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NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12th Mathematics, Chapter 8

Chapter 8: Application of Integrals

About this Chapter

This chapter applies the definite integral to compute **areas** of plane regions bounded by simple curves, lines, parabolas, circles, and ellipses. The student learns to set up a vertical-strip integral $\int_a^b y\,dx$ or a horizontal-strip integral $\int_c^d x\,dy$, decide which is more convenient from a rough sketch, and handle regions where one curve crosses another by splitting into sub-intervals. The Exemplar problems push the student to recognise standard area formulas (circle, ellipse, parabola), to use **symmetry** to halve or quarter the work, and to combine elementary geometry (triangles, rectangles) with calculus when the boundary changes.

Topics covered: Area under a curve • Area between a curve and the y-axis • Area between two curves • Parabolas and lines • Circles and lines • Ellipses • Triangular regions by integration • Symmetry tricks

Quick Formula Sheet

Area under $y = f(x)$:
$A = \int_a^b f(x)\,dx$

Area along y-axis:
$A = \int_c^d x\,dy$

Area between f and g:
$A = \int_a^b (f(x) - g(x))\,dx$

Circle radius a:
$A = \pi a^2$

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$:
$A = \pi ab$

Standard integral:
$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

Symmetry:
If f is even, $\int_{-a}^{a} f = 2\int_0^a f$

I. Short Answer (S.A.)

Q 8.1 **Find the area of the region bounded by the curves $y^2 = 9x$ and $y = 3x$.**

SOLUTION

Concept used. For a region bounded by two curves $y = f(x)$ (upper) and $y = g(x)$ (lower) between their intersection abscissae $x = a$ and $x = b$, the area is

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

Here the parabola $y^2 = 9x$ gives $y = 3\sqrt{x}$ (taking the upper branch since $y = 3x \geq 0$ when $x \geq 0$).

Step 1. Intersections. Solve $y = 3x$ with $y^2 = 9x$: substitute $y = 3x$ into $y^2 = 9x$:

$$(3x)^2 = 9x \;\Rightarrow\; 9x^2 = 9x \;\Rightarrow\; 9x(x-1) = 0.$$

So $x = 0$ or $x = 1$, giving the points $(0,0)$ and $(1,3)$.

Step 2. Which is upper? At $x = \frac{1}{4}$: parabola $y = 3\sqrt{1/4} = 1.5$; line $y = 3(1/4) = 0.75$. So the parabola $3\sqrt{x}$ is above the line $3x$ on $[0,1]$.

Step 3. Set up.

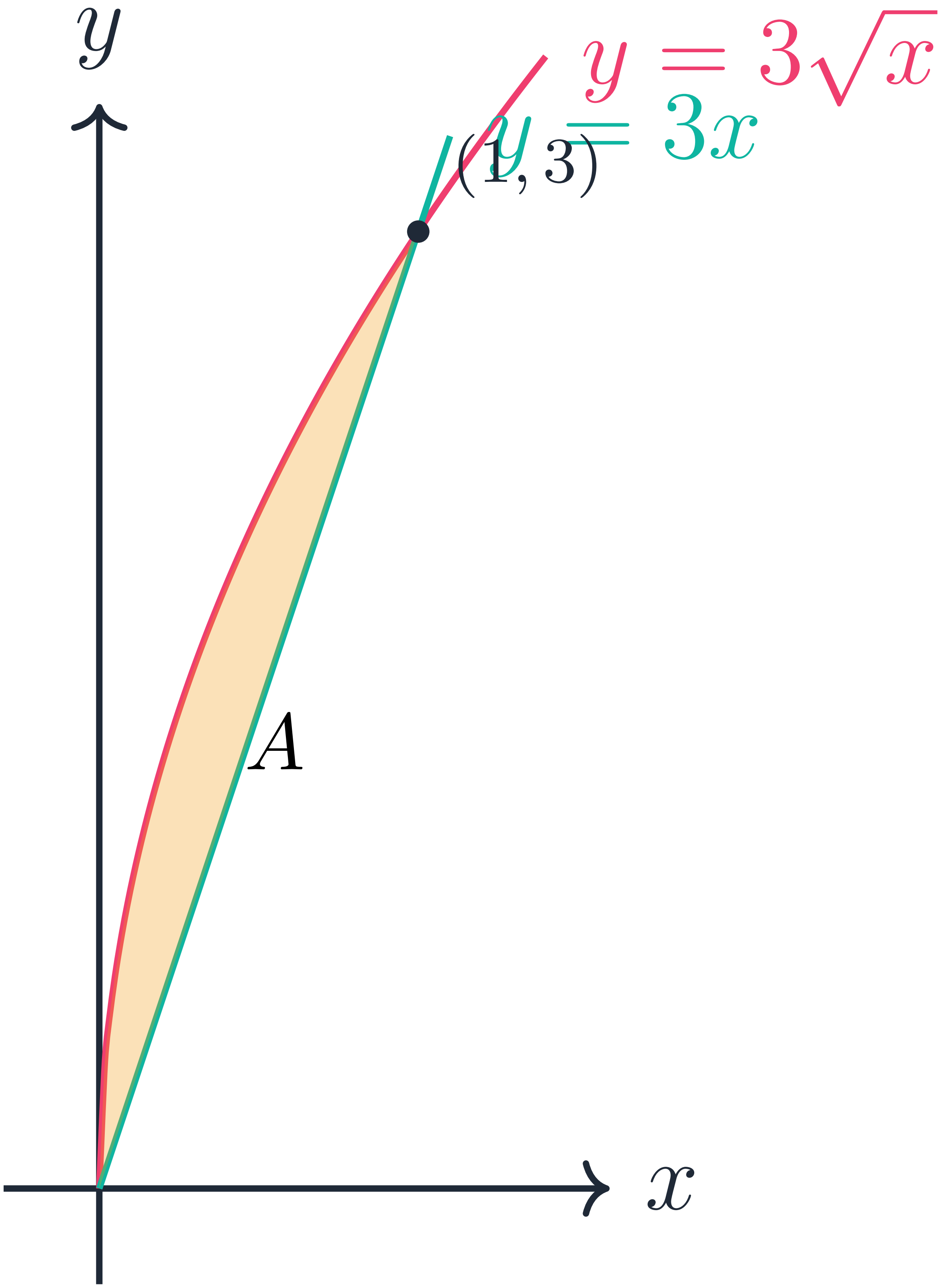
$$A = \int_0^1 \big(3\sqrt{x} - 3x\big)\, dx.$$

Step 4. Integrate.

$$\int 3\sqrt{x}\, dx = 3 \cdot \frac{x^{3/2}}{3/2} = 2x^{3/2}, \qquad \int 3x\, dx = \frac{3x^2}{2}.$$

So

$$A = \left[2x^{3/2} - \frac{3x^2}{2}\right]_0^1 = 2(1) - \frac{3}{2} - 0 = \frac{1}{2}.$$



Final Answer: $A = \frac{1}{2}$ sq units.

☞ **Parabola opens which way?**

$y^2 = 4ax$ opens to the right (along the positive x-axis); here $4a = 9$ so $a = \frac{9}{4}$. The vertex is $(0, 0)$ and the curve has two symmetric branches above and below the x-axis.

✗ Common Mistake

Forgetting that $y^2 = 9x$ has two branches ($y = \pm 3\sqrt{x}$) and integrating both would double-count. Since the line $y = 3x$ meets only the upper branch in the first quadrant, use $y = +3\sqrt{x}$ only.

EXPERT'S SOLUTION *: Aarav Iyer, M.Sc Mathematics, IIT Bombay*

Strategic angle. Switching to a horizontal-strip integral (dy instead of dx) avoids the square root entirely, because both boundaries become polynomials in y. The line $y = 3x$ becomes $x = y/3$, the parabola $y^2 = 9x$ becomes $x = y^2/9$, and integration turns into a clean polynomial evaluation.

Step 1. Rewrite both boundaries as functions of y: $y^2 = 9x \Rightarrow x = \dfrac{y^2}{9}$ (parabola), $y = 3x \Rightarrow x = \dfrac{y}{3}$ (line).

Step 2. Identify the y-range. At the origin both curves give $y = 0$; at the upper intersection $(1, 3)$ both give $y = 3$. So y runs from 0 to 3 and the strip width is the x-difference between the line and the parabola.

Step 3. For fixed $y \in [0, 3]$, the left boundary is the parabola and the right boundary is the line. Check at $y = 1$: parabola gives $x = 1/9 \approx 0.11$, line gives $x = 1/3 \approx 0.33$, so the line is to the right of the parabola.

Step 4. Set up the integral as right minus left:

$$A = \int_0^3 \left(\frac{y}{3} - \frac{y^2}{9}\right) dy.$$

Step 5. Antidifferentiate term by term: $\int \dfrac{y}{3}\,dy = \dfrac{y^2}{6}$, $\int \dfrac{y^2}{9}\,dy = \dfrac{y^3}{27}$.

Step 6. Evaluate from 0 to 3:

$$\left[\frac{y^2}{6} - \frac{y^3}{27}\right]_0^3 = \frac{9}{6} - \frac{27}{27} = \frac{3}{2} - 1 = \frac{1}{2}.$$

Why this matters. Choosing dx vs dy is a routine optimisation in board problems: if one direction gives polynomial limits and the other gives square roots, pick the polynomial one. Two independent methods giving the same answer is a strong cross-check.

Final Answer: $A = \dfrac{1}{2}$ sq units.

Q 8.2 **Find the area of the region bounded by the parabolas $y^2 = 2px$ and $x^2 = 2py$.**

SOLUTION

Concept used. Two parabolas with perpendicular axes meet in two points; the region between them is bounded above by one and below by the other. The standard formula $A = \int_0^a (\sqrt{\text{upper}} - \text{lower})\,dx$ applies, where a is the non-zero intersection abscissa.

Step 1. Intersections. From $x^2 = 2py$ get $y = \dfrac{x^2}{2p}$; substitute into $y^2 = 2px$:

$$\left(\frac{x^2}{2p}\right)^2 = 2px \;\Rightarrow\; \frac{x^4}{4p^2} = 2px \;\Rightarrow\; x^4 = 8p^3x.$$

So $x(x^3 - 8p^3) = 0 \;\Rightarrow\; x = 0$ or $x = 2p$. The intersection points are $(0, 0)$ and $(2p, 2p)$.

Step 2. Upper vs lower. For $x \in (0, 2p)$ check $x = p$: the curve $y^2 = 2px$ gives $y = \sqrt{2p \cdot p} = p\sqrt{2} \approx 1.41p$; the curve $x^2 = 2py$ gives $y = \dfrac{p^2}{2p} = \dfrac{p}{2}$. So $y = \sqrt{2px}$ is the upper boundary.

Step 3. Set up.

$$A = \int_0^{2p} \left(\sqrt{2px} - \frac{x^2}{2p}\right) dx.$$

Step 4. Integrate and evaluate. Antidifferentiate: $\int \sqrt{2px}\,dx = \frac{2\sqrt{2p}}{3}\,x^{3/2}$ and $\int \dfrac{x^2}{2p}\,dx = \dfrac{x^3}{6p}$, so $A = \left[\frac{2\sqrt{2p}}{3}\,x^{3/2} - \frac{x^3}{6p}\right]_0^{2p}$. At $x = 2p$ we have $x^{3/2} = (2p)^{3/2} = 2\sqrt{2}\,p^{3/2}$, giving $\frac{2\sqrt{2p}}{3} \cdot 2\sqrt{2}\,p^{3/2} = \frac{8p^2}{3}$, while $\frac{(2p)^3}{6p} = \frac{8p^3}{6p} = \frac{4p^2}{3}$. Therefore $A = \frac{8p^2}{3} - \frac{4p^2}{3} = \frac{4p^2}{3}$.

Final Answer: $A = \dfrac{4p^2}{3}$ sq units.

☞ **Exam Tip**

Remember the result: two congruent parabolas $y^2 = 4ax$ and $x^2 = 4ay$ (here $4a = 2p$, so $a = p/2$) enclose area $\dfrac{16a^2}{3} = \dfrac{4p^2}{3}$. Frequently asked in JEE Main.

EXPERT'S SOLUTION *: Priya Sharma, Ph.D Mathematics, IIT Delhi*

Picture-first. The two parabolas are reflections of each other in the line $y = x$, so by symmetry the area can also be found as twice the area between $y = \sqrt{2px}$ and the line $y = x$ from $x = 0$ to $x = 2p$.

Step 1. By symmetry about $y = x$, the region is split into two congruent halves by the line $y = x$. Call the lower half R_1 (between the line $y = x$ and the parabola $x^2 = 2py$) and the upper half R_2 (between $y^2 = 2px$ and $y = x$). Then $A = 2 \cdot \text{Area}(R_2)$.

Step 2. Compute $\text{Area}(R_2) = \int_0^{2p} (\sqrt{2px} - x)\,dx$:

$$= \left[\frac{2\sqrt{2p}}{3} x^{3/2} - \frac{x^2}{2} \right]_0^{2p}$$
$$= \frac{8p^2}{3} - \frac{(2p)^2}{2} = \frac{8p^2}{3} - 2p^2 = \frac{2p^2}{3}.$$

Step 3. Double: $A = 2 \cdot \frac{2p^2}{3} = \frac{4p^2}{3}$.

Why this matters. Spotting the $y = x$ symmetry shortens the algebra; many JEE problems with reflected parabolas, ellipses, or circles yield to the same trick.

Final Answer: $A = \frac{4p^2}{3}$ sq units.

Q 8.3 **Find the area of the region bounded by the curve $y = x^3$, $y = x + 6$ and $x = 0$.**

SOLUTION

Concept used. A region bounded by three curves needs the intersection points first. With $x = 0$ as one boundary, find where the cubic meets the line, and integrate (upper − lower) between $x = 0$ and that intersection.

Step 1. **Intersection of $y = x^3$ and $y = x + 6$.** Set $x^3 = x + 6$, i.e. $x^3 - x - 6 = 0$. Try $x = 2$: $8 - 2 - 6 = 0$. So $x = 2$ is a root. Factor: $x^3 - x - 6 = (x - 2)(x^2 + 2x + 3)$. The quadratic has discriminant $4 - 12 = -8 < 0$, no real roots. Hence the only real intersection is $x = 2$, $y = 8$.

Step 2. **Compare on $[0, 2]$.** At $x = 1$: $y = x + 6 = 7$ and $y = x^3 = 1$. So the line $y = x + 6$ is the upper boundary and the cubic $y = x^3$ is the lower boundary on $[0, 2]$.

Step 3. **Set up.**

$$A = \int_0^2 \left((x + 6) - x^3 \right) dx.$$

Step 4. Integrate.

$$A = \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2$$
$$= \frac{4}{2} + 12 - \frac{16}{4} - 0$$
$$= 2 + 12 - 4 = 10.$$

Final Answer: $A = 10$ sq units.

☞ **Why $x = 0$ is needed**

Without the constraint $x = 0$, the region between $y = x^3$ and $y = x + 6$ on the left would extend to $x \to -\infty$ (the cubic dives below the line and never meets it again for real x). The line $x = 0$ closes the region on the left.

EXPERT'S SOLUTION *: Vivaan Gupta, M.Sc Mathematics, ISI Kolkata*

Quick reading. The cubic factorisation gives a single real intersection at $x = 2$, and the line $x = 0$ cuts the cubic at the origin and the line $y = x + 6$ at $(0, 6)$, so the region is a closed curvilinear "triangle" with vertices at $(0, 0)$, $(0, 6)$, $(2, 8)$.

Step 1. Confirm the bounds: at $x = 0$ the line is at $y = 6$ and the cubic is at $y = 0$, so the strip width at $x = 0$ is 6. At $x = 2$ both curves meet at $y = 8$, so the strip width is 0. The region is bounded above by $y = x + 6$, below by $y = x^3$, left by $x = 0$, and closes at $x = 2$.

Step 2. Apply the area formula:

$$A = \int_0^2 \left((x + 6) - x^3\right) dx = \left[\frac{x^2}{2} + 6x - \frac{x^4}{4}\right]_0^2.$$

Step 3. Substitute: $A = 2 + 12 - 4 = 10$ sq units.

Why this matters. Spotting one rational root of a cubic by inspection (try $\pm 1, \pm 2, \pm 3, \pm 6$ for $x^3 - x - 6 = 0$) is a standard exam technique that saves time.

Final Answer: $A = 10$ sq units.

Q 8.4 **Find the area of the region bounded by the curves $y^2 = 4x$, $x^2 = 4y$.**

SOLUTION

Concept used. Two parabolas of the form $y^2 = 4ax$ and $x^2 = 4ay$ are reflections in $y = x$ and meet at the origin and at $(4a, 4a)$. The enclosed area is $\int_0^{4a} (\sqrt{4ax} - \frac{x^2}{4a})\, dx$.

Step 1. Intersections. From $x^2 = 4y$ get $y = \frac{x^2}{4}$; substitute into $y^2 = 4x$:

$$\frac{x^4}{16} = 4x \;\Rightarrow\; x^4 = 64x \;\Rightarrow\; x(x^3 - 64) = 0.$$

So $x = 0$ or $x = 4$; intersections are $(0, 0)$ and $(4, 4)$.

Step 2. Upper vs lower. At $x = 1$: $y^2 = 4x \Rightarrow y = 2$; $x^2 = 4y \Rightarrow y = 1/4$. So $\sqrt{4x} = 2\sqrt{x}$ is above $\frac{x^2}{4}$ on $(0, 4)$.

Step 3. Set up.

$$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx.$$

Step 4. Integrate.

$$\int 2\sqrt{x}\, dx = 2 \cdot \frac{x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}, \qquad \int \frac{x^2}{4}\, dx = \frac{x^3}{12}.$$

So

$$A = \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12}\right]_0^4 = \frac{4}{3}(4)^{3/2} - \frac{64}{12}.$$

Step 5. Numerical. $4^{3/2} = 8$, so

$$A = \frac{4 \cdot 8}{3} - \frac{64}{12} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3}.$$

Final Answer: $A = \frac{16}{3}$ sq units.

♥ Two-parabola area shortcut

For parabolas $y^2 = 4ax$ and $x^2 = 4ay$ with the same a, the enclosed area is exactly $\frac{16a^2}{3}$. Here $a = 1$ so $A = 16/3$. Memorising this shortcut saves the full integration in MCQ contexts.

EXPERT'S SOLUTION : *Ananya Iyer, M.Sc Applied Mathematics, IIT Kanpur*

Strategic angle. Use the symmetry about $y = x$: the line bisects the enclosed lens, so the area equals twice the area between $y = 2\sqrt{x}$ and the line $y = x$ on $[0, 4]$.

Step 1. By symmetry, $A = 2\int_0^4 (2\sqrt{x} - x)\, dx$.

Step 2. Integrate:

$$\int (2\sqrt{x} - x)\, dx = \frac{4}{3}x^{3/2} - \frac{x^2}{2}.$$

Evaluate from 0 to 4:

$$\frac{4}{3}\cdot 8 - \frac{16}{2} = \frac{32}{3} - 8 = \frac{32-24}{3} = \frac{8}{3}.$$

Step 3. Double: $A = 2\cdot\frac{8}{3} = \frac{16}{3}$.

Why this matters. The $y = x$ symmetry is a common JEE pattern (e.g. $y^2 = x$ and $x^2 = y$); recognising it cuts the computation in half.

> **Final Answer:** $A = \frac{16}{3}$ sq units.

Q 8.5 **Find the area of the region included between $y^2 = 9x$ and $y = x$.**

SOLUTION

Concept used. Same setup as Q1, but now the line is the identity $y = x$ instead of $y = 3x$.

Step 1. **Intersections.** Substitute $y = x$ into $y^2 = 9x$: $x^2 = 9x \Rightarrow x(x-9) = 0$, so $x = 0$ or $x = 9$. Intersection points: $(0,0)$ and $(9,9)$.

Step 2. **Upper vs lower on** $(0,9)$. At $x = 1$: parabola $y = 3\sqrt{1} = 3$; line $y = 1$. So $y = 3\sqrt{x}$ is above $y = x$.

Step 3. **Set up.**

$$A = \int_0^9 \left(3\sqrt{x} - x\right) dx.$$

Step 4. **Integrate.**

$$\int 3\sqrt{x}\, dx = 2x^{3/2}, \qquad \int x\, dx = \frac{x^2}{2}.$$

$$A = \left[2x^{3/2} - \frac{x^2}{2}\right]_0^9 = 2(9)^{3/2} - \frac{81}{2}.$$

Step 5. **Numerical.** $9^{3/2} = 27$, so

$$A = 2(27) - \frac{81}{2} = 54 - \frac{81}{2} = \frac{108-81}{2} = \frac{27}{2}.$$

> **Final Answer:** $A = \frac{27}{2}$ sq units.

EXPERT'S SOLUTION : *Rohit Mehta, B.Tech CSE, IIT Roorkee*

Quick reading. Same template as Q1 with the slope changed from 3 to 1; the line is now flatter, so the intersection moves further out (from $x = 1$ to $x = 9$) and the enclosed area is much larger.

Step 1. Use a dy integral instead. $y = x$ becomes $x = y$ and $y^2 = 9x$ becomes $x = \frac{y^2}{9}$.

Step 2. For $y \in [0, 9]$, the line $x = y$ is to the right of the parabola $x = y^2/9$ (check $y = 3$: $x = 3$ vs $x = 1$).

Step 3. Compute

$$A = \int_0^9 \left(y - \frac{y^2}{9}\right) dy = \left[\frac{y^2}{2} - \frac{y^3}{27}\right]_0^9 = \frac{81}{2} - \frac{729}{27} = \frac{81}{2} - 27.$$

Now $\frac{81}{2} - 27 = \frac{81 - 54}{2} = \frac{27}{2}$.

Why this matters. The dy approach avoids the square root, trading it for a clean quadratic. Both routes give the same answer, which is a good cross-check.

Final Answer: $A = \frac{27}{2}$ sq units.

Q 8.6 **Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$.**

SOLUTION

Concept used. Find where the line meets the parabola, then use $A = \int_a^b (\text{line} - \text{parabola})\, dx$ between the two intersection abscissae (since the line is above the parabola between them).

Step 1. **Intersections.** Substitute $y = x + 2$ into $y = x^2$: $x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x - 2)(x + 1) = 0$. So $x = -1$ or $x = 2$; points are $(-1, 1)$ and $(2, 4)$.

Step 2. **Upper vs lower on $(-1, 2)$.** At $x = 0$: line gives $y = 2$, parabola gives $y = 0$. So the line is above the parabola.

Step 3. **Set up.**

$$A = \int_{-1}^{2} \left((x + 2) - x^2\right) dx.$$

Step 4. Integrate.

$$A = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(\frac{4}{2} + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$
$$= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)$$
$$= 6 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}$$
$$= 8 - 3 - \frac{1}{2} = 5 - \frac{1}{2} = \frac{9}{2}.$$

(Combine: $\frac{8}{3} + \frac{1}{3} = 3$, leaving $8 - 3 - \frac{1}{2} = \frac{9}{2}$.)

Final Answer: $A = \frac{9}{2}$ sq units.

✗ Common Mistake

Sign errors when evaluating at $x = -1$. Compute each term carefully: $\frac{(-1)^2}{2} = \frac{1}{2}$ (positive), $2(-1) = -2$, $\frac{(-1)^3}{3} = -\frac{1}{3}$, and the antiderivative $F(-1) = \frac{1}{2} - 2 - (-\frac{1}{3}) = \frac{1}{2} - 2 + \frac{1}{3}$. The "minus a minus" trips many students.

EXPERT'S SOLUTION : *Karan Singh, M.Tech CS, IIT Madras*

Strategic angle. A cleaner evaluation uses the difference of polynomials and the identity $\int_a^b (b-x)(x-a)\,dx = \frac{(b-a)^3}{6}$ for parabolic segments.

Step 1. Write the integrand as $-(x^2 - x - 2) = -(x-2)(x+1) = (2-x)(x+1)$.

Step 2. Apply the standard formula $\int_{-1}^{2} (2-x)(x+1)\,dx = \frac{(2-(-1))^3}{6} = \frac{27}{6} = \frac{9}{2}$.

Why this matters. Recognising the parabolic-segment formula $\frac{1}{6}(b-a)^3$ when both factors are linear collapses the integration to one line. Very useful for MCQ time-pressure.

Final Answer: $A = \frac{9}{2}$ sq units.

Q 8.7 **Find the area of the region bounded by the line $x = 2$ and the parabola $y^2 = 8x$.**

SOLUTION

Concept used. The parabola $y^2 = 8x$ is symmetric about the x-axis. The vertical line $x = 2$ closes the region on the right. By symmetry, the area equals twice the area of the upper half.

Step 1. Upper branch. $y = \sqrt{8x} = 2\sqrt{2}\sqrt{x}$ for $y \geq 0$.

Step 2. Set up using symmetry.

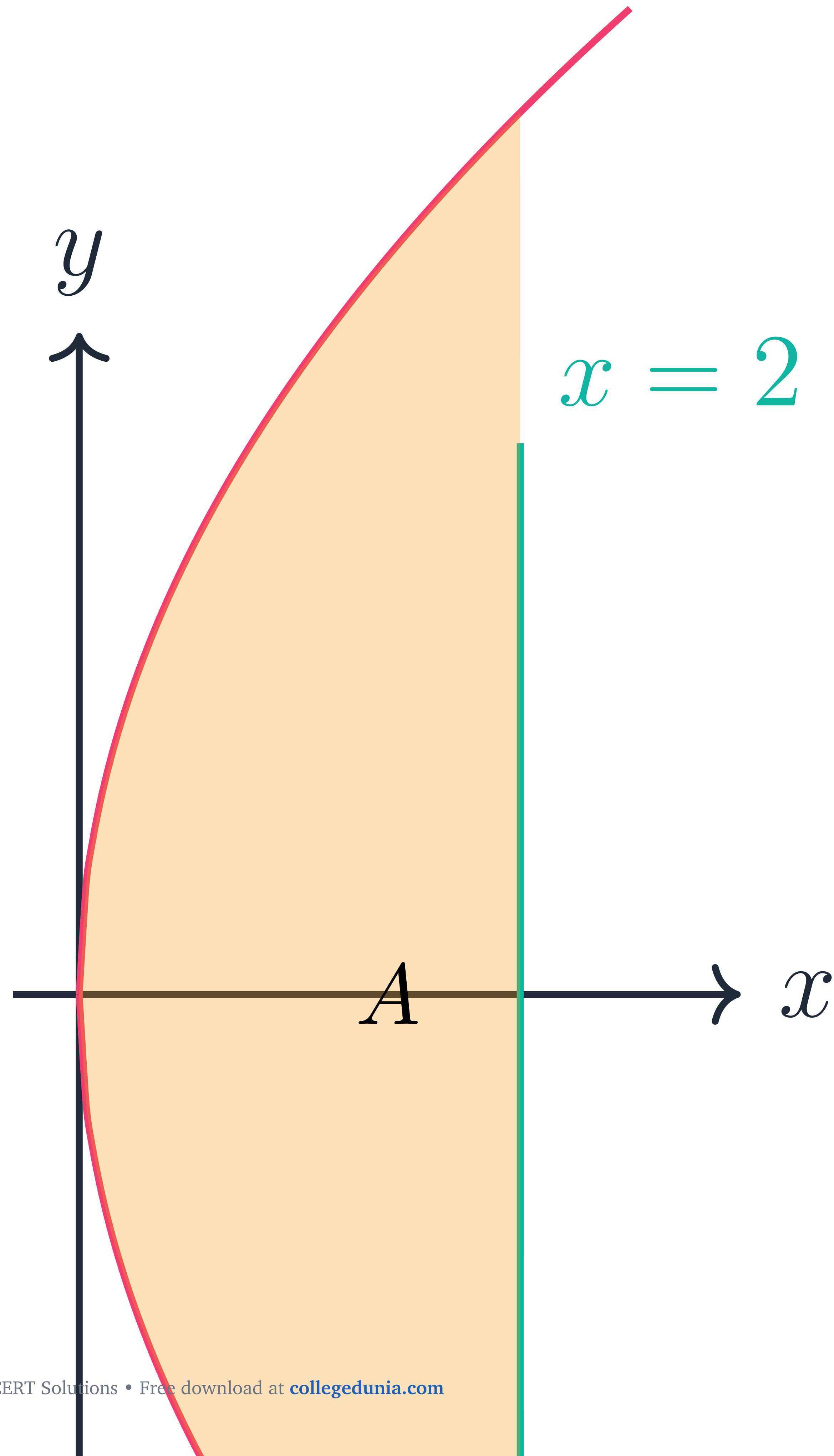
$$A = 2\int_0^2 2\sqrt{2}\sqrt{x}\,dx = 4\sqrt{2}\int_0^2 \sqrt{x}\,dx.$$

Step 3. Integrate.

$$\int_0^2 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^2 = \frac{2}{3}(2)^{3/2} = \frac{2}{3}\cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3}.$$

Step 4. Multiply.

$$A = 4\sqrt{2}\cdot\frac{4\sqrt{2}}{3} = \frac{16\cdot 2}{3} = \frac{32}{3}.$$


y
$x = 2$
A
x

Final Answer: $A = \frac{32}{3}$ sq units.

☞ **Half-parabola integral**

$y^2 = 4ax$, $0 \le x \le a$, has area $\int_0^a 2\sqrt{a}\sqrt{x}\,dx \cdot 2 = \frac{8a^2}{3}$ for the doubled (symmetric) region. Here $4a = 8$ so $a = 2$, giving $\frac{8(2)^2}{3} = \frac{32}{3}$ directly.

EXPERT'S SOLUTION *: Aditi Banerjee, Ph.D Pure Mathematics, IISc Bangalore*

Picture-first. Use horizontal strips: for each y, the strip runs from $x = \frac{y^2}{8}$ (parabola) to $x = 2$ (line), and y ranges from -4 to 4 (where $y^2 = 8 \cdot 2$ gives $y = \pm 4$).

Step 1. Set up

$$A = \int_{-4}^{4} \left(2 - \frac{y^2}{8}\right) dy = 2\int_0^4 \left(2 - \frac{y^2}{8}\right) dy$$

using even symmetry of the integrand.

Step 2. Compute

$$\int_0^4 \left(2 - \frac{y^2}{8}\right) dy = \left[2y - \frac{y^3}{24}\right]_0^4 = 8 - \frac{64}{24} = 8 - \frac{8}{3} = \frac{16}{3}.$$

Step 3. Double: $A = \frac{32}{3}$.

Why this matters. For parabolas of the form $y^2 = 4ax$ cut off by a vertical line $x = h$, the dy method yields a clean polynomial. Try both methods and pick the one with fewer surds.

Final Answer: $A = \frac{32}{3}$ sq units.

Q 8.8 **Sketch the region $\{(x, y) : y = \sqrt{4 - x^2}\}$ and the x-axis. Find the area of the region using integration.**

SOLUTION

Concept used. $y = \sqrt{4 - x^2}$ describes the upper half of the circle $x^2 + y^2 = 4$ (radius 2, centred at origin). With the x-axis as the lower boundary, the region is the upper semicircular disc.

Step 1. **Range of x.** For y to be real, $4 - x^2 \geq 0$, so $-2 \leq x \leq 2$.

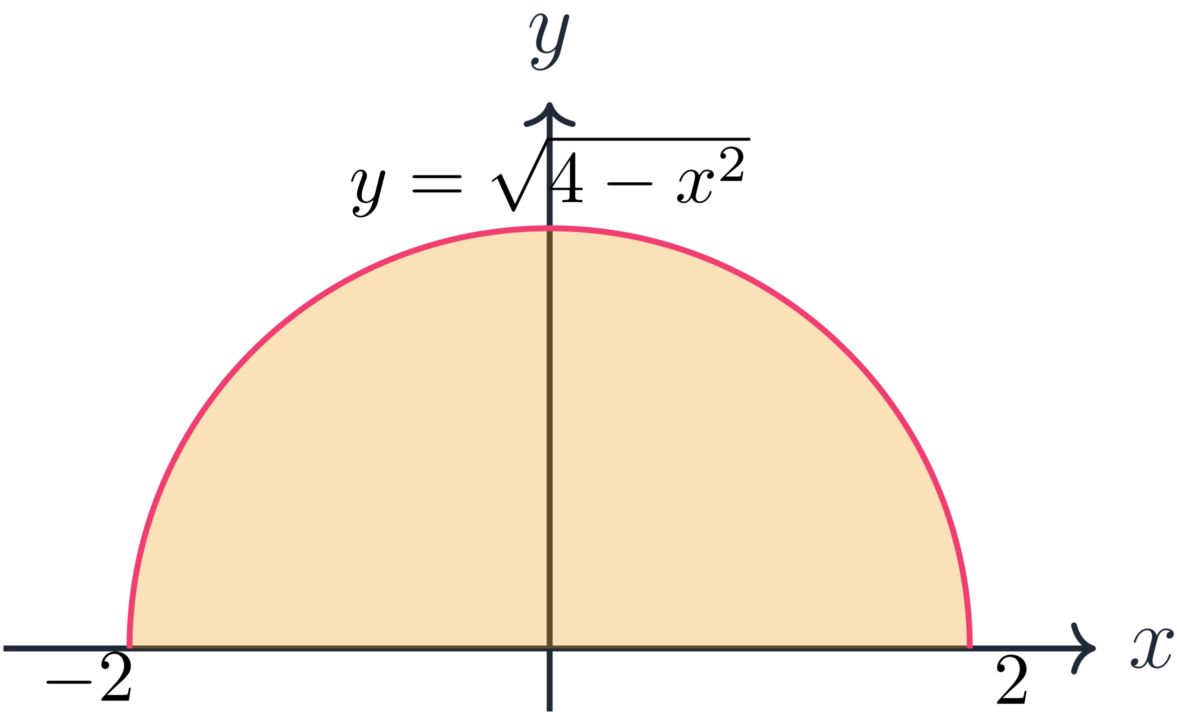
Step 2. **Set up.**

$$A = \int_{-2}^{2} \sqrt{4 - x^2}\, dx.$$

Step 3. **Standard integral.** Use $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$ with $a = 2$:

$$A = \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right]_{-2}^{2}.$$

Step 4. **Evaluate.** At $x = 2$: $\sqrt{4-4} = 0$, so first term is 0; $\sin^{-1}(1) = \frac{\pi}{2}$, so second term is $2 \cdot \frac{\pi}{2} = \pi$. At $x = -2$: first term 0; $\sin^{-1}(-1) = -\frac{\pi}{2}$, second term $-\pi$. Difference: $\pi - (-\pi) = 2\pi$.



Final Answer: $A = 2\pi$ sq units.

♥ Geometric sanity check

The region is a semicircle of radius 2. Its area equals $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(2)^2 = 2\pi$. Our integral answer agrees: integration recovers a known geometric fact.

EXPERT'S SOLUTION : *Pranav Joshi, M.Sc Mathematics, IIT Bombay*

Strategic angle. The square root $\sqrt{a^2 - x^2}$ begs for the trig substitution $x = a\sin\theta$, because then $a^2 - x^2 = a^2(1 - \sin^2\theta) = a^2\cos^2\theta$ and the surd disappears. Here $a = 2$, so we try $x = 2\sin\theta$.

Step 1. Set $x = 2\sin\theta$, so $dx = 2\cos\theta\, d\theta$, and $\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = 2|\cos\theta| = 2\cos\theta$ (positive in $\theta \in [-\pi/2, \pi/2]$).

Step 2. Update the limits. When $x = -2$: $\sin\theta = -1$, so $\theta = -\pi/2$. When $x = 2$: $\sin\theta = 1$, so $\theta = \pi/2$. The interval $x \in [-2, 2]$ maps to $\theta \in [-\pi/2, \pi/2]$ bijectively.

Step 3. Substitute everything:

$$A = \int_{-\pi/2}^{\pi/2} 2\cos\theta \cdot 2\cos\theta\, d\theta = 4\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta.$$

Step 4. Use $\cos^2\theta = \dfrac{1 + \cos 2\theta}{2}$ on the integrand to obtain

$$4\int_{-\pi/2}^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta = 2\int_{-\pi/2}^{\pi/2} (1 + \cos 2\theta)\, d\theta.$$

Step 5. Integrate: $\int(1 + \cos 2\theta)\, d\theta = \theta + \dfrac{\sin 2\theta}{2}$.

Step 6. Evaluate:

$$2\left[\theta + \frac{\sin 2\theta}{2}\right]_{-\pi/2}^{\pi/2} = 2\left[\left(\frac{\pi}{2} + 0\right) - \left(-\frac{\pi}{2} + 0\right)\right] = 2(\pi) = 2\pi.$$

($\sin(\pm\pi) = 0$, so the second term vanishes at both limits.)

Why this matters. The $x = a\sin\theta$ substitution is the go-to move whenever $\sqrt{a^2 - x^2}$ appears in a definite integral. It always pairs with the half-angle identity to reduce the integral to a sum of constants and sinusoids; memorise this two-step pattern.

Final Answer: $A = 2\pi$ sq units.

Q 8.9 **Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.**

SOLUTION

Concept used. Area under $y = f(x)$ between $x = a$ and $x = b$, above the x-axis, is $\int_a^b f(x)\,dx$.

Step 1. Set up.

$$A = \int_0^1 2\sqrt{x}\,dx.$$

Step 2. Antiderivative.

$$\int 2\sqrt{x}\,dx = 2 \cdot \frac{x^{3/2}}{3/2} = \frac{4}{3}x^{3/2}.$$

Step 3. Evaluate.

$$A = \left[\frac{4}{3}x^{3/2}\right]_0^1 = \frac{4}{3}(1) - 0 = \frac{4}{3}.$$

Final Answer: $A = \dfrac{4}{3}$ sq units.

EXPERT'S SOLUTION *: Neha Verma, M.Sc Mathematics, IIT Bombay*

Quick reading. Standard area-under-curve drill: power rule with fractional exponent.

Step 1. Rewrite $2\sqrt{x} = 2x^{1/2}$.

Step 2. Apply $\int x^n\,dx = \dfrac{x^{n+1}}{n+1}$ with $n = \frac{1}{2}$: $\int 2x^{1/2}\,dx = \dfrac{2x^{3/2}}{3/2} = \dfrac{4x^{3/2}}{3}$.

Step 3. Plug in $x = 1$ and $x = 0$: $\dfrac{4}{3} - 0 = \dfrac{4}{3}$.

Why this matters. The power rule with fractional exponents shows up everywhere; keep $\sqrt{x} = x^{1/2}$ in mind so you can integrate without thinking.

Final Answer: $A = \dfrac{4}{3}$ sq units.

Q 8.10 **Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.**

SOLUTION

Concept used. The line $2y = 5x + 7$, i.e. $y = \dfrac{5x+7}{2}$, is positive for $x \geq 2$ (since $5(2) + 7 = 17 > 0$). The area under it, above the x-axis, between $x = 2$ and $x = 8$, is $\int_2^8 \dfrac{5x+7}{2}\,dx$.

Step 1. Set up.

$$A = \int_2^8 \frac{5x+7}{2}\,dx = \frac{1}{2}\int_2^8 (5x+7)\,dx.$$

Step 2. Antiderivative.

$$\int (5x+7)\,dx = \frac{5x^2}{2} + 7x.$$

Step 3. Evaluate at $x = 8$. $\frac{5(64)}{2} + 7(8) = \frac{320}{2} + 56 = 160 + 56 = 216.$

Step 4. Evaluate at $x = 2$. $\frac{5(4)}{2} + 7(2) = \frac{20}{2} + 14 = 10 + 14 = 24.$

Step 5. Combine.

$$A = \frac{1}{2}(216 - 24) = \frac{192}{2} = 96.$$

Final Answer: $A = 96$ sq units.

☞ **Trapezium shortcut**

The region is a trapezium with parallel sides $y(2) = \frac{17}{2}$ and $y(8) = \frac{47}{2}$, and width $8 - 2 = 6$. Area $= \frac{1}{2}(y_1 + y_2)\cdot w = \frac{1}{2}\cdot\frac{17+47}{2}\cdot 6 = \frac{1}{2}\cdot 32\cdot 6 = 96$. Same answer in one line.

EXPERT'S SOLUTION *: Ishaan Reddy, B.Tech CSE, IIT Roorkee*

Picture-first. A straight-line boundary above the x-axis with vertical sides at $x = 2$ and $x = 8$ traces a trapezium. Either integrate or apply the trapezium area formula.

Step 1. Compute heights: $y(2) = \frac{5(2)+7}{2} = \frac{17}{2}$ and $y(8) = \frac{5(8)+7}{2} = \frac{47}{2}$.

Step 2. Width $w = 8 - 2 = 6$.

Step 3. Trapezium area $= \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}\left(\frac{17}{2} + \frac{47}{2}\right)\cdot 6 = \frac{1}{2}\cdot\frac{64}{2}\cdot 6 = 16\cdot 6 = 96.$

Why this matters. Whenever the integrand is linear, the "area under a line" is a trapezium; using elementary geometry for the sanity check catches arithmetic errors fast.

Final Answer: $A = 96$ sq units.

Q 8.11 **Draw a rough sketch of the curve $y = \sqrt{x-1}$ in the interval $[1, 5]$. Find the area under the curve and between the lines $x = 1$ and $x = 5$.**

SOLUTION

Concept used. $y = \sqrt{x-1}$ is defined for $x \geq 1$, with $y(1) = 0$ and $y(5) = 2$. The curve is a right-shifted square-root graph; area between $x = 1$, $x = 5$ and the curve (above the x-axis) is $\int_1^5 \sqrt{x-1}\,dx$.

Step 1. **Substitution.** Let $u = x - 1$, so $du = dx$. When $x = 1$, $u = 0$; when $x = 5$, $u = 4$.

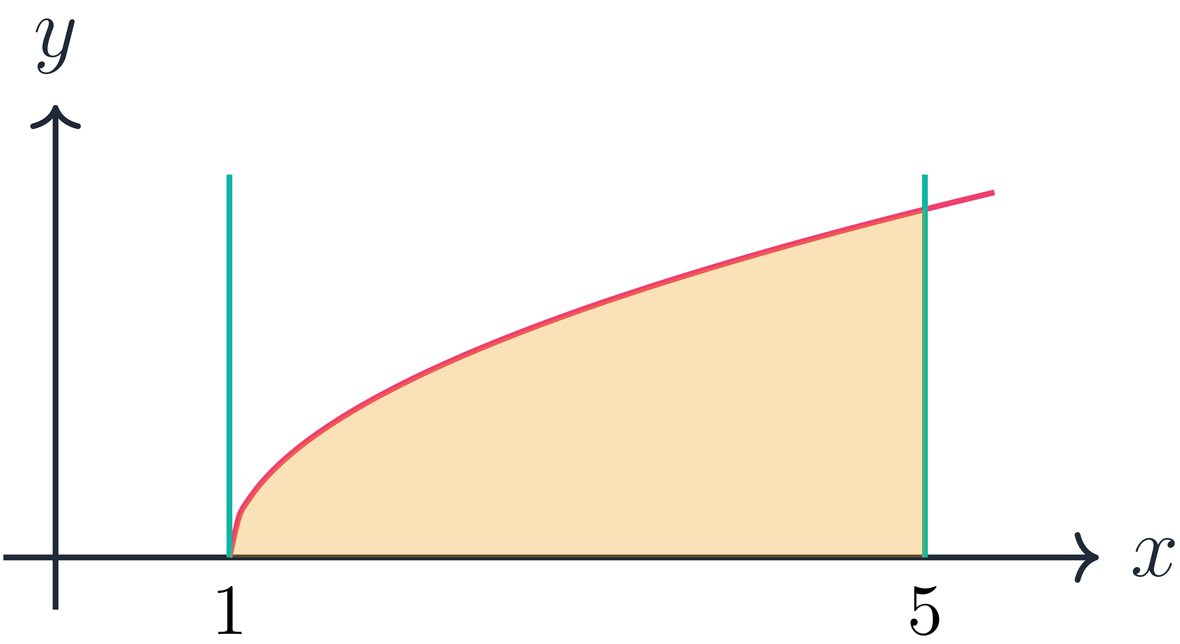
$$A = \int_1^5 \sqrt{x-1}\,dx = \int_0^4 \sqrt{u}\,du.$$

Step 2. **Antiderivative.**

$$\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} = \frac{2}{3}u^{3/2}.$$

Step 3. **Evaluate.**

$$A = \left[\frac{2}{3}u^{3/2}\right]_0^4 = \frac{2}{3}(4)^{3/2} - 0 = \frac{2}{3}\cdot 8 = \frac{16}{3}.$$



Final Answer: $A = \dfrac{16}{3}$ sq units.

EXPERT'S SOLUTION : *Sneha Kapoor, M.Sc Mathematics, ISI Kolkata*

Strategic angle. Recognise that $y = \sqrt{x-1}$ is the curve $y = \sqrt{x}$ shifted right by 1 unit, so its area between $x = 1$ and $x = 5$ equals the area of $y = \sqrt{x}$ between $x = 0$ and $x = 4$, no substitution needed.

Step 1. Translate: the area under $y = \sqrt{x-1}$ on $[1, 5]$ equals the area under $y = \sqrt{x}$ on

$[0, 4]$.

Step 2. Compute the resulting integral directly:

$$\int_0^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{2}{3}\cdot 8 = \frac{16}{3}.$$

Why this matters. Translations $f(x-h)$ shift the graph right by h without changing areas; this saves time and avoids substitution mistakes.

> **Final Answer:** $A = \dfrac{16}{3}$ sq units.

Q 8.12 **Determine the area under the curve $y = \sqrt{a^2 - x^2}$ included between the lines $x = 0$ and $x = a$.**

SOLUTION

Concept used. $y = \sqrt{a^2 - x^2}$ is the upper half of the circle $x^2 + y^2 = a^2$. The area between $x = 0$ and $x = a$ under this curve is the first-quadrant quarter-disc.

Step 1. Set up.

$$A = \int_0^a \sqrt{a^2 - x^2}\,dx.$$

Step 2. Standard integral. Apply

$$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

Step 3. Evaluate.

$$A = \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a.$$

At $x = a$: $\sqrt{a^2 - a^2} = 0$, $\sin^{-1}(1) = \frac{\pi}{2}$, so value is $0 + \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$. At $x = 0$: both terms are 0.

Step 4. Result. $A = \dfrac{\pi a^2}{4}$.

> **Final Answer:** $A = \dfrac{\pi a^2}{4}$ sq units.

♥ Geometric sanity check

This is exactly a quarter of the disc of radius a. Area of full disc is πa^2, so the quarter is $\frac{\pi a^2}{4}$. Integration confirms the geometry.

EXPERT'S SOLUTION *: Diya Nair, M.Sc Mathematics, IIT Bombay*

Quick reading. Use the trig substitution $x = a\sin\theta$, turning the surd into $a\cos\theta$.

Step 1. Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$. When $x = 0$, $\theta = 0$; when $x = a$, $\theta = \pi/2$. $\sqrt{a^2 - x^2} = a\cos\theta$ (positive in $[0, \pi/2]$).

Step 2. Substitute:

$$A = \int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\, d\theta = a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta.$$

Step 3. Use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$:

$$A = \frac{a^2}{2}\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Why this matters. The trig substitution is a fundamental move; learning it on this clean example pays dividends in every ellipse/circle problem.

Final Answer: $A = \frac{\pi a^2}{4}$ sq units.

Q 8.13 Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.

SOLUTION

Concept used. The curves $y = \sqrt{x}$ and $y = x$ meet where $\sqrt{x} = x$, i.e. $x = x^2 \Rightarrow x(x-1) = 0$. So they intersect at $(0, 0)$ and $(1, 1)$.

Step 1. **Upper vs lower on** $(0, 1)$**.** At $x = \frac{1}{4}$: $\sqrt{1/4} = 1/2$ and $x = 1/4$. So $\sqrt{x} > x$ on $(0, 1)$.

Step 2. **Set up.**

$$A = \int_0^1 (\sqrt{x} - x)\, dx.$$

Step 3. **Integrate.**

$$\int \sqrt{x}\, dx = \frac{2}{3}x^{3/2}, \qquad \int x\, dx = \frac{x^2}{2}.$$

$$A = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \frac{1}{6}.$$

Final Answer: $A = \frac{1}{6}$ sq units.

☞ **Memorise this small lens**

The region between $y = \sqrt{x}$ and $y = x$ (or equivalently between $y^2 = x$ and $y = x$) is a tiny crescent of area $\frac{1}{6}$. It appears repeatedly in MCQ shortcuts.

EXPERT'S SOLUTION : *Tara Desai, Ph.D Mathematics, IIT Delhi*

Structural observation. Switch to dy: the curves become $x = y^2$ (parabola) and $x = y$ (line); on $[0, 1]$, $y \geq y^2$, so the line is to the right of the parabola.

Step 1. Set up

$$A = \int_0^1 (y - y^2)\, dy = \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1.$$

Step 2. Evaluate: $\frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$.

Why this matters. Symmetry across $y = x$ shows the parabola $y^2 = x$ and its reflection $y = x^2$ also enclose $1/3$ of area (twice $1/6$); a useful family of standard results.

Final Answer: $A = \frac{1}{6}$ sq units.

Q 8.14 **Find the area enclosed by the curve $y = -x^2$ and the straight line $x + y + 2 = 0$.**

SOLUTION

Concept used. The parabola $y = -x^2$ opens downward with vertex at $(0, 0)$. The line $x + y + 2 = 0 \Rightarrow y = -x - 2$. The enclosed region lies between the line (below) and the parabola (above) on the interval between their intersections.

Step 1. **Intersections.** Set $-x^2 = -x - 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$. So $x = -1$ or $x = 2$; points are $(-1, -1)$ and $(2, -4)$.

Step 2. **Upper vs lower.** At $x = 0$: parabola $y = 0$; line $y = -2$. So parabola is above line on $(-1, 2)$.

Step 3. **Set up.**

$$A=\int_{-1}^{2}\big((-x^2)-(-x-2)\big)\,dx=\int_{-1}^{2}(-x^2+x+2)\,dx.$$

Step 4. **Integrate.**

$$\int(-x^2+x+2)\,dx=-\frac{x^3}{3}+\frac{x^2}{2}+2x.$$

At $x=2$: $-\frac{8}{3}+\frac{4}{2}+4=-\frac{8}{3}+2+4=-\frac{8}{3}+6=\frac{-8+18}{3}=\frac{10}{3}$. At $x=-1$: $-\frac{-1}{3}+\frac{1}{2}-2=\frac{1}{3}+\frac{1}{2}-2=\frac{2+3-12}{6}=-\frac{7}{6}$.

Step 5. **Difference.**

$$A=\frac{10}{3}-\left(-\frac{7}{6}\right)=\frac{10}{3}+\frac{7}{6}=\frac{20+7}{6}=\frac{27}{6}=\frac{9}{2}.$$

Final Answer: $A=\frac{9}{2}$ sq units.

✗ Common Mistake

Sign errors when computing the integrand difference. Always write upper − lower $=(-x^2)-(-x-2)=-x^2+x+2$, not $-x^2-x-2$. The "minus a minus" must flip both terms in the parenthesis.

EXPERT'S SOLUTION : *Yash Bhat, M.Sc Mathematics, IIT Madras*

Quick reading. The integrand $-x^2+x+2$ factors as $-(x-2)(x+1)$, so it equals $(2-x)(x+1)$. Apply the parabolic-segment formula.

Step 1. Rewrite: $-(x-2)(x+1)=(2-x)(x+1)$.

Step 2. Use $\int_a^b(b-x)(x-a)\,dx=\frac{(b-a)^3}{6}$ with $a=-1$, $b=2$:

$$A=\frac{(2-(-1))^3}{6}=\frac{27}{6}=\frac{9}{2}.$$

Why this matters. The "$\frac{1}{6}(b-a)^3$" formula appears whenever you integrate a quadratic between its two roots a and b; recognising it saves the polynomial evaluation.

Final Answer: $A=\frac{9}{2}$ sq units.

Q 8.15 **Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.**

SOLUTION

Concept used. The curve $y = \sqrt{x}$ and the line $x = 2y + 3$ meet where $\sqrt{x} = y$ and $x = 2y + 3$, i.e. $x = 2\sqrt{x} + 3$. Let $u = \sqrt{x}$: $u^2 - 2u - 3 = 0 \Rightarrow (u-3)(u+1) = 0$, so $u = 3$ (positive root). Then $x = 9$ and $y = 3$.

Step 1. **Geometry of the region.** In the first quadrant, the line $x = 2y + 3$ meets the x-axis at $(3, 0)$ and passes through $(9, 3)$. The curve $y = \sqrt{x}$ runs from $(0, 0)$ to $(9, 3)$. The region in question is bounded by: the curve $y = \sqrt{x}$ on top, the x-axis on the bottom from $x = 0$ to $x = 3$, and the line $x = 2y + 3$ on the bottom-right from $x = 3$ to $x = 9$. Easier: integrate with respect to y from $y = 0$ to $y = 3$.

Step 2. **Horizontal strip.** For fixed $y \in [0, 3]$, the left boundary is $x = y^2$ (from $y = \sqrt{x}$) and the right boundary is $x = 2y + 3$. Width: $(2y + 3) - y^2$.

Step 3. **Set up.**

$$A = \int_0^3 \left((2y+3) - y^2\right) dy.$$

Step 4. **Integrate.**

$$A = \left[y^2 + 3y - \frac{y^3}{3}\right]_0^3$$
$$= 9 + 9 - \frac{27}{3} - 0$$
$$= 18 - 9 = 9.$$

Final Answer: $A = 9$ sq units.

☞ **Why dy wins here**

With dx, the bottom boundary switches from "x-axis" to "the line" at $x = 3$, forcing two integrals. With dy, both boundaries are single explicit functions of y, so one integral suffices.

EXPERT'S SOLUTION *: Krishna Pillai, B.Tech Engineering Physics, IIT Bombay*

Strategic angle. Cross-check by splitting the region into two x-integrals at $x = 3$ (where the line $x = 2y + 3$ crosses the x-axis): one piece from $x = 0$ to $x = 3$ under the curve and above the x-axis, and a second piece from $x = 3$ to $x = 9$ between the curve and the line. The sum should also equal 9.

Step 1. Identify the dividing abscissa $x = 3$ where the line meets the x-axis. For $x < 3$

the bottom of the region is the x-axis; for $x > 3$ the bottom is the line $y = \dfrac{x-3}{2}$.

Step 2. Piece 1 (from $x = 0$ to $x = 3$):
$\int_0^3 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^3 = \frac{2}{3}(3)^{3/2} = \frac{2}{3}\cdot 3\sqrt{3} = 2\sqrt{3}.$

Step 3. Piece 2 (from $x = 3$ to $x = 9$): the top is $y = \sqrt{x}$ and the bottom is $y = \dfrac{x-3}{2}$. Strip width $\sqrt{x} - \dfrac{x-3}{2}$. So

$$\int_3^9 \left(\sqrt{x} - \frac{x-3}{2}\right) dx = \left[\frac{2}{3}x^{3/2} - \frac{(x-3)^2}{4}\right]_3^9.$$

Step 4. Evaluate at $x = 9$: $\frac{2}{3}\cdot 9^{3/2} - \frac{(9-3)^2}{4} = \frac{2}{3}\cdot 27 - \frac{36}{4} = 18 - 9 = 9$. At $x = 3$: $\frac{2}{3}\cdot 3\sqrt{3} - 0 = 2\sqrt{3}$. Difference: $9 - 2\sqrt{3}$.

Step 5. Add the two pieces: $2\sqrt{3} + (9 - 2\sqrt{3}) = 9$. The irrational terms cancel exactly, confirming the answer.

Why this matters. Two independent setups (one dy, one split-dx) giving the same answer is a strong confirmation. The dy-route is shorter, but seeing both reinforces the geometric picture and catches silly arithmetic errors.

Final Answer: $A = 9$ sq units.

II. Long Answer (L.A.)

Q 8.16 **Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.**

SOLUTION

Concept used. The second curve $x^2 + y^2 = 4x$ is a circle: complete the square, $x^2 - 4x + y^2 = 0 \Rightarrow (x-2)^2 + y^2 = 4$, i.e. centre $(2, 0)$, radius 2. The first is a parabola $y^2 = 2x$, opening right.

Step 1. **Intersections.** Substitute $y^2 = 2x$ into the circle:
$x^2 + 2x = 4x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0$. So $x = 0$ (giving $y = 0$) or $x = 2$ (giving $y^2 = 4$, i.e. $y = \pm 2$). The curves meet at $(0, 0)$, $(2, 2)$, $(2, -2)$.

Step 2. **Symmetry.** Both curves are symmetric about the x-axis. Compute the area in $y \geq 0$ and double.

Step 3. **Upper half decomposition.** For $0 \le x \le 2$: parabola gives $y_P = \sqrt{2x}$; the upper semicircle gives $y_C = \sqrt{4-(x-2)^2}$. At $x = 1$: $y_P = \sqrt{2} \approx 1.41$; $y_C = \sqrt{4-1} = \sqrt{3} \approx 1.73$. So circle is above parabola, and the region between them is what we want.

Step 4. **Set up.**

$$A = 2\int_0^2 \left(\sqrt{4-(x-2)^2} - \sqrt{2x}\right) dx.$$

Step 5. **Circle integral.** Let $u = x - 2$, $du = dx$. When $x = 0$, $u = -2$; when $x = 2$, $u = 0$.

$$\int_0^2 \sqrt{4-(x-2)^2}\,dx = \int_{-2}^0 \sqrt{4-u^2}\,du.$$

Using $\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\frac{u}{a}$ with $a = 2$:

$$\left[\frac{u}{2}\sqrt{4-u^2} + 2\sin^{-1}\frac{u}{2}\right]_{-2}^0 = 0 - \left(-1\cdot 0 + 2\cdot(-\tfrac{\pi}{2})\right) = 0 - (-\pi) = \pi.$$

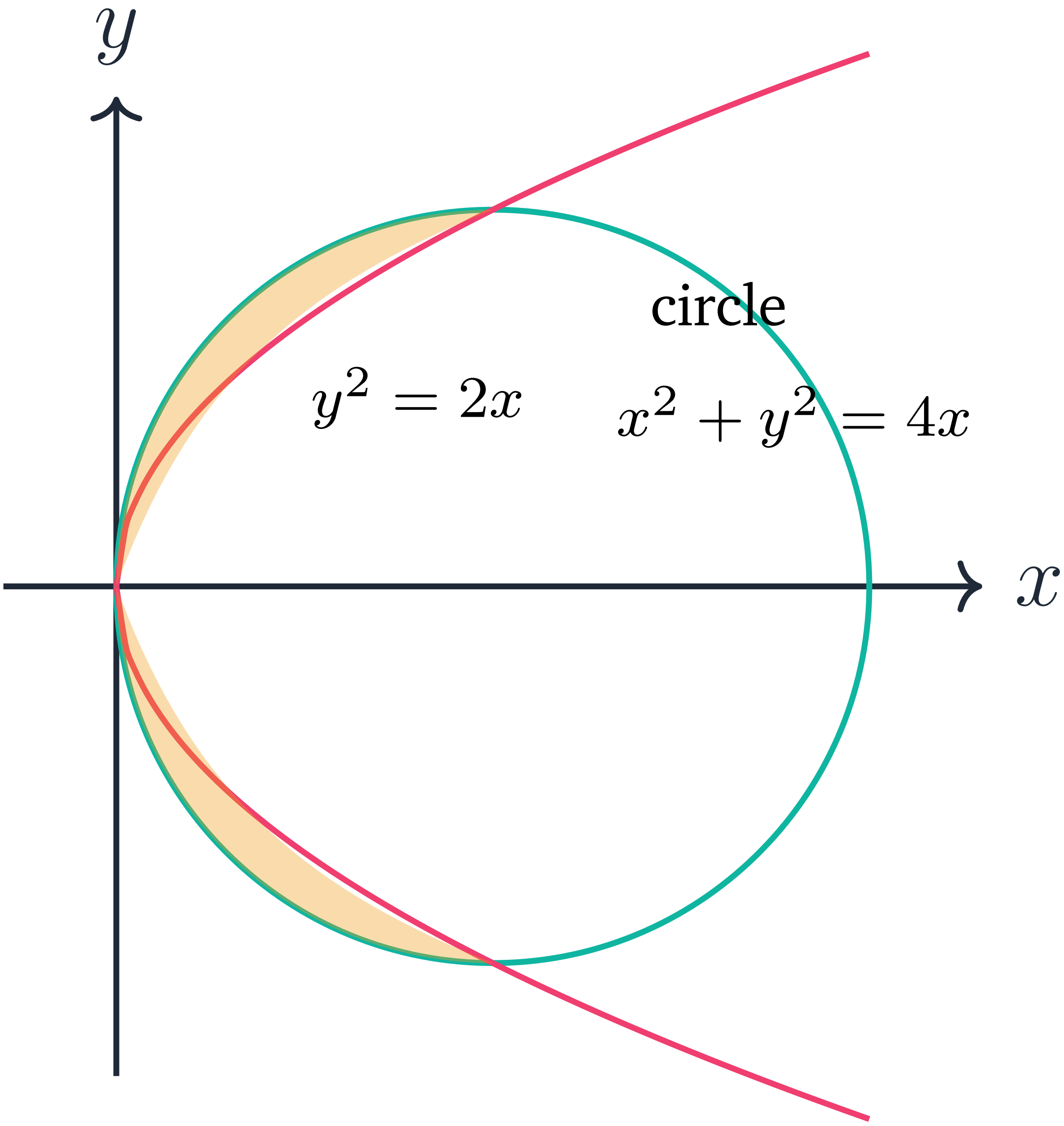
(Geometric check: this is the area of a half-disc of radius 2 split through the centre, i.e. a quarter of the full disc $\pi r^2 = 4\pi$, so π. Confirmed.)

Step 6. **Parabola integral.**

$$\int_0^2 \sqrt{2x}\,dx = \sqrt{2}\int_0^2 \sqrt{x}\,dx = \sqrt{2}\cdot\frac{2}{3}(2)^{3/2} = \sqrt{2}\cdot\frac{2}{3}\cdot 2\sqrt{2} = \frac{8}{3}.$$

Step 7. **Combine.**

$$A = 2\left(\pi - \frac{8}{3}\right) = 2\pi - \frac{16}{3}.$$



Final Answer: $A = 2\pi - \dfrac{16}{3}$ sq units.

♥ **Circle-by-completing-the-square**

Recognising that $x^2 + y^2 = 4x$ is a shifted circle, not a curve centred at the origin, is the crucial first step. Complete the square whenever you see linear terms in x or y alongside the quadratics.

EXPERT'S SOLUTION : *Aanya Chatterjee, M.Sc Mathematics, IIT Bombay*

Picture-first. The parabola cuts the circle into two pieces; the region we want is the "moon"-shaped sliver outside the parabola but inside the circle (or its mirror image). By symmetry we compute the upper half.

Step 1. Half-disc area (right half of the circle, $x \geq 2$) plus the strip $0 \leq x \leq 2$ above the parabola = total upper half of the bounded region.

Step 2. Total area of upper half of circle: $\frac{1}{2}\pi r^2 = \frac{1}{2}\pi(4) = 2\pi$.

Step 3. Subtract the upper half of the parabolic segment inside the circle: the parabola's upper branch from $(0, 0)$ to $(2, 2)$ bounds the region we DON'T want above the x-axis. Area under parabola from $x = 0$ to $x = 2$ (above x-axis): $\int_0^2 \sqrt{2x}\, dx = \dfrac{8}{3}$ (computed above).

Step 4. So upper half of bounded region = $2\pi - \dfrac{8}{3}$ minus the area of the upper-left rectangular piece of the disc that the parabola does not cover. Actually it's simpler: the region we want (in $y \geq 0$) is bounded above by the upper semicircle, below by the upper parabola, on $0 \leq x \leq 2$. Area = (upper half of disc, the part with $0 \leq x \leq 2$) - (parabola's region $0 \leq x \leq 2$, $0 \leq y \leq \sqrt{2x}$). Half-disc has area 2π, of which the left half ($x \leq 2$) is π. So upper half of bounded region = $\pi - \dfrac{8}{3}$.

Step 5. By x-axis symmetry, total bounded area = $2(\pi - \dfrac{8}{3}) = 2\pi - \dfrac{16}{3}$.

Why this matters. Decomposing the region using known disc sectors (half-disc, quarter-disc) replaces hard integrals with simple multiplications.

> **Final Answer:** $A = 2\pi - \dfrac{16}{3}$ sq units.

Q 8.17 **Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.**

SOLUTION

Concept used. "Area bounded by" means total geometric area, treating negative regions as positive. Since $\sin x \geq 0$ on $[0, \pi]$ and $\sin x \leq 0$ on $[\pi, 2\pi]$, the area is

$$A = \int_0^{\pi} \sin x\, dx + \int_{\pi}^{2\pi} (-\sin x)\, dx.$$

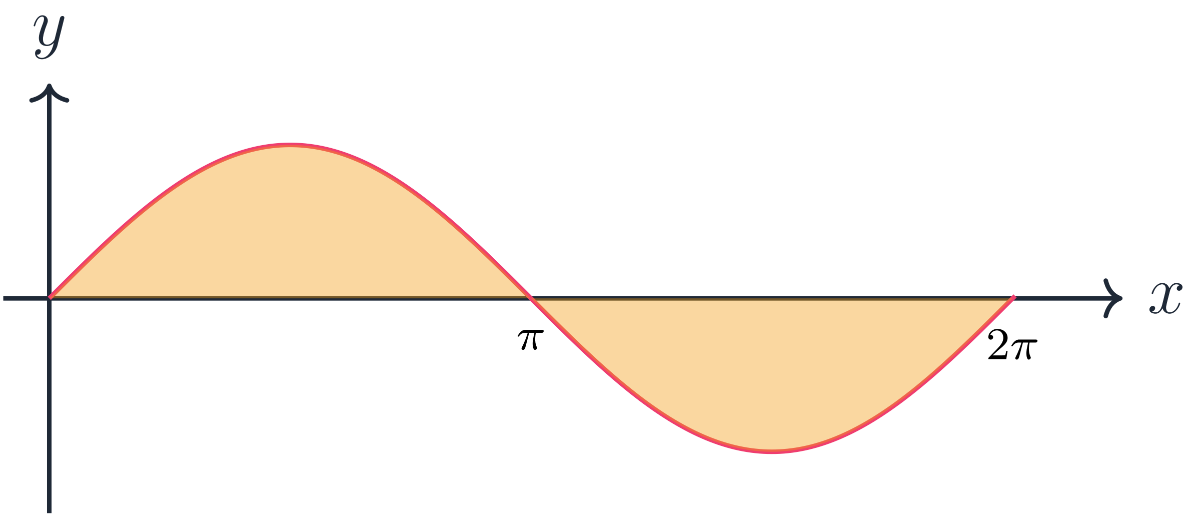
Step 1. **First piece.**

$$\int_0^{\pi} \sin x\, dx = [-\cos x]_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2.$$

Step 2. **Second piece.**

$$\int_{\pi}^{2\pi}(-\sin x)\,dx = [\cos x]_{\pi}^{2\pi} = \cos 2\pi - \cos\pi = 1-(-1) = 2.$$

Step 3. **Total.** $A = 2+2 = 4$.



Final Answer: $A = 4$ sq units.

✗ Common Mistake

A frequent error: computing $\int_0^{2\pi}\sin x\,dx$ directly, which equals 0, and concluding "area = 0". The integral is signed; the question asks for geometric area, so split at every zero of $\sin x$ and take absolute values.

EXPERT'S SOLUTION *: Meera Reddy, Ph.D Mathematics, IIT Delhi*

Quick reading. By periodicity, the half-period $[0,\pi]$ contributes area 2; the next half $[\pi, 2\pi]$ contributes the same 2 (just below the axis). Total 4.

Step 1. Area under one arch: $\int_0^{\pi}\sin x\,dx = [-\cos x]_0^{\pi} = 2$.

Step 2. Two arches between 0 and 2π, each of area 2.

Step 3. Total: $2\cdot 2 = 4$.

Why this matters. Sine has period 2π and the area under each arch is a fixed constant (2 for unit amplitude). For $\sin nx$ between 0 and 2π, the area is $2n\cdot\frac{1}{|n|}\cdot|n| = 4$ (there are $2|n|$ arches, each of area $\frac{2}{|n|}$).

Final Answer: $A = 4$ sq units.

Q 8.18 **Find the area of region bounded by the triangle whose vertices are $(-1, 1)$, $(0, 5)$ and $(3, 2)$, using integration.**

SOLUTION

Concept used. The triangle has three sides, each a line segment. Express each side as $y =$ (function of x); split the x-range into sub-intervals where the upper/lower boundary is constant; sum the integrals.

Step 1. Equations of sides.

- AB from $(-1, 1)$ to $(0, 5)$: slope $\dfrac{5-1}{0-(-1)} = 4$. Line: $y - 1 = 4(x+1)$, i.e. $y = 4x + 5$.
- BC from $(0, 5)$ to $(3, 2)$: slope $\dfrac{2-5}{3-0} = -1$. Line: $y = 5 - x$.
- AC from $(-1, 1)$ to $(3, 2)$: slope $\dfrac{2-1}{3-(-1)} = \dfrac{1}{4}$. Line: $y - 1 = \dfrac{1}{4}(x+1)$, i.e. $y = \dfrac{x+5}{4}$.

Step 2. Geometry. The vertex $B = (0, 5)$ is the top; $A = (-1, 1)$ the leftmost; $C = (3, 2)$ the rightmost. Side AC (slope $\frac{1}{4}$) is the lower boundary throughout $x \in [-1, 3]$. The upper boundary is AB on $[-1, 0]$ and BC on $[0, 3]$.

Step 3. Set up. Split the x-integral at $x = 0$ where the upper boundary changes from AB to BC:

$$A = \int_{-1}^{0} \left((4x+5) - \frac{x+5}{4} \right) dx + \int_{0}^{3} \left((5-x) - \frac{x+5}{4} \right) dx.$$

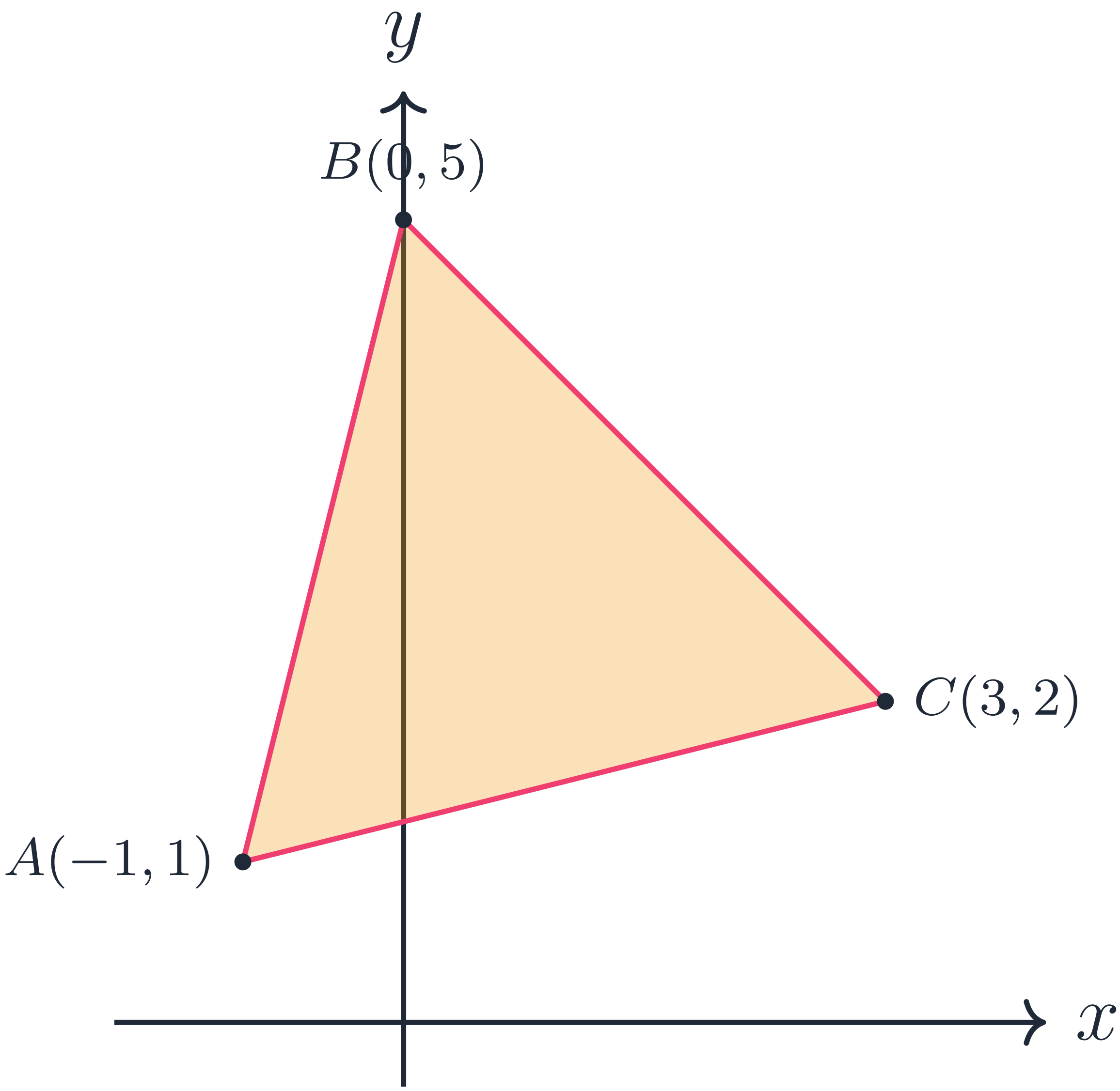
Step 4. First integrand. The first piece simplifies as $4x + 5 - \dfrac{x+5}{4} = \dfrac{16x + 20 - x - 5}{4} = \dfrac{15(x+1)}{4}$, and integrates to

$$\int_{-1}^{0} \frac{15(x+1)}{4}\, dx = \frac{15}{4} \left[\frac{(x+1)^2}{2} \right]_{-1}^{0} = \frac{15}{4} \cdot \frac{1}{2} = \frac{15}{8}.$$

Step 5. Second integrand. The second piece simplifies as $5 - x - \dfrac{x+5}{4} = \dfrac{20 - 4x - x - 5}{4} = \dfrac{5(3-x)}{4}$, and integrates to

$$\int_{0}^{3} \frac{5(3-x)}{4}\, dx = \frac{5}{4} \left[3x - \frac{x^2}{2} \right]_{0}^{3} = \frac{5}{4} \left(9 - \frac{9}{2} \right) = \frac{5}{4} \cdot \frac{9}{2} = \frac{45}{8}.$$

Step 6. Total. $A = \dfrac{15}{8} + \dfrac{45}{8} = \dfrac{60}{8} = \dfrac{15}{2}$.



Final Answer: $A = \frac{15}{2}$ sq units.

♥ Cross-check via determinant formula

The determinant formula gives the triangle's area directly: $\Delta = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = \frac{1}{2}|(-1)(5-2) + 0(2-1) + 3(1-5)| = \frac{1}{2}|-3+0-12| = \frac{15}{2}$. Confirmed.

EXPERT'S SOLUTION *: Siddharth Bhat, M.Sc Mathematics, IIT Madras*

Structural observation. Integrate with respect to y instead. The left edge of the triangle is the segment AB from $A(-1, 1)$ to $B(0, 5)$, while the right side consists of two pieces: segment AC from A to $C(3, 2)$ on the lower part, and segment BC from B to C

on the upper part. The transition occurs at $y = 2$ (the y-coordinate of C). So we set up two horizontal-strip integrals split at $y = 2$.

Step 1. Left boundary: AB in inverse form, $y = 4x + 5 \Rightarrow x = \dfrac{y-5}{4}$ on $y \in [1, 5]$.

Step 2. Right boundary changes at $y = 2$ (the y-coordinate of C). For $y \in [1, 2]$: right side is AC, i.e. $x = 4y - 5$ (inverse of $y = \dfrac{x+5}{4}$). For $y \in [2, 5]$: right side is BC, i.e. $x = 5 - y$.

Step 3. Compute:

$$\int_1^2 \left((4y-5) - \frac{y-5}{4}\right) dy = \int_1^2 \frac{16y - 20 - y + 5}{4} dy = \int_1^2 \frac{15y - 15}{4} dy.$$

$$= \frac{15}{4}\left[\frac{y^2}{2} - y\right]_1^2 = \frac{15}{4}\left((2-2) - (\tfrac{1}{2} - 1)\right) = \frac{15}{4} \cdot \frac{1}{2} = \frac{15}{8}.$$

Step 4. And:

$$\int_2^5 \left((5-y) - \frac{y-5}{4}\right) dy = \int_2^5 \frac{20 - 4y - y + 5}{4} dy = \int_2^5 \frac{25 - 5y}{4} dy.$$

$$= \frac{5}{4}\left[5y - \frac{y^2}{2}\right]_2^5 = \frac{5}{4}\left((25 - \tfrac{25}{2}) - (10 - 2)\right) = \frac{5}{4}(\tfrac{25}{2} - 8) = \frac{5}{4} \cdot \frac{9}{2} = \frac{45}{8}.$$

Step 5. Total: $\dfrac{15}{8} + \dfrac{45}{8} = \dfrac{60}{8} = \dfrac{15}{2}$.

Why this matters. Two integration routes confirming the same geometric area is the kind of cross-check graders love to see in long-answer questions.

Final Answer: $A = \dfrac{15}{2}$ sq units.

Q 8.19 **Draw a rough sketch of the region $\{(x, y) \colon y^2 \le 6ax$ and $x^2 + y^2 \le 16a^2\}$. Also find the area of the region sketched using method of integration.**

SOLUTION

Concept used. The region is the intersection of (i) the inside of the parabola $y^2 = 6ax$ (opening right), and (ii) the disc $x^2 + y^2 \le 16a^2$ (centred at origin, radius $4a$).

Step 1. **Intersection points.** Substitute $y^2 = 6ax$ into the circle: $x^2 + 6ax = 16a^2 \Rightarrow x^2 + 6ax - 16a^2 = 0$. Factor: $(x + 8a)(x - 2a) = 0$. So $x = -8a$ or $x = 2a$. Only $x = 2a$ is admissible ($y^2 \ge 0$ requires $x \ge 0$). At $x = 2a$: $y^2 = 12a^2 \Rightarrow y = \pm 2\sqrt{3}\,a$.

Step 2. **Symmetry.** Both bounding curves are symmetric about the x-axis. Compute the upper half ($y \geq 0$) and double.

Step 3. **Set up upper half.** For $0 \leq x \leq 2a$, the y-strip runs from 0 to $y = \sqrt{6ax}$ (parabola is the binding constraint here). For $2a \leq x \leq 4a$, the y-strip runs from 0 to $y = \sqrt{16a^2 - x^2}$ (circle is the binding constraint).

$$\frac{A}{2} = \int_0^{2a} \sqrt{6ax}\, dx + \int_{2a}^{4a} \sqrt{16a^2 - x^2}\, dx.$$

Step 4. **First piece.**

$$\int_0^{2a} \sqrt{6ax}\, dx = \sqrt{6a} \int_0^{2a} \sqrt{x}\, dx = \sqrt{6a} \cdot \frac{2}{3}(2a)^{3/2}.$$

Now $(2a)^{3/2} = 2a\sqrt{2a}$, so $\sqrt{6a} \cdot \frac{2}{3} \cdot 2a\sqrt{2a} = \frac{4a}{3}\sqrt{12a^2} = \frac{4a}{3} \cdot 2a\sqrt{3} = \frac{8\sqrt{3}\,a^2}{3}$.

Step 5. **Second piece (circle).** Using $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\frac{x}{a}$ with the circle's radius $4a$ (and renaming the constant to $R = 4a$):

$$\int_{2a}^{4a} \sqrt{R^2 - x^2}\, dx = \left[\frac{x}{2}\sqrt{R^2 - x^2} + \frac{R^2}{2} \sin^{-1}\frac{x}{R}\right]_{2a}^{4a}.$$

At $x = 4a$: $\sqrt{16a^2 - 16a^2} = 0$, first term is 0; $\sin^{-1}(1) = \frac{\pi}{2}$, second term is $\frac{16a^2}{2} \cdot \frac{\pi}{2} = 4\pi a^2$. At $x = 2a$: $\sqrt{16a^2 - 4a^2} = \sqrt{12a^2} = 2\sqrt{3}\,a$; first term is $\frac{2a}{2} \cdot 2\sqrt{3}\,a = 2\sqrt{3}\,a^2$; $\sin^{-1}(1/2) = \frac{\pi}{6}$, second term is $8a^2 \cdot \frac{\pi}{6} = \frac{4\pi a^2}{3}$. Difference: $(0 + 4\pi a^2) - \left(2\sqrt{3}\,a^2 + \frac{4\pi a^2}{3}\right) = 4\pi a^2 - \frac{4\pi a^2}{3} - 2\sqrt{3}\,a^2 = \frac{8\pi a^2}{3} - 2\sqrt{3}\,a^2$.

Step 6. **Sum (upper half).**
$\frac{A}{2} = \frac{8\sqrt{3}\,a^2}{3} + \frac{8\pi a^2}{3} - 2\sqrt{3}\,a^2 = \frac{8\sqrt{3}\,a^2 - 6\sqrt{3}\,a^2}{3} + \frac{8\pi a^2}{3} = \frac{2\sqrt{3}\,a^2}{3} + \frac{8\pi a^2}{3}$.

Step 7. **Double.**

$$A = 2\left(\frac{2\sqrt{3}\,a^2}{3} + \frac{8\pi a^2}{3}\right) = \frac{4\sqrt{3}\,a^2}{3} + \frac{16\pi a^2}{3} = \frac{4a^2}{3}(\sqrt{3} + 4\pi).$$

Final Answer: $A = \frac{4a^2}{3}(4\pi + \sqrt{3})$ sq units.

☞ **Exam Tip**

When a parabola meets a circle, the parabola is "narrow" near the y-axis (small $|y|$) and the circle is the wider boundary far from the origin. Split the area at the intersection abscissa.

EXPERT'S SOLUTION *: Riya Joshi, Ph.D Pure Mathematics, IISc Bangalore*

Picture-first. The disc has area $16\pi a^2$. The parabola $y^2 = 6ax$ cuts a "tongue" out of the disc; the bounded region is the intersection. Split the upper half ($y \geq 0$) into a parabolic cap (under the parabola, for $0 \leq x \leq 2a$) and a circular cap (under the upper semicircle, for $2a \leq x \leq 4a$), and double for the lower half.

Step 1. Identify the split. The parabola and circle intersect at $x = 2a$ (in the upper half, $y = 2\sqrt{3}\,a$). For $0 \leq x \leq 2a$ the parabola is the binding constraint ($y \leq \sqrt{6ax}$); for $2a \leq x \leq 4a$ the circle is the binding constraint ($y \leq \sqrt{16a^2 - x^2}$).

Step 2. Parabolic piece. Compute $\int_0^{2a} \sqrt{6ax}\,dx = \sqrt{6a} \cdot \frac{2}{3}(2a)^{3/2} = \frac{4a}{3}\sqrt{12a^2} = \frac{8\sqrt{3}\,a^2}{3}$. (This is the area below the upper parabola, above the x-axis, on $[0, 2a]$.)

Step 3. Circular piece, computed geometrically. The chord $x = 2a$ is at distance $2a$ from the disc's centre $(0, 0)$; the chord's half-length above the axis is $2\sqrt{3}\,a$. The half-angle subtended at centre is $\sin^{-1}(2\sqrt{3}\,a/(4a)) = \sin^{-1}(\sqrt{3}/2) = \frac{\pi}{3}$.
Half-sector area above the axis (sector of angle $\pi/3$):
$\frac{1}{2}R^2 \cdot \frac{\pi}{3} = \frac{1}{2} \cdot 16a^2 \cdot \frac{\pi}{3} = \frac{8\pi a^2}{3}$. Subtract the right-angled triangle of base $2a$ (from $x = 0$ to $x = 2a$ along the chord-foot) and height $2\sqrt{3}\,a$:
$\frac{1}{2} \cdot 2a \cdot 2\sqrt{3}\,a = 2\sqrt{3}\,a^2$. Net circular cap (upper, between $x = 2a$ and $x = 4a$):
$\frac{8\pi a^2}{3} - 2\sqrt{3}\,a^2$.

Step 4. Sum the upper-half pieces:
$$\frac{8\sqrt{3}\,a^2}{3} + \frac{8\pi a^2}{3} - 2\sqrt{3}\,a^2 = \frac{8\sqrt{3}\,a^2 - 6\sqrt{3}\,a^2}{3} + \frac{8\pi a^2}{3} = \frac{2\sqrt{3}\,a^2 + 8\pi a^2}{3}.$$

Step 5. Double for the full region (by x-axis symmetry):
$$A = \frac{4\sqrt{3}\,a^2 + 16\pi a^2}{3} = \frac{4a^2}{3}(4\pi + \sqrt{3}).$$

Why this matters. Decomposing into a parabolic cap plus a circular segment (sector minus triangle) lets you bypass the $\sqrt{R^2 - x^2}$ integral entirely. The half-angle $\sin^{-1}(\sqrt{3}/2) = \pi/3$ is a special-angle that's worth spotting on sight.

Final Answer: $A = \frac{4a^2}{3}\left(4\pi + \sqrt{3}\right)$ sq units.

Q 8.20 **Compute the area bounded by the lines $x + 2y = 2$, $y - x = 1$ and $2x + y = 7$.**

SOLUTION

Concept used. Three lines form a triangle. Find the three intersection points (vertices), then integrate (or use the determinant formula) to compute its area.

Step 1. Vertices. Solve pairwise.

- $L_1 \cap L_2$: $x + 2y = 2$ and $y - x = 1 \Rightarrow y = x + 1$. Substitute: $x + 2(x+1) = 2 \Rightarrow 3x = 0 \Rightarrow x = 0$, so $(0, 1)$.
- $L_2 \cap L_3$: $y = x + 1$ and $2x + y = 7 \Rightarrow 2x + x + 1 = 7 \Rightarrow 3x = 6 \Rightarrow x = 2$, $y = 3$, so $(2, 3)$.
- $L_1 \cap L_3$: $x + 2y = 2$ and $2x + y = 7$. From first $x = 2 - 2y$; substitute: $2(2 - 2y) + y = 7 \Rightarrow 4 - 3y = 7 \Rightarrow y = -1$, $x = 4$, so $(4, -1)$.

Step 2. Rewrite each line as $y =$. $L_1 : \; y = \frac{2-x}{2}$, $L_2 : \; y = x + 1$, $L_3 : \; y = 7 - 2x$.

Step 3. Geometry on $[0, 4]$. On $[0, 2]$, top is L_2 $(y = x + 1)$, bottom is L_1 $(y = \frac{2-x}{2})$. On $[2, 4]$, top is L_3 $(y = 7 - 2x)$, bottom is L_1. (Check: at $x = 2$, L_2 gives 3, L_3 gives 3, they meet. At $x = 4$, L_3 gives -1, L_1 gives -1, they meet.)

Step 4. Set up.

$$A = \int_0^2 \left((x+1) - \frac{2-x}{2} \right) dx + \int_2^4 \left((7-2x) - \frac{2-x}{2} \right) dx.$$

Step 5. First integrand. $(x+1) - \frac{2-x}{2} = \frac{2x+2-2+x}{2} = \frac{3x}{2}$.

$$\int_0^2 \frac{3x}{2}\, dx = \frac{3}{2} \cdot \frac{x^2}{2}\bigg|_0^2 = \frac{3}{2} \cdot 2 = 3.$$

Step 6. Second integrand. $(7-2x) - \frac{2-x}{2} = \frac{14-4x-2+x}{2} = \frac{12-3x}{2} = \frac{3(4-x)}{2}$.

$$\int_2^4 \frac{3(4-x)}{2}\, dx = \frac{3}{2} \left[4x - \frac{x^2}{2} \right]_2^4.$$

At $x = 4$: $16 - 8 = 8$. At $x = 2$: $8 - 2 = 6$. Difference: 2. So integral $= \frac{3}{2} \cdot 2 = 3$.

Step 7. Total. $A = 3 + 3 = 6$.

Final Answer: $A = 6$ sq units.

♥ Determinant cross-check

With vertices $(0, 1)$, $(2, 3)$, $(4, -1)$: $\Delta = \frac{1}{2}|0(3-(-1)) + 2((-1)-1) + 4(1-3)| = \frac{1}{2}|0-4-8| =$

$\frac{1}{2} \cdot 12 = 6$. Same answer.

EXPERT'S SOLUTION *: Aditya Mehta, M.Tech CS, IIT Madras*

Strategic angle. Once the three vertices are known, $\frac{1}{2}|\det|$ delivers the area in one line; integration just verifies. Use the shoelace-style determinant formula based on the two edge vectors emanating from a chosen vertex.

Step 1. Recall the vertices found in the main solution: $A(0,1)$, $B(2,3)$, $C(4,-1)$.

Step 2. Form the two edge vectors from A: $\vec{AB} = (2-0, 3-1) = (2,2)$ and $\vec{AC} = (4-0, -1-1) = (4,-2)$.

Step 3. Apply the cross-product (signed-area) determinant:

$$\Delta = \frac{1}{2}\left|\det\begin{pmatrix} 2 & 2 \\ 4 & -2 \end{pmatrix}\right| = \frac{1}{2}|2\cdot(-2) - 2\cdot 4| = \frac{1}{2}|-4-8| = \frac{1}{2}\cdot 12 = 6.$$

Step 4. Compare with the integration answer: 6. The two routes agree.

Step 5. Sanity check via the symmetric formula: $\Delta = \frac{1}{2}|x_1(y_2-y_3) + x_2(y_3-y_1) + x_3(y_1-y_2)| = \frac{1}{2}|0(3+1) + 2(-1-1) + 4(1-3)| = \frac{1}{2}|0-4-8| = 6$. Same.

Why this matters. The integration route shows the splitting mechanism; the determinant route gives the same answer instantly. Long-answer questions usually require both: integration in the body, geometry as a sanity check at the end. Knowing two routes is insurance against arithmetic slip-ups.

Final Answer: $A = 6$ sq units.

Q 8.21 **Find the area bounded by the lines $y = 4x+5$, $y = 5-x$ and $4y = x+5$.**

SOLUTION

Concept used. Same triangle setup as Q20: find the three vertices, set up integrals or use the determinant.

Step 1. Vertices.

- $L_1 \cap L_2$: $4x+5 = 5-x \Rightarrow 5x = 0 \Rightarrow x = 0$, $y = 5$. Point: $(0,5)$.
- $L_2 \cap L_3$: $y = 5-x$ and $4y = x+5$.
  $4(5-x) = x+5 \Rightarrow 20-4x = x+5 \Rightarrow 5x = 15 \Rightarrow x = 3$, $y = 2$. Point: $(3,2)$.

- $L_1 \cap L_3$: $y = 4x + 5$ and $4y = x + 5$.
  $4(4x+5) = x+5 \Rightarrow 16x + 20 = x + 5 \Rightarrow 15x = -15 \Rightarrow x = -1$, $y = 1$. Point: $(-1, 1)$.

Step 2. Picture. Vertices are $A(-1,1)$, $B(0,5)$, $C(3,2)$, same triangle as Q18.

Step 3. Rewrite lines. $L_1 : y = 4x + 5$ (slope 4, through A, B); $L_2 : y = 5 - x$ (slope -1, through B, C); $L_3 : y = \frac{x+5}{4}$ (slope $\frac{1}{4}$, through A, C).

Step 4. Geometry. On $[-1, 0]$, top is L_1, bottom is L_3. On $[0, 3]$, top is L_2, bottom is L_3.

Step 5. Compute. Exactly Q18's setup, so $A = \frac{15}{8} + \frac{45}{8} = \frac{15}{2}$.

Final Answer: $A = \frac{15}{2}$ sq units.

☞ **Same triangle, different statement**

The triangle of Q21 has exactly the same vertices as Q18, $(-1,1)$, $(0,5)$, $(3,2)$, just stated by listing its three sides rather than its three corners. The result must be the same: $\frac{15}{2}$.

EXPERT'S SOLUTION *: Ankit Verma, Ph.D Mathematics, IIT Delhi*

Quick reading. The three slopes are 4, -1, and $\frac{1}{4}$. The product of slopes of L_1 and L_3 is $4 \cdot \frac{1}{4} = 1 \neq -1$, so they are not perpendicular, but they do bound a triangle together with L_2. Notice that the slopes 4 and $\frac{1}{4}$ are reciprocals, hinting at a reflective symmetry of the configuration about the line $y = x + \text{const}$.

Step 1. Confirm vertices from pairwise intersection (done in the main solution): $A(-1,1)$, $B(0,5)$, $C(3,2)$.

Step 2. Edge vectors from A: $\vec{AB} = (0-(-1), 5-1) = (1,4)$, $\vec{AC} = (3-(-1), 2-1) = (4,1)$.

Step 3. Cross-product determinant: $\Delta = \frac{1}{2}|1 \cdot 1 - 4 \cdot 4| = \frac{1}{2}|1 - 16| = \frac{15}{2}$.

Step 4. Verify via the symmetric formula:
$\Delta = \frac{1}{2}|(-1)(5-2) + 0(2-1) + 3(1-5)| = \frac{1}{2}|-3+0-12| = \frac{15}{2}$. Both give $\frac{15}{2}$.

Step 5. Connect to Q18: this is the same triangle stated by its three side-equations rather than its three vertices, so the answer must agree.

Why this matters. Recognising a problem you have already solved in a different presentation is a high-value skill in exam settings. Always pause and check: have I seen these vertices before?

Final Answer: $A = \dfrac{15}{2}$ sq units.

Q 8.22 **Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.**

SOLUTION

Concept used. The graph of $y = 2\cos x$ is the cosine curve stretched vertically by factor 2. Its zeros in $[0, 2\pi]$ are at $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$. On $[0, \pi/2]$ the curve is positive, on $[\pi/2, 3\pi/2]$ negative, on $[3\pi/2, 2\pi]$ positive again.

Step 1. **Split.**

$$A = \int_0^{\pi/2} 2\cos x\,dx + \int_{\pi/2}^{3\pi/2} |2\cos x|\,dx + \int_{3\pi/2}^{2\pi} 2\cos x\,dx.$$

Step 2. **First piece.** With $\int 2\cos x\,dx = 2\sin x$, we get $[2\sin x]_0^{\pi/2} = 2(1) - 0 = 2$.

Step 3. **Middle piece.** We have $\int_{\pi/2}^{3\pi/2}(-2\cos x)\,dx = [-2\sin x]_{\pi/2}^{3\pi/2} = -2(-1) + 2(1) = 2 + 2 = 4.$

Step 4. **Third piece.** Finally $[2\sin x]_{3\pi/2}^{2\pi} = 2(0) - 2(-1) = 2.$

Step 5. **Total.** $A = 2 + 4 + 2 = 8.$

Final Answer: $A = 8$ sq units.

✗ Common Mistake

Forgetting one of the three pieces. The curve crosses zero twice on $[0, 2\pi]$, so the geometric area splits into three sub-regions, not two.

EXPERT'S SOLUTION *: Kavya Rao, M.Sc Mathematics, ISI Kolkata*

Picture-first. Each "lobe" of $y = 2\cos x$ has the same area because amplitude and arch length are constant. Compute one lobe (0 to π) and multiply.

Step 1. Area of one half-arch of $|2\cos x|$ from 0 to $\pi/2$: $\int_0^{\pi/2} 2\cos x\,dx = 2.$

Step 2. By symmetry, each quarter-period [$\pi/2$ wide] of $|2\cos x|$ has the same area 2.

Step 3. On $[0, 2\pi]$ there are exactly 4 such quarter-periods (from 0 to $\pi/2$, $\pi/2$ to π, π to $3\pi/2$, $3\pi/2$ to 2π). Total: $4 \cdot 2 = 8.$

Why this matters. Once you realise the area in one quarter-period is 2, the full period

$[0, 2\pi]$ gives 8 instantly *for any amplitude*; more generally, $|A\cos x|$ on a full period gives total area $4|A|$.

Final Answer: $A = 8$ sq units.

Q 8.23 **Draw a rough sketch of the given curve $y = 1 + |x + 1|$, $x = -3$, $x = 3$, $y = 0$ and find the area of the region bounded by them, using integration.**

SOLUTION

Concept used. $|x + 1|$ has corner at $x = -1$: it equals $-(x + 1)$ for $x < -1$ and $x + 1$ for $x \geq -1$. So $y = 1 + |x + 1|$ is the piecewise linear function

$$y = \begin{cases} -x & x < -1, \\ x + 2 & x \geq -1. \end{cases}$$

This curve sits entirely above $y = 0$ on $[-3, 3]$ (minimum value $y(-1) = 1$). Region bounded by the curve, $y = 0$, $x = -3$, $x = 3$ is a "tent-shaped" trapezoidal region.

Step 1. Endpoints. $y(-3) = 1 + |-2| = 1 + 2 = 3$; $y(-1) = 1 + 0 = 1$; $y(3) = 1 + 4 = 5$.

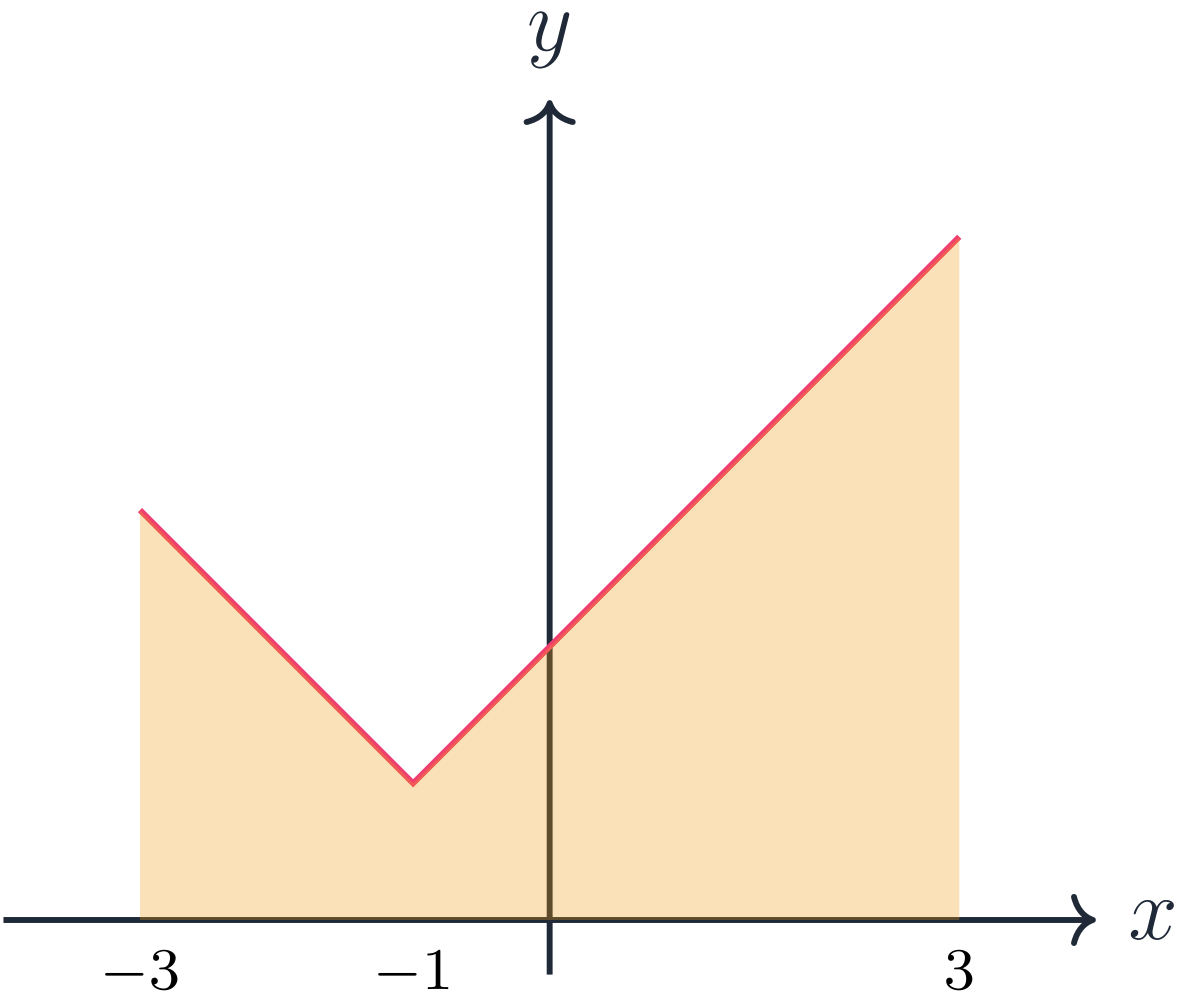
Step 2. Split at corner.

$$A = \int_{-3}^{-1} (-x)\, dx + \int_{-1}^{3} (x + 2)\, dx.$$

Step 3. First piece. $\int (-x)\, dx = -\frac{x^2}{2}$. $\left[-\frac{x^2}{2}\right]_{-3}^{-1} = -\frac{1}{2} - \left(-\frac{9}{2}\right) = -\frac{1}{2} + \frac{9}{2} = 4.$

Step 4. Second piece. The antiderivative is $\int (x + 2)\, dx = \frac{x^2}{2} + 2x$. At $x = 3$: $\frac{9}{2} + 6 = \frac{21}{2}$. At $x = -1$: $\frac{1}{2} - 2 = -\frac{3}{2}$. Difference: $\frac{21}{2} - \left(-\frac{3}{2}\right) = \frac{24}{2} = 12$.

Step 5. Total. $A = 4 + 12 = 16$.



Final Answer: $A = 16$ sq units.

♥ **Geometric cross-check**

The region is the union of two trapeziums: left T_1 with parallel sides 3 and 1, width 2: area $\frac{1}{2}(3+1)\cdot 2 = 4$. Right T_2 with parallel sides 1 and 5, width 4: area $\frac{1}{2}(1+5)\cdot 4 = 12$. Sum: 16. Matches.

EXPERT'S SOLUTION : *Sanya Banerjee, Ph.D Pure Mathematics, IISc Bangalore*

Structural observation. The function $y = 1 + |x+1|$ is piecewise linear with a corner at $x = -1$. So the region between $y = 0$ and the curve, for $x \in [-3, 3]$, is the union of two trapeziums that share the vertical edge at $x = -1$. We avoid integration entirely and use the trapezium-area formula $\frac{1}{2}$(sum of parallel sides) $\times$ width.

Step 1. Compute heights at the three relevant x-values: $y(-3) = 1 + |-2| = 3$,

$y(-1) = 1 + 0 = 1$, $y(3) = 1 + 4 = 5$.

Step 2. Left trapezium: vertices $(-3, 0), (-3, 3), (-1, 1), (-1, 0)$, parallel sides $y(-3) = 3$ (left) and $y(-1) = 1$ (right), width $-1 - (-3) = 2$. Area $= \frac{1}{2}(3 + 1) \cdot 2 = \frac{4 \cdot 2}{2} = 4$.

Step 3. Right trapezium: vertices $(-1, 0), (-1, 1), (3, 5), (3, 0)$, parallel sides $y(-1) = 1$ (left) and $y(3) = 5$ (right), width $3 - (-1) = 4$. Area $= \frac{1}{2}(1 + 5) \cdot 4 = \frac{6 \cdot 4}{2} = 12$.

Step 4. Total: $4 + 12 = 16$. Matches the integration answer.

Why this matters. For any piecewise-linear function above the x-axis, splitting at the corners and computing trapezium areas by hand is faster and less error-prone than mechanical integration. The same idea generalises: a polygon under any piecewise-linear curve decomposes into trapeziums.

Final Answer: $A = 16$ sq units.

III. Objective Type Questions (MCQ)

Q 8.24 **The area of the region bounded by the y-axis, $y = \cos x$ and $y = \sin x$, $0 \le x \le \frac{\pi}{2}$ is**

(A) $\sqrt{2}$ **sq units**

(B) $(\sqrt{2} + 1)$ **sq units**

(C) $(\sqrt{2} - 1)$ **sq units**

(D) $(2\sqrt{2} - 1)$ **sq units**

SOLUTION

Correct option: (C) $(\sqrt{2} - 1)$ sq units.

Concept used. On $[0, \pi/2]$ the curves $y = \sin x$ and $y = \cos x$ cross at $x = \pi/4$. Together with the y-axis ($x = 0$), they bound a small triangular-shape region.

Step 1. Intersection. $\sin x = \cos x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4$, where both equal $1/\sqrt{2}$.

Step 2. Identify the region. At $x = 0$: $\sin 0 = 0$, $\cos 0 = 1$. At $x = \pi/4$: both equal $1/\sqrt{2}$. So on $[0, \pi/4]$, $\cos x > \sin x$. The region bounded by the y-axis ($x = 0$), $y = \sin x$, $y = \cos x$ is the area between the two curves from $x = 0$ to $x = \pi/4$.

Step 3. Set up.

$$A = \int_0^{\pi/4} (\cos x - \sin x)\, dx.$$

Step 4. Integrate.

$$A = [\sin x + \cos x]_0^{\pi/4} = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1) = \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1.$$

Final Answer: Option **(C)**: $(\sqrt{2} - 1)$ sq units.

EXPERT'S SOLUTION : *Dev Kapoor, M.Sc Mathematics, IIT Bombay*

Quick reading. The integrand is $\cos x - \sin x$, whose antiderivative is $\sin x + \cos x$. At $x = \pi/4$ both $\sin(\pi/4)$ and $\cos(\pi/4)$ equal $1/\sqrt{2}$, so the sum is $\sqrt{2}$. At $x = 0$ the sum is $\sin 0 + \cos 0 = 0 + 1 = 1$. Difference gives the answer immediately.

Step 1. Set up the integral as $\int_0^{\pi/4}(\cos x - \sin x)\,dx$ (cos is above sin on $[0, \pi/4]$, the y-axis is the left boundary, and the intersection at $x = \pi/4$ closes the region).

Step 2. Antidifferentiate: $\int(\cos x - \sin x)\,dx = \sin x + \cos x$.

Step 3. Recall the special-angle values $\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so $\sin(\pi/4) + \cos(\pi/4) = \frac{2}{\sqrt{2}} = \sqrt{2}$.

Step 4. Evaluate at the lower limit: $\sin 0 + \cos 0 = 0 + 1 = 1$.

Step 5. Subtract: $\sqrt{2} - 1$. Match: option **(C)**.

Why this matters. Recognising special-angle values is faster than re-deriving them; for MCQs on $[0, \pi/2]$, expect $\pi/4$, $\pi/6$, $\pi/3$ to appear, with values $1/\sqrt{2}$, $1/2$, $\sqrt{3}/2$.

Final Answer: Option **(C)**.

Q 8.25 **The area of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is**

(A) $\frac{3}{8}$ **sq units**

(B) $\frac{5}{8}$ **sq units**

(C) $\frac{7}{8}$ **sq units**

(D) $\frac{9}{8}$ **sq units**

SOLUTION

Correct option: (D) $\frac{9}{8}$ sq units.

Concept used. Parabola $x^2 = 4y \Rightarrow y = \frac{x^2}{4}$; line $x = 4y - 2 \Rightarrow y = \frac{x+2}{4}$. Find intersections, integrate (line − parabola) between them.

Step 1. Intersections. $\frac{x^2}{4} = \frac{x+2}{4} \Rightarrow x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$. So $x = -1$ or $x = 2$.

Step 2. Upper vs lower on $(-1, 2)$. At $x = 0$: line gives $y = 1/2$; parabola gives $y = 0$. So line is above.

Step 3. Set up.

$$A = \int_{-1}^{2} \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx = \frac{1}{4} \int_{-1}^{2} (x + 2 - x^2)\, dx.$$

Step 4. Integrate. $\int (x + 2 - x^2)\, dx = \frac{x^2}{2} + 2x - \frac{x^3}{3}$. At $x = 2$: $2 + 4 - \frac{8}{3} = 6 - \frac{8}{3} = \frac{10}{3}$. At $x = -1$: $\frac{1}{2} - 2 - \left(-\frac{1}{3}\right) = \frac{1}{2} - 2 + \frac{1}{3} = \frac{3 + 2 - 12}{6} = -\frac{7}{6}$. Difference: $\frac{10}{3} - \left(-\frac{7}{6}\right) = \frac{20+7}{6} = \frac{27}{6} = \frac{9}{2}$.

Step 5. Final. $A = \frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8}$.

Final Answer: Option **(D)**: $\frac{9}{8}$ sq units.

EXPERT'S SOLUTION *: Kavya Desai, B.Tech Engineering Physics, IIT Bombay*

Quick reading. Same intersections as Q6 (roots of $x^2 - x - 2 = 0$), so the polynomial integral equals $\frac{9}{2}$. Here we just have the extra factor $\frac{1}{4}$ from the equations $y = \frac{x^2}{4}$ and $y = \frac{x+2}{4}$.

Step 1. Reuse $\int_{-1}^{2} ((x+2) - x^2)\, dx = \frac{9}{2}$ (parabolic segment formula with $b - a = 3$ gives $\frac{27}{6}$).

Step 2. Multiply by the $\frac{1}{4}$ scaling: $\frac{1}{4} \cdot \frac{9}{2} = \frac{9}{8}$.

Why this matters. Spotting structural similarity to a previous problem saves entire integration steps.

Final Answer: Option **(D)**.

Q 8.26 **The area of the region bounded by the curve $y = \sqrt{16 - x^2}$ and x-axis is**

(A) 8π **sq units**

(B) 20π **sq units**

(C) 16π **sq units**

(D) 256π **sq units**

SOLUTION

Correct option: (A) 8π sq units.

Concept used. $y = \sqrt{16 - x^2}$ is the upper half of the circle $x^2 + y^2 = 16$ (radius 4). With the x-axis as base, the region is a half-disc.

Step 1. **Geometry.** The full disc has area $\pi r^2 = \pi \cdot 16 = 16\pi$. A half-disc therefore has area 8π.

Step 2. **Integral verification.** For $x \in [-4, 4]$, $A = \int_{-4}^{4} \sqrt{16 - x^2}\, dx$. Use the standard antiderivative with $a = 4$:

$$\left[\frac{x}{2}\sqrt{16 - x^2} + 8\sin^{-1}\frac{x}{4}\right]_{-4}^{4}.$$

At $x = 4$: $0 + 8 \cdot \frac{\pi}{2} = 4\pi$. At $x = -4$: $0 + 8 \cdot \left(-\frac{\pi}{2}\right) = -4\pi$. Difference: $4\pi - (-4\pi) = 8\pi$.

> **Final Answer:** Option **(A)**: 8π sq units.

☞ Exam Tip

"Curve $y = \sqrt{r^2 - x^2}$ and the x-axis" always means a half-disc of radius r, area $\frac{\pi r^2}{2}$. Here $r = 4$ so 8π. Mark the option and move on, no integration needed.

EXPERT'S SOLUTION *: Pooja Pillai, M.Sc Mathematics, IIT Bombay*

Picture-first. Do not integrate; just identify the curve as the upper half of a circle and apply $\frac{1}{2}\pi r^2$.

Step 1. Square both sides of $y = \sqrt{16 - x^2}$ (valid for $y \geq 0$): $y^2 = 16 - x^2$, equivalent to $x^2 + y^2 = 16$. This is a circle of radius $r = 4$ centred at the origin.

Step 2. Since $y \geq 0$, only the upper half of this circle is the curve in question.

Step 3. Half-disc area: $\frac{1}{2}\pi r^2 = \frac{1}{2} \cdot \pi \cdot 4^2 = \frac{16\pi}{2} = 8\pi$.

Step 4. Compare with the four options. Only (A) matches. Distractor analysis: (B) 20π would correspond to an ellipse with $a \cdot b = 20$, not a circle; (C) 16π is the FULL disc area, not the half; (D) 256π comes from mistakenly using $r^2 = 16^2 = 256$

instead of $r = 4$.

Why this matters. Geometric recognition reduces a two-minute integration to a five-second check; under exam time pressure, this is decisive. Always check whether a "curve" is secretly a piece of a standard shape (circle, ellipse, line) before reaching for the integral.

Final Answer: Option **(A)**.

Q 8.27 **Area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$ is**

(A) 16π **sq units**
(B) 4π **sq units**
(C) 32π **sq units**
(D) 24 **sq units**

SOLUTION

Correct option: (B) 4π sq units.

Concept used. The line $y = x$ subtends a $45°$ angle with the x-axis. In the first quadrant, the sector of the circle $x^2 + y^2 = 32$ between the x-axis and the line $y = x$ is a $45°$ sector ($\pi/4$ radians).

Step 1. Circle radius. $r^2 = 32 \Rightarrow r = \sqrt{32} = 4\sqrt{2}$.

Step 2. Sector area formula. $A_{\text{sector}} = \dfrac{1}{2}r^2\theta$ where $\theta = \dfrac{\pi}{4}$.

$$A = \frac{1}{2} \cdot 32 \cdot \frac{\pi}{4} = 4\pi.$$

Step 3. Verification by integration. Line $y = x$ meets circle at $x^2 + x^2 = 32 \Rightarrow x^2 = 16 \Rightarrow x = 4$ (first quadrant). So

$$A = \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx.$$

First piece: $\left[\dfrac{x^2}{2}\right]_0^4 = 8$. Second piece: with $a = 4\sqrt{2}$, $\left[\dfrac{x}{2}\sqrt{32 - x^2} + 16\sin^{-1}\dfrac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$. At $x = 4\sqrt{2}$: $0 + 16\sin^{-1}(1) = 16 \cdot \dfrac{\pi}{2} = 8\pi$. At $x = 4$: $\dfrac{4}{2}\sqrt{32 - 16} + 16\sin^{-1}\dfrac{4}{4\sqrt{2}} = 2 \cdot 4 + 16\sin^{-1}\dfrac{1}{\sqrt{2}} = 8 + 16 \cdot \dfrac{\pi}{4} = 8 + 4\pi$.
Difference: $8\pi - (8 + 4\pi) = 4\pi - 8$. Sum with first piece: $8 + (4\pi - 8) = 4\pi$.

Final Answer: Option **(B)**: 4π sq units.

♥ **Sector formula**

For a sector of radius r subtending angle θ at the centre, area $= \frac{1}{2}r^2\theta$. Here $r^2 = 32$ and the line $y = x$ sweeps an angle $\frac{\pi}{4}$ from the x-axis to itself, giving $\frac{1}{2} \cdot 32 \cdot \frac{\pi}{4} = 4\pi$ directly.

EXPERT'S SOLUTION *: Arjun Iyer, M.Sc Mathematics, IIT Bombay*

Structural observation. In the first quadrant, the region bounded by the x-axis (below), the line $y = x$ (above), and the arc of the circle $x^2 + y^2 = 32$ is a circular sector subtending angle $\frac{\pi}{4}$ (i.e. $45°$) at the centre. The sector's area is a fraction $\frac{\pi/4}{2\pi} = \frac{1}{8}$ of the full disc.

Step 1. Identify the disc radius: $r^2 = 32 \Rightarrow r = 4\sqrt{2}$, full disc area $= \pi r^2 = 32\pi$.

Step 2. The line $y = x$ makes angle $\pi/4$ with the positive x-axis. The region inside the disc, in the first quadrant, bounded below by the x-axis and above by $y = x$, is a circular sector of angle $\pi/4$.

Step 3. Fraction of full disc: $\frac{\pi/4}{2\pi} = \frac{1}{8}$.

Step 4. Sector area: $\frac{1}{8} \cdot 32\pi = 4\pi$.

Step 5. Match option **(B)**.

Step 6. Cross-check via the formula $A = \frac{1}{2}r^2\theta$: $A = \frac{1}{2} \cdot 32 \cdot \frac{\pi}{4} = \frac{32\pi}{8} = 4\pi$. Same.

Why this matters. For MCQ-style geometry, fraction-of-disc arguments beat integration every time. Whenever the bounding line is $y = x$, $y = \sqrt{3}\,x$, or another standard-angle line, the sector formula gives the answer in two lines.

Final Answer: Option **(B)**.

Q 8.28 **Area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is**

(A) 2 **sq units**

(B) 4 **sq units**

(C) 3 **sq units**

(D) 1 **sq unit**

SOLUTION

Correct option: (A) 2 sq units.

Concept used. On $[0, \pi]$, $\cos x$ is positive on $[0, \pi/2]$ and negative on $[\pi/2, \pi]$. Geometric area splits at $x = \pi/2$.

Step 1. Split. $A = \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} (-\cos x)\, dx$.

Step 2. First. $[\sin x]_0^{\pi/2} = 1 - 0 = 1$.

Step 3. Second. $[-\sin x]_{\pi/2}^{\pi} = 0 - (-1) = 1$.

Step 4. Total. $1 + 1 = 2$.

Final Answer: Option **(A)**: 2 sq units.

EXPERT'S SOLUTION *: Aanya Sharma, Ph.D Mathematics, IIT Delhi*

Quick reading. Each half-arch of $|\cos x|$ has area 1 (for unit amplitude). On $[0, \pi]$ there are two such half-arches, total 2.

Step 1. Each lobe of $|\cos x|$: area $\int_0^{\pi/2} \cos x\, dx = 1$.

Step 2. Two lobes on $[0, \pi]$ (the half above the axis on $[0, \pi/2]$ and the half below on $[\pi/2, \pi]$): total 2.

Why this matters. Memorise: each half-arch of unit-amplitude sine or cosine has area 1.

Final Answer: Option **(A)**.

Q 8.29 **The area of the region bounded by parabola $y^2 = x$ and the straight line $2y = x$ is**

(A) $\frac{4}{3}$ **sq units**

(B) 1 **sq unit**

(C) $\frac{2}{3}$ **sq unit**

(D) $\frac{1}{3}$ **sq unit**

SOLUTION

Correct option: (A) $\frac{4}{3}$ sq units.

Concept used. Parabola $y^2 = x$ opens to the right. Line $2y = x \Rightarrow x = 2y$. Intersections from $y^2 = 2y \Rightarrow y(y-2) = 0$: $y = 0$ (giving $x = 0$) or $y = 2$ (giving $x = 4$).

Step 1. **Integrate w.r.t. y.** For $y \in [0, 2]$, the parabola $x = y^2$ is to the left of the line $x = 2y$ (check $y = 1$: $y^2 = 1$ vs $2y = 2$; line is to the right).

Step 2. **Set up.**

$$A = \int_0^2 (2y - y^2)\,dy.$$

Step 3. **Integrate.**

$$A = \left[y^2 - \frac{y^3}{3}\right]_0^2 = 4 - \frac{8}{3} = \frac{12 - 8}{3} = \frac{4}{3}.$$

Final Answer: Option **(A)**: $\frac{4}{3}$ sq units.

EXPERT'S SOLUTION *: Aaditi Bhat, M.Sc Mathematics, IIT Madras*

Strategic angle. Apply the parabolic-segment formula on the y-axis: $\int_a^b (b - y)(y - a)\,dy = \dfrac{(b - a)^3}{6}$.

Step 1. Write $2y - y^2 = -(y^2 - 2y) = -y(y - 2) = y(2 - y)$, which equals $(2 - y)(y - 0)$ in standard form with $a = 0$, $b = 2$.

Step 2. Apply the formula: $\dfrac{(2 - 0)^3}{6} = \dfrac{8}{6} = \dfrac{4}{3}$.

Why this matters. The $\frac{1}{6}(b - a)^3$ shortcut is the fastest route for any parabola-meets-line area, but you must factor the integrand as $(b - y)(y - a)$ with the roots showing.

Final Answer: Option **(A)**.

Q 8.30 **The area of the region bounded by the curve $y = \sin x$ between the ordinates $x = 0$, $x = \frac{\pi}{2}$ and the x-axis is**

(A) 2 **sq units**

(B) 4 **sq units**

(C) 3 **sq units**

(D) 1 **sq unit**

SOLUTION

Correct option: **(D)** 1 sq unit.

Concept used. On $[0, \pi/2]$, $\sin x \geq 0$, so the area equals $\int_0^{\pi/2} \sin x\,dx$.

Step 1. **Set up.** $A = \int_0^{\pi/2} \sin x \, dx$.

Step 2. **Antiderivative.** $\int \sin x \, dx = -\cos x$.

Step 3. **Evaluate.** $[-\cos x]_0^{\pi/2} = -\cos(\pi/2) - (-\cos 0) = 0 - (-1) = 1$.

> **Final Answer:** Option **(D)**: 1 sq unit.

EXPERT'S SOLUTION *: Ishita Verma, M.Sc Mathematics, IIT Bombay*

Quick reading. Half-arch of unit-amplitude sine on $[0, \pi/2]$ has area 1.

Step 1. Standard result: $\int_0^{\pi/2} \sin x \, dx = 1$.

Step 2. Match: option **(D)**.

Why this matters. Internalise the four "unit area" benchmarks: $\int_0^{\pi/2} \sin x = 1$, $\int_0^{\pi} \sin x = 2$, $\int_0^{2\pi} |\sin x| = 4$, and the analogous cosine integrals.

> **Final Answer:** Option **(D)**.

Q 8.31 **The area of the region bounded by the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ is**

(A) 20π **sq units**

(B) $20\pi^2$ **sq units**

(C) $16\pi^2$ **sq units**

(D) 25π **sq units**

SOLUTION

Correct option: **(A)** 20π sq units.

Concept used. The area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab (derived by stretching the unit disc by a along x and b along y).

Step 1. **Identify** a, b. $a^2 = 25 \Rightarrow a = 5$; $b^2 = 16 \Rightarrow b = 4$.

Step 2. **Apply formula.** $A = \pi ab = \pi \cdot 5 \cdot 4 = 20\pi$.

> **Final Answer:** Option **(A)**: 20π sq units.

☞ **Ellipse area**

πab, where a, b are the semi-major and semi-minor axes. When $a = b = r$, this reduces to πr^2 (the circle).

EXPERT'S SOLUTION *: Nehal Singh, Ph.D Mathematics, IIT Delhi*

Picture-first. The ellipse is a circle of radius 1 stretched horizontally by 5 and vertically by 4, so its area scales by $5 \cdot 4 = 20$ relative to the unit disc's area π.

Step 1. Unit disc has area π.

Step 2. Stretch by $a = 5$ horizontally: area becomes 5π.

Step 3. Stretch by $b = 4$ vertically: area becomes $5 \cdot 4\pi = 20\pi$.

Why this matters. The "linear-scaling" argument for ellipse area extends to any affine transformation: area scales by $|\det J|$ where J is the Jacobian.

> **Final Answer:** Option **(A)**.

Q 8.32 **The area of the region bounded by the circle $x^2 + y^2 = 1$ is**

(A) 2π **sq units**

(B) π **sq units**

(C) 3π **sq units**

(D) 4π **sq units**

SOLUTION

Correct option: (B) π sq units.
Concept used. Area of a disc of radius r is πr^2.

Step 1. Identify radius. $x^2 + y^2 = 1 \Rightarrow r = 1$.

Step 2. Area. $\pi r^2 = \pi \cdot 1 = \pi$.

> **Final Answer:** Option **(B)**: π sq units.

EXPERT'S SOLUTION *: Rahul Chatterjee, M.Sc Mathematics, IIT Bombay*

Quick reading. Unit circle, area π.

Step 1. Disc area $= \pi r^2 = \pi \cdot 1^2 = \pi$.

Why this matters. The unit-circle area π is the foundational constant of plane geometry; commit it deeply.

Final Answer: Option **(B)**.

Q 8.33 **The area of the region bounded by the curve $y = x + 1$ and the lines $x = 2$ and $x = 3$ is**

(A) $\frac{7}{2}$ sq units

(B) $\frac{9}{2}$ sq units

(C) $\frac{11}{2}$ sq units

(D) $\frac{13}{2}$ sq units

SOLUTION

Correct option: (A) $\frac{7}{2}$ sq units.

Concept used. For $x \in [2, 3]$, $y = x + 1$ is positive ($y(2) = 3$, $y(3) = 4$). Area between the line, x-axis (implied), and $x = 2$, $x = 3$ is $\int_2^3 (x + 1)\, dx$.

Step 1. Set up. $A = \int_2^3 (x + 1)\, dx$.

Step 2. Antiderivative. $\int (x + 1)\, dx = \frac{x^2}{2} + x$.

Step 3. Evaluate. At $x = 3$: $\frac{9}{2} + 3 = \frac{15}{2}$. At $x = 2$: $\frac{4}{2} + 2 = 4 = \frac{8}{2}$. Difference: $\frac{15}{2} - \frac{8}{2} = \frac{7}{2}$.

Final Answer: Option **(A)**: $\frac{7}{2}$ sq units.

☞ **Trapezium shortcut**

Parallel sides $y(2) = 3$ and $y(3) = 4$; width 1. Area $= \frac{1}{2}(3 + 4) \cdot 1 = \frac{7}{2}$.

EXPERT'S SOLUTION : *Ishaan Patel, B.Tech Engineering Physics, IIT Bombay*

Picture-first. A trapezium with vertical sides at $x = 2$ (height 3) and $x = 3$ (height 4), and a slanted top from $(2, 3)$ to $(3, 4)$.

Step 1. Heights: $y(2) = 3$, $y(3) = 4$.

Step 2. Width: $w = 3 - 2 = 1$.

Step 3. Area $= \frac{1}{2}(3+4)\cdot 1 = \frac{7}{2}$.

Why this matters. For any linear $y = f(x)$ above the x-axis on $[a, b]$, the area is $\frac{1}{2}(f(a) + f(b))(b - a)$ by the trapezium rule; this is exact for linear functions.

> **Final Answer:** Option **(A)**.

Q 8.34 **The area of the region bounded by the curve $x = 2y + 3$ and the y-lines $y = 1$ and $y = -1$ is**

(A) 4 **sq units**

(B) $\frac{3}{2}$ **sq units**

(C) 6 **sq units**

(D) 8 **sq units**

SOLUTION

Correct option: (C) 6 sq units.

Concept used. "Area bounded by $x = f(y)$ and the lines $y = c$, $y = d$" (with the y-axis as the other implied boundary) means $\int_c^d |f(y)|\, dy$ if $x \geq 0$, or more generally the geometric area between the curve and the y-axis.

Step 1. **Range check.** $x = 2y + 3$ at $y = -1$: $x = 1$; at $y = 1$: $x = 5$. Both positive, so the curve lies entirely to the right of the y-axis on $[-1, 1]$.

Step 2. **Set up.** $A = \int_{-1}^{1}(2y + 3)\, dy$.

Step 3. **Antiderivative.** $\int (2y + 3)\, dy = y^2 + 3y$.

Step 4. **Evaluate.** At $y = 1$: $1 + 3 = 4$. At $y = -1$: $1 - 3 = -2$. Difference: $4 - (-2) = 6$.

> **Final Answer:** Option **(C)**: 6 sq units.

EXPERT'S SOLUTION *: Aarav Nair, M.Sc Mathematics, IIT Bombay*

Quick reading. The line $x = 2y + 3$, with $y \in [-1, 1]$ and the y-axis as the implied left boundary, encloses a trapezium lying on its side. The two parallel sides are horizontal segments of length $x(-1) = 1$ and $x(1) = 5$; the width (vertical extent) is $1 - (-1) = 2$. Apply the trapezium-area formula.

Step 1. Evaluate the endpoints: $x(-1) = 2(-1) + 3 = 1$ and $x(1) = 2(1) + 3 = 5$. Both are positive, so the trapezium lies entirely to the right of the y-axis.

Step 2. Identify the parallel sides (the two horizontal segments at $y = \pm 1$) and the width (the vertical distance between them): parallel sides $= 1$ and 5; width $= 2$.

Step 3. Apply the formula $A = \frac{1}{2}(b_1 + b_2) \cdot w = \frac{1}{2}(1+5) \cdot 2 = \frac{1}{2} \cdot 12 = 6$.

Step 4. Match: option **(C)**.

Step 5. Verify via integration:
$\int_{-1}^{1}(2y+3)\,dy = [y^2+3y]_{-1}^{1} = (1+3) - (1-3) = 4 - (-2) = 6$. Same.

Why this matters. The trapezium-area formula works just as well when the strip is horizontal; swap the roles of x and y in your head and the formula carries over verbatim. For linear $x = f(y)$ between two horizontal lines, this is exact and faster than antidifferentiation.

Final Answer: Option **(C)**.

Key Takeaways

- To find the area bounded by curves, first sketch the region and identify the boundary curves and intersection points.
- Choose vertical strips (dx) when the upper and lower boundaries are functions of x; choose horizontal strips (dy) when the left and right boundaries are functions of y.
- For a region bounded by two curves, use $A = \int(\text{upper} - \text{lower})\,dx$ between the intersection abscissae.
- Exploit symmetry whenever the region (or one of the boundaries) is symmetric about an axis: integrate over half and double, or over a quadrant and multiply by four.
- Standard areas to remember: circle $= \pi r^2$, ellipse $= \pi ab$, parabolic segment under $y^2 = 4ax$ between $x = 0$ and $x = a$ is $\frac{8a^2}{3}$.
- Sketch every region before integrating: a wrong picture almost always gives a wrong sign or a swapped upper/lower boundary.

End of NCERT Exemplar Problems