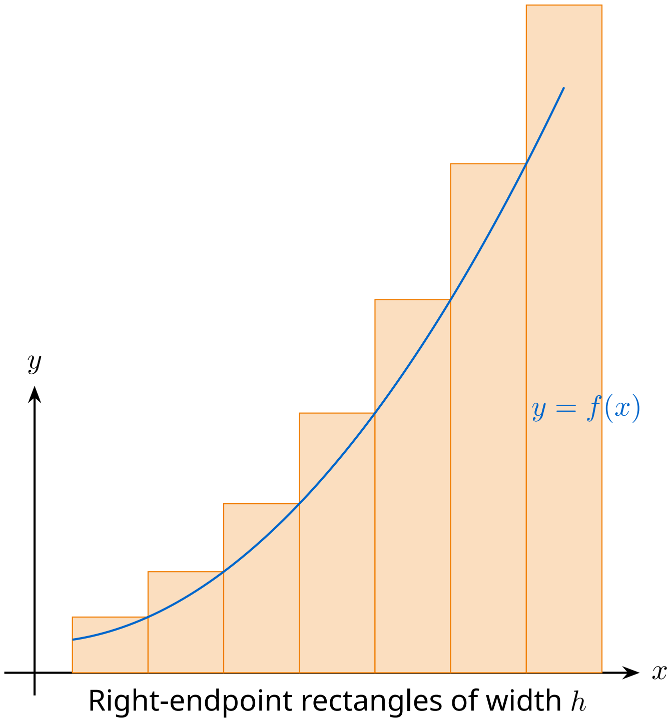
For a continuous function f on $[a, b]$, partition the interval into n equal sub-intervals of width $h = (b - a)/n$. Sample f at the right end-points $x_k = a + kh$, $k = 1, 2, \ldots, n$.

Definite Integral as a Riemann Sum

$$\int_a^b f(x)\, dx = \lim_{n \to \infty} h \sum_{k=1}^{n} f(a + kh), \qquad h = \frac{b - a}{n}.$$

Equivalently,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h \left[f(a) + f(a + h) + \cdots + f(a + (n - 1)h) \right].$$



Right-endpoint rectangles of width h

Two algebraic identities make these sums tractable:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Worked example. Evaluate $\int_0^2 x^2\,dx$ as a limit.
Here $a = 0$, $b = 2$, $h = 2/n$. Then

$$\begin{aligned}\int_0^2 x^2\,dx &= \lim_{n\to\infty} \frac{2}{n}\sum_{k=1}^{n}(2k/n)^2 \\ &= \lim_{n\to\infty} \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\ &= \frac{8}{3}.\end{aligned}$$

A direct anti-derivative check: $\left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$. The two methods agree — which is exactly what the Fundamental Theorem promises.

> **Quick Tip**
>
> In JEE problems where you're asked to recognise a limit as an integral, look for $\frac{1}{n}\sum f\left(\frac{k}{n}\right)$. This equals $\int_0^1 f(x)\,dx$.

9 Quick Reference Summary

9.1 Master formula list

Indefinite Integrals — Full Table

Integrand	Integral (omit $+C$)
$x^n,\ n \neq -1$	$\dfrac{x^{n+1}}{n+1}$
$\dfrac{1}{x}$	$\log\lvert x\rvert$
e^x, a^x	e^x, $\dfrac{a^x}{\log a}$
$\sin x$, $\cos x$	$-\cos x$, $\sin x$
$\sec^2 x$, $\operatorname{cosec}^2 x$	$\tan x$, $-\cot x$
$\sec x \tan x$, $\operatorname{cosec} x \cot x$	$\sec x$, $-\operatorname{cosec} x$
$\tan x$, $\cot x$	$\log\lvert\sec x\rvert$, $\log\lvert\sin x\rvert$
$\sec x$, $\operatorname{cosec} x$	$\log\lvert\sec x + \tan x\rvert$, $\log\lvert\operatorname{cosec} x - \cot x\rvert$
$\dfrac{1}{\sqrt{1-x^2}}$, $\dfrac{1}{1+x^2}$	$\sin^{-1} x$, $\tan^{-1} x$
$\dfrac{1}{x^2-a^2}$	$\dfrac{1}{2a}\log\left\lvert\dfrac{x-a}{x+a}\right\rvert$
$\dfrac{1}{a^2-x^2}$	$\dfrac{1}{2a}\log\left\lvert\dfrac{a+x}{a-x}\right\rvert$
$\dfrac{1}{x^2+a^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$
$\dfrac{1}{\sqrt{x^2-a^2}}$	$\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a}$
$\dfrac{1}{\sqrt{x^2+a^2}}$	$\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert$
$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert$
$\sqrt{a^2-x^2}$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

9.2 Comparison: trigonometric integral patterns

Integrand type	Strategy	Substitution / Identity
$\sin^m x \cos^n x$, m or n odd	Peel one factor, use Pythagoras	$t = \cos x$ or $\sin x$
$\sin^m x \cos^n x$, both even	Double-angle reduction	$\cos^2 x = \frac{1+\cos 2x}{2}$
$\sin mx \cos nx$ etc. products	Product-to-sum	$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$
$\tan^n x$, $\sec^n x$	Pull $\sec^2 x = 1 + \tan^2 x$	$t = \tan x$
$\dfrac{1}{a + b\sin x + c\cos x}$	Weierstrass	$t = \tan(x/2)$ (JEE)
$\dfrac{1}{a\sin^2 x + b\cos^2 x}$	Divide by $\cos^2 x$	$t = \tan x$

9.3 Properties of definite integrals — one-line summary

- P_1: Swap limits $\Rightarrow$ flip sign.
- P_2: Cut the interval at any $c \in [a, b]$.
- P_3, P_4: Replace x by $a + b - x$ (or $a - x$). *Use first* for trig integrals on $[0, a]$.
- P_5, P_6: Decompose $[0, 2a]$. Symmetric ($f(2a - x) = f(x)$) doubles, anti-symmetric vanishes.
- P_7: Symmetric interval $[-a, a]$. Even doubles, odd vanishes.

Final Checklist Before Computing

1. Is the integrand continuous on the integration interval? (If not, the integral may not exist.)
2. Can the integrand be split or simplified algebraically first? (linearity)
3. Is there an obvious substitution? (inner-function-with-derivative-present)
4. For products: which of L–I–A–T–E goes first? (by-parts)
5. For rational functions: divide if improper, then partial fractions.
6. For definite integrals: check symmetry properties before computing.

Real-World Application

Integration is the mathematics of accumulation. The total charge that flowed through a wire over a time interval is the integral of the current. The work done by a variable force along a path is the integral of force-along-displacement. The probability that a continuous random variable falls in $[a, b]$ is the integral of its density function. Every one of these is the same chapter you just finished.

End of Chapter 7 — Integrals