Application of Integrals

From geometry we know how to find areas of simple shapes - triangles, rectangles, trapezia, circles etc. But these formulae fail for areas ~~between~~ enclosed by curves. There we need Integral Calculus.

In this Chapter

1. Area under a curve $y = f(x)$ or $x = g(y)$.
2. Area between two curves.
3. Sign convention - curve below x-axis.
4. Standard regions : circle, ellipse, parabola , lines + arcs of curves.

Recall

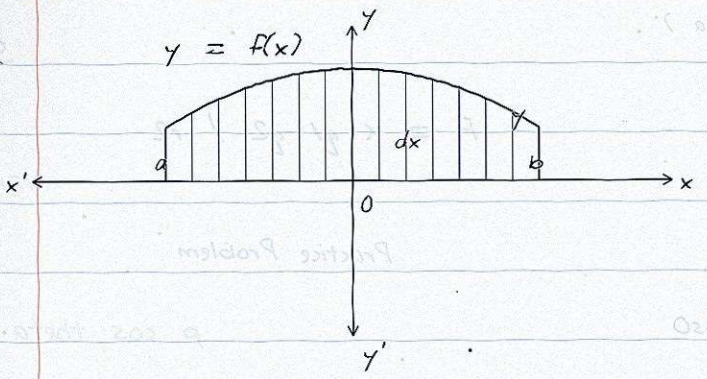
Definite integral from previous chapter - evaluated by Fundamental Theorem of Calc:

$\boxed{\text{integral\_a} \quad b \; f(x)\,dx = F(b)}$ - <- F(a) is any anti-
<-derivative of f

Idea : think of the region as a sum of thin strips ; let strip width -> 0 ; sum becomes a definite integral.

Area under Simple Curves

Region bounded by curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$.



Formula (Vertical Strips)

A = integral_a b y dx = integral_a b f(x) dx <- thin strip
<- dA = y dx

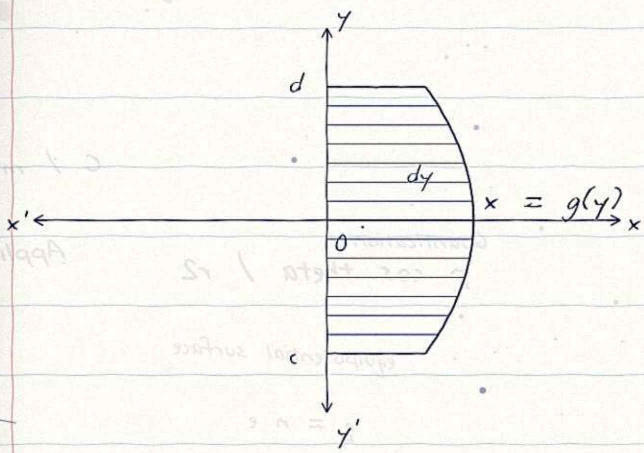
Reason : strip at x has height $y = f(x)$, width dx ; area $dA = y\,dx$. Summing all strips from $x = a$ to $x = b$ gives the total:

A = lim (sum y dx) = integral_a b f(x) dx

Required : $f(x) >= 0$ in $[a, b]$.

Area w.r.t. y-axis

Region bounded by $x = g(y)$, the y-axis and the lines $y = c$, $y = d$ $(d > c)$.



Formula (Horizontal Strips)

$\boxed{A = \text{integral\_c} \quad d \; x \; dy} = \text{integral\_c} \; d \; g(y) \; dy$

<- strip width

<- dy , height x

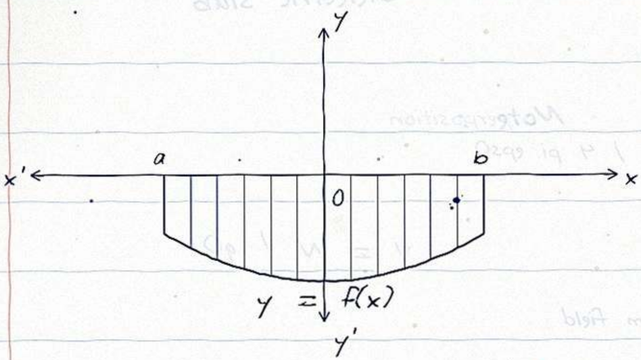
Use horizontal strips when the curve is given as $x = g(y)$ OR when the region is easier to bound by $y = c$, $y = d$ rather than $x = a$, $x = b$.

Required : $g(y) >= 0$ in $[c, d]$.

Sign Convention

Case 1 : f(x) <= 0 in [a, b] .

Then the curve lies below the x-axis.



Definite integral comes out negative , but area is always ~~negative~~ positive . So take

A = integral_a b f(x) dx

<- absolute value
<-gives true area

Case 2 : Curve crosses x-axis

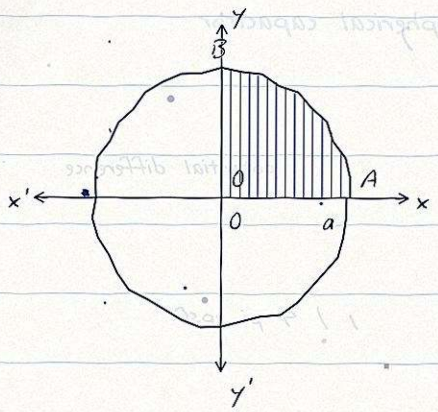
If part of curve is above and part below x-axis , split at the zeros of f .

$A = A_1 + A_2$ (Fig 8.4)

Never just add - they would cancel !

Example 1 : Area of a Circle

Find area enclosed by $x^2 + y^2 = a^2$.



Setup

Symmetry about both axes :

Total area = 4 x (area in 1st quadrant)

Working

$A = 4 \int_0^a y \, dx$, $y = \text{root}(a^2 - x^2)$

Standard integral :

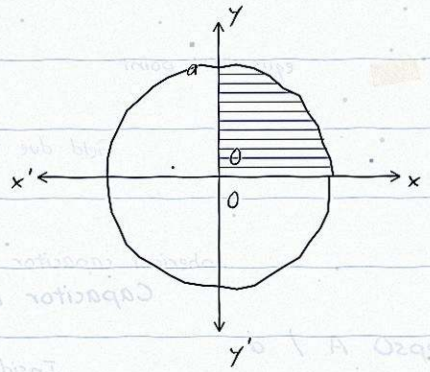
integral $\text{root}(a^2-x^2)\,dx = (x/2)\,\text{root}(a^2-x^2) +$
$(a^2/2)\,\sin^{-1}(x/a)$

$\boxed{A = \text{pi } a \quad 2}$ <-matches the <-known formula

Alt Method (Horizontal Strips)

Same circle, but integrate w.r.t. y.

From $x^2 + y^2 = a^2$: $x = \text{root}(a^2 - y^2)$



Working

$A = 4 \int_0^a x \, dy = 4 \int_0^a \text{root}(a^2 - y^2) \, dy$

By symmetry, same standard integral :

$A = 4 \, [\, (y/2) \, \text{root}(a^2 - y^2) + (a^2/2) \sin^{-1}(y/a) \,]$ from 0 to a

$A = 4 \, (\, a^2/2 \, . \, \text{pi}/2 \,) = \text{pi} \; a^2$

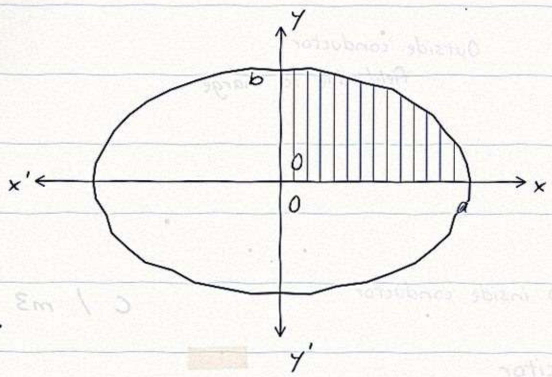
$\boxed{A = \text{pi} \; a \quad 2}$ <-same answer - <-both methods

Moral : choose strips along axis of symmetry.

Scanned with CamScanner

Example 2 : Area of an Ellipse

Find area enclosed by $x^2/a^2 + y^2/b^2 = 1$.



Working

From eqn : $y = +/- (b/a) \text{ root}(a^2 - x^2)$

$$A = 4 \int_0^a (b/a) \text{ root}(a^2 - x^2) \, dx$$

$$= (4b/a) \left[(x/2) \text{ root}(a^2 - x^2) + (a^2/2) \sin^{-1}(x/a) \right] \text{ from } 0 \text{ to } a$$

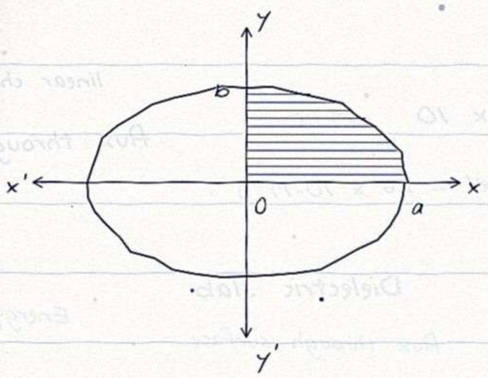
$$= (4b/a) \, . \, (a^2/2)(pi/2) = pi \; a \; b$$

$$\boxed{A = pi \; a \; b}$$

<-when a = b ,
<-becomes pi a 2

Ellipse - Alt (Horizontal Strips)

From eqn : $x = +/- \ (a/b) \ \text{root}(b^2 - y^2)$



Working

$A = 4 \int_0^b x \, dy = (4a/b) \int_0^b \text{root}(b^2 - y^2) \, dy$

$= (4a/b)[\ (y/2) \ \text{root}(b^2 - y^2) +$

$(b^2/2) \sin^{-1}(y/b) \ ]$ from 0 to b

$= (4a/b)(b^2/2)(\text{pi}/2) = \text{pi} \ a \ b$

$\boxed{A = \text{pi} \ a \ b}$ <-same answer

Confirms : area is intrinsic , not axis-dependent.

Standard Areas - Quick Recap

(i) Circle $x^2 + y^2 = a^2$

$\boxed{A = \pi a^2}$ <-radius a

(ii) Ellipse $x^2/a^2 + y^2/b^2 = 1$

$\boxed{A = \pi a b}$ <-semi-axes a, b

(iii) $y = mx + c$, x-axis, $x = a$, $x = b$:

$$A = \int_a^b (mx + c)\, dx$$
$$= m(b^2 - a^2)/2 + c(b - a)$$

(iv) Parabola $y^2 = 4ax$, $0 \le x \le h$:

$$A = 2 \int_0^h \sqrt{4ax}\, dx$$
$$= (8/3) \sqrt{a} \cdot h^{(3/2)}$$

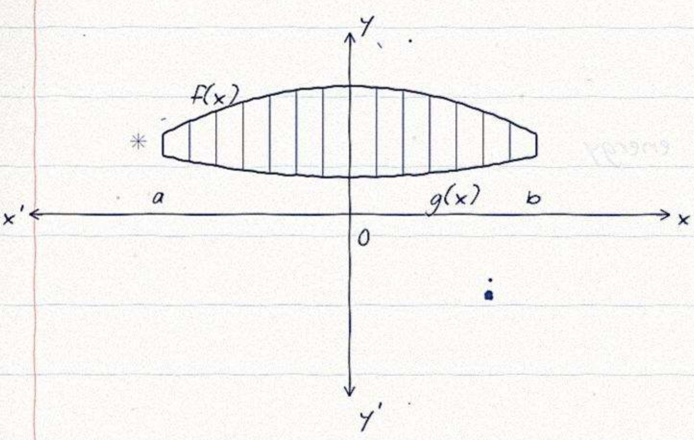
(v) $y = \sqrt{x}$, x-axis, $x = 0$, $x = 4$:

$$A = \int_0^4 \sqrt{x}\, dx = (2/3)(8) = 16/3$$

Memorise these - saves time in MCQs.

Area between Two Curves

Region bounded between $y = f(x)$ and $y = g(x)$, with $f(x) >= g(x)$ in $[a, b]$.



Formula

$$\boxed{A = \text{integral\_a} \quad b \, [\, f(x) - g(x) \,] \, dx}$$ (upper) - <-(lower)

Strip height = top curve - bottom curve

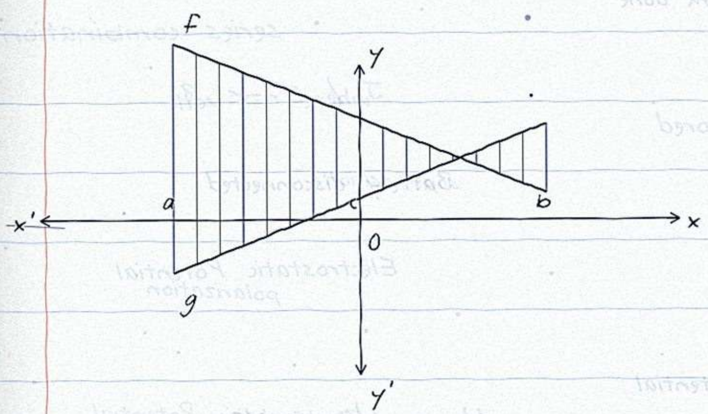
= $f(x) - g(x)$; width = dx

If curves intersect, the limits a, b are the x-coords of points of ~~contact~~ intersection .

When Curves Cross Each Other

If f(x) >= g(x) on [a, c] but
f(x) <= g(x) on [c, b] , split at x = c :

A = integral_a c (f - g) dx + integral_c b always (g - f)
<- upper - low



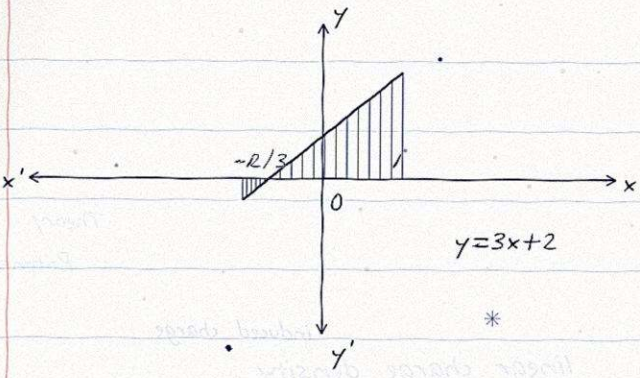
Tip

1. Find intersection by f(x) = g(x) .
2. In each sub-interval check which is bigger (plug in a test x) .
3. Use A = integral f - g dx in short.

Equivalent : $A = \int_a^b f - g \quad dx$.

Example : Line crossing x-axis

Find area bounded by $y = 3x + 2$, the x-axis and the ordinates $x = -1$, $x = 1$.



Working

Line meets x-axis when $3x + 2 = 0 \to x = -2/3$

Below x-axis on $[-1, -2/3]$, above on $[-2/3, 1]$

$$A = \int_{-1}^{-2/3} (3x+2)\,dx + \int_{-2/3}^{1} (3x+2)\,dx$$

$$= [3x^2/2 + 2x] \text{ from } -1 \text{ to } -2/3 \quad + \ldots$$

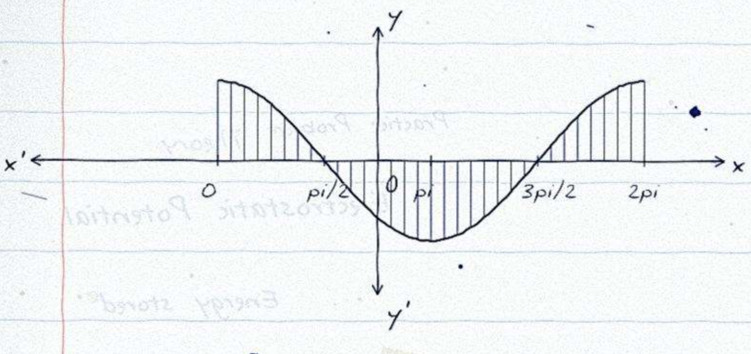
$$= -1/6 - (-1/2) \quad + (5/2 - (-2/3))$$

$$= 1/3 + (5/2 + 2/3) = 1/3 + 19/6$$

$$\boxed{A = 13/3 \text{ sq units}}$$

Example : Area under $y = \cos x$

Find area bounded by $y = \cos x$ between $x = 0$ and $x = 2\pi$.



Working

cos x is + on $(0, \pi/2)$; - on $(\pi/2, 3\pi/2)$; + on $(3\pi/2, 2\pi)$. Split accordingly :

$$A = \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{3\pi/2} \cos x \, dx$$

$$+ \int_{3\pi/2}^{2\pi} \cos x \, dx$$

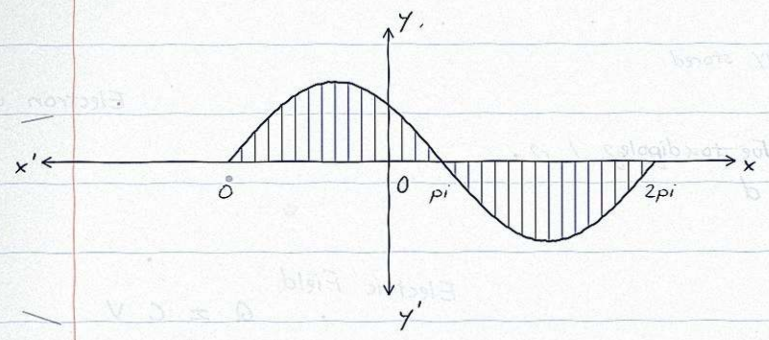
$$= [\sin x] : (1 - 0) + (-1 - 1) + (0 - (-1))$$

$$= 1 + 2 + 1$$

$$\boxed{A = 4 \text{ sq units}}$$

Example : Area under y = sin x

Find area bounded by $y = \sin x$ between $x = 0$ and $x = 2\pi$ (Misc Q 3).



Working

$\sin x \geq 0$ on $[0, \pi]$; $\sin x \leq 0$ on $[\pi, 2\pi]$

$$A = \int_0^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} \sin x \, dx$$

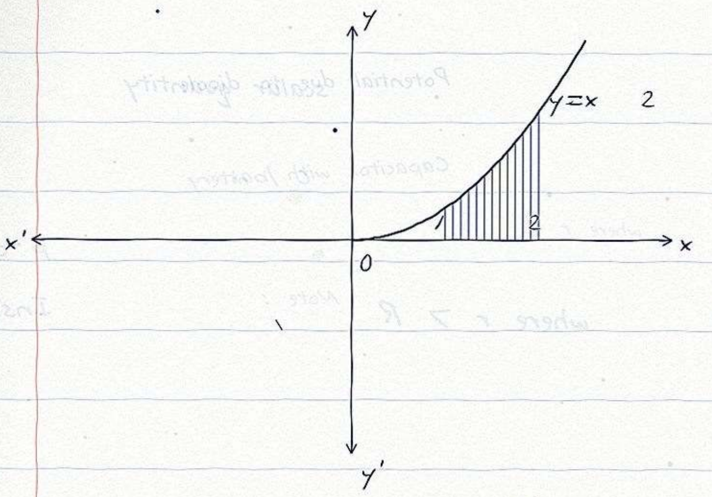
$$= [-\cos x] \text{ (0 to pi)} + [-\cos x] \text{ (pi to 2pi)}$$

$$= (1 - (-1)) + (-1 - 1)$$
$$= 2 + 2$$

$\boxed{A = 4 \text{ sq units}}$ <- same as <- cos x case

Example : $y = x \quad 2$, $x = 1$ to 2

Find area between $y = x^2$, x-axis ,
and the lines $x = 1$, $x = 2$.



Working

$$A = \int_1^2 x^2 \, dx = [\, x^3 / 3 \,] \text{ from } 1 \text{ to } 2$$

$$= 8/3 - 1/3 = 7/3$$

$$\boxed{A = 7/3 \text{ sq units}}$$

Always check : $y >= 0$ on the interval ?
Here $x^2 >= 0$ always - no mod needed.

Example : y = x 4 , x = 1 to 5

Find area between $y = x^4$, x-axis , and the lines $x = 1$, $x = 5$.

Setup

$y = x^4$ is >= 0 for all real x . So strip height = y , width = dx , no sign issue.

Working

$$A = \int_1^5 x^4 \, dx$$

= [x^5 / 5] from 1 to 5

= (5^5 - 1^5) / 5

= (3125 - 1) / 5 = 3124 / 5

$$\boxed{A = 3124 / 5 \text{ sq units}}$$

<-= 624.8

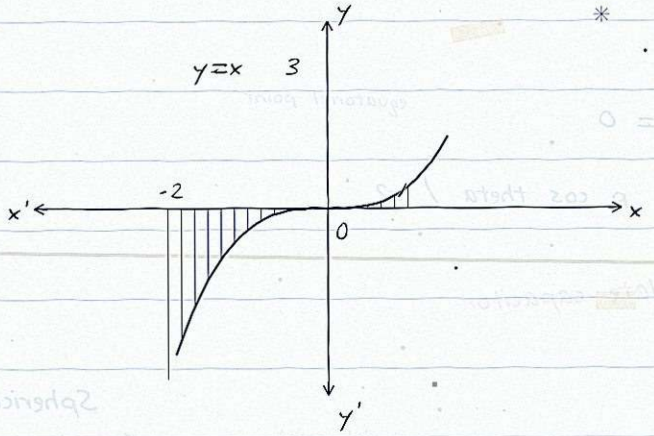
Note

Generally : integral_a b x^n dx = (b^{n+1} - a^{n+1}) / (n + 1) ,

(case n = -1 gives log x .)

Example : y = x 3 , x = -2 to 1

Area bounded by $y = x^3$, x-axis, and $x = -2$, $x = 1$. (Misc Q 4)



Working

$x^3 < 0$ on $[-2, 0]$; > 0 on $[0, 1]$

$$A = \int_{-2}^{0} x^3\,dx. + \int_{0}^{1} x^3\,dx$$

$$= [x^4/4](-2 \text{ to } 0) + [x^4/4](0 \text{ to } 1)$$

$$= 0 - 4 + (1/4 - 0) = 4 + 1/4$$

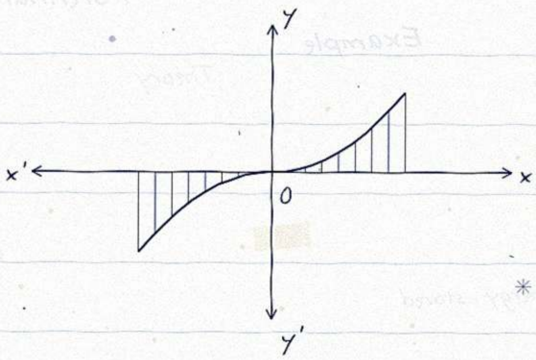
$$\boxed{A = 17/4 \text{ sq units}}$$

Example : $y = x \, . \, mod(x)$

Area bounded by $y = x \, mod(x)$, x-axis ,
and $x = -1$, $x = 1$. (Misc Q 5)

Hint - Rewrite

$y = x \, mod(x) = \{ x^2 , \quad x >= 0$
$\qquad\qquad\qquad\quad \{ -x \ 2 , \quad x < 0$



Working

$A = \int_{-1}^{0} (-x^2) \, dx \quad + \int_{0}^{1} x^2 \, dx$

$= [-x^3/3](-1 \text{ to } 0) \quad + [x^3/3](0 \text{ to } 1)$

$= 0 - 1/3 \quad + (1/3 - 0)$

$= 1/3 + 1/3$

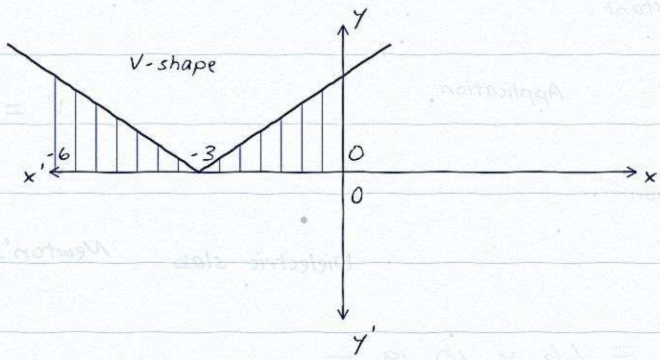
$\boxed{A = 2/3 \text{ sq units}}$

Example : $y = \mathrm{mod}(x + 3)$

Sketch $y = \mathrm{mod}(x + 3)$ and evaluate
$\int_{-6}^{0} \mathrm{mod}(x + 3)\, dx$. (Misc Q 2)

Re-express

$$\mathrm{mod}(x + 3) = -(x + 3), \quad x <= -3$$
$$= (x + 3), \quad x >= -3$$



Working

$$I = \int_{-6}^{-3} -(x+3)\, dx + \int_{-3}^{0} (x+3)\, dx$$

$$= [-(x+3)^2/2] \; (-6 \text{ to } -3)$$
$$+ [(x+3)^2/2] \; (-3 \text{ to } 0)$$
$$= (0 - (-9/2)) + (9/2 - 0) = 9/2 + 9/2$$

$$\boxed{I = 9}$$

Quick MCQs (Exercise 8.1)

Q3 : 1st-quadrant area of circle

$x^2 + y^2 = 4 \ , \ x = 0 \ , x = 2$. Find A.

$$A = \int_0^2 \text{root}(4 - x^2) \, dx$$

$$= [(x/2) \ \text{root}(4-x^2) + 2 \ \sin^{-1}(x/2)]$$

from 0 to 2

$$= (0 + 2 \ . \ \text{pi}/2) - 0 = \text{pi}$$

$$\boxed{A = \text{pi} \quad (\text{Option A})}$$

Q4 : $y \quad 2 = 4x$, y-axis, $y = 3$.

$x = y^2/4$. Region $0 <= y <= 3$.

$$A = \int_0^3 x \, dy = \int_0^3 y^2/4 \, dy$$

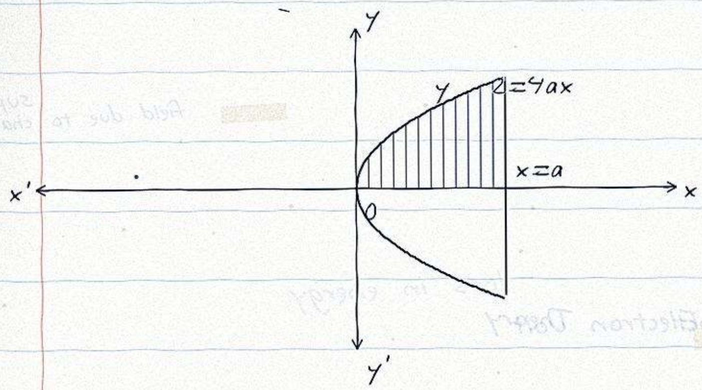
$$= [\ y^3 \ / \ 12 \ ] \text{ from } 0 \text{ to } 3$$

$$= 27 \ / \ 12 = 9 \ / \ 4$$

$$\boxed{A = 9/4 \quad (\text{Option B})}$$

Parabola $y^2 = 4ax$

Area between the parabola and latus rectum $x = a$ $(a > 0)$.



Working

By symmetry about x-axis :

$A = 2 \int_0^a y \, dx = 2 \int_0^a 2 \text{ root}(ax) \, dx$

$= 4 \text{ root}(a) \, [\, 2 x^{3/2}/3 \,]$ from 0 to a

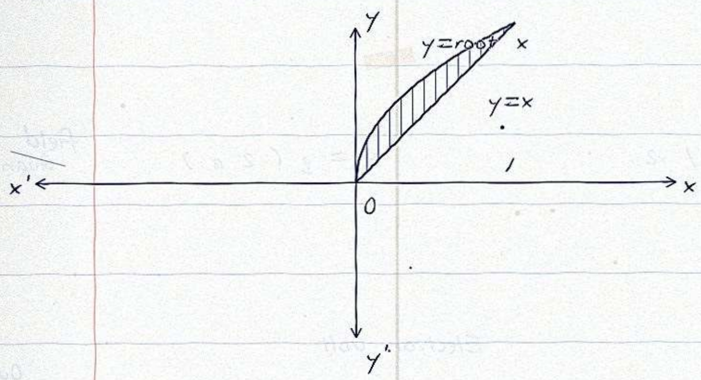
$= (8/3) \text{ root}(a) \cdot a^{3/2} = (8/3)\, a^2$

$\boxed{A = (8/3)\, a^2 \text{ sq units}}$

<- JEE
<- favourite

Area between Line & Parabola

Find area between $y = x$ and $y^2 = x$.



Working

Intersection : $x = \text{root } x \implies x = 0, 1$

On $[0, 1]$: $\text{root } x \geq x$, so

$$A = \int_0^1 (\text{root } x - x)\, dx$$

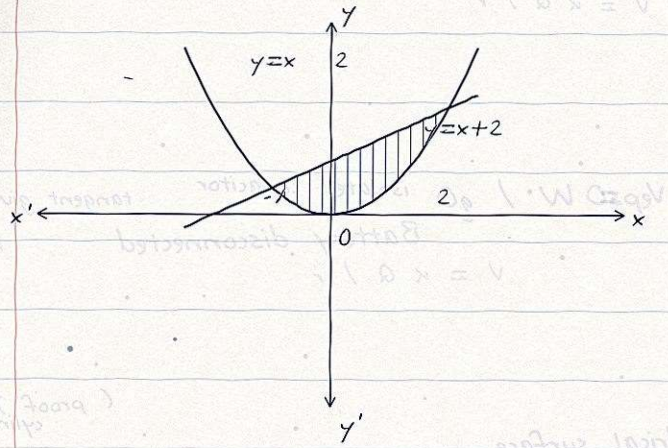
$$= \left[2x^{3/2}/3 - x^2/2 \right] 0 \text{ to } 1$$

$$= 2/3 - 1/2$$

$$\boxed{A = 1/6 \text{ sq units}}$$

Parabola $y = x \quad 2$ & Line $y = x + 2$

Find area enclosed by these curves.



Working

Intersection : $x^2 = x + 2 \Rightarrow x = -1, 2$

Line is above parabola on $[-1, 2]$:

$$A = \int_{-1}^{2} ((x + 2) - x^2)\, dx$$

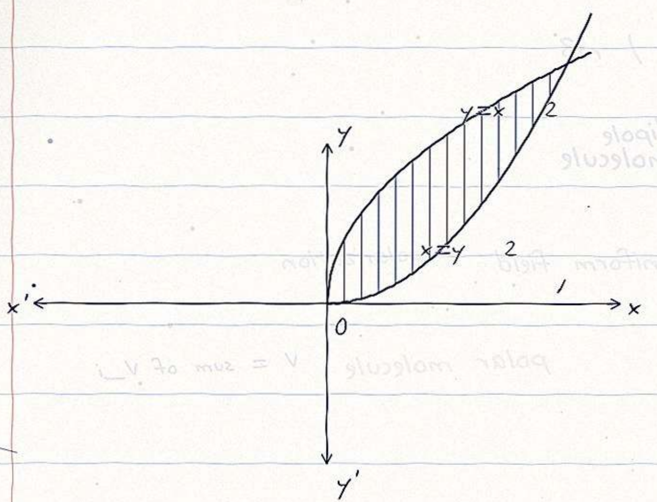
$$= [\, x^2/2 + 2x - x^3/3 \,]\ -1 \text{ to } 2$$

$$= (2 + 4 - 8/3) - (1/2 - 2 + 1/3)$$

$$\boxed{A = 9/2 \text{ sq units}}$$

Two Parabolas : y=x 2 , x=y 2

Find area enclosed by both parabolas.



Working

Solve x^2 = root x => x = 0 , 1

On (0, 1) : root x > x^2 . So

$A = \int_0^1$ (root x - x^2) dx

= [2 $x^{3/2}/3$ - $x^3/3$] 0 to 1

= 2/3 - 1/3

$\boxed{A = 1/3 \text{ sq units}}$

How to Tackle Any Area Problem

Five-Step Recipe

1. Sketchthe curves carefully.
2. Identifythe region & its boundaries.
3. Findpoints of intersection (limits).
4. Choosevertical or horizontal strips.
5. Integrateand take . if needed.

Strip-direction Tip

* Vertical strip (dx) : if region bounded by vertical lines $x = a, x = b$.
* Horizontal strip (dy) : if bounded by horizontal lines $y = c, y = d$.
* Symmetry first : ~~usually~~ often quarters the work.

Common Slip-ups

(a) Forgetting . when curve is below axis.

(b) Wrong limits - check by substitution !

(c) Splitting at wrong intersection point.

Real-life Applications

Where these formulas matter

1. Land surveying - irregular plots .
2. Engineering : cross-section of beams .
3. Physics : work = integral F dx ,
   i.e. area under F vs x graph .
4. Economics : consumer / producer surplus
   = area between demand & supply curves.
5. Statistics : probability = area under
   the probability-density function .

Chapter Summary

$\boxed{A = \text{integral\_a} \quad b \ f(x)\ dx , \quad f(x)}$ >= 0

$\boxed{A = \text{integral\_c} \quad d \ g(y)\ dy , \quad g(y)}$ >= 0

$\boxed{A = \text{integral\_a} \quad b \quad f - g}$ dx <- two-curve
<- area

Constants worth remembering :

circle pi a 2 ; ellipse pi a b ;

parabola-latus (8/3) a 2 .

Historical Note

Integral Calculus traces back to ancient Greek Method of Exhaustion - Eudoxus (440 B.C.) and Archimedes (300 B.C.) used it to compute areas of plane regions.

17th century revolution

* Newton (1665) - theory of fluxions, anti-derivative as inverse of tangent.
* Leibnitz (1684-86) - introduced the symbol ∫ integral for summing thin areas.
* Cauchy (early 19C) - gave the rigour via the concept of limit.

Key Idea

Differentiation & integration are inverse operations - Fund. Thm of Calc.

$$\boxed{d/dx \left(\text{integral\_a} \quad x \; f(t) \; dt \right) = f(x)}$$ <- FTC - <- Part I

---- END OF CHAPTER 8 ----

Practice : Ex 8.1 + Misc Ex on Ch 8.