Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 8: Application of Integrals

What this chapter covers: using the definite integral to compute areas. The rationalised NCERT keeps two themes — the area under a simple curve bounded by lines and the area between two curves. We additionally include classical regions (circle, ellipse, parabola), modulus-function areas and parametric-curve areas as *[JEE/NEET Extension]*, since these remain frequently tested in entrance exams.

1 Introduction: From Sums to Areas

In Class 11 we computed areas of rectangles, triangles, trapezia and circles using elementary formulas. Those formulas cover only straight-edged or perfectly circular regions. Most regions a student meets in physics, statistics or engineering are bounded by *curves* — and elementary geometry has nothing to say about those.

The definite integral, introduced in Chapter 7 as the limit of a Riemann sum, is exactly the tool we need. It adds up infinitely many thin rectangular strips to fill a curved region. Chapter 8 turns that machinery into a working method for computing the area under a curve, between two curves, and bounded by mixed line-and-curve configurations.

1.1 The fundamental idea: strip and sum

Imagine a region R in the xy-plane whose upper boundary is the curve $y = f(x)$ ($f \geq 0$), whose lower boundary is the x-axis, and whose sides are the vertical lines $x = a$ and $x = b$. Slice R into n vertical strips of equal width Δx. Each strip is almost a rectangle: width Δx, height $f(x)$. Its area is $f(x)\,\Delta x$. Adding up all the strips and letting $\Delta x \to 0$ gives

$$\text{Area}(R) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx.$$

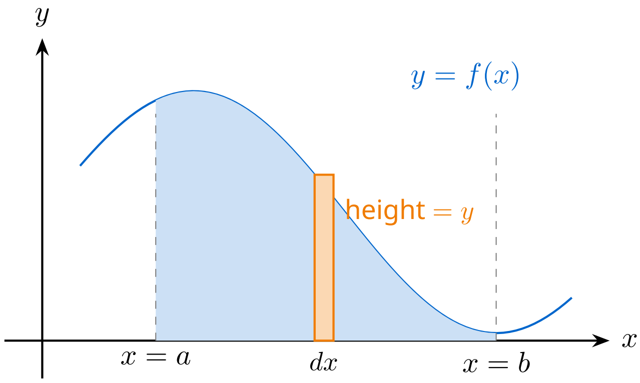
The Fundamental Theorem of Calculus then lets us evaluate this integral via an antiderivative — so a problem that began as a geometric area is solved by indefinite integration.

Area Under a Curve — Vertical Strip

For $f(x) \geq 0$ on $[a, b]$, the area bounded by $y = f(x)$, the x-axis and the lines $x = a$, $x = b$ is

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx.$$

The infinitesimal strip $dA = y\,dx$ has width dx and height y.



The Strip is the Engine

Every area calculation in this chapter reduces to: **(i)** draw the region, **(ii)** pick the variable of integration (x or y), **(iii)** write down the elementary strip dA in that variable, **(iv)** integrate over the right limits. The answer follows.

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1.2 Horizontal strip: area in terms of y

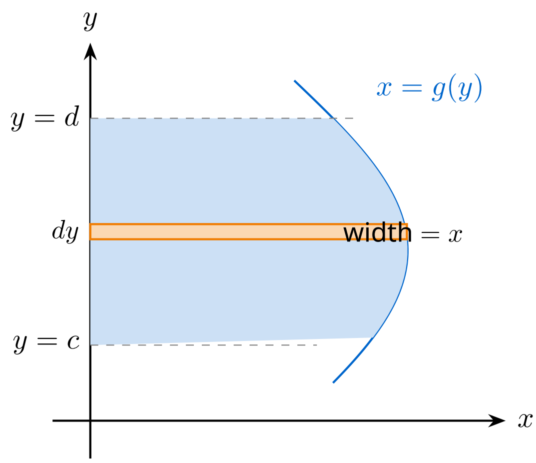
Some regions are awkward to slice vertically. If the boundary is given as $x = g(y)$ — a curve expressed with x as a function of y — it is far easier to use *horizontal* strips: each strip has height dy and width $x = g(y)$.

Area Under a Curve — Horizontal Strip

For $g(y) \geq 0$ on $[c, d]$, the area bounded by $x = g(y)$, the y-axis and the lines $y = c$, $y = d$ is

$$A = \int_c^d x\,dy = \int_c^d g(y)\,dy.$$

The strip $dA = x\,dy$.



Vertical or Horizontal?

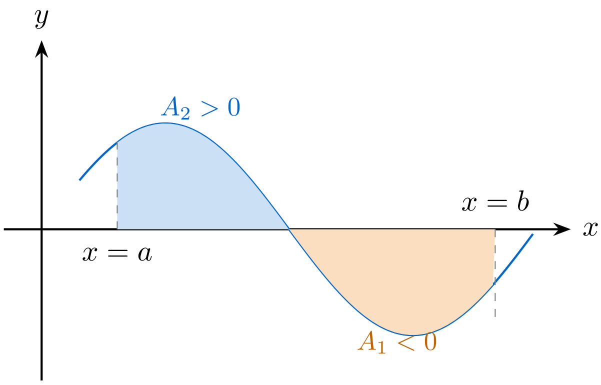
Look at the equation form. If the curve is $y = f(x)$ (y *explicit in* x), use **vertical strips**. If the curve is $x = g(y)$ (x *explicit in* y), use **horizontal strips**. For curves like $y^2 = 4ax$ (parabola opening right), $x = y^2/(4a)$ — horizontal strips are usually cleaner.

1.3 When the curve dips below the x-axis

If $f(x) < 0$ on part of $[a, b]$, the integral $\int_a^b f(x)\,dx$ contributes a *negative* number for that part. But area is a non-negative geometric quantity. The standard fix is to integrate the absolute value, or break the interval at the zeros of f and use $|\cdot|$ on each piece:

$$A = \int_a^b |f(x)|\,dx = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|,$$

where c is the point at which f changes sign.



So the total area $A = |A_1| + A_2$, not $A_1 + A_2$.

Common Mistake

Writing the area as $\int_a^b f(x)\,dx$ without checking the sign of f on the interval is the single most common error in this chapter. If the curve crosses the x-axis inside $[a, b]$, you **must** split the integral at the zero and take absolute values, or integrate $|f(x)|$ directly.

2 Area Under Simple Curves

This section works the formulas of Section 1 on the curves that the NCERT rationalised syllabus retains: lines, polynomial curves, and the standard form $y^2 = 4ax$ when it appears alongside a line. We also include the circle and ellipse as a *[JEE/NEET Extension]* since these closed-form areas are heavily tested.

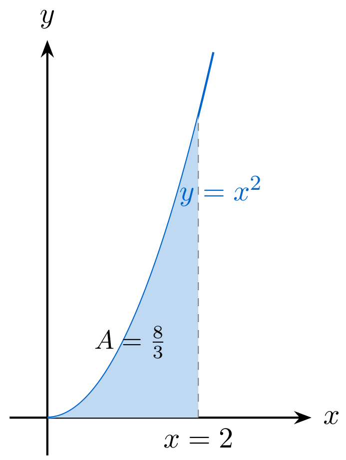
2.1 Area between a curve and the x-axis

For $f \geq 0$, the recipe is mechanical: write down $\int_a^b f(x)\,dx$ and integrate. The art is choosing the right limits.

Worked Example. Find the area bounded by $y = x^2$, the x-axis, $x = 0$ and $x = 2$.

$$A = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

The region is a parabolic sliver in the first quadrant; area $= 8/3$ square units.



2.2 Area bounded by a line and the axes

The simplest curved-area problem is no curve at all — a line and the axes. Although elementary geometry handles it, integration shows that the same answer falls out of the strip method.

Worked Example. Find the area enclosed by $y = 3x + 2$, the x-axis, $x = -1$ and $x = 1$ (NCERT Example 3, rephrased).

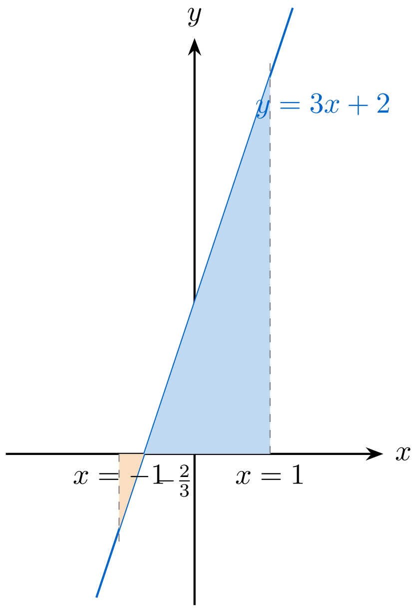
The line $y = 3x + 2$ meets the x-axis at $x = -2/3$. For $x \in (-1, -2/3)$ it lies *below* the axis; for $x \in (-2/3, 1)$ it lies above. So the area is

$$A = \left| \int_{-1}^{-2/3} (3x + 2)\, dx \right| + \int_{-2/3}^{1} (3x + 2)\, dx.$$

Compute each piece:

$$\int_{-1}^{-2/3} (3x + 2)\, dx = \left[\frac{3x^2}{2} + 2x \right]_{-1}^{-2/3} = -\frac{1}{6}, \qquad \int_{-2/3}^{1} (3x + 2)\, dx = \frac{25}{6}.$$

So $A = \frac{1}{6} + \frac{25}{6} = \frac{26}{6} = \frac{13}{3}$ square units.



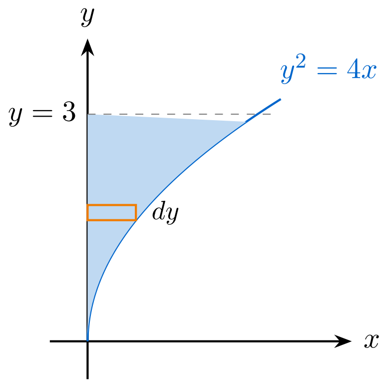
2.3 Area bounded by a curve, the y-axis and horizontal lines

When the bounding curve is naturally expressed as $x = g(y)$ — e.g., $y^2 = 4ax$ rewritten as $x = y^2/(4a)$ — horizontal strips are easier.

Worked Example. Find the area bounded by $y^2 = 4x$, the y-axis and $y = 3$ (NCERT Exercise 8.1 Q4 style).

The parabola opens rightward; for $y \in [0, 3]$ the strip runs from $x = 0$ (the y-axis) to $x = y^2/4$. So

$$A = \int_0^3 \frac{y^2}{4}\, dy = \left[\frac{y^3}{12}\right]_0^3 = \frac{27}{12} = \frac{9}{4} \text{ sq. units.}$$



Choosing the Strip Direction

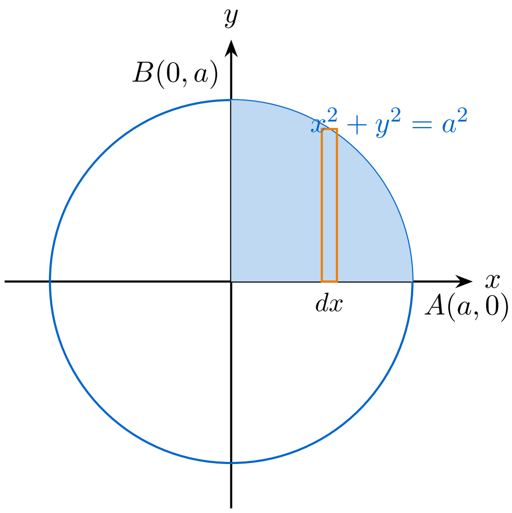
A vertical strip turns the area into $\int y\, dx$; a horizontal strip turns it into $\int x\, dy$. Pick the form in which the integrand is simpler. For $y = f(x)$ shaped regions vertical wins; for $x = g(y)$ shaped regions, horizontal.

2.4 Area enclosed by a circle [JEE/NEET Extension]

The circle $x^2 + y^2 = a^2$ has area πa^2 by elementary geometry. Showing it by integration is a rite of passage — and the technique generalises to ellipses and conics that are *not* covered by an elementary formula.

By symmetry, the whole area equals four times the area of the first-quadrant quarter. In the first quadrant, $y = \sqrt{a^2 - x^2}$ on $[0, a]$, so

$$A = 4\int_0^a \sqrt{a^2 - x^2}\,dx = 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi a^2.$$



Horizontal-strip alternative. The same area, computed with horizontal strips: $A = 4\int_0^a \sqrt{a^2 - y^2}\,dy = \pi a^2$. The integral is identical in form — only the dummy variable changes. This symmetry is the standard "Why?" the NCERT margin asks in Example 1.

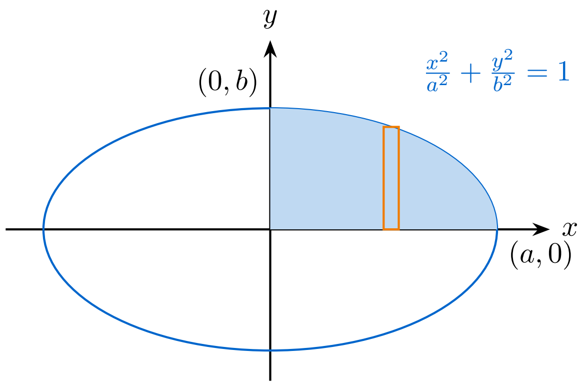
> **Symmetry Shortcut**
>
> For any region symmetric about both axes, compute the area in the first quadrant and multiply by 4. For symmetry about one axis only, multiply by 2. Always exploit symmetry before setting up an integral — it cuts the work in half (or quarter).

2.5 Area enclosed by an ellipse [JEE/NEET Extension]

For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the same symmetry-and-vertical-strip method gives

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = \frac{4b}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \pi ab.$$

When $a = b$ this reduces to πa^2, the circle case. The ellipse area πab is one of the most-cited derivations in JEE Main.



Classical Closed-Region Areas

- Circle of radius a: $A = \pi a^2$
- Ellipse with semi-axes a, b: $A = \pi ab$
- Quarter circle in 1^{st} quadrant: $A = \frac{\pi a^2}{4}$
- Semicircle (above x-axis): $A = \frac{\pi a^2}{2}$

3 Area Between Two Curves

The second NCERT theme: regions bounded *above and below* by two curves rather than by a curve and the x-axis. The strip-and-sum idea is unchanged — the only twist is that each strip now has height $y_{\text{top}} - y_{\text{bot}}$ instead of just y.

3.1 The general formula

Let $f(x) \geq g(x)$ on $[a, b]$. The region bounded by $y = f(x)$ above, $y = g(x)$ below, and the vertical lines $x = a$, $x = b$ has area

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

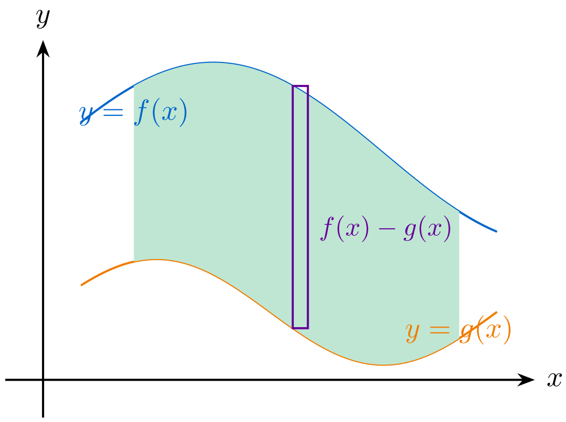
If the two curves intersect at $x = a$ and $x = b$, no vertical-line boundary is needed — the curves close the region themselves, and the same integral applies. The limits a, b are the x-coordinates of the intersection points.

Area Between Two Curves

For $f \geq g$ on $[a, b]$,

$$A = \int_a^b (y_{\text{top}} - y_{\text{bot}})\, dx = \int_a^b [f(x) - g(x)]\, dx.$$

Analogously, when integrating horizontally: $A = \int_c^d (x_{\text{right}} - x_{\text{left}})\, dy$.



3.2 Worked Example: area between $y = x$ and $y = x^2$

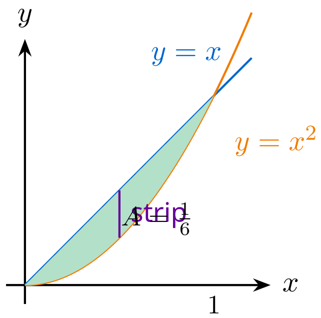
Find the area bounded by $y = x$ (line) and $y = x^2$ (parabola).

Step 1 — intersections. Set $x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1$.

Step 2 — which is on top? On $(0, 1)$, take a test point $x = 0.5$: $y = x$ gives 0.5, $y = x^2$ gives 0.25. So the line is above the parabola.

Step 3 — integrate.

$$A = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$



Three-Step Recipe

(1) Solve $f(x) = g(x)$ to find intersection points — these set your limits. **(2)** Use a test point to decide which curve is on top. **(3)** Integrate $(y_{\text{top}} - y_{\text{bot}})$ from the smaller intersection to the larger. Forget any of these and you will get a sign error or wrong limits.

3.3 Worked Example: area between $y^2 = x$ and $y = x$

Find the area bounded by $y^2 = x$ and $y = x$.

Intersections. Substitute $y = x$ into $y^2 = x$: $x^2 = x \Rightarrow x = 0, 1$. So the points are $(0, 0)$ and $(1, 1)$.

Which is on top? On $(0, 1)$, $y = \sqrt{x}$ from the parabola gives (e.g. at $x = 0.25$) $y = 0.5$; the line gives $y = 0.25$. The *parabola* sits above the line on this interval.

$$A = \int_0^1 (\sqrt{x} - x)\, dx = \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}.$$

So the area is $\frac{1}{6}$ square units — identical to the previous example, as expected by the symmetry $y \leftrightarrow x$.

3.4 Area bounded by a line and a parabola

This is a JEE staple. Take the parabola $y^2 = 4ax$ and the line through specific points; locate intersections, decide orientation, integrate.

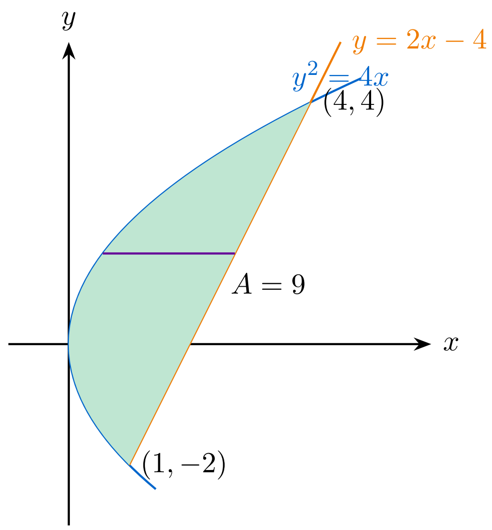
Worked Example. Find the area bounded by $y^2 = 4x$ and the line $y = 2x - 4$.

Intersections. From the line, $x = (y+4)/2$. Substitute into $y^2 = 4x$: $y^2 = 2(y+4) \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow y = 4, -2$. Intersection points: $(4, 4)$ and $(1, -2)$.

The region is awkward to slice vertically (the parabola has two y-values for each x), so use **horizontal strips**. The right boundary is the line $x = (y + 4)/2$; the left boundary is the parabola $x = y^2/4$. So

$$A = \int_{-2}^{4} \left[\frac{y+4}{2} - \frac{y^2}{4}\right] dy = \left[\frac{(y+4)^2}{4} - \frac{y^3}{12}\right]_{-2}^{4}.$$

Evaluate: at $y = 4$, value $= 16 - 16/3 = 32/3$; at $y = -2$, value $= 1 + 2/3 = 5/3$. Difference: $32/3 - 5/3 = 27/3 = 9$ sq. units.



3.5 Area of the region bounded by $x^2+y^2 = 4$ and $y = x$ [JEE/NEET Extension]

Find the area of the smaller region cut off from the disc $x^2+y^2 = 4$ by the line $y = x$, lying *above* the line and below the upper semicircle, in the first quadrant.

Intersections. $x^2 + x^2 = 4 \Rightarrow x = \pm\sqrt{2}$. In the first quadrant, $x = \sqrt{2}$, $y = \sqrt{2}$.

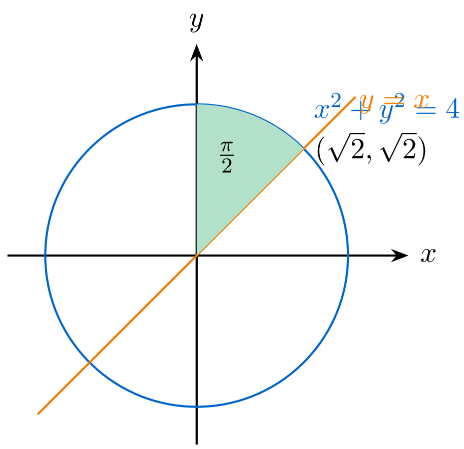
For $x \in [0, \sqrt{2}]$, top is the line $y = x$, bottom is $y = 0$. For $x \in [\sqrt{2}, 2]$, top is the circle $y = \sqrt{4 - x^2}$, bottom is $y = 0$. Adding:

$$A = \int_0^{\sqrt{2}} x\,dx + \int_{\sqrt{2}}^{2} \sqrt{4 - x^2}\,dx.$$

First integral: $[x^2/2]_0^{\sqrt{2}} = 1$. Second integral, using $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$ with $a = 2$:

$$\int_{\sqrt{2}}^{2} \sqrt{4 - x^2}\,dx = \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\tfrac{x}{2}\right]_{\sqrt{2}}^{2} = (0 + \pi) - (1 + \tfrac{\pi}{2}) = \tfrac{\pi}{2} - 1.$$

Total: $A = 1 + (\pi/2 - 1) = \pi/2$ square units.



Real-World Application

Wedges of circular regions show up in radar coverage maps, pizza-slice problems in operations research and antenna directivity charts. The radial line + arc set-up here is the prototype.

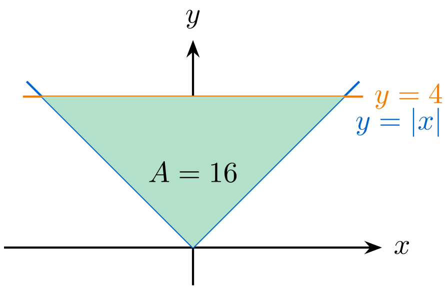
3.6 Area of a region involving modulus [JEE/NEET Extension]

Modulus functions split the integrand at the point where the inside expression changes sign, then sum the pieces.

Worked Example. Find the area bounded by $y = |x|$ and the line $y = 4$.

The graph of $y = |x|$ is a V; the line $y = 4$ caps it at $x = \pm 4$. The region is a triangle with vertices $(-4, 4)$, $(4, 4)$, $(0, 0)$. By integration:

$$A = \int_{-4}^{4} \left[4 - |x|\right] dx = 2\int_0^4 (4 - x)\,dx = 2\left[4x - \tfrac{x^2}{2}\right]_0^4 = 2(16 - 8) = 16.$$

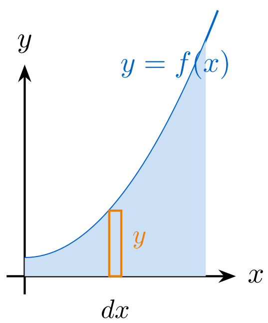


4 Comparing Strip Choices and Strategy

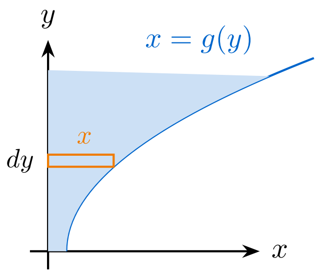
By now we have seen vertical strips, horizontal strips, single curves and double-curve regions. Before turning to harder problems it is worth pausing to lay out a comparison so the student can pick the right tool at sight.

4.1 Vertical vs horizontal strip — side-by-side

Vertical strip ($dA = y\,dx$)	**Horizontal strip ($dA = x\,dy$)**
Use when curves are given as $y = f(x)$.	Use when curves are given as $x = g(y)$.
Limits are x-values; integrate w.r.t. x.	Limits are y-values; integrate w.r.t. y.
Strip height = y (or $y_{\text{top}} - y_{\text{bot}}$).	Strip width = x (or $x_{\text{right}} - x_{\text{left}}$).
Natural for $y = x^2$, $y = \sin x$, linear curves.	Natural for $y^2 = 4ax$, $x = y^3$, sideways parabolas.
Awkward when curve is multi-valued in y.	Awkward when curve is multi-valued in x.



Vertical



Horizontal

4.2 Strategy flowchart for any area problem

1. **Sketch the region.** Always. Marking the curves and intersections by hand prevents 90% of mistakes.
2. **Find intersection points.** Equate the boundary curves; solve for both coordinates if needed.
3. **Check sign** of the integrand on the interval. If the curve dips below the axis,

split and take $|\cdot|$.

4. **Choose strip direction.** Pick the form in which the integrand is simpler — usually the direction matching the explicit form of the curves.
5. **Exploit symmetry.** If the region is symmetric about an axis, compute one side and double; about both, compute the first quadrant and multiply by 4.
6. **Evaluate the integral** using standard antiderivatives.

Sketch First, Always

The boards reward students who draw a clean labelled diagram alongside the calculation. The marker can follow your reasoning, and shading the region forces you to identify y_{top} and y_{bot} correctly. Never start an area integral without the picture.

4.3 Sign conventions and pitfalls

Three Recurring Errors

(i) Forgetting $|\cdot|$ when the curve drops below the axis. Always check the sign of $f(x)$ on $[a, b]$ before writing $\int f$.
(ii) Reversing y_{top} and y_{bot}. A negative answer almost always means you swapped which curve is on top. Take $|A|$, or fix the order.
(iii) Wrong limits after substitution. If you change variables (e.g. $u = x^2$), the limits change too. Update them before evaluating, or change back before substituting.

"Top minus Bottom, Right minus Left"

Vertical strip: $y_{\text{top}} - y_{\text{bot}}$ (positive height). Horizontal strip: $x_{\text{right}} - x_{\text{left}}$ (positive width). The mnemonic "**T–B, R–L**" — always larger minus smaller — prevents sign errors.

5 More Worked Examples

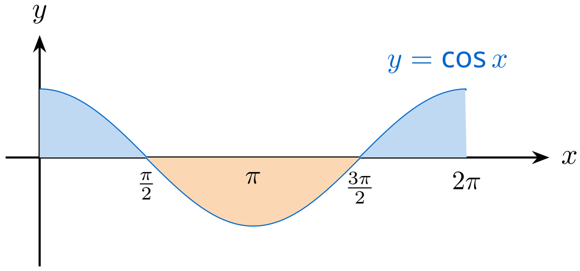
The exercises in NCERT Chapter 8 mostly recombine the four pillars: line + axes, curve + axes, curve + curve, classical conic. This section runs a representative spread of slightly harder problems — the kind that appear in board paper second-section "5-mark" questions, and in JEE Main numerical-answer-type problems.

5.1 Example: $y = \cos x$ over $[0, 2\pi]$ (NCERT Example 4)

Find the area bounded by the curve $y = \cos x$ between $x = 0$ and $x = 2\pi$.

On $[0, \pi/2]$, $\cos x \geq 0$; on $[\pi/2, 3\pi/2]$, $\cos x \leq 0$; on $[3\pi/2, 2\pi]$, $\cos x \geq 0$. Split:

$$A = \int_0^{\pi/2} \cos x\,dx + \left|\int_{\pi/2}^{3\pi/2} \cos x\,dx\right| + \int_{3\pi/2}^{2\pi} \cos x\,dx$$
$$= [\sin x]_0^{\pi/2} + |[\sin x]_{\pi/2}^{3\pi/2}| + [\sin x]_{3\pi/2}^{2\pi}$$
$$= 1 + 2 + 1 = 4 \text{ sq. units.}$$



5.2 Example: area bounded by $y = x|x|$, the x-axis and $x = \pm 1$

By definition, $y = x|x|$ equals x^2 for $x \geq 0$ and $-x^2$ for $x < 0$. The region splits at $x = 0$:

$$A = \int_{-1}^{0} |-x^2|\,dx + \int_0^1 x^2\,dx = \int_{-1}^{0} x^2\,dx + \int_0^1 x^2\,dx = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}.$$

5.3 Example: area of region enclosed by $y = x^3$, x-axis, $x = -2$, $x = 1$

The cubic crosses the axis at $x = 0$. On $(-2, 0)$ it is negative; on $(0, 1)$ it is positive.

$$A = \left|\int_{-2}^{0} x^3\,dx\right| + \int_0^1 x^3\,dx = \left|\tfrac{-16}{4}\right| + \tfrac{1}{4} = 4 + \tfrac{1}{4} = \tfrac{17}{4} \text{ sq. units.}$$

5.4 Example: area enclosed by the parabola $y^2 = 4ax$ and its latus rectum [JEE]

The latus rectum is the vertical line $x = a$. The parabola opens rightward with vertex at origin. By symmetry about the x-axis, the enclosed area is $2\int_0^a 2\sqrt{ax}\,dx$ (taking $y = 2\sqrt{ax}$ for the upper half):

$$A = 2\int_0^a 2\sqrt{ax}\,dx = 4\sqrt{a} \cdot \tfrac{2}{3}a^{3/2} = \tfrac{8a^2}{3} \text{ sq. units.}$$

5.5 Example: $y = x^2$, $y = |x|$ in $[-1, 1]$

Set $x^2 = |x| \Rightarrow x^2 = \pm x \Rightarrow x = 0, \pm 1$. By symmetry, compute on $[0, 1]$ and double: on $(0, 1)$, $|x| = x \geq x^2$, so $y_{\text{top}} = x$, $y_{\text{bot}} = x^2$.

$$A = 2\int_0^1 (x - x^2)\,dx = 2\left[\tfrac{x^2}{2} - \tfrac{x^3}{3}\right]_0^1 = 2 \cdot \tfrac{1}{6} = \tfrac{1}{3}.$$

5.6 Example: area under $y = e^x$ from 0 to 1 [JEE Extension]

The exponential is positive on $[0, 1]$, so no sign issue:

$$A = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1 \approx 1.718 \text{ sq. units.}$$

5.7 Example: hyperbola wedge — area bounded by $xy = 4$, $x = 1$, $x = 4$ [JEE Extension]

On $[1, 4]$, $y = 4/x \geq 0$, so

$$A = \int_1^4 \tfrac{4}{x}\,dx = 4[\ln x]_1^4 = 4\ln 4 = 8\ln 2 \text{ sq. units.}$$

This is the basic "area under a rectangular hyperbola" — one of the standard JEE shortcuts is that the area under $y = k/x$ between $x = p$ and $x = q$ is $k\ln(q/p)$.

The Logarithm IS an Area

The natural log $\ln t$ is, by one definition, $\int_1^t \frac{1}{x}\,dx$. So computing areas under $1/x$ *is* computing logarithms. This is the unifying view: definite integrals don't just *measure* areas, certain elementary functions *are* areas.

5.8 Example: area enclosed by $y = \sin x$ between 0 and 2π

Mirror of the $\cos x$ example: $\sin x \geq 0$ on $[0, \pi]$, ≤ 0 on $[\pi, 2\pi]$.

$$A = \int_0^\pi \sin x\,dx + \left|\int_\pi^{2\pi} \sin x\,dx\right| = 2 + 2 = 4 \text{ sq. units.}$$

5.9 Example: triangle by three lines [board favourite]

Find the area of the triangle whose sides have equations $y = 2x + 1$, $y = 3x + 1$, $x = 4$.

Vertices. Solve pairwise: $y = 2x + 1$ and $y = 3x + 1$ meet at $(0, 1)$. $y = 2x + 1$ and $x = 4$ meet at $(4, 9)$. $y = 3x + 1$ and $x = 4$ meet at $(4, 13)$.

For $x \in [0, 4]$, top line is $y = 3x + 1$, bottom is $y = 2x + 1$:

$$A = \int_0^4 \big[(3x + 1) - (2x + 1)\big]\,dx = \int_0^4 x\,dx = 8 \text{ sq. units.}$$

6 Parametric and Polar Curves [JEE/NEET Extension]

The rationalised NCERT stops at Cartesian curves. Entrance exams routinely give regions bounded by parametric curves (cycloid, astroid) or polar curves (cardioid, rose). The formulas are direct generalisations of the strip method.

6.1 Area enclosed by a parametric curve

If a closed curve is given by $x = x(t)$, $y = y(t)$ for $t \in [\alpha, \beta]$, traced once counter-clockwise, the area enclosed is

$$A = \int_{\alpha}^{\beta} y(t)\, x'(t)\, dt = \int_{\alpha}^{\beta} y\, \frac{dx}{dt}\, dt.$$

Worked Example: area enclosed by the astroid $x = a\cos^3 t$, $y = a\sin^3 t$.

By symmetry, compute the first-quadrant arc ($t : \pi/2 \to 0$) and multiply by 4.

$$A = 4\int_0^{\pi/2} y\,|x'(t)|\, dt = 4\int_0^{\pi/2} a\sin^3 t \cdot 3a\cos^2 t \sin t\, dt = 12a^2 \int_0^{\pi/2} \sin^4 t\cos^2 t\, dt.$$

Using the standard reduction $\int_0^{\pi/2} \sin^4 t \cos^2 t\, dt = \frac{\pi}{32}$:

$$A = 12a^2 \cdot \tfrac{\pi}{32} = \tfrac{3\pi a^2}{8}.$$

6.2 Area enclosed by a polar curve

For a curve $r = r(\theta)$ in polar form, the elementary "pie-slice" has area $\frac{1}{2}r^2\, d\theta$. So:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta.$$

Example: area of the cardioid $r = a(1 + \cos\theta)$.

$$A = \frac{1}{2}\int_0^{2\pi} a^2(1+\cos\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \frac{3\pi a^2}{2}.$$

Real-World Application

Cardioid microphones used in podcasting are named after this curve: the polar pickup pattern of the microphone literally traces a cardioid, with maximum sensitivity in front (where r is largest) and a null directly behind it.

6.3 Area of a hyperbola sector [JEE Extension]

For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the area between the curve, the x-axis, $x = a$ (vertex) and $x = a\cosh u$ is $\frac{1}{2}ab \cdot u$ (the hyperbolic-angle interpretation). The asymptotic infinite area — $A = \int_a^{\infty} \frac{b}{a}\sqrt{x^2 - a^2}\, dx$ — diverges, mirroring the unbounded character of the curve.

Closed-Form Areas to Memorize

For competitive exams, memorize:

- Circle: πa^2
- Ellipse: πab
- Astroid: $\frac{3\pi a^2}{8}$
- Cardioid: $\frac{3\pi a^2}{2}$
- Area between $y^2 = 4ax$ and latus rectum $x = a$: $\frac{8a^2}{3}$
- Area of parabolic segment cut by chord through focus perpendicular to axis: $\frac{8a^2}{3}$ (same as latus-rectum result)

6.4 Area under a curve given parametrically: cycloid arch

The cycloid $x = a(t - \sin t)$, $y = a(1 - \cos t)$ traces one arch as $t : 0 \to 2\pi$. The area under one arch:

$$A = \int_0^{2\pi} y\,x'(t)\,dt = \int_0^{2\pi} a(1 - \cos t)\cdot a(1 - \cos t)\,dt = a^2\int_0^{2\pi}(1 - \cos t)^2\,dt = 3\pi a^2.$$

So one arch of a rolling-circle cycloid has area exactly $3\pi a^2$ — three times the area of the generating circle.

7 Quick Reference Summary

A single page distilling every formula a student needs for board exams plus the most-tested entrance-exam extensions.

7.1 Core formulas

Master Area Formulas

Curve and x-axis.

$$A = \int_a^b y\,dx = \int_a^b f(x)\,dx \quad (f \geq 0).$$

Curve and y-axis.

$$A = \int_c^d x\,dy = \int_c^d g(y)\,dy \quad (g \geq 0).$$

Region between two curves (with $f \geq g$ on $[a, b]$):

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Curve dipping below axis — take absolute value:

$$A = \int_a^b |f(x)|\,dx.$$

7.2 Closed-region areas to remember

Region	Area
Circle of radius a: $x^2 + y^2 = a^2$	πa^2
Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	πab
Parabola $y^2 = 4ax$ and latus rectum $x = a$	$\frac{8a^2}{3}$
$y = x^2$ and $y = x$ (or $y^2 = x$ and $y = x$)	$\frac{1}{6}$
$y = \lvert x\rvert$ and $y = c$ (capping line)	c^2
Astroid $x = a\cos^3 t$, $y = a\sin^3 t$	$\frac{3\pi a^2}{8}$
Cardioid $r = a(1 + \cos\theta)$	$\frac{3\pi a^2}{2}$
One arch of cycloid $x = a(t - \sin t), y = a(1 - \cos t)$	$3\pi a^2$
Triangle with vertices (x_i, y_i)	$\frac{1}{2}\left\lvert \sum (x_i y_{i+1} - x_{i+1} y_i)\right\rvert$

7.3 Standard antiderivatives needed

Most-Used Antiderivatives in Area Problems

$$\int \sqrt{a^2 - x^2}\,dx = \tfrac{x}{2}\sqrt{a^2 - x^2} + \tfrac{a^2}{2}\sin^{-1}\tfrac{x}{a} + C.$$

$$\int \sqrt{x^2 - a^2}\,dx = \tfrac{x}{2}\sqrt{x^2 - a^2} - \tfrac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C.$$

$$\int \sqrt{x^2 + a^2}\,dx = \tfrac{x}{2}\sqrt{x^2 + a^2} + \tfrac{a^2}{2}\ln\left|x + \sqrt{x^2 + a^2}\right| + C.$$

$$\int \tfrac{1}{x}\,dx = \ln|x| + C, \qquad \int e^x dx = e^x + C.$$

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \tfrac{1}{2}B\left(\tfrac{m+1}{2}, \tfrac{n+1}{2}\right) \quad \text{(Beta function, JEE shortcut).}$$

7.4 Step-by-step algorithm

Six-Step Area Algorithm

1. Sketch the region; label all curves and intersection points.
2. Find intersections by solving simultaneous equations.
3. Determine sign of integrand; split interval if curve crosses axis.
4. Choose strip direction — vertical if curves are $y = f(x)$, horizontal if $x =$

$g(y)$.

5. Set limits and write the integral

$$A = \int (\text{top} - \text{bot})\, d(\text{strip variable}).$$

6. Evaluate; use symmetry whenever possible.

7.5 Common test-question patterns

- "Find the area between $y = f(x)$ and the x-axis on $[a, b]$." $\Rightarrow$ single integral, watch sign.
- "Find the area enclosed by $f(x)$ and $g(x)$." $\Rightarrow$ solve $f = g$ for limits, integrate $|f - g|$.
- "Find the area enclosed by a closed curve (circle, ellipse, astroid)." $\Rightarrow$ use symmetry and quadrant trick.
- "Find the area bounded by a parabola and a chord (line, latus rectum)." $\Rightarrow$ horizontal strips usually best.
- "Find the area enclosed by a modulus function and a line." $\Rightarrow$ split at the kink, sum the pieces.

Where Areas Matter

The area under a velocity-time graph is total distance travelled. The area under a force-displacement curve is work done. The area under a probability density is a probability. The area under a marginal-cost curve is total cost. Whenever a rate is plotted against its independent variable, the area carries direct physical meaning — which is why Chapter 8 underlies almost every quantitative science a Class 12 student will encounter.

"SLIP" for Area Problems

Sketch the region. **L**imits from intersections. **I**dentify top and bottom. **P**ut into the integral. Follow SLIP and the problem solves itself.

7.6 Final note: what to revise the night before the exam

For the board exam, the question on Application of Integrals is almost always either (a) area between a parabola/circle and a line, or (b) area between two simple curves like $y = x^2$ and $y = x$. Practice these two until they are mechanical. For JEE, add the modulus and parametric examples from Section 6. Carry the closed-form area table in your head — knowing πa^2, πab, and the $\frac{1}{6}$ result alone lets you solve a surprising fraction of multiple-choice questions in seconds.

The chapter is short on theorems and long on technique: every problem is the same strip-and-integrate idea applied to a new region. Master that one idea and the rest is execution.