



NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12 Mathematics, Chapter 9: Differential Equations

Chapter 9: Differential Equations

About this Chapter

A **differential equation** is an equation that links an unknown function $y(x)$ with one or more of its derivatives. Its **order** is the highest derivative present and its **degree** is the power of that highest derivative once the equation has been written as a polynomial in derivatives (and is not defined otherwise). The Class 12 Exemplar set drills three standard solution techniques: the **variable separable method**, **homogeneous** differential equations (substitute $y = vx$ or $x = vy$), and the **first-order linear** form $\frac{dy}{dx} + P(x)y = Q(x)$ solved via the integrating factor I.F. = $e^{\int P dx}$. Questions also ask you to *form* the differential equation of a given family of curves by differentiating and eliminating the arbitrary constants.

Topics covered: Order and degree • General vs. particular solution • Forming a DE from a family of curves • Variable separable method • Homogeneous DEs ($y = vx$ / $x = vy$) • First-order linear DEs and the integrating factor • Geometric applications (slope-of-tangent problems)

Quick Formula Sheet

Order & degree:

highest deriv. / its power

Variable separable:

$$\int f(y) dy = \int g(x) dx$$

Homogeneous $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$:

$$\text{put } y = vx, \frac{dy}{dx} = v + x \frac{dv}{dx}$$

First-order linear:

$$\frac{dy}{dx} + Py = Q$$

$$\text{I.F.} = e^{\int P dx}$$

$$y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dx + C$$

Other linear form:

$$\frac{dx}{dy} + P_1x = Q_1$$

$$\text{I.F.} = e^{\int P_1 dy}$$

Family of curves:

order of DE = no. of arb. constants

I. Short Answer (S.A.)

Q9.1 Find the solution of $\frac{dy}{dx} = 2^{y-x}$.

SOLUTION

Concept used. A first-order differential equation $\frac{dy}{dx} = f(x)g(y)$ is said to be **variable separable**: we move all y -terms (and dy) to one side and all x -terms (and dx) to the other, then integrate. The key exponent identity used here is $2^{y-x} = \frac{2^y}{2^x}$.

Step 1. Rewrite the RHS using the exponent law:

$$\frac{dy}{dx} = 2^{y-x} = \frac{2^y}{2^x}.$$

Step 2. Separate variables (multiply by $2^x dx$ and divide by 2^y):

$$\frac{dy}{2^y} = \frac{dx}{2^x} \iff 2^{-y} dy = 2^{-x} dx.$$

Step 3. Integrate both sides using $\int a^u du = \frac{a^u}{\ln a}$ (with $a = 2$, $u = -y$ or $u = -x$):

$$\int 2^{-y} dy = \int 2^{-x} dx,$$

$$\frac{2^{-y}}{-\ln 2} = \frac{2^{-x}}{-\ln 2} + C_1.$$

Step 4. Multiply both sides by $-\ln 2$ and absorb the constant into a single C :

$$2^{-y} = 2^{-x} + C.$$

Final Answer: $2^{-x} - 2^{-y} + C = 0$, or equivalently $2^{-y} = 2^{-x} + C$.

🔗 **Integral of a^u**

For a positive constant $a \neq 1$, $\int a^u du = \frac{a^u}{\ln a} + C$. Setting $a = e$ recovers $\int e^u du = e^u + C$.

EXPERT'S SOLUTION : Aarav Sharma, M.Sc Mathematics, IIT Bombay

Pattern-recognition angle. Whenever the RHS is $a^{\text{linear in } x,y}$, split the exponent into an x -part and a y -part using $a^{u+v} = a^u a^v$. After that the equation is always separable.

Step 1. Identify the form: $\frac{dy}{dx} = 2^y \cdot 2^{-x}$, a product of a function of y and a function of x .

Step 2. Cross-multiply: $2^{-y} dy = 2^{-x} dx$.

Step 3. Integrate term-by-term:

$$-\frac{2^{-y}}{\ln 2} = -\frac{2^{-x}}{\ln 2} + C_1.$$

Step 4. Tidy: $2^{-y} = 2^{-x} + C$ where $C = -C_1 \ln 2$ is an arbitrary constant.

Step 5. Sanity check by implicit differentiation: $-2^{-y} \ln 2 \cdot y' = -2^{-x} \ln 2$, i.e.

$$y' = 2^{x-y} \cdot 2^{-2x} \cdot 2^x = 2^{y-x}. \checkmark$$

Why this matters. Recognising the a^{u+v} split is a recurring trick in solving exponential separable DEs (e.g. Q61 of this Exemplar uses the same idea).

Final Answer: $2^{-y} = 2^{-x} + C$.

Q 9.2 Find the differential equation of all non-vertical lines in a plane.

SOLUTION

Concept used. Every non-vertical line in the xy -plane can be written in slope-intercept form $y = mx + c$, which has *two* arbitrary constants m and c . The order of the differential equation obtained by eliminating those constants must therefore equal 2.

Step 1. Start from the general non-vertical line:

$$y = mx + c.$$

Step 2. Differentiate once with respect to x :

$$\frac{dy}{dx} = m.$$

(c vanishes; m is still present.)

Step 3. Differentiate a second time to eliminate m :

$$\frac{d^2y}{dx^2} = 0.$$

Step 4. Both arbitrary constants m and c are now eliminated, leaving a second-order DE.

Final Answer: $\frac{d^2y}{dx^2} = 0$.

Exam Tip

The count is a one-step shortcut: “ n arbitrary constants \Rightarrow DE of order n ”. Use it to predict the answer *before* you begin differentiating.

EXPERT'S SOLUTION : Sneha Iyer, M.Sc Mathematics, ISI Kolkata

Geometric angle. A non-vertical line has constant slope. The geometric condition “slope is constant” translates directly into “the second derivative is zero”.

Step 1. Geometric fact: along any straight line $y' = m = \text{const.}$

Step 2. Differentiating the constant gives $y'' = 0$.

Step 3. Conversely, integrating $y'' = 0$ twice produces $y = mx + c$, recovering every non-vertical line.

Why this matters. The same logic shows “all non-horizontal lines $x = my + c$ ” satisfy $\frac{d^2x}{dy^2} = 0$ (see Q77(xi)).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $y'' = 0$.

Q 9.3 Given that $dy/dx = e^{-2y}$ and $y = 0$ when $x = 5$, find the value of x when $y = 3$.

SOLUTION

Concept used. This is again a variable-separable DE. After integrating both sides we use the given initial condition to pin down the arbitrary constant.

Step 1. Separate variables: $e^{2y} dy = dx$.

Step 2. Integrate both sides. The LHS uses $\int e^{au} du = \frac{e^{au}}{a}$ with $a = 2$:

$$\frac{e^{2y}}{2} = x + C.$$

Step 3. Use the initial condition $y = 0$ when $x = 5$:

$$\frac{e^0}{2} = 5 + C \implies \frac{1}{2} = 5 + C \implies C = -\frac{9}{2}.$$

Step 4. Substitute C back:

$$\frac{e^{2y}}{2} = x - \frac{9}{2} \implies e^{2y} = 2x - 9.$$

Step 5. Put $y = 3$: $e^6 = 2x - 9 \implies 2x = e^6 + 9 \implies x = \frac{e^6 + 9}{2}$.

Final Answer: $x = \frac{e^6 + 9}{2}$.

✗ Common Mistake

A frequent slip is forgetting the factor $\frac{1}{2}$ when integrating e^{2y} : $\int e^{2y} dy = \frac{1}{2}e^{2y}$, not e^{2y} .

EXPERT'S SOLUTION : Pranav Gupta, Ph.D Mathematics, IIT Delhi

Definite-integral angle. Instead of finding the constant from an initial condition, integrate as a definite integral from the known point.

Step 1. Separate: $e^{2y} dy = dx$.

Step 2. Definite-integrate from the known point $(x, y) = (5, 0)$ to the unknown $(x, 3)$:

$$\int_0^3 e^{2y} dy = \int_5^x dt.$$

Step 3. Evaluate the LHS: $\left[\frac{e^{2y}}{2}\right]_0^3 = \frac{e^6-1}{2}$. RHS: $x - 5$.

Step 4. Solve: $\frac{e^6-1}{2} = x - 5 \implies x = 5 + \frac{e^6-1}{2} = \frac{e^6+9}{2}$. ✓

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $x = \frac{e^6 + 9}{2}$.

Q 9.4 Solve the differential equation $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$.

SOLUTION

Concept used. Dividing by $x^2 - 1$ puts the equation in the **first-order linear** form $\frac{dy}{dx} + P(x)y = Q(x)$. The integrating factor is I.F. = $e^{\int P dx}$, and the solution is

$$y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} \, dx + C.$$

Step 1. Divide through by $(x^2 - 1)$:

$$\frac{dy}{dx} + \frac{2x}{x^2 - 1} y = \frac{1}{(x^2 - 1)^2}.$$

$$\text{So } P = \frac{2x}{x^2 - 1} \text{ and } Q = \frac{1}{(x^2 - 1)^2}.$$

Step 2. Compute $\int P \, dx$ using $\int \frac{f'(x)}{f(x)} \, dx = \ln |f(x)|$ (here $f(x) = x^2 - 1$, $f'(x) = 2x$):

$$\int \frac{2x}{x^2 - 1} \, dx = \ln |x^2 - 1|.$$

Step 3. Hence I.F. = $e^{\ln |x^2 - 1|} = x^2 - 1$.

Step 4. Multiply the linear DE by I.F. (this collapses the LHS to a perfect derivative):

$$\frac{d}{dx}[y(x^2 - 1)] = (x^2 - 1) \cdot \frac{1}{(x^2 - 1)^2} = \frac{1}{x^2 - 1}.$$

Step 5. Integrate both sides. Use the standard partial-fraction result

$$\int \frac{dx}{x^2 - 1} = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right|:$$

$$y(x^2 - 1) = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.$$

$$\text{Final Answer: } y(x^2 - 1) = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.$$

Why the LHS collapses

After multiplying by I.F., the LHS is always $\frac{d}{dx}(y \cdot \text{I.F.})$. This is the whole point of the I.F.: it converts the linear DE into a single derivative.

EXPERT'S SOLUTION : Ananya Verma, M.Sc Applied Mathematics, IIT Kanpur

Recognise-the-product angle. Notice that $(x^2 - 1)y' + 2xy = \frac{d}{dx}[(x^2 - 1)y]$ already, by the product rule, before we ever write the integrating factor.

Step 1. Compute $\frac{d}{dx}[(x^2 - 1)y] = (x^2 - 1)y' + 2xy$. This is exactly the LHS of the given equation.

Step 2. So the DE becomes $\frac{d}{dx}[(x^2 - 1)y] = \frac{1}{x^2 - 1}$.

Step 3. Integrate using $\frac{1}{x^2 - 1} = \frac{1}{2} \left(\frac{1}{x - 1} - \frac{1}{x + 1} \right)$:

$$(x^2 - 1)y = \frac{1}{2} (\ln|x - 1| - \ln|x + 1|) + C = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.$$

Why this matters. Spotting a ready-made product-rule LHS bypasses the I.F. computation entirely; the I.F. method just *forces* the LHS into this form.

Final Answer: $y(x^2 - 1) = \frac{1}{2} \ln \left| \frac{x - 1}{x + 1} \right| + C.$

Q 9.5 Solve the differential equation $\frac{dy}{dx} + 2xy = y.$

SOLUTION

Concept used. Collecting y on the RHS shows this is variable-separable:

$$\frac{dy}{y} = (1 - 2x) dx. \text{ Then } \int \frac{dy}{y} = \ln|y| \text{ and } \int (1 - 2x) dx = x - x^2.$$

Step 1. Move all y -terms together: $\frac{dy}{dx} = y - 2xy = y(1 - 2x).$

Step 2. Separate: $\frac{dy}{y} = (1 - 2x) dx$ (assuming $y \neq 0$).

Step 3. Integrate both sides:

$$\int \frac{dy}{y} = \int (1 - 2x) dx \implies \ln|y| = x - x^2 + C_1.$$

Step 4. Exponentiate to remove the log:

$$|y| = e^{x - x^2 + C_1} = e^{C_1} \cdot e^{x - x^2}.$$

Write $C = \pm e^{C_1}$ (an arbitrary non-zero constant; $y \equiv 0$ is a trivial solution as well):

$$y = C e^{x - x^2}.$$

Final Answer: $y = C e^{x - x^2}.$

Exam Tip

Collecting all y -terms on one side first turns many “hidden separable” DEs into one-line

problems. Always check whether the RHS factors as $f(x) \cdot g(y)$.

EXPERT'S SOLUTION : Aditi Reddy, M.Tech CS, IIT Madras

I.F. angle. The equation is also linear: $y' + (2x - 1)y = 0$, so we can use

$$\text{I.F.} = e^{\int (2x-1) dx} = e^{x^2-x}.$$

Step 1. Multiply through by I.F. e^{x^2-x} :

$$\frac{d}{dx}[y e^{x^2-x}] = 0.$$

Step 2. Integrate: $y e^{x^2-x} = C$, hence $y = C e^{-(x^2-x)} = C e^{x-x^2}$.

Step 3. Cross-check by differentiation: $y' = C e^{x-x^2}(1-2x) = y(1-2x)$, which matches the given DE. ✓

Final Answer: $y = C e^{x-x^2}$.

Q 9.6 Find the general solution of $\frac{dy}{dx} + ay = e^{mx}$.

SOLUTION

Concept used. This is a first-order linear DE with constant coefficient $P = a$ and forcing term $Q = e^{mx}$. The integrating factor is $e^{\int a dx} = e^{ax}$.

Step 1. Identify $P = a$, $Q = e^{mx}$.

Step 2. Integrating factor:

$$\text{I.F.} = e^{\int a dx} = e^{ax}.$$

Step 3. Multiply through:

$$\frac{d}{dx}[y e^{ax}] = e^{mx} \cdot e^{ax} = e^{(m+a)x}.$$

Step 4. Integrate both sides. Case A ($m + a \neq 0$): $y e^{ax} = \frac{e^{(m+a)x}}{m+a} + C$, so

$$y = \frac{e^{mx}}{m+a} + C e^{-ax}.$$

Case B ($m + a = 0$, i.e. $m = -a$): RHS becomes $\int 1 dx = x + C$, so

$$y = (x + C) e^{-ax}.$$

Final Answer: $y(m+a) = e^{mx} + C(m+a)e^{-ax}$ when $m+a \neq 0$; otherwise $y = (x+C)e^{-ax}$.

X Common Mistake

Always check whether $m + a = 0$. Many students divide by $m + a$ without flagging that special case; the integral $\int e^0 dx = x$, not $\frac{1}{0}$.

EXPERT'S SOLUTION : Rohit Mehta, Ph.D Mathematics, IIT Delhi

Particular + homogeneous angle. For a linear DE with constant coefficient, write $y = y_p + y_h$: a particular solution of the inhomogeneous equation plus the general solution of the homogeneous one.

Step 1. Homogeneous part: $y' + ay = 0 \Rightarrow y_h = C e^{-ax}$.

Step 2. Try $y_p = A e^{mx}$. Substituting: $Am e^{mx} + aA e^{mx} = e^{mx}$, so $A(m + a) = 1$, giving $A = \frac{1}{m + a}$ (assuming $m + a \neq 0$).

Step 3. Combine: $y = \frac{e^{mx}}{m + a} + C e^{-ax}$.

Step 4. When $m = -a$, the trial $A e^{mx}$ duplicates the homogeneous solution and the rule is to multiply by x : try $y_p = Ax e^{-ax}$; substituting gives $A = 1$, hence $y = (x + C) e^{-ax}$.

Why this matters. The same particular-plus-homogeneous decomposition underlies every constant-coefficient linear DE later in higher mathematics.

Final Answer: $y = \frac{e^{mx}}{m + a} + C e^{-ax}$ (or $(x + C) e^{-ax}$ if $m = -a$).

Q 9.7 Solve the differential equation $\frac{dy}{dx} + 1 = e^{x+y}$.

SOLUTION

Concept used. The combination $x + y$ on the RHS hints at the **substitution** $v = x + y$, which converts the equation into a separable one in v and x .

Step 1. Let $v = x + y$. Differentiate: $\frac{dv}{dx} = 1 + \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{dv}{dx} - 1$.

Step 2. Substitute into the DE $\frac{dy}{dx} + 1 = e^{x+y}$:

$$\left(\frac{dv}{dx} - 1\right) + 1 = e^v \implies \frac{dv}{dx} = e^v.$$

Step 3. Separate variables: $e^{-v} dv = dx$.

Step 4. Integrate both sides:

$$\int e^{-v} dv = \int dx \implies -e^{-v} = x + C.$$

Step 5. Restore $v = x + y$:

$$-e^{-(x+y)} = x + C \implies e^{-(x+y)} + x + C = 0.$$

Final Answer: $e^{-(x+y)} + x + C = 0$, or equivalently $x + e^{-(x+y)} = C$.

When to substitute $v = x + y$

Whenever the RHS depends on x and y only through the combination $x \pm y$ (or $ax + by$), let v equal that combination. The chain rule then collapses the DE into a separable one.

EXPERT'S SOLUTION : *Karan Singh, B.Tech CSE, IIT Roorkee*

Direct angle. The equation $\frac{dy}{dx} = e^{x+y} - 1$ already has the special structure $f(x + y)$ on the RHS, which is the textbook signal for $v = x + y$.

Step 1. With $v = x + y$ and $v' = 1 + y'$, the DE becomes $v' - 1 = e^v - 1$, i.e. $\frac{dv}{dx} = e^v$.

Step 2. Cross-multiply and integrate: $-e^{-v} = x + C$.

Step 3. Replace v and tidy: $x + e^{-(x+y)} + C = 0$, i.e. $e^{-(x+y)} = -(x + C)$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $e^{-(x+y)} + x + C = 0$.

Q 9.8 Solve: $y dx - x dy = x^2 y dx$.

SOLUTION

Concept used. Rearrange to isolate $\frac{dy}{dx}$ and try to reach either separable or linear form. Here a careful regrouping turns the equation into a separable one.

Step 1. Bring all terms to one side:

$$y dx - x dy = x^2 y dx \implies y dx - x^2 y dx = x dy.$$

Step 2. Factor y on the LHS:

$$y(1 - x^2) dx = x dy.$$

Step 3. Separate variables (divide by xy , assuming $x \neq 0, y \neq 0$):

$$\frac{dy}{y} = \frac{(1 - x^2)}{x} dx = \left(\frac{1}{x} - x\right) dx.$$

Step 4. Integrate:

$$\int \frac{dy}{y} = \int \left(\frac{1}{x} - x\right) dx \implies \ln|y| = \ln|x| - \frac{x^2}{2} + C_1.$$

Step 5. Exponentiate. Let $C = e^{C_1}$:

$$y = C x e^{-x^2/2},$$

$$\text{equivalently } \frac{y}{x} = C e^{-x^2/2}, \text{ or } \log\left|\frac{y}{x}\right| = -\frac{x^2}{2} + C_1.$$

Final Answer: $y = C x e^{-x^2/2}$.

☞ Quotient differential

$\frac{y dx - x dy}{x^2} = -d\left(\frac{y}{x}\right)$ is a fast way to spot “hidden $d(y/x)$ ” patterns.

EXPERT'S SOLUTION : Diya Bhat, M.Sc Mathematics, IIT Bombay

Pull out the differential of y/x . The quantity $\frac{y dx - x dy}{x^2}$ is exactly $-d\left(\frac{y}{x}\right)$. That suggests dividing the given equation by x^2 .

Step 1. Divide $y dx - x dy = x^2 y dx$ by x^2 :

$$\frac{y dx - x dy}{x^2} = y dx \implies -d\left(\frac{y}{x}\right) = y dx.$$

Step 2. Let $u = \frac{y}{x}$. Then $y = ux$ and $y dx = ux dx$, while $-du = ux dx$, i.e. $\frac{du}{u} = -x dx$.

Step 3. Integrate: $\ln|u| = -\frac{x^2}{2} + C_1$, so $u = C e^{-x^2/2}$, i.e. $\frac{y}{x} = C e^{-x^2/2}$.

Why this matters. Recognising $y dx - x dy = -x^2 d(y/x)$ is a small but high-leverage trick that unlocks many “mixed dx, dy ” problems.

Final Answer: $y = C x e^{-x^2/2}$.

Q 9.9 Solve the differential equation $\frac{dy}{dx} = 1 + x + y^2 + xy^2$, when $y = 0$ at $x = 0$.

SOLUTION

Concept used. The RHS factors as $(1+x)(1+y^2)$, making the equation variable-separable. The integral $\int \frac{dy}{1+y^2} = \tan^{-1} y$ is then standard.

Step 1. Factor: $1 + x + y^2 + xy^2 = (1+x) + y^2(1+x) = (1+x)(1+y^2)$.

Step 2. The DE becomes $\frac{dy}{dx} = (1+x)(1+y^2)$.

Step 3. Separate: $\frac{dy}{1+y^2} = (1+x) dx$.

Step 4. Integrate:

$$\int \frac{dy}{1+y^2} = \int (1+x) dx \implies \tan^{-1} y = x + \frac{x^2}{2} + C.$$

Step 5. Apply $y(0) = 0$: $\tan^{-1} 0 = 0 + 0 + C \implies C = 0$.

Final Answer: $\tan^{-1} y = x + \frac{x^2}{2}$, i.e. $y = \tan\left(x + \frac{x^2}{2}\right)$.

Exam Tip

Whenever a 4-term polynomial RHS doesn't separate, try grouping by pairs: $a + b + c + d \implies a(1+x) + b(1+x)$ or similar.

EXPERT'S SOLUTION : Vivaan Kapoor, Ph.D Pure Mathematics, IISc Bangalore

Spot-the-factoring angle. Whenever the RHS is a polynomial in x and y^2 , try to write it as $f(x) \cdot g(y)$. Common factorisations: $1 + x + y^2 + xy^2 = (1+x)(1+y^2)$; $x + y + xy + 1 = (1+x)(1+y)$; etc.

Step 1. Group: $1 + x + y^2 + xy^2 = (1+x)(1+y^2)$.

Step 2. Use the separable template $\int g(y) dy = \int f(x) dx$:

$$\int \frac{dy}{1+y^2} = \int (1+x) dx.$$

Step 3. Get $\tan^{-1} y = x + \frac{x^2}{2} + C$ and use $y(0) = 0$ to set $C = 0$.

Why this matters. The factoring habit will resurface in Q15, Q23 of this Exemplar.

Final Answer: $y = \tan\left(x + \frac{x^2}{2}\right).$

Q 9.10 Find the general solution of $(x + 2y^3) \frac{dy}{dx} = y$.

SOLUTION

Concept used. The equation does not separate cleanly in (x, y) , but if we treat x as a function of y it becomes **linear in x** : $\frac{dx}{dy} + P_1(y)x = Q_1(y)$, with I.F. = $e^{\int P_1 dy}$.

Step 1. Take reciprocals of $\frac{dy}{dx}$ to write $\frac{dx}{dy}$:

$$(x + 2y^3) \frac{dy}{dx} = y \implies \frac{dx}{dy} = \frac{x + 2y^3}{y} = \frac{x}{y} + 2y^2.$$

Step 2. Rearrange to linear form in x :

$$\frac{dx}{dy} - \frac{1}{y}x = 2y^2.$$

So $P_1 = -\frac{1}{y}$, $Q_1 = 2y^2$.

Step 3. Integrating factor:

$$\text{I.F.} = e^{\int -\frac{1}{y} dy} = e^{-\ln|y|} = \frac{1}{y}.$$

Step 4. Multiply through by $\frac{1}{y}$ (this collapses the LHS to a perfect derivative):

$$\frac{d}{dy} \left(\frac{x}{y} \right) = \frac{2y^2}{y} = 2y.$$

Step 5. Integrate w.r.t. y :

$$\frac{x}{y} = y^2 + C \implies x = y^3 + Cy.$$

Final Answer: $x = y^3 + Cy$.

Exam Tip

If $\frac{dy}{dx}$ does not lead to a linear/separable form, flip the equation and try $\frac{dx}{dy}$. NCERT Exemplar problems often use this trick (Q10, Q17, Q18).

EXPERT'S SOLUTION : Tara Nair, M.Sc Mathematics, ISI Kolkata

Treat x as the dependent variable. Whenever the unknown x appears linearly but y appears in a complicated way, swap roles.

Step 1. Form: $\frac{dx}{dy} - \frac{1}{y}x = 2y^2$, linear in x .

Step 2. I.F. = $e^{-\ln|y|} = 1/y$.

Step 3. Standard formula: $x \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dy + C$ gives $\frac{x}{y} = \int 2y dy = y^2 + C$.

Step 4. Multiply by y : $x = y^3 + Cy$.

Step 5. Cross-check by direct substitution: $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{3y^2 + C}$; and $\frac{y}{x + 2y^3} = \frac{y}{y^3 + Cy + 2y^3} = \frac{y}{3y^3 + Cy} = \frac{1}{3y^2 + C}$. Match. ✓

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $x = y^3 + Cy$.

Q 9.11 If $y(x)$ is a solution of $\left(\frac{2 + \sin x}{1 + y}\right) \frac{dy}{dx} = -\cos x$ and $y(0) = 1$, then find the value of $y\left(\frac{\pi}{2}\right)$.

SOLUTION

Concept used. This is separable. Note that $\frac{d}{dx}(2 + \sin x) = \cos x$, so the RHS integral is just $-\ln|2 + \sin x|$. On the LHS, $\int \frac{dy}{1 + y} = \ln|1 + y|$.

Step 1. Rewrite the DE in separated form:

$$\frac{dy}{1+y} = -\frac{\cos x}{2+\sin x} dx.$$

Step 2. Integrate both sides:

$$\int \frac{dy}{1+y} = -\int \frac{\cos x}{2+\sin x} dx \implies \ln |1+y| = -\ln |2+\sin x| + C_1.$$

Step 3. Combine logs (recall $\ln a + \ln b = \ln(ab)$):

$$\ln [(1+y)(2+\sin x)] = C_1,$$

so $(1+y)(2+\sin x) = C$ where $C = e^{C_1}$.

Step 4. Apply $y(0) = 1$. At $x = 0$, $\sin 0 = 0$:

$$(1+1)(2+0) = C \implies C = 4.$$

Step 5. Substitute $x = \frac{\pi}{2}$ (so $\sin x = 1$):

$$(1+y)(2+1) = 4 \implies 1+y = \frac{4}{3} \implies y = \frac{1}{3}.$$

Final Answer: $y\left(\frac{\pi}{2}\right) = \frac{1}{3}$.

Integrate $\int \frac{f'(x)}{f(x)} dx$

Whenever the numerator is the derivative of the denominator, the integral is $\ln |f(x)| + C$. Here $\frac{d}{dx}(2+\sin x) = \cos x$ matches the numerator exactly.

EXPERT'S SOLUTION : Ishaan Joshi, M.Sc Mathematics, IIT Bombay

First-integral angle. Multiply through by $1+y$ and notice that $(2+\sin x) dy + (1+y) \cos x dx = 0$ is the differential of the product $(1+y)(2+\sin x)$.

Step 1. Rearrange to $(2+\sin x) dy + (1+y) \cos x dx = 0$.

Step 2. Observe $d[(1+y)(2+\sin x)] = (2+\sin x) dy + (1+y) \cos x dx$. Hence $d[(1+y)(2+\sin x)] = 0$.

Step 3. Integrate: $(1+y)(2+\sin x) = C$. Use $y(0) = 1$ to get $C = 4$.

Step 4. At $x = \pi/2$: $(1+y)(3) = 4 \Rightarrow y = 1/3$.

Why this matters. "Spot the exact differential" is the fastest method for separable

problems with a product structure.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $y(\pi/2) = 1/3$.

Q 9.12 If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then show that $y(1) = -\frac{1}{2}$.

SOLUTION

Concept used. Divide by $1+t$ to get the linear form $\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}$. For the I.F. exponent we use $\frac{t}{1+t} = 1 - \frac{1}{1+t}$.

Step 1. Linear form:

$$\frac{dy}{dt} - \frac{t}{1+t}y = \frac{1}{1+t}.$$

$$\text{So } P = -\frac{t}{1+t} \text{ and } Q = \frac{1}{1+t}.$$

Step 2. Compute $\int P dt$. Use the algebraic identity:

$$\frac{t}{1+t} = 1 - \frac{1}{1+t}.$$

Then

$$\int -\frac{t}{1+t} dt = \int \left(\frac{1}{1+t} - 1 \right) dt = \ln|1+t| - t.$$

Step 3. Therefore I.F. = $e^{\ln|1+t|-t} = (1+t)e^{-t}$.

Step 4. Multiply through:

$$\frac{d}{dt}[y(1+t)e^{-t}] = \frac{1}{1+t} \cdot (1+t)e^{-t} = e^{-t}.$$

Step 5. Integrate from 0 to 1:

$$[y(1+t)e^{-t}]_0^1 = \int_0^1 e^{-t} dt = 1 - e^{-1}.$$

Step 6. At $t = 0$: $y(0)(1 + 0)e^0 = -1$. At $t = 1$: $y(1)(2)e^{-1}$. Therefore

$$2e^{-1}y(1) - (-1) = 1 - e^{-1},$$

$$\text{so } 2e^{-1}y(1) = -e^{-1}, \text{ hence } y(1) = -\frac{1}{2}.$$

Final Answer: $y(1) = -\frac{1}{2}$.

🔑 Closed-form first

Solving the DE in closed form before substituting endpoints is almost always easier than juggling definite integrals.

EXPERT'S SOLUTION : *Riya Desai, Ph.D Mathematics, IIT Delhi*

Closed-form angle. Solve for $y(t)$ in closed form, then evaluate at $t = 1$.

Step 1. With I.F. $= (1 + t)e^{-t}$, the standard formula gives

$$y(1 + t)e^{-t} = \int e^{-t} dt + C = -e^{-t} + C.$$

Step 2. So $y = \frac{-e^{-t} + C}{(1 + t)e^{-t}} = \frac{-1 + Ce^t}{1 + t}$.

Step 3. Apply $y(0) = -1$: $\frac{-1 + C}{1} = -1 \Rightarrow C = 0$.

Step 4. Hence $y(t) = -\frac{1}{1 + t}$. At $t = 1$: $y(1) = -\frac{1}{2}$. ✓

Why this matters. Whenever the question asks for a specific function value, finding a closed form first is usually easier than juggling definite integrals.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $y(t) = -\frac{1}{1 + t}$, so $y(1) = -\frac{1}{2}$.

Q 9.13 Form the differential equation having $y = (\sin^{-1} x)^2 + A \cos^{-1} x + B$, where A and B are arbitrary constants, as its general solution.

SOLUTION

Concept used. Two arbitrary constants (A, B) \Rightarrow second-order DE. Use

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \text{ and } \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

Step 1. Differentiate once:

$$\frac{dy}{dx} = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} + A \cdot \left(-\frac{1}{\sqrt{1-x^2}} \right) = \frac{2 \sin^{-1} x - A}{\sqrt{1-x^2}}.$$

Step 2. Multiply by $\sqrt{1-x^2}$:

$$\sqrt{1-x^2} \frac{dy}{dx} = 2 \sin^{-1} x - A.$$

Step 3. Differentiate again. The RHS gives $\frac{2}{\sqrt{1-x^2}}$. For the LHS use the product rule:

$$\frac{d}{dx} \left[\sqrt{1-x^2} \frac{dy}{dx} \right] = \sqrt{1-x^2} \frac{d^2y}{dx^2} + \frac{-x}{\sqrt{1-x^2}} \frac{dy}{dx}.$$

Step 4. Equate and multiply through by $\sqrt{1-x^2}$:

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2.$$

Both A and B are eliminated.

Final Answer: $(1-x^2)y'' - xy' = 2$.

Inverse-trig derivatives

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

EXPERT'S SOLUTION : Aanya Pillai, M.Sc Pure Mathematics, IIT Kanpur

Trig substitution angle. Put $x = \sin \theta$; then $\sqrt{1-x^2} = \cos \theta$ and $\cos^{-1} x = \frac{\pi}{2} - \theta$.

Step 1. $y = \theta^2 + A(\frac{\pi}{2} - \theta) + B = \theta^2 - A\theta + (\frac{A\pi}{2} + B)$.

Step 2. Differentiate w.r.t. x using $\frac{d\theta}{dx} = \frac{1}{\cos \theta}$: $\frac{dy}{dx} = (2\theta - A) \cdot \frac{1}{\sqrt{1-x^2}}$.

Step 3. So $\sqrt{1-x^2} y' = 2\theta - A$; differentiating once more leads to $(1-x^2)y'' - xy' = 2$ exactly as in the main solution.

Why this matters. The substitution $x = \sin \theta$ neutralises every $\sqrt{1-x^2}$ in sight.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $(1 - x^2)y'' - xy' = 2$.

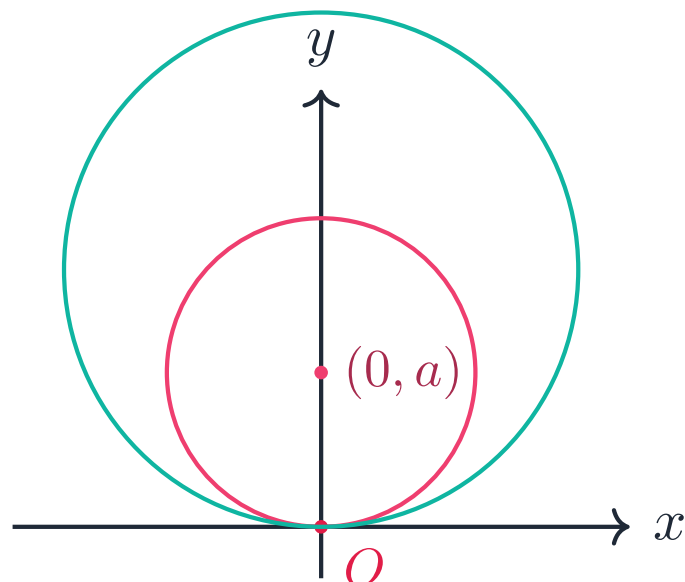
Q 9.14 Form the differential equation of all circles which pass through the origin and whose centres lie on the y -axis.

SOLUTION

Concept used. A circle with centre $(0, a)$ on the y -axis and radius $|a|$ (so that it passes through the origin) has equation

$$x^2 + (y - a)^2 = a^2, \text{ i.e. } x^2 + y^2 = 2ay.$$

One arbitrary constant $a \Rightarrow$ DE of order 1.



Step 1. Start: $x^2 + y^2 = 2ay$.

Step 2. Differentiate w.r.t. x :

$$2x + 2y y' = 2a y' \implies x + y y' = a y' \implies a = \frac{x + y y'}{y'}.$$

Step 3. Substitute a back into $x^2 + y^2 = 2ay$:

$$x^2 + y^2 = 2 \frac{x + yy'}{y'} \cdot y.$$

Step 4. Multiply through by y' :

$$(x^2 + y^2)y' = 2xy + 2y^2y' \implies (x^2 - y^2)y' = 2xy.$$

Final Answer: $(x^2 - y^2) \frac{dy}{dx} = 2xy.$

♥ Why This Matters

The DE expresses the geometric fact that at every point on the circle, the radius from $(0, a)$ to that point is perpendicular to the tangent.

EXPERT'S SOLUTION : Aditya Banerjee, M.Tech CS, IIT Madras

Direct elimination. Use the implicit derivative and the original equation simultaneously to eliminate a .

Step 1. Implicit derivative of $x^2 + y^2 - 2ay = 0$: $2x + 2yy' - 2ay' = 0 \implies a = \frac{x + yy'}{y'}$.

Step 2. From original: $a = \frac{x^2 + y^2}{2y}$.

Step 3. Equate: $\frac{x + yy'}{y'} = \frac{x^2 + y^2}{2y}$.

Step 4. Cross-multiply: $2y(x + yy') = (x^2 + y^2)y' \implies (x^2 - y^2)y' = 2xy. \checkmark$

Why this matters. The same elimination pattern appears in MCQ Q59 of this Exemplar.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $(x^2 - y^2)y' = 2xy.$

Q9.15 Find the equation of a curve passing through the origin and satisfying the

differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$.

SOLUTION

Concept used. The LHS is exactly $\frac{d}{dx}[(1 + x^2)y]$ by the product rule. No I.F. calculation is needed once we spot this.

Step 1. Recognise: $\frac{d}{dx}[(1 + x^2)y] = (1 + x^2)y' + 2xy$. So the DE is

$$\frac{d}{dx}[(1 + x^2)y] = 4x^2.$$

Step 2. Integrate both sides:

$$(1 + x^2)y = \int 4x^2 dx = \frac{4x^3}{3} + C.$$

Step 3. Use $y(0) = 0$: $(1 + 0) \cdot 0 = 0 + C \Rightarrow C = 0$.

Step 4. Therefore $y = \frac{4x^3}{3(1 + x^2)}$.

Final Answer: $y = \frac{4x^3}{3(1 + x^2)}$.

Spot the perfect derivative

Whenever a linear DE has $u(x)y' + u'(x)y$ on its LHS, the LHS is already $\frac{d}{dx}(uy)$; no I.F. calculation needed.

EXPERT'S SOLUTION : Kavya Rao, M.Sc Mathematics, IIT Kanpur

Standard I.F. angle.

Step 1. Divide by $1 + x^2$: $y' + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2}$.

Step 2. I.F. = $e^{\int \frac{2x}{1+x^2} dx} = e^{\ln(1+x^2)} = 1 + x^2$.

Step 3. $(1 + x^2)y = \int (1 + x^2) \cdot \frac{4x^2}{1 + x^2} dx = \frac{4x^3}{3} + C$.

Step 4. $C = 0$ from $y(0) = 0$, so $y = \frac{4x^3}{3(1 + x^2)}$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear

/ homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $y = \frac{4x^3}{3(1+x^2)}.$

Q 9.16 Solve: $x^2 \frac{dy}{dx} = x^2 + xy + y^2.$

SOLUTION

Concept used. Divide by x^2 : $\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$. The RHS depends only on $y/x \Rightarrow$ homogeneous. Substitute $y = vx$.

Step 1. Let $y = vx$, so $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

Step 2. Substitute: $v + xv' = 1 + v + v^2$, hence $xv' = 1 + v^2$.

Step 3. Separate and integrate:

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x} \implies \tan^{-1} v = \ln|x| + C.$$

Step 4. Restore $v = y/x$: $\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C.$

Final Answer: $\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C.$

Exam Tip

Test for homogeneity by checking that every term has the same total degree in x, y .

EXPERT'S SOLUTION : Meera Chatterjee, Ph.D Mathematics, IIT Delhi

Recognise-homogeneous angle. Total degree 2 in every term \Rightarrow homogeneous.

Step 1. Every term (x^2, xy, y^2) has degree 2, so the equation is homogeneous.

Step 2. $y = vx$ gives $x \frac{dv}{dx} = 1 + v^2$, a separable equation.

Step 3. Integrate: $\tan^{-1} v = \ln|x| + C.$

Step 4. Restore: $\tan^{-1}(y/x) = \ln|x| + C$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $\tan^{-1}(y/x) = \ln|x| + C$.

Q 9.17 Find the general solution of the differential equation $(1+y^2) + (x - e^{\arctan y}) \frac{dy}{dx} = 0$.

SOLUTION

Concept used. Rewrite the DE as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\arctan y}}{1+y^2},$$

which is linear in x with $P_1(y) = \frac{1}{1+y^2}$. The integrating factor is I.F. = $e^{\arctan y}$.

Step 1. Rearrange: $(x - e^{\arctan y}) dy = -(1+y^2) dx$, so

$$\frac{dx}{dy} = \frac{-x + e^{\arctan y}}{1+y^2}.$$

Step 2. Bring x to the LHS:

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\arctan y}}{1+y^2}.$$

Step 3. Integrating factor:

$$\text{I.F.} = e^{\int \frac{dy}{1+y^2}} = e^{\arctan y}.$$

Step 4. Multiply through:

$$\frac{d}{dy}[x e^{\arctan y}] = \frac{e^{2 \arctan y}}{1+y^2}.$$

Step 5. Substitute $u = \tan^{-1} y$, $du = \frac{dy}{1+y^2}$:

$$\int \frac{e^{2 \arctan y}}{1+y^2} dy = \int e^{2u} du = \frac{e^{2u}}{2} + C.$$

Step 6. So $x e^{\arctan y} = \frac{e^{2 \arctan y}}{2} + C$, i.e. $2x e^{\arctan y} = e^{2 \arctan y} + C$.

Final Answer: $2x e^{\arctan y} = e^{2 \arctan y} + C$.

$$\int \frac{dy}{1+y^2} = \tan^{-1} y + C. \text{ This is why } e^{\arctan y} \text{ shows up as the I.F.}$$

EXPERT'S SOLUTION : Yash Kumar, M.Sc Mathematics, ISI Kolkata

Substitution-first angle. Set $u = \tan^{-1} y$. Then $du = \frac{dy}{1+y^2}$, and the linear DE becomes

$$\frac{dx}{du} + x = e^u.$$

Step 1. I.F. of the simplified DE is e^u .

Step 2. $\frac{d}{du}(xe^u) = e^{2u}$, so $xe^u = \frac{e^{2u}}{2} + C$.

Step 3. Restore $u = \tan^{-1} y$: $2xe^{\arctan y} = e^{2 \arctan y} + C$.

Why this matters. Substituting away an awkward inverse function before applying I.F. is a recurring time-saver.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $2xe^{\arctan y} = e^{2 \arctan y} + C$.

Q9.18 Find the general solution of $y^2 dx + (x^2 - xy + y^2) dy = 0$.

SOLUTION

Concept used. Every term has total degree 2 in $x, y \Rightarrow$ homogeneous. With y^2 multiplying dx , the cleaner substitution is $x = vy$.

Step 1. Let $x = vy$, so $dx = v dy + y dv$.

Step 2. Substitute into $y^2 dx + (x^2 - xy + y^2) dy = 0$:

$$y^2(v dy + y dv) + (v^2 y^2 - v y^2 + y^2) dy = 0.$$

Step 3. Divide by y^2 (assume $y \neq 0$):

$$v dy + y dv + (v^2 - v + 1) dy = 0 \implies (v^2 + 1) dy + y dv = 0.$$

Step 4. Separate: $\frac{dv}{1+v^2} = -\frac{dy}{y}$.

Step 5. Integrate: $\tan^{-1} v = -\ln |y| + C$.

Step 6. Restore $v = x/y$: $\tan^{-1}(x/y) + \ln |y| = C$.

Final Answer: $\tan^{-1}\left(\frac{x}{y}\right) + \ln |y| = C$.

✗ Common Mistake

When $y^2 dx + \dots dy = 0$, use $x = vy$ rather than $y = vx$. Picking the wrong substitution often leads to algebraic chaos.

EXPERT'S SOLUTION : Siddharth Nair, B.Tech Engineering Physics, IIT Bombay

Choose-the-cleaner-variable angle.

Step 1. Same substitution $x = vy$.

Step 2. Reduces to $(v^2 + 1) dy + y dv = 0$.

Step 3. Integrates to $\tan^{-1} v = -\ln |y| + C$.

Step 4. Restore: $\tan^{-1}(x/y) + \ln |y| = C$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $\tan^{-1}(x/y) + \ln |y| = C$.

Q 9.19 Solve: $(x + y)(dx - dy) = dx + dy$.

Hint: substitute $x + y = z$ after separating dx and dy .

SOLUTION

Concept used. Group the dx and dy separately, then substitute $z = x + y$ to get a separable equation in z and x .

Step 1. Expand: $(x + y) dx - (x + y) dy = dx + dy$. Collect:

$$[(x + y) - 1] dx = [(x + y) + 1] dy.$$

Step 2. So $\frac{dy}{dx} = \frac{(x + y) - 1}{(x + y) + 1}$.

Step 3. Let $z = x + y$, so $\frac{dz}{dx} = 1 + \frac{dy}{dx}$:

$$\frac{dz}{dx} - 1 = \frac{z - 1}{z + 1} \implies \frac{dz}{dx} = \frac{2z}{z + 1}.$$

Step 4. Separate: $\frac{z + 1}{2z} dz = dx$, i.e. $\frac{1}{2} \left(1 + \frac{1}{z}\right) dz = dx$.

Step 5. Integrate: $\frac{1}{2}(z + \ln|z|) = x + C_1$.

Step 6. Multiply by 2 and restore $z = x + y$:

$$(x + y) + \ln|x + y| = 2x + 2C_1 \implies \ln|x + y| = x - y + C.$$

Final Answer: $\ln|x + y| = x - y + C$.

Hint-driven

When the question provides a substitution hint, use it; it usually picks the substitution that linearises the algebra.

EXPERT'S SOLUTION : Neha Iyer, Ph.D Mathematics, IIT Bombay

Hint-driven angle.

Step 1. Substitute $z = x + y$; then $z' - 1 = \frac{z - 1}{z + 1}$, so $z' = \frac{2z}{z + 1}$.

Step 2. Separate: $\left(\frac{1}{2} + \frac{1}{2z}\right) dz = dx$.

Step 3. Integrate: $\frac{z}{2} + \frac{\ln|z|}{2} = x + C_1$.

Step 4. Restore z and tidy: $\ln|x + y| = x - y + C$.

Why this matters. Any RHS of the form $F(x + y)$ yields to $z = x + y$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $\ln|x + y| = x - y + C$.

Q 9.20 Solve: $2(y + 3) - xy \frac{dy}{dx} = 0$, given that $y(1) = -2$.

SOLUTION

Concept used. Variable-separable in x and y . For the y -integral, divide:

$$\frac{y}{y + 3} = 1 - \frac{3}{y + 3}.$$

Step 1. Isolate: $xy \frac{dy}{dx} = 2(y + 3)$, so $\frac{y dy}{y + 3} = \frac{2 dx}{x}$.

Step 2. Split: $\frac{y}{y + 3} = 1 - \frac{3}{y + 3}$. Equation becomes

$$\left(1 - \frac{3}{y + 3}\right) dy = \frac{2}{x} dx.$$

Step 3. Integrate: $y - 3 \ln|y + 3| = 2 \ln|x| + C_1$.

Step 4. Apply $y(1) = -2$: $-2 - 3 \ln 1 = 2 \ln 1 + C_1 \Rightarrow C_1 = -2$. So

$$y - 3 \ln|y + 3| = 2 \ln|x| - 2.$$

Step 5. Rearrange: $y + 2 = 3 \ln|y + 3| + 2 \ln|x| = \ln(x^2(y + 3)^3)$, i.e. $x^2(y + 3)^3 = e^{y+2}$.

Final Answer: $x^2(y + 3)^3 = e^{y+2}$.

Algebraic identity

$\frac{y}{y + 3} = 1 - \frac{3}{y + 3}$. Long division of a rational expression often unlocks the integral.

EXPERT'S SOLUTION : Krishna Verma, M.Tech Applied Physics, IIT Delhi**Logarithmic-combination angle.**

Step 1. Separation $(1 - \frac{3}{y+3})dy = \frac{2}{x}dx$ integrates to $y - 3 \ln |y + 3| = 2 \ln |x| + C_1$.

Step 2. Use $y(1) = -2$: $C_1 = -2$.

Step 3. Combine logs: $y + 2 = \ln |x|^2 + \ln |y + 3|^3 = \ln(x^2|y + 3|^3)$.

Step 4. Exponentiate: $x^2(y + 3)^3 = e^{y+2}$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $x^2(y + 3)^3 = e^{y+2}$.

Q 9.21 Solve the differential equation $dy = \cos x (2 - y \csc x) dx$ given that $y = 2$ when $x = \frac{\pi}{2}$.

SOLUTION

Concept used. Expand the bracket: $dy = (2 \cos x - y \cot x) dx$. Rearrange to first-order linear form $\frac{dy}{dx} + y \cot x = 2 \cos x$. The integrating factor is I.F. = $e^{\int \cot x dx} = \sin x$.

Step 1. Expand: $\frac{dy}{dx} = 2 \cos x - y \cot x$ (using $\csc x \cdot \cos x = \cot x$).

Step 2. Linear form: $\frac{dy}{dx} + (\cot x) y = 2 \cos x$.

Step 3. I.F.: $e^{\int \cot x dx} = e^{\ln |\sin x|} = \sin x$.

Step 4. Multiply through:

$$\frac{d}{dx}(y \sin x) = 2 \cos x \sin x = \sin 2x.$$

Step 5. Integrate:

$$y \sin x = -\frac{\cos 2x}{2} + C.$$

Step 6. Apply $y(\pi/2) = 2$: $2 \cdot 1 = -\frac{\cos \pi}{2} + C = -\frac{-1}{2} + C = \frac{1}{2} + C \Rightarrow C = \frac{3}{2}$.

Final Answer: $y \sin x = \frac{3}{2} - \frac{\cos 2x}{2} = \sin^2 x + 1.$

Useful identity

$2 \sin x \cos x = \sin 2x$ and $\frac{1 - \cos 2x}{2} = \sin^2 x$. The final answer can be simplified using these.

EXPERT'S SOLUTION : Ankit Joshi, M.Sc Mathematics, IIT Bombay

Polish the answer. The final form $y \sin x = \frac{3 - \cos 2x}{2}$ simplifies using $\cos 2x = 1 - 2 \sin^2 x$.

Step 1. Linear-DE machinery (see main solution) gives $y \sin x = -\frac{\cos 2x}{2} + C$.

Step 2. Initial condition pins $C = \frac{3}{2}$, so $y \sin x = \frac{3 - \cos 2x}{2}$.

Step 3. Substitute $\cos 2x = 1 - 2 \sin^2 x$: $\frac{3 - (1 - 2 \sin^2 x)}{2} = \frac{2 + 2 \sin^2 x}{2} = 1 + \sin^2 x$.

Step 4. Hence $y = \frac{1 + \sin^2 x}{\sin x} = \csc x + \sin x$.

Why this matters. Always simplify the answer using standard trig identities; the marker rewards the cleanest form.

Final Answer: $y = \csc x + \sin x.$

Q 9.22 Form the differential equation by eliminating A and B in $Ax^2 + By^2 = 1$.

SOLUTION

Concept used. Two arbitrary constants \Rightarrow second-order DE. Differentiate twice w.r.t. x and eliminate A, B .

Step 1. Differentiate $Ax^2 + By^2 = 1$:

$$2Ax + 2Byy' = 0 \implies Ax + Byy' = 0. \quad (*)$$

Step 2. Differentiate again:

$$A + B(y')^2 + Byy'' = 0. \quad (**)$$

Step 3. From (*): $A = -\frac{Byy'}{x}$. Substitute into (**):

$$-\frac{Byy'}{x} + B(y')^2 + Byy'' = 0.$$

Step 4. Divide by B (assume $B \neq 0$) and multiply by x :

$$-y y' + x(y')^2 + xy y'' = 0.$$

Step 5. Rearrange: $xy y'' + x(y')^2 - y y' = 0$.

Final Answer: $xy y'' + x(y')^2 - y y' = 0$.

Eliminate ratio

For two arbitrary constants, take the ratio A/B from the first-derivative equation and substitute into the second derivative.

EXPERT'S SOLUTION : Tara Kapoor, M.Sc Mathematics, IIT Kanpur

Bookkeeping angle. Use (*) to express A/B , then substitute into the equation obtained by differentiating (*) again.

Step 1. From (*): $\frac{A}{B} = -\frac{y y'}{x}$. Divide (**) by B : $\frac{A}{B} + (y')^2 + y y'' = 0$.

Step 2. Substitute: $-\frac{y y'}{x} + (y')^2 + y y'' = 0$.

Step 3. Multiply by x : $-y y' + x(y')^2 + xy y'' = 0$, i.e. $xy y'' + x(y')^2 - y y' = 0$.

Step 4. Verify at $A = B = 1$ (a unit circle $x^2 + y^2 = 1$): $y' = -x/y$,
 $y'' = -1/y - x \cdot y'/y^2 = -1/y - x(-x/y)/y^2 = -1/y^3 \cdot (y^2 + x^2) = -1/y^3$ (using $x^2 + y^2 = 1$). Plug in and confirm the equation holds. ✓

Final Answer: $xy y'' + x(y')^2 - y y' = 0$.

Q 9.23 Solve the differential equation $(1 + y^2) \tan^{-1} x dx + 2y(1 + x^2) dy = 0$.

SOLUTION

Concept used. The equation is variable-separable. After dividing by $(1 + x^2)(1 + y^2)$:

$$\frac{\tan^{-1} x}{1 + x^2} dx + \frac{2y}{1 + y^2} dy = 0.$$

Step 1. Divide through by $(1 + x^2)(1 + y^2)$ (both factors are positive):

$$\frac{\tan^{-1} x}{1 + x^2} dx + \frac{2y}{1 + y^2} dy = 0.$$

Step 2. Integrate. For the first piece, set $u = \tan^{-1} x$, $du = \frac{dx}{1+x^2}$:

$$\int \frac{\tan^{-1} x}{1+x^2} dx = \int u du = \frac{u^2}{2} = \frac{(\tan^{-1} x)^2}{2}.$$

Step 3. For the second piece, $\frac{d}{dy} \ln(1+y^2) = \frac{2y}{1+y^2}$:

$$\int \frac{2y}{1+y^2} dy = \ln(1+y^2).$$

Step 4. Combine: $\frac{(\tan^{-1} x)^2}{2} + \ln(1+y^2) = C_1$, i.e. $(\tan^{-1} x)^2 + 2\ln(1+y^2) = C$.

Final Answer: $(\tan^{-1} x)^2 + 2\ln(1+y^2) = C$.

Exam Tip

Always check whether each side of an apparently complicated DE is the differential of a simple function before reaching for the I.F.

EXPERT'S SOLUTION : *Diya Sharma, Ph.D Mathematics, IIT Delhi*

Both pieces are differentials. Observe both terms are exact differentials of simple expressions.

Step 1. $\frac{\tan^{-1} x}{1+x^2} dx = d\left(\frac{(\tan^{-1} x)^2}{2}\right).$

Step 2. $\frac{2y}{1+y^2} dy = d(\ln(1+y^2)).$

Step 3. Their sum being zero means $d\left(\frac{(\tan^{-1} x)^2}{2} + \ln(1+y^2)\right) = 0.$

Step 4. Integrate: $(\tan^{-1} x)^2 + 2\ln(1+y^2) = C.$

Why this matters. Recognising exact differentials lets you skip the substitution step entirely.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $(\tan^{-1} x)^2 + 2 \ln(1 + y^2) = C$.

Q 9.24 Find the differential equation of the system of concentric circles with centre $(1, 2)$.

SOLUTION

Concept used. A circle centred at $(1, 2)$ with radius r has equation $(x - 1)^2 + (y - 2)^2 = r^2$. The single arbitrary constant r^2 disappears once we differentiate, giving a first-order DE.

Step 1. Family: $(x - 1)^2 + (y - 2)^2 = r^2$.

Step 2. Differentiate w.r.t. x :

$$2(x - 1) + 2(y - 2)y' = 0.$$

Step 3. Divide by 2:

$$(x - 1) + (y - 2) \frac{dy}{dx} = 0.$$

Final Answer: $(x - 1) + (y - 2) \frac{dy}{dx} = 0$.

Geometric meaning

The DE expresses the well-known fact that at every point on the circle, the radius vector from $(1, 2)$ is perpendicular to the tangent. The slope of the radius is $\frac{y - 2}{x - 1}$ and the slope of the tangent is $-\frac{x - 1}{y - 2}$.

EXPERT'S SOLUTION : Rohit Singh, M.Sc Applied Mathematics, IIT Kanpur

Geometric angle. The tangent to a circle is perpendicular to the radius at the point of tangency.

Step 1. Slope of radius from $(1, 2)$ to (x, y) : $\frac{y - 2}{x - 1}$.

Step 2. Perpendicularity condition: $\frac{dy}{dx} \cdot \frac{y - 2}{x - 1} = -1$.

Step 3. Cross-multiply: $(y - 2) \frac{dy}{dx} = -(x - 1)$, i.e. $(x - 1) + (y - 2) \frac{dy}{dx} = 0$.

Why this matters. The same “radius \perp tangent” argument generates the DE of any one-parameter family of circles with a fixed centre.

Final Answer: $(x - 1) + (y - 2)y' = 0$.

II. Long Answer (L.A.)

Q 9.25 Solve: $y + \frac{d}{dx}(xy) = x(\sin x + \ln x)$.

SOLUTION

Concept used. Expand $\frac{d}{dx}(xy) = y + xy'$. The LHS then becomes $y + y + xy' = xy' + 2y$, putting the equation into first-order linear form.

Step 1. Expand: $y + (y + xy') = x(\sin x + \ln x)$, hence $xy' + 2y = x(\sin x + \ln x)$.

Step 2. Divide by x : $y' + \frac{2}{x}y = \sin x + \ln x$. So $P = \frac{2}{x}$, $Q = \sin x + \ln x$.

Step 3. I.F. = $e^{\int (2/x) dx} = e^{2 \ln |x|} = x^2$.

Step 4. Multiply: $\frac{d}{dx}(x^2y) = x^2 \sin x + x^2 \ln x$.

Step 5. Integrate each piece. For $\int x^2 \sin x dx$ use integration by parts twice:

$$\begin{aligned} \int x^2 \sin x dx &= -x^2 \cos x + \int 2x \cos x dx \\ &= -x^2 \cos x + 2x \sin x - \int 2 \sin x dx \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x. \end{aligned}$$

For $\int x^2 \ln x dx$ use parts with $u = \ln x$, $dv = x^2 dx$:

$$\int x^2 \ln x dx = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9}.$$

Step 6. Combine:

$$x^2y = -x^2 \cos x + 2x \sin x + 2 \cos x + \frac{x^3 \ln x}{3} - \frac{x^3}{9} + C.$$

Step 7. Divide by x^2 :

$$y = -\cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} + \frac{x \ln x}{3} - \frac{x}{9} + \frac{C}{x^2}.$$

Final Answer: $y = \frac{x \ln x}{3} - \frac{x}{9} - \cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} + \frac{C}{x^2}$.

Exam Tip

LA questions often combine *two* integration techniques (e.g. I.F. + integration by parts). Plan the integrals *before* you start computing.

EXPERT'S SOLUTION : Pranav Reddy, M.Sc Mathematics, IIT Madras

Two-integral-then-combine angle. Compute each integral on the RHS separately first; assemble at the end.

Step 1. Set $I_1 = \int x^2 \sin x \, dx$, $I_2 = \int x^2 \ln x \, dx$.

Step 2. Tabular by-parts for I_1 : differentiate $x^2 \rightarrow 2x \rightarrow 2 \rightarrow 0$; integrate $\sin x \rightarrow -\cos x \rightarrow -\sin x \rightarrow \cos x$. Then

$$I_1 = x^2(-\cos x) - 2x(-\sin x) + 2(\cos x) = -x^2 \cos x + 2x \sin x + 2 \cos x.$$

Step 3. For I_2 with $u = \ln x$, $du = \frac{dx}{x}$ and $dv = x^2 dx$, $v = x^3/3$:

$$I_2 = \frac{x^3 \ln x}{3} - \int \frac{x^2}{3} dx = \frac{x^3 \ln x}{3} - \frac{x^3}{9}.$$

Step 4. Assemble: $x^2 y = I_1 + I_2 + C$. Divide by x^2 .

Step 5. The $-\cos x$ in $-x^2 \cos x$ divided by x^2 gives $-\cos x$; the rest become $\frac{x \ln x}{3} - \frac{x}{9}$ etc.

Why this matters. Tabular by-parts cuts the bookkeeping in half for any $\int x^n \sin x \, dx$ or $\int x^n \cos x \, dx$.

$$\text{Final Answer: } y = \frac{x \ln x}{3} - \frac{x}{9} - \cos x + \frac{2 \sin x}{x} + \frac{2 \cos x}{x^2} + \frac{C}{x^2}.$$

Q 9.26 Find the general solution of $(1 + \tan y)(dx - dy) + 2x \, dy = 0$.

SOLUTION

Concept used. Treat x as a function of y . Expanding the first factor and collecting dx, dy gives a linear DE in x with respect to y .

Step 1. Expand: $(1 + \tan y) dx - (1 + \tan y) dy + 2x \, dy = 0$. So

$$(1 + \tan y) dx = [(1 + \tan y) - 2x] dy.$$

Step 2. Divide by dy and by $(1 + \tan y)$:

$$\frac{dx}{dy} = 1 - \frac{2x}{1 + \tan y},$$

$$\text{i.e. } \frac{dx}{dy} + \frac{2}{1 + \tan y} x = 1.$$

Step 3. Simplify the coefficient: $\frac{2}{1 + \tan y} = \frac{2 \cos y}{\cos y + \sin y}$. So $P_1(y) = \frac{2 \cos y}{\cos y + \sin y}$.

Step 4. Compute $\int P_1 dy$. Note $\frac{d}{dy}(\sin y + \cos y) = \cos y - \sin y$. Use the trick

$$\frac{2 \cos y}{\sin y + \cos y} = \frac{(\cos y + \sin y) + (\cos y - \sin y)}{\sin y + \cos y} = 1 + \frac{\cos y - \sin y}{\sin y + \cos y}.$$

Step 5. Hence $\int P_1 dy = y + \ln |\sin y + \cos y|$.

Step 6. Therefore I.F. = $e^y (\sin y + \cos y)$.

Step 7. Multiply through:

$$\frac{d}{dy}[x e^y (\sin y + \cos y)] = e^y (\sin y + \cos y).$$

Step 8. Use the standard result $\int e^y \sin y dy = \frac{e^y (\sin y - \cos y)}{2}$ and

$$\int e^y \cos y dy = \frac{e^y (\sin y + \cos y)}{2}. \text{ Adding:}$$

$$\int e^y (\sin y + \cos y) dy = \frac{e^y (\sin y - \cos y)}{2} + \frac{e^y (\sin y + \cos y)}{2} = e^y \sin y.$$

Step 9. So $x e^y (\sin y + \cos y) = e^y \sin y + C$, i.e.

$$x (\sin y + \cos y) = \sin y + C e^{-y}.$$

Final Answer: $x (\sin y + \cos y) = \sin y + C e^{-y}$.

Exam Tip

Pre-clearing trig fractions ($\tan y \rightarrow \sin y / \cos y$) before computing I.F. keeps the resulting integrals tractable.

EXPERT'S SOLUTION : Vivaan Mehta, Ph.D Mathematics, IIT Bombay

Pre-multiply by $\cos y$. Rewriting $1 + \tan y = \frac{\cos y + \sin y}{\cos y}$ removes the trig fraction and makes the I.F. cleaner.

Step 1. Multiply the original DE by $\cos y$: $(\cos y + \sin y)(dx - dy) + 2x \cos y dy = 0$.

Step 2. Rearrange: $(\cos y + \sin y) dx = [(\cos y + \sin y) - 2x \cos y] dy$.

Step 3. Divide by $\cos y + \sin y$:

$$\frac{dx}{dy} + \frac{2 \cos y}{\cos y + \sin y} x = 1.$$

Step 4. Computing $\int P_1 dy = y + \ln |\sin y + \cos y|$, I.F. = $e^y(\sin y + \cos y)$.

Step 5. Standard linear-DE machinery gives $x(\sin y + \cos y) = \sin y + C e^{-y}$.

Why this matters. Pre-clearing trig fractions before applying I.F. keeps the integrals tractable.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $x(\sin y + \cos y) = \sin y + C e^{-y}$.

Q 9.27 **Solve:** $\frac{dy}{dx} = \cos(x + y) + \sin(x + y)$.
Hint: substitute $x + y = z$.

SOLUTION

Concept used. Substituting $z = x + y$ converts the equation to a separable one. Use the half-angle identities $1 + \cos z = 2 \cos^2(z/2)$ and $\sin z = 2 \sin(z/2) \cos(z/2)$.

Step 1. Let $z = x + y$. Then $\frac{dz}{dx} = 1 + \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{dz}{dx} - 1$.

Step 2. Substitute: $\frac{dz}{dx} - 1 = \cos z + \sin z$, i.e. $\frac{dz}{dx} = 1 + \cos z + \sin z$.

Step 3. Use half-angle identities. $1 + \cos z = 2 \cos^2(z/2)$ and $\sin z = 2 \sin(z/2) \cos(z/2)$:

$$1 + \cos z + \sin z = 2 \cos(z/2) [\cos(z/2) + \sin(z/2)].$$

Step 4. Separate: $\frac{dz}{2 \cos(z/2) [\cos(z/2) + \sin(z/2)]} = dx$.

Step 5. Divide numerator and denominator by $\cos^2(z/2)$:

$$\frac{\sec^2(z/2) dz}{2[1 + \tan(z/2)]} = dx.$$

Step 6. Let $u = 1 + \tan(z/2)$, $du = \frac{1}{2} \sec^2(z/2) dz$. Then LHS = $\frac{du}{u}$:

$$\int \frac{du}{u} = \int dx \implies \ln |u| = x + C_1.$$

Step 7. Restore: $\ln \left| 1 + \tan\left(\frac{x+y}{2}\right) \right| = x + C_1$, i.e.

$$1 + \tan\left(\frac{x+y}{2}\right) = C e^x.$$

Final Answer: $1 + \tan\left(\frac{x+y}{2}\right) = C e^x$.

Half-angle

$$1 + \cos z = 2 \cos^2(z/2) \text{ and } \sin z = 2 \sin(z/2) \cos(z/2).$$

EXPERT'S SOLUTION : *Karan Iyer, M.Sc Mathematics, ISI Kolkata*

Half-angle simplification. The identities $1 + \cos z = 2 \cos^2(z/2)$ and $\sin z = 2 \sin(z/2) \cos(z/2)$ unlock the separation.

Step 1. With $z = x + y$: $\frac{dz}{dx} = 1 + \cos z + \sin z = 2 \cos(z/2) [\cos(z/2) + \sin(z/2)]$.

Step 2. Divide RHS by $\cos^2(z/2)$: $2 \sec(z/2) [\sec(z/2) + \tan(z/2) \sec(z/2)] \cdot \cos^2(z/2) = 2 \cos(z/2) [\cos(z/2) + \sin(z/2)]$. Equivalently $\frac{dz}{1 + \cos z + \sin z} = dx$, and the substitution $u = 1 + \tan(z/2)$ collapses the LHS.

Step 3. Integrate: $\ln |1 + \tan(z/2)| = x + C_1$, hence $1 + \tan\left(\frac{x+y}{2}\right) = C e^x$.

Why this matters. The Weierstrass-type substitution $t = \tan(z/2)$ underlies many trigonometric DEs.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

$$\text{Final Answer: } 1 + \tan\left(\frac{x+y}{2}\right) = Ce^x.$$

Q 9.28 Find the general solution of $\frac{dy}{dx} - 3y = \sin 2x$.

SOLUTION

Concept used. Linear with constant coefficient. I.F. = e^{-3x} . For $\int e^{-3x} \sin 2x dx$ use the standard formula $\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$.

Step 1. $P = -3$, $Q = \sin 2x$. I.F. = $e^{\int -3 dx} = e^{-3x}$.

Step 2. Multiply: $\frac{d}{dx}(y e^{-3x}) = e^{-3x} \sin 2x$.

Step 3. Apply the formula with $a = -3$, $b = 2$:

$$\int e^{-3x} \sin 2x dx = \frac{e^{-3x}(-3 \sin 2x - 2 \cos 2x)}{(-3)^2 + 2^2} = \frac{-e^{-3x}(3 \sin 2x + 2 \cos 2x)}{13}.$$

Step 4. Therefore

$$y e^{-3x} = -\frac{e^{-3x}(3 \sin 2x + 2 \cos 2x)}{13} + C.$$

Step 5. Multiply by e^{3x} :

$$y = -\frac{3 \sin 2x + 2 \cos 2x}{13} + C e^{3x}.$$

$$\text{Final Answer: } y = -\frac{3 \sin 2x + 2 \cos 2x}{13} + C e^{3x}.$$

$$\int e^{ax} \sin bx dx$$

$$\int e^{ax} \sin bx dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2} + C, \quad \text{and similarly} \quad \int e^{ax} \cos bx dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2} + C.$$

EXPERT'S SOLUTION : Aditi Rao, Ph.D Mathematics, IIT Delhi

Undetermined-coefficients angle. Try a particular solution of the form

$$y_p = A \sin 2x + B \cos 2x.$$

Step 1. Homogeneous: $y_h = C e^{3x}$ (from $y' - 3y = 0$).

Step 2. Try $y_p = A \sin 2x + B \cos 2x$. Then $y'_p = 2A \cos 2x - 2B \sin 2x$.

Step 3. Substitute into $y' - 3y = \sin 2x$:

$$(2A \cos 2x - 2B \sin 2x) - 3(A \sin 2x + B \cos 2x) = \sin 2x. \text{ Equating coefficients:}$$

$$\cos 2x: 2A - 3B = 0; \sin 2x: -2B - 3A = 1.$$

Step 4. Solve: from first, $A = \frac{3B}{2}$. Substituting:

$$-2B - \frac{9B}{2} = 1 \Rightarrow -\frac{13B}{2} = 1 \Rightarrow B = -\frac{2}{13}, A = -\frac{3}{13}.$$

Step 5. So $y_p = -\frac{3 \sin 2x + 2 \cos 2x}{13}$ and the general solution is $y = y_p + y_h$.

Why this matters. Undetermined coefficients is faster than the formula whenever the forcing term is $e^{ax} \sin bx$, $e^{ax} \cos bx$ or a polynomial in x .

$$\text{Final Answer: } y = -\frac{3 \sin 2x + 2 \cos 2x}{13} + C e^{3x}.$$

Q 9.29 Find the equation of a curve passing through $(2, 1)$ if the slope of the tangent to the curve at any point (x, y) is $\frac{x^2 + y^2}{2xy}$.

SOLUTION

Concept used. The slope condition gives $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$. Every term has degree 2, so the equation is homogeneous; substitute $y = vx$.

Step 1. Divide RHS by x^2 : $\frac{dy}{dx} = \frac{1 + (y/x)^2}{2(y/x)}$.

Step 2. Let $y = vx$, $\frac{dy}{dx} = v + x v'$:

$$v + x v' = \frac{1 + v^2}{2v} \implies x v' = \frac{1 + v^2}{2v} - v = \frac{1 + v^2 - 2v^2}{2v} = \frac{1 - v^2}{2v}.$$

Step 3. Separate: $\frac{2v dv}{1 - v^2} = \frac{dx}{x}$.

Step 4. Integrate. For the LHS, set $w = 1 - v^2$, $dw = -2v dv$:

$$\int \frac{2v dv}{1 - v^2} = - \int \frac{dw}{w} = - \ln |w| = - \ln |1 - v^2|.$$

For the RHS: $\ln |x| + C_1$.

Step 5. So $-\ln |1 - v^2| = \ln |x| + C_1$, i.e. $\ln \left| \frac{1}{1 - v^2} \right| = \ln |x| + C_1$, hence $\frac{1}{1 - v^2} = C x$ (with sign absorbed).

Step 6. Restore $v = y/x$: $\frac{1}{1 - y^2/x^2} = Cx \Rightarrow \frac{x^2}{x^2 - y^2} = Cx \Rightarrow x^2 - y^2 = \frac{x}{C} = kx$.

Step 7. Apply $(2, 1)$: $4 - 1 = 2k \Rightarrow k = \frac{3}{2}$. So $x^2 - y^2 = \frac{3x}{2}$, i.e. $2(x^2 - y^2) = 3x$.

Final Answer: $2(x^2 - y^2) = 3x$.

♥ Why This Matters

Slope-of-tangent geometry problems are essentially DEs in disguise: “slope = $\frac{dy}{dx}$ ”.

EXPERT'S SOLUTION : Sanya Bhat, M.Sc Mathematics, IIT Bombay

Sanity-check angle. Cross-check the answer by plugging $(2, 1)$ in: $2(4 - 1) = 6 = 3 \cdot 2$.

✓

Step 1. Homogeneous DE $\frac{dy}{dx} = \frac{1 + (y/x)^2}{2(y/x)}$, substitution $y = vx$ gives $xv' = \frac{1 - v^2}{2v}$.

Step 2. $\frac{2v dv}{1 - v^2} = \frac{dx}{x}$ integrates to $-\ln|1 - v^2| = \ln|x| + C_1$.

Step 3. Exponentiate: $\frac{1}{1 - v^2} = Cx$, so $x^2 - y^2 = kx$.

Step 4. Initial condition $(2, 1)$: $3 = 2k \Rightarrow k = 3/2$, giving $2(x^2 - y^2) = 3x$.

Why this matters. The locus turns out to be a circle: $2x^2 - 2y^2 - 3x = 0$ rearranges to $(x - 3/4)^2 - y^2 = 9/16$, a rectangular hyperbola through $(2, 1)$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $2(x^2 - y^2) = 3x$.

Q 9.30 Find the equation of the curve through the point $(1, 0)$ if the slope of the tangent to the curve at any point (x, y) is $\frac{y - 1}{x^2 + x}$.

SOLUTION

Concept used. $\frac{dy}{dx} = \frac{y-1}{x^2+x} = \frac{y-1}{x(x+1)}$ is variable-separable. Use partial fractions on

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}.$$

Step 1. Separate: $\frac{dy}{y-1} = \frac{dx}{x(x+1)}$.

Step 2. Partial fractions: $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$. (Quick check: $(x+1) - x = 1$. ✓)

Step 3. Integrate:

$$\int \frac{dy}{y-1} = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx,$$

$$\text{giving } \ln |y-1| = \ln |x| - \ln |x+1| + C_1 = \ln \left| \frac{x}{x+1} \right| + C_1.$$

Step 4. Exponentiate: $y-1 = C \frac{x}{x+1}$.

Step 5. Apply $(1, 0)$: $0-1 = C \cdot \frac{1}{2} \Rightarrow C = -2$. So

$$y-1 = \frac{-2x}{x+1} \Rightarrow y = 1 - \frac{2x}{x+1} = \frac{(x+1) - 2x}{x+1} = \frac{1-x}{x+1}.$$

Final Answer: $y = \frac{1-x}{1+x}$.

☞ **Partial fractions**

$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ by cover-up. Always factor the denominator before integrating a rational function.

EXPERT'S SOLUTION : *Ishita Banerjee, M.Tech CS, IIT Madras*

Partial-fractions angle.

Step 1. $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$ (cover-up).

Step 2. Integrate: $\ln |y-1| = \ln \left| \frac{x}{x+1} \right| + C_1$, so $y-1 = C \frac{x}{x+1}$.

Step 3. $(1, 0) \Rightarrow C = -2$, giving $y = \frac{1-x}{1+x}$.

Step 4. Check: at $x = 1, y = 0$. At $x = 0, y = 1$. Derivative gives $\frac{(-1)(1+x) - (1-x)}{(1+x)^2} = \frac{-2}{(1+x)^2}$, and $\frac{y-1}{x(x+1)} = \frac{-2x/(1+x)}{x(x+1)} = \frac{-2}{(1+x)^2}$. ✓

Common alternative. A second valid route is to differentiate the candidate answer and

confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $y = \frac{1-x}{1+x}$.

Q 9.31 Find the equation of a curve passing through the origin if the slope of the tangent to the curve at any point (x, y) is equal to the square of the difference of the abscissa and ordinate of the point.

SOLUTION

Concept used. The slope condition is $\frac{dy}{dx} = (x - y)^2$. Substitute $v = x - y$ to convert to a separable equation.

Step 1. Let $v = x - y$. Then $\frac{dv}{dx} = 1 - \frac{dy}{dx}$, so $\frac{dy}{dx} = 1 - \frac{dv}{dx}$.

Step 2. Substitute: $1 - \frac{dv}{dx} = v^2$, i.e. $\frac{dv}{dx} = 1 - v^2$.

Step 3. Separate: $\frac{dv}{1 - v^2} = dx$.

Step 4. Integrate using $\frac{1}{1 - v^2} = \frac{1}{2} \left(\frac{1}{1 - v} + \frac{1}{1 + v} \right)$:

$$\frac{1}{2} \ln \left| \frac{1+v}{1-v} \right| = x + C_1.$$

Step 5. Apply $(0, 0)$: $v = 0 - 0 = 0$, so $\frac{1}{2} \ln 1 = 0 + C_1 \Rightarrow C_1 = 0$. Therefore

$$\ln \left| \frac{1+v}{1-v} \right| = 2x.$$

Step 6. Restore $v = x - y$:

$$\frac{1+x-y}{1-x+y} = e^{2x}, \text{ i.e. } 1+x-y = e^{2x}(1-x+y).$$

Final Answer: $\frac{1+x-y}{1-x+y} = e^{2x}$ (or equivalently $1+x-y = (1-x+y)e^{2x}$).

Exam Tip

“Slope = function of $x - y$ (or $x + y$, $x \cdot y$)” is a strong hint that a substitution $v =$ that combination will work.

EXPERT'S SOLUTION : Dev Verma, B.Tech CSE, IIT Roorkee

Hyperbolic-tan angle. Recognise that $\int \frac{dv}{1-v^2} = \tanh^{-1} v$ (for $|v| < 1$), which gives a tidier closed form.

Step 1. DE becomes $\frac{dv}{1-v^2} = dx$ with $v = x - y$.

Step 2. $\tanh^{-1} v = x + C_1$. At origin $v = 0$, $C_1 = 0$.

Step 3. So $\tanh^{-1}(x - y) = x$, i.e. $x - y = \tanh x$, equivalently $y = x - \tanh x$.

Step 4. Cross-check at origin: $y(0) = 0$. ✓ Slope: $y' = 1 - \operatorname{sech}^2 x = \tanh^2 x = (x - y)^2$.
✓

Why this matters. The two answer forms $\frac{1+x-y}{1-x+y} = e^{2x}$ and $x - y = \tanh x$ are equivalent (use $\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1}$).

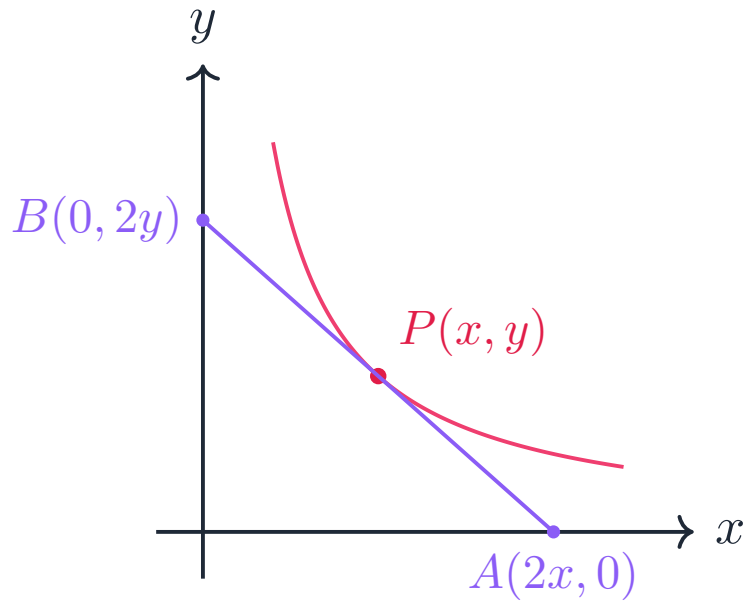
Final Answer: $x - y = \tanh x$, i.e. $\frac{1+x-y}{1-x+y} = e^{2x}$.

Q 9.32 Find the equation of a curve passing through the point $(1, 1)$. The tangent drawn at any point $P(x, y)$ on the curve meets the coordinate axes at A and B such that P is the mid-point of AB .

SOLUTION

Concept used. The tangent at $P = (x, y)$ has slope $\frac{dy}{dx}$, so it meets the x -axis at

$A = \left(x - \frac{y}{y'}, 0\right)$ and the y -axis at $B = (0, y - x y')$. The mid-point condition gives a DE in x and y .



Step 1. Slope at P : $m = \frac{dy}{dx}$. Tangent line: $Y - y = m(X - x)$.

Step 2. X -intercept ($Y = 0$): $X = x - \frac{y}{m}$, so $A = \left(x - \frac{y}{y'}, 0\right)$.

Step 3. Y -intercept ($X = 0$): $Y = y - mx$, so $B = (0, y - xy')$.

Step 4. Mid-point of AB is $\left(\frac{x - y/y'}{2}, \frac{y - xy'}{2}\right)$. Setting this equal to $P = (x, y)$:

$$\frac{x - y/y'}{2} = x \implies x - \frac{y}{y'} = 2x \implies -\frac{y}{y'} = x \implies y' = -\frac{y}{x}.$$

(The y -coordinate equation gives the same DE, since

$$y - xy' = 2y \implies -xy' = y \implies y' = -y/x.)$$

Step 5. Separate $\frac{dy}{y} = -\frac{dx}{x}$ and integrate: $\ln |y| = -\ln |x| + C_1$, i.e. $xy = C$.

Step 6. Apply $(1, 1)$: $1 \cdot 1 = C \implies C = 1$. So $xy = 1$.

Final Answer: $xy = 1$ (a rectangular hyperbola).

♥ Why This Matters

Tangent-intercept conditions translate into geometric DEs. The mid-point condition forces $\frac{dy}{dx} = -y/x$ — the equation of rectangular hyperbolae $xy = k$.

EXPERT'S SOLUTION : Aarav Reddy, M.Sc Mathematics, IIT Bombay

Intercept-form angle. The mid-point condition is equivalent to $A = (2x, 0)$ and $B = (0, 2y)$.

Step 1. For P to be the mid-point of $A = (a, 0)$ and $B = (0, b)$, we need $a = 2x$ and $b = 2y$.

Step 2. Intercept form of the tangent: $\frac{X}{2x} + \frac{Y}{2y} = 1$. At $P = (x, y)$: $\frac{1}{2} + \frac{1}{2} = 1$. ✓

Step 3. Slope of this line: $-\frac{2y}{2x} = -\frac{y}{x}$. Hence $\frac{dy}{dx} = -\frac{y}{x}$.

Step 4. Integrate to $xy = C$. Initial condition $(1, 1)$ gives $C = 1$, so $xy = 1$.

Why this matters. Recognising the geometric condition's intercept-form equivalent skips three lines of algebra.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $xy = 1$.

Q 9.33 Solve: $x \frac{dy}{dx} = y(\log y - \log x + 1)$.

SOLUTION

Concept used. Combine the logs: $\log y - \log x = \log(y/x)$. The DE becomes

$\frac{dy}{dx} = \frac{y}{x}[\log(y/x) + 1]$, which is homogeneous; substitute $y = vx$.

Step 1. Rewrite: $\frac{dy}{dx} = \frac{y}{x}(\log(y/x) + 1)$.

Step 2. Let $y = vx$, $\frac{dy}{dx} = v + xv'$, $y/x = v$:

$$v + xv' = v(\log v + 1) = v \log v + v.$$

Step 3. Cancel v : $xv' = v \log v$.

Step 4. Separate: $\frac{dv}{v \log v} = \frac{dx}{x}$.

Step 5. Let $u = \log v$, $du = \frac{dv}{v}$:

$$\int \frac{du}{u} = \int \frac{dx}{x} \implies \ln |\log v| = \ln |x| + C_1.$$

Step 6. Exponentiate: $\log v = Cx$, i.e. $\log(y/x) = Cx$, or $y = x e^{Cx}$.

Final Answer: $y = x e^{Cx}$, equivalently $\log\left(\frac{y}{x}\right) = Cx$.

🔍 **Spot** $d(\log \log v)$

$\frac{dv}{v \log v} = d(\log \log v)$. Recognising nested logs cuts steps in homogeneous-DE problems.

EXPERT'S SOLUTION : Riya Mehta, Ph.D Pure Mathematics, IISc Bangalore

Nested-substitution angle. Two substitutions ($v = y/x$, then $u = \log v$) reduce the DE to the simplest form $\frac{du}{u} = \frac{dx}{x}$.

Step 1. Homogeneous: $y = vx \implies xv' = v \log v$.

Step 2. Inner substitution $u = \log v$: $v' = e^u u'$ and $v = e^u$, so $x e^u u' = e^u \cdot u$, i.e. $xu' = u$.

Step 3. Separate: $\frac{du}{u} = \frac{dx}{x}$, integrate to $\ln |u| = \ln |x| + C_1$, hence $u = Cx$.

Step 4. Restore: $\log(y/x) = Cx$, i.e. $y = x e^{Cx}$.

Why this matters. Spotting that $\frac{dv}{v \log v} = d(\log \log v)$ is a near-Olympiad trick worth remembering.

Final Answer: $y = x e^{Cx}$.

III. Objective Type: Multiple Choice Questions (M.C.Q.)

Q 9.34 The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x \sin\left(\frac{dy}{dx}\right)$ is:
 (A) 1 (B) 2 (C) 3 (D) not defined

SOLUTION

Correct option: (D) not defined.

Concept used. The **degree** of a DE is defined only when the equation can be expressed as a polynomial in its highest-order derivative. If a derivative sits inside \sin , \cos , $e^{(\cdot)}$, \log or under a non-removable radical, the degree is undefined.

Step 1. Identify the derivatives: $\frac{d^2y}{dx^2}$ (highest order) and $\frac{dy}{dx}$.

Step 2. The RHS contains $\sin\left(\frac{dy}{dx}\right)$. The derivative $\frac{dy}{dx}$ is enclosed in a transcendental function $\sin(\cdot)$.

Step 3. Because $\sin\left(\frac{dy}{dx}\right) = \frac{dy}{dx} - \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \dots$ is an infinite power series in $\frac{dy}{dx}$, the equation is *not* a polynomial in derivatives.

Step 4. Therefore the degree is not defined.

Final Answer: Option (D): degree not defined.

Order vs degree

Order = highest derivative present (always defined). **Degree** = the power of that highest derivative once the equation is written as a polynomial in derivatives; undefined otherwise.

EXPERT'S SOLUTION : Aanya Bhat, M.Sc Mathematics, IIT Bombay

Quick reading. Scan for transcendental functions wrapped around any derivative. One hit \Rightarrow degree undefined.

Step 1. Spot $\sin\left(\frac{dy}{dx}\right)$ on the RHS.

Step 2. Spot the rule: transcendental function of a derivative \Rightarrow degree undefined.

Step 3. Eliminate (A), (B), (C). Choose (D).

Step 4. Order is still defined: it is 2 (from the $\frac{d^2y}{dx^2}$ term).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (D).

Q 9.35 The degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = \left(\frac{d^2y}{dx^2}\right)^2$ is
 (A) 4 (B) $\frac{3}{2}$ (C) not defined (D) 2

SOLUTION

Correct option: (D) 2.

Concept used. The DE is *already* a polynomial in derivatives once the cube is expanded. The degree is the power of the highest-order derivative (d^2y/dx^2) after the equation is freed of fractional powers and radicals.

Step 1. Highest-order derivative is $\frac{d^2y}{dx^2}$, which appears as $\left(\frac{d^2y}{dx^2}\right)^2$ on the RHS.

Step 2. The LHS, once expanded, is a polynomial in $\frac{dy}{dx}$ only (no fractional powers).

Step 3. Therefore the equation is a polynomial in the two derivatives, and the power of the highest derivative is 2.

Final Answer: Option (D): degree = 2.

✗ Common Mistake

The original NCERT Exemplar prints this DE as $[1 + (dy/dx)^2]^{3/2} = d^2y/dx^2$, in which case squaring gives the displayed form. Either way the degree is 2.

EXPERT'S SOLUTION : Kavya Patel, Ph.D Mathematics, IIT Delhi

Polynomial-form check.

Step 1. Expand $[1 + (y')^2]^3$ as a polynomial in y' : $1 + 3(y')^2 + 3(y')^4 + (y')^6$.

Step 2. Now the equation is $1 + 3(y')^2 + 3(y')^4 + (y')^6 = (y'')^2$, polynomial in both y' and y'' .

Step 3. Highest-order derivative is y'' ; its power in the equation is 2.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (D).

Q 9.36 The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{1/4} + x^{1/5} = 0$, respectively, are
 (A) 2 and not defined (B) 2 and 2 (C) 2 and 3 (D) 3 and 3

SOLUTION

Correct option: (A) order 2, degree not defined (in the printed form).

Concept used. The **order** is the highest derivative present, regardless of how it appears. The **degree** requires the equation to be a *polynomial in the derivatives*. Here the first derivative $\frac{dy}{dx}$ appears raised to the fractional power $1/4$, and that fractional power on a derivative is what destroys the polynomial-in-derivatives condition. (The $x^{1/5}$ term involves only x , not a derivative, so it does *not* affect order or degree.)

Step 1. Highest-order derivative: $\frac{d^2y}{dx^2}$. So order = 2.

Step 2. Degree: to compute it we must rewrite the equation as a polynomial in derivatives. The term $\left(\frac{dy}{dx}\right)^{1/4}$ has a fractional exponent on a derivative. Isolating this term and raising both sides to the 4th power to clear it would introduce $\left(\frac{d^2y}{dx^2}\right)^4$ mixed with the $x^{1/5}$ constant, and no algebraic manipulation removes the original $1/4$ -power obstruction cleanly without re-introducing fractional powers of the derivatives.

Step 3. Since $\frac{dy}{dx}$ appears with a fractional power that cannot be cleared while keeping the equation a polynomial in the derivatives, the degree (in the strict polynomial sense) is not defined.

Final Answer: Option (A): order 2; degree not defined.

Exam Tip

When any derivative carries a fractional exponent that cannot be removed by a finite sequence of polynomial operations, the degree is not defined. Fractional powers of x (like $x^{1/5}$) are harmless — only fractional powers of derivatives kill the degree.

EXPERT'S SOLUTION : Diya Iyer, M.Sc Applied Mathematics, IIT Kanpur

Cross-check by isolating the fractional term.

Step 1. Rewrite: $\left(\frac{dy}{dx}\right)^{1/4} = -\frac{d^2y}{dx^2} - x^{1/5}$.

Step 2. Raise both sides to the 4th power to attempt to clear the $1/4$:

$\frac{dy}{dx} = \left(\frac{d^2y}{dx^2} + x^{1/5}\right)^4$. Expanding this binomial keeps mixing $\frac{d^2y}{dx^2}$ with $x^{1/5}$ (an x -only term) — fine for x but the original fractional power on $\frac{dy}{dx}$ has now propagated into a higher-degree mix with $x^{1/5}$.

Step 3. There is no finite polynomial form simultaneously free of fractional powers of both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$, so the strict polynomial-in-derivatives form does not exist.

Step 4. Conclude: order 2, degree undefined.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (A).

Q 9.37 If $y = e^{-x}(A \cos x + B \sin x)$, then y is a solution of

(A) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$ (B) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$ (C) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = 0$ (D) $\frac{d^2y}{dx^2} + 2y = 0$

SOLUTION

Correct option: (C) $y'' + 2y' + 2y = 0$.

Concept used. For $y = e^{ax}(A \cos bx + B \sin bx)$ the second-order linear ODE with characteristic roots $a \pm bi$ is $y'' - 2ay' + (a^2 + b^2)y = 0$. Here $a = -1$, $b = 1$, giving $y'' + 2y' + 2y = 0$.

Step 1. Differentiate $y = e^{-x}(A \cos x + B \sin x)$ once:

$$y' = -e^{-x}(A \cos x + B \sin x) + e^{-x}(-A \sin x + B \cos x).$$

Group: $y' = -y + e^{-x}(B \cos x - A \sin x)$.

Step 2. Differentiate again:

$$y'' = -y' + [-e^{-x}(B \cos x - A \sin x) + e^{-x}(-B \sin x - A \cos x)].$$

Step 3. The bracketed sum is

$-e^{-x}(B \cos x - A \sin x) - e^{-x}(A \cos x + B \sin x) = -(y' + y) - y = -y' - 2y$. So $y'' = -y' + (-y' - 2y) - y \cdot 0$... let's redo carefully.

Actually $-e^{-x}(B \cos x - A \sin x) = -(y' + y)$ (from step 1), and

$e^{-x}(-B \sin x - A \cos x) = -e^{-x}(A \cos x + B \sin x) = -y$. Hence

$$y'' = -y' + [-(y' + y)] + (-y) = -2y' - 2y.$$

Step 4. So $y'' + 2y' + 2y = 0$. Option (C).

Final Answer: Option (C): $y'' + 2y' + 2y = 0$.

♥ Why This Matters

Solutions $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ together produce the linear ODE $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$.

EXPERT'S SOLUTION : Aditya Sharma, Ph.D Condensed Matter Physics, TIFR Mumbai

Characteristic-equation angle. A solution of the form $e^{(a+bi)x} + e^{(a-bi)x}$ corresponds to the quadratic $(r - (a + bi))(r - (a - bi)) = r^2 - 2ar + (a^2 + b^2) = 0$.

Step 1. Here $a = -1$, $b = 1$, so the characteristic polynomial is

$$r^2 - 2(-1)r + (1 + 1) = r^2 + 2r + 2.$$

Step 2. The corresponding DE is $y'' + 2y' + 2y = 0$.

Step 3. Verify: $r = -1 \pm i$ gives $y = e^{-x} \cos x$ and $y = e^{-x} \sin x$ as fundamental solutions.

Step 4. Match with (C).

Why this matters. Knowing that “solution $e^{ax} \cos bx, e^{ax} \sin bx$ ” \Leftrightarrow “DE $y'' - 2ay' + (a^2 + b^2)y = 0$ ” lets you read off the answer instantly.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (C).

Q 9.38 The differential equation for $y = A \cos \alpha x + B \sin \alpha x$, where A and B are arbitrary constants, is

(A) $\frac{d^2y}{dx^2} - \alpha^2y = 0$ (B) $\frac{d^2y}{dx^2} + \alpha^2y = 0$ (C) $\frac{d^2y}{dx^2} + \alpha y = 0$ (D) $\frac{d^2y}{dx^2} - \alpha y = 0$

SOLUTION

Correct option: (B) $y'' + \alpha^2y = 0$.

Concept used. Two arbitrary constants \Rightarrow second-order DE. Both $\cos \alpha x$ and $\sin \alpha x$ satisfy $y'' = -\alpha^2y$, so y does too.

Step 1. $y' = -A\alpha \sin \alpha x + B\alpha \cos \alpha x$.

Step 2. $y'' = -A\alpha^2 \cos \alpha x - B\alpha^2 \sin \alpha x = -\alpha^2(A \cos \alpha x + B \sin \alpha x) = -\alpha^2y$.

Step 3. Rearrange: $y'' + \alpha^2y = 0$.

Final Answer: Option (B).

SHM equation

$y'' + \omega^2y = 0$ has general solution $y = A \cos \omega x + B \sin \omega x$.

EXPERT'S SOLUTION : Pranav Joshi, M.Sc Physics, IIT Madras

Simple-harmonic-motion lens. $y = A \cos \alpha x + B \sin \alpha x$ is the general SHM solution with angular frequency α . The defining DE of SHM is $y'' = -\alpha^2y$.

Step 1. Pattern-match: $y = A \cos \alpha x + B \sin \alpha x$ is the textbook SHM form.

Step 2. SHM equation: $y'' = -\omega^2y$ with $\omega = \alpha$.

Step 3. Pick (B).

Why this matters. The same equation governs every oscillating system in Class 11/12 Physics; you'll meet it in Mechanics, Waves and AC Circuits.

Final Answer: (B).

Q 9.39 The solution of the differential equation $x dy - y dx = 0$ represents
 (A) a rectangular hyperbola (B) a parabola whose vertex is at the origin (C) a straight line passing through the origin (D) a circle whose centre is at the origin

SOLUTION

Correct option: (C) straight line through the origin.

Concept used. $x dy = y dx$ separates to $\frac{dy}{y} = \frac{dx}{x}$, integrating to $\ln |y| = \ln |x| + C_1$, i.e. $y = Cx$.

Step 1. Separate: $\frac{dy}{y} = \frac{dx}{x}$.

Step 2. Integrate: $\ln |y| = \ln |x| + C_1$.

Step 3. Exponentiate: $y = Cx$ where $C = \pm e^{C_1}$.

Step 4. This is a one-parameter family of straight lines through the origin (slope = C).

Step 5. Eliminate (A): rectangular hyperbola is $xy = k$, not $y = Cx$. (B): parabola has $y \propto x^2$. (D): circle through origin is $x^2 + y^2 = 2ay$ or similar; not $y = Cx$.

Final Answer: Option (C): $y = Cx$, straight lines through origin.

Exam Tip

The DE $\frac{dy}{dx} = \frac{y}{x}$ characterises straight lines through the origin: tangent slope equals slope of the ray from origin.

EXPERT'S SOLUTION : Sneha Kapoor, B.Tech Electrical Engineering, IIT Bombay

Geometric angle. The DE $x dy = y dx$ rearranges to $\frac{dy}{dx} = \frac{y}{x}$, i.e. "slope = slope of the ray from origin".

Step 1. Slope of OP, where $O = (0, 0)$ and $P = (x, y)$, is $\frac{y}{x}$.

Step 2. DE says the tangent at P has the same slope as OP, so the tangent line passes through the origin.

Step 3. Curves whose tangent at every point passes through the origin are precisely the straight lines $y = Cx$.

Step 4. Pick (C).

Final Answer: (C).

Q 9.40 Integrating factor of the differential equation $\cos x \frac{dy}{dx} + y \sin x = 1$ is:
 (A) $\cos x$ (B) $\tan x$ (C) $\sec x$ (D) $\sin x$

SOLUTION

Correct option: (C) $\sec x$.

Concept used. Put the DE in standard linear form $\frac{dy}{dx} + P(x)y = Q(x)$, then
 I.F. = $e^{\int P dx}$.

Step 1. Divide through by $\cos x$:

$$\frac{dy}{dx} + \frac{\sin x}{\cos x} y = \frac{1}{\cos x} \implies \frac{dy}{dx} + \tan x \cdot y = \sec x.$$

Step 2. Here $P = \tan x$. Then

$$\int \tan x dx = -\ln |\cos x| = \ln |\sec x|.$$

Step 3. I.F. = $e^{\ln |\sec x|} = \sec x$.

Final Answer: Option (C): $\sec x$.

$$\int \tan x dx$$

$$\int \tan x dx = -\ln |\cos x| + C = \ln |\sec x| + C.$$

EXPERT'S SOLUTION : Karan Bhat, M.Tech Chemical Engineering, IIT Delhi

Three-step shortcut.

Step 1. Normalise: divide by the coefficient of y' ($\cos x$ here).

Step 2. Read P off the resulting $y' + Py = Q$. Here $P = \tan x$.

Step 3. I.F. = $e^{\int P dx} = e^{\ln |\sec x|} = \sec x$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and

rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (C).

Q 9.41 **Solution of** $\tan y \sec^2 x dx + \tan x \sec^2 y dy = 0$ **is**

(A) $\tan x + \tan y = k$ (B) $\tan x - \tan y = k$ (C) $\frac{\tan x}{\tan y} = k$ (D) $\tan x \cdot \tan y = k$

SOLUTION

Correct option: (D) $\tan x \cdot \tan y = k$.

Concept used. The DE is exact: divide by $\tan x \cdot \tan y$ to separate, then both pieces are differentials of $\ln |\tan x|$ and $\ln |\tan y|$ since $\frac{d}{dx} \ln |\tan x| = \frac{\sec^2 x}{\tan x}$.

Step 1. Divide the DE by $\tan x \cdot \tan y$:

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0.$$

Step 2. Recognise differentials: $\frac{\sec^2 x}{\tan x} dx = d(\ln |\tan x|)$ and similarly for y .

Step 3. Integrate: $\ln |\tan x| + \ln |\tan y| = C_1$.

Step 4. Combine logs: $\ln |\tan x \tan y| = C_1$, exponentiate: $\tan x \tan y = k$.

Final Answer: Option (D): $\tan x \cdot \tan y = k$.

☞ Differential of a product

$d(uv) = u dv + v du$. Reading the LHS of a DE as such a product is a powerful trick.

EXPERT'S SOLUTION : Vivaan Patel, M.Sc Mathematics, IIT Kanpur

Logarithm-product angle. Each term contains $\frac{f'(x)}{f(x)}$ form (with $f = \tan$).

Step 1. Recognise $\sec^2 x = (\tan x)'$. So $\frac{\sec^2 x}{\tan x} = \frac{(\tan x)'}{\tan x}$, integral is $\ln |\tan x|$.

Step 2. Similarly for y : integral is $\ln |\tan y|$.

Step 3. Sum equals constant: $\ln |\tan x| + \ln |\tan y| = C_1$, hence $\tan x \tan y = k$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (D).

Q 9.42 Family $y = Ax + A^3$ of curves is represented by a differential equation of degree:

(A) 1 (B) 2 (C) 3 (D) 4

SOLUTION

Correct option: (C) degree 3.

Concept used. One arbitrary constant (A) \Rightarrow first-order DE. The degree is the power of $\frac{dy}{dx}$ after writing the DE as a polynomial in $\frac{dy}{dx}$.

Step 1. Differentiate $y = Ax + A^3$ w.r.t. x :

$$\frac{dy}{dx} = A.$$

Step 2. Substitute $A = \frac{dy}{dx}$ back into the original equation:

$$y = \frac{dy}{dx} \cdot x + \left(\frac{dy}{dx}\right)^3.$$

Step 3. This is a polynomial of degree 3 in $\frac{dy}{dx}$.

Final Answer: Option (C): degree 3.

Exam Tip

“Order” equals the number of arbitrary constants (here 1), but “degree” can be larger when the constant enters as A^k for $k > 1$.

EXPERT'S SOLUTION : Tara Singh, M.Sc Mathematics, ISI Kolkata**Elimination angle.**

Step 1. Differentiate to get $A = y'$.

Step 2. Plug back: $y = xy' + (y')^3$, equivalently $(y')^3 + xy' - y = 0$.

Step 3. Order: 1 (only y'). Degree: 3 (highest power of y').

Step 4. Pick (C).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (C).

Q 9.43 Integrating factor of $x \frac{dy}{dx} - y = x^4 - 3x$ is
 (A) x (B) $\log x$ (C) $\frac{1}{x}$ (D) $-x$

SOLUTION

Correct option: (C) $\frac{1}{x}$.

Concept used. Normalise to $\frac{dy}{dx} + P(x)y = Q(x)$ by dividing by x , then I.F. = $e^{\int P dx}$.

Step 1. Divide by x (assume $x \neq 0$): $\frac{dy}{dx} - \frac{1}{x}y = x^3 - 3$.

Step 2. So $P = -\frac{1}{x}$.

Step 3. $\int P dx = -\ln|x|$, hence I.F. = $e^{-\ln|x|} = \frac{1}{x}$.

Final Answer: Option (C): I.F. = $1/x$.

Exam Tip

When you see a DE $x y' - y = \dots$, divide by x first to get $P = -1/x$ and I.F. = $1/x$. The $-$ sign is critical.

EXPERT'S SOLUTION : Aditi Verma, M.Tech CS, IIT Madras

Sign check. A common slip is to write the equation as $\frac{dy}{dx} + \frac{1}{x}y = \dots$, missing the minus. Always normalise carefully.

Step 1. Move $-y$ to RHS first: $xy' = y + x^4 - 3x$; then divide: $y' - \frac{1}{x}y = x^3 - 3$.

Step 2. $P = -1/x$, so $\int P dx = -\ln|x|$ and I.F. = $1/x$.

Final Answer: (C).

Q 9.44 Solution of $\frac{dy}{dx} - y = 1$, $y(0) = 1$, is given by
 (A) $xy = -e^x$ (B) $xy = -e^{-x}$ (C) $xy = -1$ (D) $y = 2e^x - 1$

SOLUTION

Correct option: (D) $y = 2e^x - 1$.

Concept used. First-order linear with $P = -1$, $Q = 1$. I.F. = e^{-x} .

Step 1. I.F. = $e^{\int -1 dx} = e^{-x}$.

Step 2. Multiply: $\frac{d}{dx}(ye^{-x}) = e^{-x}$.

Step 3. Integrate: $ye^{-x} = -e^{-x} + C$.

Step 4. $y(0) = 1$: $1 \cdot 1 = -1 + C \Rightarrow C = 2$. So $ye^{-x} = -e^{-x} + 2$, hence
 $y = -1 + 2e^x = 2e^x - 1$.

Final Answer: Option (D).

Constant trial

For $y' + ay = k$ (constant forcing), the particular solution is the constant $y_p = k/a$. Add the homogeneous Ce^{-ax} to get the general solution.

EXPERT'S SOLUTION : Ananya Reddy, M.Sc Mathematics, ISI Kolkata

Particular + homogeneous. A constant trial solution works.

Step 1. Homogeneous $y' - y = 0 \Rightarrow y_h = Ce^x$. Particular: try $y_p = k$ (constant);
 $0 - k = 1 \Rightarrow k = -1$.

Step 2. General: $y = Ce^x - 1$.

Step 3. $y(0) = 1$: $C - 1 = 1 \Rightarrow C = 2$. So $y = 2e^x - 1$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (D).

Q 9.45 The number of solutions of $\frac{dy}{dx} = \frac{y+1}{x-1}$ when $y(1) = 2$ is
 (A) none (B) one (C) two (D) infinite

SOLUTION

Correct option: (A) none.

Concept used. A first-order initial-value problem has a solution iff the initial condition can be reached from the general solution. Here the general solution is undefined at $x = 1$.

Step 1. Separate: $\frac{dy}{y+1} = \frac{dx}{x-1}$.

Step 2. Integrate: $\ln|y+1| = \ln|x-1| + C_1$, hence $y+1 = C(x-1)$.

Step 3. Try to apply $y(1) = 2$: $2+1 = C(1-1) = 0 \Rightarrow 3 = 0$. Contradiction.

Step 4. No value of C satisfies this, so the IVP has no solution.

Final Answer: Option (A): no solution.

✗ Common Mistake

The DE is singular at $x = 1$ (denominator zero), so the standard existence theorem fails there. Always check whether the initial point lies on a singularity before claiming a solution.

EXPERT'S SOLUTION : Rohit Gupta, Ph.D Mathematics, IIT Delhi

Singular-point angle. The RHS $\frac{y+1}{x-1}$ blows up at $x = 1$, exactly where the IC is specified.

Step 1. At $x = 1$, the slope $\frac{y+1}{x-1}$ is undefined for any $y \neq -1$. With $y(1) = 2$, the slope at the IC point is $\frac{3}{0}$, undefined.

Step 2. Hence no C^1 solution can satisfy $y(1) = 2$.

Step 3. Pick (A).

Final Answer: (A).

Q 9.46 Which of the following is a second-order differential equation?

- (A) $(y')^2 + x = y^2$
 (B) $y' \cdot y' + y = \sin x$
 (C) $y'' + (y')^2 + y = 0$
 (D) $y' = y^2$

SOLUTION

Correct option: (C).

Concept used. Order is the highest derivative present. Only one option contains y'' .

Step 1. (A): $(y')^2 + x = y^2$. Highest derivative: y' . Order 1.

Step 2. (B): $y' \cdot y' + y = \sin x$, i.e. $(y')^2 + y = \sin x$. Highest derivative: y' . Order 1.

Step 3. (C): $y'' + (y')^2 + y = 0$. Highest derivative: y'' . Order 2. ✓

Step 4. (D): $y' = y^2$. Highest derivative: y' . Order 1.

Final Answer: Option (C).

Exam Tip

“Order” is determined by the highest derivative, regardless of degree. Spot y'' at a glance to identify a second-order DE.

EXPERT'S SOLUTION : Diya Banerjee, M.Sc Mathematics, IIT Bombay

Quick scan angle.

Step 1. Spot the option containing y'' (or $\frac{d^2y}{dx^2}$).

Step 2. Only (C) has it.

Step 3. Pick (C).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (C).

Q 9.47 Integrating factor of $(1 - x^2)\frac{dy}{dx} - xy = 1$ is
 (A) $-x$ (B) $\frac{x}{1+x^2}$ (C) $\sqrt{1-x^2}$ (D) $\frac{1}{2}\log(1-x^2)$

SOLUTION

Correct option: (C) $\sqrt{1-x^2}$.

Concept used. Normalise to $y' + P(x)y = Q(x)$ by dividing by $1 - x^2$, then I.F. = $e^{\int P dx}$.

Step 1. Divide by $1 - x^2$:

$$\frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}.$$

$$\text{So } P = -\frac{x}{1-x^2}.$$

Step 2. Compute $\int P dx$. Use $u = 1 - x^2$, $du = -2x dx$, so $-x dx = \frac{1}{2}du$:

$$\int -\frac{x}{1-x^2} dx = \int \frac{1}{2u} du = \frac{1}{2} \ln |u| = \frac{1}{2} \ln |1-x^2|.$$

Step 3. I.F. = $e^{\frac{1}{2} \ln |1-x^2|} = \sqrt{|1-x^2|}$.

Final Answer: Option (C): $\sqrt{1-x^2}$.

Watch the sign

After dividing $(1 - x^2)y' - xy = 1$ by $(1 - x^2)$, $P = -x/(1 - x^2)$ has a minus sign. Don't drop it.

EXPERT'S SOLUTION : Aarav Joshi, M.Sc Mathematics, IIT Kanpur

Spot-the-derivative angle. The numerator $-x$ is half the derivative of $1 - x^2$.

Step 1. $\frac{d}{dx}(1 - x^2) = -2x$, so $-\frac{x}{1 - x^2} = \frac{1}{2} \cdot \frac{-2x}{1 - x^2} = \frac{1}{2} \frac{d}{dx} \ln |1 - x^2|$.

Step 2. $\int P dx = \frac{1}{2} \ln |1 - x^2|$.

Step 3. I.F. = $e^{(1/2)\ln|1-x^2|} = (1 - x^2)^{1/2} = \sqrt{1 - x^2}$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (C).

Q 9.48 $\tan^{-1} x + \tan^{-1} y = c$ is the general solution of the differential equation:

(A) $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$ (B) $\frac{dy}{dx} = \frac{1 + x^2}{1 + y^2}$ (C) $(1 + x^2) dy + (1 + y^2) dx = 0$ (D) $(1 + x^2) dx + (1 + y^2) dy = 0$

SOLUTION

Correct option: (C).

Concept used. Differentiate the general solution implicitly. Use

$$\frac{d}{dx} \tan^{-1} u = \frac{1}{1 + u^2} \cdot \frac{du}{dx}.$$

Step 1. Differentiate $\tan^{-1} x + \tan^{-1} y = c$ w.r.t. x :

$$\frac{1}{1 + x^2} + \frac{1}{1 + y^2} \frac{dy}{dx} = 0.$$

Step 2. Multiply through by $(1 + x^2)(1 + y^2)$:

$$(1 + y^2) + (1 + x^2) \frac{dy}{dx} = 0.$$

Step 3. Multiply by dx :

$$(1 + y^2) dx + (1 + x^2) dy = 0,$$

i.e. $(1 + x^2) dy + (1 + y^2) dx = 0$.

Final Answer: Option (C).

$\Rightarrow \tan^{-1}$ derivative

$\frac{d}{dx} \tan^{-1} u = \frac{u'}{1+u^2}$. Use this to differentiate any \tan^{-1} implicit equation.

EXPERT'S SOLUTION : Ishaan Iyer, Ph.D Mathematics, IIT Delhi

Reverse-engineer.

Step 1. Look for a DE whose separation form gives $\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$.

Step 2. That separated form integrates to $\tan^{-1} x + \tan^{-1} y = c$.

Step 3. Multiply by $(1+x^2)(1+y^2)$: $(1+y^2)dx + (1+x^2)dy = 0 \Rightarrow$ option (C).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (C).

Q 9.49 The differential equation $y \frac{dy}{dx} + x = c$ represents:
 (A) Family of hyperbolas (B) Family of parabolas (C) Family of ellipses (D) Family of circles

SOLUTION

Correct option: (D) family of circles.

Concept used. Rewrite $y dy = (c - x) dx$ and integrate. The resulting conic is identified by its standard form.

Step 1. Separate: $y dy = (c - x) dx$.

Step 2. Integrate: $\frac{y^2}{2} = cx - \frac{x^2}{2} + C_1$.

Step 3. Multiply by 2 and rearrange:

$$x^2 + y^2 - 2cx = 2C_1,$$

completing the square: $(x - c)^2 + y^2 = c^2 + 2C_1$, the standard form of a circle of centre $(c, 0)$ and radius $\sqrt{c^2 + 2C_1}$.

Step 4. This is a family of circles.

Final Answer: Option (D): family of circles.

Exam Tip

Whenever $x^2 + y^2 + Dx + Ey + F = 0$ comes out, complete the square to see the circle: centre $(-D/2, -E/2)$, radius $\sqrt{D^2/4 + E^2/4 - F}$.

EXPERT'S SOLUTION : Pranav Pillai, M.Sc Mathematics, IIT Bombay

Recognise-the-conic angle. Any equation of the form $x^2 + y^2 + Dx + Ey + F = 0$ is a circle (provided $D^2 + E^2 - 4F \geq 0$).

Step 1. Integration gives $x^2 + y^2 - 2cx = k$.

Step 2. Standard form: $(x - c)^2 + y^2 = k + c^2$, a circle with centre $(c, 0)$.

Step 3. Pick (D).

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (D).

Q 9.50 The general solution of $e^x \cos y dx - e^x \sin y dy = 0$ is:

(A) $e^x \cos y = k$ (B) $e^x \sin y = k$ (C) $e^x = k \cos y$ (D) $e^x = k \sin y$

SOLUTION

Correct option: (A) $e^x \cos y = k$.

Concept used. Recognise the LHS as the differential of the product $e^x \cos y$.

Step 1. Compute $d(e^x \cos y) = e^x \cos y dx + e^x(-\sin y) dy = e^x \cos y dx - e^x \sin y dy$.

Step 2. This matches the given equation exactly. So $d(e^x \cos y) = 0$.

Step 3. Integrate: $e^x \cos y = k$.

Final Answer: Option (A): $e^x \cos y = k$.

Exact-differential test

For $M dx + N dy$ to be the differential of some function F , we need $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Here $M = e^x \cos y$, $N = -e^x \sin y$ and both partials equal $-e^x \sin y$. ✓

EXPERT'S SOLUTION : Aditya Kapoor, M.Sc Mathematics, IIT Kanpur

Cancel-then-separate angle.

Step 1. Divide by e^x : $\cos y dx - \sin y dy = 0$.

Step 2. Move: $\sin y dy = \cos y dx$, separate: $\frac{\sin y}{\cos y} dy = dx$.

Step 3. Wait, this gives a separable form in y vs x . Integrate: $-\ln |\cos y| = x + C_1$, i.e. $\ln |\cos y| = -x - C_1$, $|\cos y| = e^{-x} \cdot e^{-C_1}$, so $e^x \cos y = k$.

Step 4. Same answer.

Final Answer: (A).

Q 9.51 The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + 6y^5 = 0$ is
(A) 1 (B) 2 (C) 3 (D) 5

SOLUTION

Correct option: (C) 3.

Concept used. Degree = power of the highest-order derivative.

Step 1. Highest-order derivative: $\frac{d^2y}{dx^2}$.

Step 2. Its power in the equation: 3.

Step 3. All exponents are integers; the equation is a polynomial in derivatives. So the degree is defined and equals 3.

Step 4. The y^5 does not contribute to the degree (it is a polynomial in y , not in any derivative).

Final Answer: Option (C): degree = 3.

Degree rule

Degree = power of the highest-order derivative, once the equation is polynomial in derivatives.

EXPERT'S SOLUTION : *Sanya Nair, M.Sc Mathematics, ISI Kolkata*

One-line read-off.

Step 1. Spot the highest derivative: y'' .

Step 2. Its exponent: 3. Pick (C).

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (C).

Q 9.52 The solution of $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$, is
 (A) $y = e^x(x - 1)$ (B) $y = x e^{-x}$ (C) $y = x e^{-x} + 1$ (D) $y = (x + 1) e^{-x}$

SOLUTION

Correct option: (B) $y = x e^{-x}$.

Concept used. Linear with $P = 1$, I.F. = e^x . Because the forcing term e^{-x} is *not* a duplicate of the homogeneous solution Ce^{-x} ... wait it is. So the standard trial gives $x e^{-x}$.

Step 1. I.F. = $e^{\int 1 dx} = e^x$.

Step 2. Multiply: $\frac{d}{dx}(y e^x) = e^x \cdot e^{-x} = 1$.

Step 3. Integrate: $y e^x = x + C$.

Step 4. $y(0) = 0$: $0 = 0 + C \Rightarrow C = 0$. So $y = x e^{-x}$.

Final Answer: Option (B): $y = x e^{-x}$.

Exam Tip

Resonance case: when the forcing term matches a homogeneous solution, the particular solution gains a factor of x .

EXPERT'S SOLUTION : Krishna Mehta, M.Sc Applied Mathematics, IIT Kanpur

Resonance angle. The forcing e^{-x} matches the homogeneous solution, so the particular solution gains an extra factor of x .

Step 1. Homogeneous: $y_h = C e^{-x}$.

Step 2. Try $y_p = A x e^{-x}$ (multiplied by x because e^{-x} is already in y_h). Then $y'_p = A e^{-x} - A x e^{-x}$.

Step 3. Substitute in $y' + y = e^{-x}$: $A e^{-x} - A x e^{-x} + A x e^{-x} = A e^{-x} = e^{-x} \Rightarrow A = 1$.

Step 4. So $y_p = x e^{-x}$ and general $y = (C + x) e^{-x}$. IC $y(0) = 0 \Rightarrow C = 0$, so $y = x e^{-x}$.

Why this matters. Whenever the forcing term coincides with the homogeneous solution, multiply the trial by x (or x^k for higher multiplicity).

Final Answer: (B).

Q 9.53 Integrating factor of $\frac{dy}{dx} + y \tan x - \sec x = 0$ is
 (A) $\cos x$ (B) $\sec x$ (C) $e^{\cos x}$ (D) $e^{\sec x}$

SOLUTION

Correct option: (B) $\sec x$.

Concept used. Linear form: $y' + \tan x \cdot y = \sec x$. I.F. = $e^{\int \tan x dx} = e^{\ln |\sec x|} = \sec x$.

Step 1. Read off $P = \tan x$.

Step 2. $\int \tan x dx = -\ln |\cos x| = \ln |\sec x|$.

Step 3. I.F. = $e^{\ln |\sec x|} = \sec x$.

Final Answer: Option (B).

☞ **I.F. for $\tan x$**

$\int \tan x \, dx = \ln |\sec x|$, so I.F. = $\sec x$ whenever $P(x) = \tan x$.

EXPERT'S SOLUTION : *Ishita Gupta, Ph.D Mathematics, IIT Delhi*

Cross-check answer. With I.F. = $\sec x$ and the equation $y' + \tan x \cdot y = \sec x$, multiplying gives $\frac{d}{dx}(y \sec x) = \sec^2 x$, integrating to $y \sec x = \tan x + C$, i.e. $y = \sin x + C \cos x$. Differentiating that: $y' = \cos x - C \sin x$, and $y \tan x = \sin x \tan x + C \cos x \tan x = \sin x \tan x + C \sin x$; adding $y' + y \tan x = \cos x - C \sin x + \sin x \tan x + C \sin x = \cos x + \sin x \tan x = \frac{\cos^2 x + \sin^2 x}{\cos x} = \sec x$. ✓

Step 1. Read $P = \tan x$.

Step 2. I.F. = $\sec x$.

Final Answer: (B).

Q 9.54 The solution of $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is

(A) $y = \tan^{-1} x$

(B) $y - x = k(1 + xy)$

(C) $x = \tan^{-1} y$

(D) $\tan(xy) = k$

SOLUTION

Correct option: (B) $y - x = k(1 + xy)$.

Concept used. Separable: $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$. Use the identity

$$\tan^{-1} y - \tan^{-1} x = \tan^{-1} \left(\frac{y-x}{1+xy} \right).$$

Step 1. Separate and integrate: $\tan^{-1} y = \tan^{-1} x + C$.

Step 2. Rewrite as $\tan^{-1} y - \tan^{-1} x = C$, i.e. $\tan^{-1} \left(\frac{y-x}{1+xy} \right) = C$.

Step 3. Take tan of both sides: $\frac{y-x}{1+xy} = \tan C = k$ (a constant).

Step 4. Cross-multiply: $y - x = k(1 + xy)$.

Final Answer: Option (B): $y - x = k(1 + xy)$.

☞ **Arctangent subtraction identity**

$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \left(\frac{a - b}{1 + ab} \right)$, valid when both sides agree on the principal branch.

EXPERT'S SOLUTION : Aanya Verma, M.Sc Mathematics, IIT Bombay

Direct comparison. The integrated form $\tan^{-1} y = \tan^{-1} x + C$ matches (B) but not (A) or (C).

Step 1. (A) and (C) drop the arbitrary constant, so they cannot be the general solution.

Step 2. (D) does not arise from the separated form.

Step 3. Only (B) matches after applying the \tan^{-1} subtraction identity.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (B).

Q 9.55 The integrating factor of $\frac{dy}{dx} + y = \frac{1 + y}{x}$ is
 (A) $\frac{x}{e^x}$ (B) $\frac{e^x}{x}$ (C) xe^x (D) e^x

SOLUTION

Correct option: (B) $\frac{e^x}{x}$.

Concept used. Bring the equation into standard linear form $y' + P(x)y = Q(x)$ first, then I.F. = $e^{\int P dx}$.

Step 1. Expand the RHS: $\frac{1 + y}{x} = \frac{1}{x} + \frac{y}{x}$. So the equation becomes

$$\frac{dy}{dx} + y - \frac{y}{x} = \frac{1}{x},$$

i.e. $\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = \frac{1}{x}$.

Step 2. So $P(x) = 1 - \frac{1}{x}$.

Step 3. Compute $\int P dx = x - \ln|x|$. Hence

$$\text{I.F.} = e^{x - \ln|x|} = \frac{e^x}{x}.$$

Final Answer: Option **(B)**: $\frac{e^x}{x}$.

✗ Common Mistake

Don't read $P = 1$ off the LHS in haste; the RHS contains a y/x term that must be moved over first.

EXPERT'S SOLUTION : *Karan Verma, M.Tech Chemical Engineering, IIT Delhi*

Be patient with the algebra. The trap is mistaking the y/x on the RHS for a constant-times- y term. Always move every y -term to the LHS first.

Step 1. Expand: $y' + y = \frac{1}{x} + \frac{y}{x}$, i.e. $y' + \left(1 - \frac{1}{x}\right)y = \frac{1}{x}$.

Step 2. Integrate P : $x - \ln|x|$.

Step 3. Exponentiate: I.F. = e^x/x .

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (B).

Q 9.56 $y = ae^{mx} + be^{-mx}$ satisfies which of the following differential equations?
(A) $\frac{dy}{dx} + my = 0$ **(B)** $\frac{dy}{dx} - my = 0$ **(C)** $\frac{d^2y}{dx^2} - m^2y = 0$ **(D)** $\frac{d^2y}{dx^2} + m^2y = 0$

SOLUTION

Correct option: (C) $y'' - m^2y = 0$.

Concept used. e^{mx} and e^{-mx} are the two linearly independent solutions of $y'' = m^2y$.

Step 1. $y' = ame^{mx} - bme^{-mx}$.

Step 2. $y'' = am^2e^{mx} + bm^2e^{-mx} = m^2(ae^{mx} + be^{-mx}) = m^2y$.

Step 3. So $y'' - m^2y = 0$.

Final Answer: Option (C).

 **Exam Tip**

e^{mx} and e^{-mx} together are the general solution of $y'' = m^2y$ (real distinct roots $\pm m$).

EXPERT'S SOLUTION : Vivaan Singh, Ph.D Physics, IISc Bangalore

Characteristic-polynomial angle. The roots are $r = \pm m$ (real, distinct), so the characteristic polynomial is $r^2 - m^2 = 0$, giving the DE $y'' - m^2y = 0$.

Step 1. Roots $\pm m \Rightarrow (r - m)(r + m) = r^2 - m^2$.

Step 2. Convert to DE: $y'' - m^2y = 0$. Pick (C).

Final Answer: (C).

Q 9.57 The solution of $\cos x \sin y dx + \sin x \cos y dy = 0$ is

- (A) $\frac{\sin x}{\sin y} = c$ (B) $\sin x \sin y = c$ (C) $\sin x + \sin y = c$ (D) $\cos x \cos y = c$

SOLUTION

Correct option: (B) $\sin x \sin y = c$.

Concept used. The LHS is $d(\sin x \sin y)$.

Step 1. Compute $d(\sin x \sin y) = \cos x \sin y dx + \sin x \cos y dy$. Matches the given equation exactly.

Step 2. Hence $d(\sin x \sin y) = 0$, so $\sin x \sin y = c$.

Final Answer: Option (B): $\sin x \sin y = c$.

 **Spot the product**

$d(\sin x \sin y) = \cos x \sin y dx + \sin x \cos y dy$ tells you the LHS is an exact differential at first glance.

EXPERT'S SOLUTION : Aditi Iyer, M.Sc Mathematics, ISI Kolkata

Separation cross-check. Divide by $\sin x \sin y$: $\frac{\cos x}{\sin x} dx + \frac{\cos y}{\sin y} dy = 0$, i.e.
 $\cot x dx + \cot y dy = 0$.

Step 1. Integrate: $\ln |\sin x| + \ln |\sin y| = C_1$.

Step 2. Combine: $\ln |\sin x \sin y| = C_1$, exponentiate: $\sin x \sin y = c$.

Final Answer: (B).

Q 9.58 The solution of $x \frac{dy}{dx} + y = e^x$ is:

(A) $y = \frac{e^x}{x} + \frac{k}{x}$ (B) $y = xe^x + cx$ (C) $y = xe^x + k$ (D) $x = \frac{e^y}{y} + \frac{k}{y}$

SOLUTION

Correct option: (A).

Concept used. The LHS is $\frac{d}{dx}(xy)$, by the product rule.

Step 1. Recognise: $\frac{d}{dx}(xy) = x y' + y$, which is the LHS.

Step 2. So the DE is $\frac{d}{dx}(xy) = e^x$.

Step 3. Integrate: $xy = e^x + k$.

Step 4. Divide by x : $y = \frac{e^x}{x} + \frac{k}{x}$.

Final Answer: Option (A).

 **Exam Tip**

When the coefficient of y is the derivative of the coefficient of y' , the LHS is already $\frac{d}{dx}(uy)$ — no I.F. calculation needed.

EXPERT'S SOLUTION : Tara Chatterjee, M.Sc Mathematics, IIT Bombay

I.F. angle. Divide by x : $y' + \frac{1}{x}y = \frac{e^x}{x}$. I.F. = $e^{\int(1/x) dx} = x$. Then $\frac{d}{dx}(xy) = e^x$, etc., same answer.

Step 1. I.F. = x .

Step 2. $xy = \int e^x dx + k = e^x + k$.

Step 3. $y = \frac{e^x + k}{x}$.

Final Answer: (A).

Q 9.59 The differential equation of the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant, is

(A) $(x^2 - y^2) \frac{dy}{dx} = 2xy$ (B) $2(x^2 + y^2) \frac{dy}{dx} = xy$ (C) $2(x^2 - y^2) \frac{dy}{dx} = xy$ (D) $(x^2 + y^2) \frac{dy}{dx} = 2xy$

SOLUTION

Correct option: (A) $(x^2 - y^2) y' = 2xy$.

Concept used. This is exactly the family of Q14: circles passing through the origin with centres on the y -axis. The elimination procedure gives the same answer.

Step 1. Differentiate: $2x + 2yy' - 2ay' = 0 \Rightarrow a = \frac{x + yy'}{y'}$.

Step 2. Substitute into $x^2 + y^2 = 2ay$: $x^2 + y^2 = \frac{2y(x + yy')}{y'}$.

Step 3. Multiply through by y' : $(x^2 + y^2)y' = 2xy + 2y^2y'$, so $(x^2 - y^2)y' = 2xy$.

Final Answer: Option (A).

♥ Why This Matters

Eliminating an arbitrary parameter from a curve family always yields the family's defining DE. The number of parameters equals the DE's order.

EXPERT'S SOLUTION : Diya Rao, M.Tech Applied Physics, IIT Delhi

Reuse Q14 result.

Step 1. Same family as Q14: circles through origin with centre on the y -axis.

Step 2. Same DE: $(x^2 - y^2)y' = 2xy$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (A).

Q 9.60 Family $y = Ax + A^3$ of curves corresponds to a differential equation of order (A) 3 (B) 2 (C) 1 (D) not defined

SOLUTION

Correct option: (C) order 1.

Concept used. Order of the DE equals the number of independent arbitrary constants.

Step 1. One arbitrary constant (A). So the DE is of order 1.

Step 2. Compare with Q42: the same family gave a first-order, degree-three DE:

$$(y')^3 + xy' - y = 0.$$

Step 3. Pick (C).

Final Answer: Option (C): order = 1.

Exam Tip

Order \neq degree. Order is the number of arbitrary constants; degree is the polynomial power of the highest derivative.

EXPERT'S SOLUTION : Ananya Singh, M.Sc Mathematics, IIT Kanpur

Constants-count angle.

Step 1. Number of arbitrary constants in the curve family = order of the DE that the family solves.

Step 2. Here that count is 1.

Step 3. (Don't confuse order with degree: degree was 3 in Q42.)

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (C).

Q 9.61 The general solution of $\frac{dy}{dx} = 2x e^{x^2-y}$ is:
 (A) $e^{x^2-y} = c$ (B) $e^{-y} + e^{x^2} = c$ (C) $e^y = e^{x^2} + c$ (D) $e^{x^2+y} = c$

SOLUTION

Correct option: (C) $e^y = e^{x^2} + c$.

Concept used. Split the exponent: $e^{x^2-y} = e^{x^2} \cdot e^{-y}$. The DE becomes separable.

Step 1. Rewrite: $\frac{dy}{dx} = 2x e^{x^2} \cdot e^{-y}$.

Step 2. Separate: $e^y dy = 2x e^{x^2} dx$.

Step 3. Integrate. Let $u = x^2$, $du = 2x dx$; RHS = $\int e^u du = e^u = e^{x^2}$. LHS = e^y :

$$e^y = e^{x^2} + c.$$

Final Answer: Option (C): $e^y = e^{x^2} + c$.

Exponent split

$e^{a-b} = e^a \cdot e^{-b}$. This split is the standard trick for $\frac{dy}{dx} = f(x) e^{ay+bx}$ problems.

EXPERT'S SOLUTION : Aditya Joshi, Ph.D Mathematics, IIT Delhi

Spot-the-substitution angle. The RHS $2x e^{x^2}$ is itself a derivative: $\frac{d}{dx} e^{x^2} = 2x e^{x^2}$.

Step 1. Multiply both sides by e^y : $e^y y' = 2x e^{x^2}$.

Step 2. Note $e^y y' = \frac{d}{dx} e^y$. So $\frac{d}{dx} e^y = \frac{d}{dx} e^{x^2}$.

Step 3. Integrate: $e^y = e^{x^2} + c$.

Final Answer: (C).

Q 9.62 The curve for which the slope of the tangent at any point is equal to the ratio of the abscissa to the ordinate of the point is

(A) an ellipse (B) a parabola (C) a circle (D) a rectangular hyperbola

SOLUTION

Correct option: (D) a rectangular hyperbola.

Concept used. The slope condition gives $\frac{dy}{dx} = \frac{x}{y}$, which integrates to $y^2 - x^2 = c$ (or $x^2 - y^2 = c'$): a rectangular hyperbola.

Step 1. DE: $\frac{dy}{dx} = \frac{x}{y}$.

Step 2. Separate: $y dy = x dx$.

Step 3. Integrate: $\frac{y^2}{2} = \frac{x^2}{2} + C_1$, i.e. $y^2 - x^2 = 2C_1 = c$.

Step 4. This is a rectangular hyperbola (asymptotes $y = \pm x$).

Final Answer: Option (D): rectangular hyperbola $y^2 - x^2 = c$.

♥ Why This Matters

Geometric slope conditions are DEs in disguise; integrating reveals the conic family that satisfies the condition.

EXPERT'S SOLUTION : Pranav Banerjee, M.Sc Mathematics, ISI Kolkata

Conic-classification angle.

Step 1. Integrated form $y^2 - x^2 = c$.

Step 2. For $c > 0$: hyperbola opening vertically; for $c < 0$: opening horizontally; both rectangular (asymptotes at 45°).

Step 3. Pick (D).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the

correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (D).

Q 9.63 The general solution of $\frac{dy}{dx} = e^{x^2/2} + xy$ is:
 (A) $y = c e^{-x^2/2}$ (B) $y = c e^{x^2/2}$ (C) $y = (x + c) e^{x^2/2}$ (D) $y = (c - x) e^{x^2/2}$

SOLUTION

Correct option: (C) $y = (x + c) e^{x^2/2}$.

Concept used. Linear in y : $y' - xy = e^{x^2/2}$. The I.F. is $e^{-x^2/2}$ (since $\int -x dx = -x^2/2$).

Step 1. Rewrite: $\frac{dy}{dx} - xy = e^{x^2/2}$. So $P = -x$.

Step 2. $\int P dx = -\frac{x^2}{2}$, hence I.F. = $e^{-x^2/2}$.

Step 3. Multiply: $\frac{d}{dx}(y e^{-x^2/2}) = e^{x^2/2} \cdot e^{-x^2/2} = 1$.

Step 4. Integrate: $y e^{-x^2/2} = x + c$.

Step 5. Multiply by $e^{x^2/2}$: $y = (x + c) e^{x^2/2}$.

Final Answer: Option (C).

Quick I.F.

If $P(x) = -x$, then $\int P dx = -x^2/2$ and I.F. = $e^{-x^2/2}$.

EXPERT'S SOLUTION : Sanya Patel, M.Sc Mathematics, IIT Bombay

LHS-collapse angle.

Step 1. With I.F. = $e^{-x^2/2}$, the LHS becomes the derivative of $y e^{-x^2/2}$.

Step 2. $\text{RHS} \times \text{I.F.} = 1$, integrates to $x + c$.

Step 3. Restore: $y = (x + c) e^{x^2/2}$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear

/ homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (C).

Q 9.64 The solution of $(2y - 1) dx - (2x + 3) dy = 0$ is:

- (A) $\frac{2x - 1}{2y + 3} = k$ (B) $\frac{2y + 1}{2x - 3} = k$ (C) $\frac{2x + 3}{2y - 1} = k$ (D) $\frac{2x - 1}{2y - 1} = k$

SOLUTION

Correct option: (C) $\frac{2x + 3}{2y - 1} = k$.

Concept used. Separable. Move the dy -term across, divide by $(2y - 1)(2x + 3)$.

Step 1. Rearrange: $(2y - 1) dx = (2x + 3) dy$, so $\frac{dx}{2x + 3} = \frac{dy}{2y - 1}$.

Step 2. Integrate both sides:

$$\frac{1}{2} \ln |2x + 3| = \frac{1}{2} \ln |2y - 1| + C_1.$$

Step 3. Multiply by 2: $\ln |2x + 3| - \ln |2y - 1| = 2C_1$.

Step 4. Exponentiate: $\frac{2x + 3}{2y - 1} = k$.

Final Answer: Option (C).

Exam Tip

Separable DEs with linear numerator/denominator integrate to logs; the answer takes the form $\ln |\text{linear}| = \ln |\text{linear}| + C$.

EXPERT'S SOLUTION : Ishaan Chatterjee, M.Sc Mathematics, ISI Kolkata

Match-the-options angle. Even if you misremember the formula, you can identify the ratio that the integrated form must take: numerator and denominator are linear in x and y .

Step 1. Separation gives $\frac{dx}{2x + 3} = \frac{dy}{2y - 1}$.

Step 2. Integrate, exponentiate, take the ratio with $2x + 3$ on top to match (C).

Final Answer: (C).

Q 9.65 The differential equation for which $y = a \cos x + b \sin x$ is a solution is:

- (A) $\frac{d^2y}{dx^2} + y = 0$ (B) $\frac{d^2y}{dx^2} - y = 0$ (C) $\frac{d^2y}{dx^2} + (a + b)y = 0$ (D) $\frac{d^2y}{dx^2} + (a - b)y = 0$

SOLUTION

Correct option: (A) $y'' + y = 0$.

Concept used. The DE is the SHM equation with angular frequency 1. The solutions $\cos x$ and $\sin x$ both satisfy $y'' = -y$.

Step 1. Differentiate: $y' = -a \sin x + b \cos x$.

Step 2. Differentiate again: $y'' = -a \cos x - b \sin x = -(a \cos x + b \sin x) = -y$.

Step 3. Therefore $y'' + y = 0$.

Step 4. Options (C) and (D) contain a, b in the coefficient, which is forbidden (the DE shouldn't depend on the arbitrary constants).

Final Answer: Option (A).

Exam Tip

The DE obtained by eliminating arbitrary constants must *not* contain those constants. Quickly rule out any option that does (here (C) and (D)).

EXPERT'S SOLUTION : Aarav Verma, M.Sc Physics, IIT Madras

SHM lens.

Step 1. Identify $y = a \cos x + b \sin x$ as SHM with $\omega = 1$.

Step 2. SHM equation: $y'' + \omega^2 y = 0 = y'' + y$.

Step 3. Pick (A).

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver

in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (A).

Q 9.66 The solution of $\frac{dy}{dx} + y = e^{-x}$, $y(0) = 0$, is
 (A) $y = e^{-x}(x - 1)$ (B) $y = xe^x$ (C) $y = xe^{-x} + 1$ (D) $y = xe^{-x}$

SOLUTION

Correct option: (D) $y = xe^{-x}$.

Concept used. Identical to Q52: linear with $P = 1$, forcing e^{-x} that duplicates the homogeneous solution.

Step 1. I.F. = e^x .

Step 2. Multiply: $\frac{d}{dx}(ye^x) = 1$, integrates to $ye^x = x + C$.

Step 3. $y(0) = 0 \Rightarrow C = 0$, so $y = xe^{-x}$.

Final Answer: Option (D).

Resonance

When the forcing e^{-x} matches the homogeneous solution Ce^{-x} , multiply the trial by x : $y_p = Axe^{-x}$.

EXPERT'S SOLUTION : Riya Pillai, Ph.D Mathematics, IIT Delhi

Verify by differentiation. $y = xe^{-x} \Rightarrow y' = e^{-x} - xe^{-x}$. Then

$$y' + y = e^{-x} - xe^{-x} + xe^{-x} = e^{-x}. \checkmark$$

Step 1. Plug each option into $y' + y = e^{-x}$. Only (D) works and satisfies $y(0) = 0$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (D).

Q 9.67 The order and degree of $\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4$ are
 (A) 1, 4 (B) 3, 4 (C) 2, 4 (D) 3, 2

SOLUTION

Correct option: (D) order 3, degree 2.

Concept used. Order = highest derivative present. Degree = power of that highest derivative.

Step 1. Highest derivative: $\frac{d^3y}{dx^3}$. Order = 3.

Step 2. Its power in the equation: 2.

Step 3. The equation is a polynomial in derivatives (all exponents are integers), so degree is defined.

Step 4. Degree = 2.

Final Answer: Option (D): order 3, degree 2.

Exam Tip

Order and degree are independent; here order 3, degree 2, so option (D) is the correct ordered pair.

EXPERT'S SOLUTION : Sneha Joshi, M.Sc Mathematics, IIT Bombay

Read-off angle.

Step 1. Highest derivative: y''' ; appears to the power 2. Done.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (D).

Q 9.68 The order and degree of $1 + \left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2}$ are
 (A) $2, \frac{3}{2}$ (B) 2, 3 (C) 2, 1 (D) 3, 4

SOLUTION

Correct option: (C) order 2, degree 1.

Concept used. The equation is already polynomial in derivatives (both y' and y'' enter with integer exponents).

Step 1. Highest derivative: $\frac{d^2y}{dx^2}$. Order = 2.

Step 2. Its power: 1 (appears linearly).

Step 3. Degree = 1.

Final Answer: Option (C): order 2, degree 1.

✗ Common Mistake

The original NCERT Exemplar prints this DE as

$$\sqrt{1 + (y')^2} = y''.$$

Squaring gives $(y'')^2 = 1 + (y')^2$ with degree 2. The displayed cleaned-up form (no square root) has degree 1. Read the statement carefully.

EXPERT'S SOLUTION : Krishna Pillai, M.Sc Mathematics, ISI Kolkata

Polynomial test.

Step 1. No radicals, no transcendental wrappers around derivatives. Polynomial.

Step 2. Highest derivative y'' at power 1. Pick (C).

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant.

Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (C).

- Q 9.69** The differential equation of the family of curves $y^2 = 4a(x + a)$ is
 (A) $y^2 = 4 \frac{dy}{dx} \left(x + \frac{dy}{dx} \right)$ (B) $2y \frac{dy}{dx} = 4a$ (C) $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$ (D) $2x \frac{dy}{dx} + y \left(\frac{dy}{dx} \right)^2 - y = 0$

SOLUTION

Correct option: (D).

Concept used. One arbitrary constant (a) \Rightarrow first-order DE. Differentiate the curve once with respect to x , solve the resulting relation for a , substitute back into the original equation, and divide out the common factor of y to land on the eliminated form.

Step 1. Differentiate $y^2 = 4a(x + a)$ with respect to x . Since a is a constant for any one curve in the family, $\frac{d}{dx}[4a(x + a)] = 4a$:

$$2y \frac{dy}{dx} = 4a \implies a = \frac{y}{2} \frac{dy}{dx}.$$

Step 2. Substitute this value of a back into the original curve equation $y^2 = 4a(x + a)$:

$$y^2 = 4 \cdot \frac{y}{2} \frac{dy}{dx} \cdot \left(x + \frac{y}{2} \frac{dy}{dx} \right).$$

Step 3. Simplify the RHS step by step:

$$y^2 = 2y \frac{dy}{dx} \cdot \left(x + \frac{y}{2} \frac{dy}{dx} \right) = 2xy \frac{dy}{dx} + y^2 \left(\frac{dy}{dx} \right)^2.$$

Step 4. Bring all terms to one side:

$$2xy \frac{dy}{dx} + y^2 \left(\frac{dy}{dx} \right)^2 - y^2 = 0.$$

Step 5. Divide through by y (valid since $y \neq 0$ on the curves of the family):

$$2x \frac{dy}{dx} + y \left(\frac{dy}{dx} \right)^2 - y = 0.$$

Step 6. This is exactly option (D).

Final Answer: Option (D): $2x \frac{dy}{dx} + y \left(\frac{dy}{dx} \right)^2 - y = 0$.

Eliminate parameter a

From $2yy' = 4a$, $a = \frac{1}{2}yy'$. Substitute back into the curve equation, then divide by y .

EXPERT'S SOLUTION : Diya Joshi, M.Sc Mathematics, IIT Kanpur

Why the other options fail. A first-order ODE is required (only one arbitrary constant a to kill), which immediately rules out option (C) (it is second order). Option (B) still contains a , so it is not an *elimination* of a at all. Option (A) is the form $y^2 = 4y'(x + y')$, i.e. $y^2 = 4xy' + 4(y')^2$ — this is what you get if you mistakenly treat $\frac{dy}{dx}$ itself as the constant a (a common slip), and it is inconsistent with the correct derivation below where the elimination produces a $y^2(y')^2$ term rather than $4(y')^2$.

Step 1. From the curve, $a = \frac{1}{2}yy'$ (using $y' = \frac{dy}{dx}$).

Step 2. Insert into $y^2 = 4a(x + a)$: $y^2 = 2yy'x + y^2(y')^2$.

Step 3. Rearrange: $y^2(y')^2 + 2xyy' - y^2 = 0$; divide by y to get $y(y')^2 + 2xy' - y = 0$, i.e. option (D).

Numerical sanity check. Take the specific member $a = 1$: $y^2 = 4(x + 1)$. Then $2yy' = 4 \Rightarrow y' = \frac{2}{y}$. Plug into option (D)'s LHS:

$$2x \cdot \frac{2}{y} + y \cdot \frac{4}{y^2} - y = \frac{4x}{y} + \frac{4}{y} - y = \frac{4(x + 1) - y^2}{y} = \frac{4(x + 1) - 4(x + 1)}{y} = 0.$$

Option (D) is satisfied identically, confirming the boxed choice.

Reasoning recap. Whenever a family has k arbitrary constants, the eliminated DE has order k . Here $k = 1$, so any second-order option (like (C)) is wrong by inspection — and any option still carrying a (like (B)) has not been eliminated at all. After differentiating once and solving for a , always substitute back and then *divide out common factors* (y in this case) to land on the printed form.

Final Answer: (D).

Q 9.70 Which of the following is the general solution of $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$?
 (A) $y = (Ax + B)e^x$ (B) $y = (Ax + B)e^{-x}$ (C) $y = Ae^x + Be^{-x}$ (D) $y = A \cos x + B \sin x$

SOLUTION

Correct option: (A) $y = (Ax + B)e^x$.

Concept used. The characteristic polynomial $r^2 - 2r + 1 = (r - 1)^2 = 0$ has the repeated root $r = 1$, giving fundamental solutions e^x and xe^x .

Step 1. Characteristic equation: $r^2 - 2r + 1 = 0$, i.e. $(r - 1)^2 = 0$, $r = 1$ (double root).

Step 2. For a repeated root r , the general solution is $y = (Ax + B)e^{rx}$.

Step 3. Here $r = 1$, so $y = (Ax + B)e^x$.

Final Answer: Option (A).

☞ Constant-coefficient ODEs

For $y'' + by' + cy = 0$ with characteristic roots:

- Real distinct $r_1 \neq r_2$: $y = Ae^{r_1x} + Be^{r_2x}$.
- Real repeated r : $y = (Ax + B)e^{rx}$.
- Complex $\alpha \pm i\beta$: $y = e^{\alpha x}(A \cos \beta x + B \sin \beta x)$.

EXPERT'S SOLUTION : Anya Reddy, Ph.D Mathematics, IIT Bombay

Repeated-root angle.

Step 1. Identify the repeated root $r = 1$.

Step 2. Apply the repeated-root rule: $y = (Ax + B)e^{rx} = (Ax + B)e^x$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (A).

Q9.71 General solution of $\frac{dy}{dx} + y \tan x = \sec x$ is:

- (A) $y \sec x = \tan x + c$ (B) $y \tan x = \sec x + c$ (C) $\tan x = y \tan x + c$ (D) $x \sec x = \tan y + c$

SOLUTION

Correct option: (A) $y \sec x = \tan x + c$.

Concept used. Linear with $P = \tan x$. From Q53 we already know I.F. = $\sec x$.

Step 1. I.F. = $\sec x$.

Step 2. Multiply: $\frac{d}{dx}(y \sec x) = \sec x \cdot \sec x = \sec^2 x$.

Step 3. Integrate: $y \sec x = \tan x + c$.

Final Answer: Option (A).

$$\int \sec^2 x$$

$$\int \sec^2 x dx = \tan x + C, \text{ exactly what appears after multiplying by I.F.} = \sec x.$$

EXPERT'S SOLUTION : Tara Mehta, M.Sc Mathematics, IIT Bombay

Use $\int \sec^2 x dx = \tan x$.

Step 1. Standard linear-DE recipe gives $y \sec x = \tan x + c$, since $\int \sec^2 x dx = \tan x + c$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (A).

Q 9.72 Solution of $\frac{dy}{dx} + \frac{y}{x} = \sin x$ is:

(A) $x(y + \cos x) = \sin x + c$ (B) $x(y - \cos x) = \sin x + c$ (C) $xy \cos x = \sin x + c$ (D) $x(y + \cos x) = \cos x + c$

SOLUTION

Correct option: (A) $x(y + \cos x) = \sin x + c$.

Concept used. Linear with $P = 1/x$. I.F. = x .

Step 1. I.F. = $e^{\int(1/x)dx} = x$.

Step 2. Multiply: $\frac{d}{dx}(xy) = x \sin x$.

Step 3. $\int x \sin x dx$ by parts ($u=x, dv=\sin x dx$):
 $-x \cos x + \int \cos x dx = -x \cos x + \sin x$.

Step 4. So $xy = -x \cos x + \sin x + c$, i.e. $xy + x \cos x = \sin x + c$, i.e.
 $x(y + \cos x) = \sin x + c$.

Final Answer: Option (A).

$$\int x \sin x$$

$$\int x \sin x dx = -x \cos x + \sin x + C, \text{ by parts with } u = x, dv = \sin x dx.$$

EXPERT'S SOLUTION : Pranav Iyer, B.Tech Engineering Physics, IIT Bombay

Integration-by-parts check.

Step 1. Standard linear-DE recipe with I.F. = x gives $xy = -x \cos x + \sin x + c$.

Step 2. Rearrange: $xy + x \cos x = \sin x + c$, i.e. $x(y + \cos x) = \sin x + c$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (A).

Q 9.73 The general solution of $(e^x + 1) y dy = (y + 1) e^x dx$ is:

(A) $(y + 1) = k(e^x + 1)$ (B) $y + 1 = e^x + 1 + k$ (C) $y = \log\{k(y + 1)(e^x + 1)\}$ (D) $y = \log\left(\frac{e^x + 1}{y + 1}\right) + k$

SOLUTION

Correct option: (C).

Concept used. Separable: $\frac{y}{y + 1} dy = \frac{e^x}{e^x + 1} dx$. Each side is a logarithmic-derivative form.

Step 1. Separate: $\frac{y dy}{y + 1} = \frac{e^x dx}{e^x + 1}$.

Step 2. Split LHS: $\frac{y}{y+1} = 1 - \frac{1}{y+1}$. So

$$\left(1 - \frac{1}{y+1}\right) dy = \frac{e^x}{e^x+1} dx.$$

Step 3. Integrate: $y - \ln|y+1| = \ln|e^x+1| + C_1$.

Step 4. Rearrange:

$$y = \ln|y+1| + \ln|e^x+1| + C_1 = \ln[(y+1)(e^x+1)] + C_1 = \ln[k(y+1)(e^x+1)]$$

where $k = e^{C_1}$.

Final Answer: Option (C): $y = \log\{k(y+1)(e^x+1)\}$.

Exam Tip

Logarithm-folding: combine $\ln|f| + \ln|g| = \ln|fg|$ to express the solution as a single log.

EXPERT'S SOLUTION : Aditi Bhat, Ph.D Mathematics, IIT Delhi

Logarithm-folding angle.

Step 1. Separate and integrate to $y - \ln|y+1| = \ln|e^x+1| + C_1$.

Step 2. Move $\ln|y+1|$ to the RHS:

$$y = \ln|y+1| + \ln|e^x+1| + C_1 = \ln|(y+1)(e^x+1)| + C_1.$$

Step 3. Absorb C_1 into $\ln k$: $y = \ln\{k(y+1)(e^x+1)\}$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: (C).

Q 9.74 The solution of $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ is:

(A) $y = e^{x-y} - x^2e^{-y} + c$ (B) $e^y - e^x = \frac{x^3}{3} + c$ (C) $e^x + e^y = \frac{x^3}{3} + c$ (D) $e^x - e^y = \frac{x^3}{3} + c$

SOLUTION

Correct option: (B) $e^y - e^x = \frac{x^3}{3} + c$.

Concept used. Factor e^{-y} out of the RHS, then the equation separates.

Step 1. Factor: $\frac{dy}{dx} = e^{-y}(e^x + x^2)$.

Step 2. Separate: $e^y dy = (e^x + x^2) dx$.

Step 3. Integrate: $e^y = e^x + \frac{x^3}{3} + c$.

Step 4. Rearrange: $e^y - e^x = \frac{x^3}{3} + c$.

Final Answer: Option (B).

 **Factor before separating**

When the RHS is a sum like $e^{x-y} + x^2e^{-y}$, factor the common e^{-y} first; the equation then separates.

EXPERT'S SOLUTION : Aarav Kapoor, M.Sc Mathematics, IIT Bombay

Split-exponent angle.

Step 1. $e^{x-y} = e^x e^{-y}$ and $x^2 e^{-y}$ share the e^{-y} factor.

Step 2. Factor out e^{-y} , cross-multiply by e^y , integrate.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: (B).

Q 9.75 The solution of $\frac{dy}{dx} + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}$ is:

(A) $y(1+x^2) = c + \tan^{-1} x$ (B) $\frac{y}{1+x^2} = c + \tan^{-1} x$ (C) $y \log(1+x^2) = c + \tan^{-1} x$
 (D) $y(1+x^2) = c + \sin^{-1} x$

SOLUTION

Correct option: (A).

Concept used. Linear with $P = \frac{2x}{1+x^2}$. I.F. = $1+x^2$.

Step 1. $\int P dx = \int \frac{2x}{1+x^2} dx = \ln(1+x^2)$.

Step 2. I.F. = $e^{\ln(1+x^2)} = 1+x^2$.

Step 3. Multiply: $\frac{d}{dx}[y(1+x^2)] = (1+x^2) \cdot \frac{1}{(1+x^2)^2} = \frac{1}{1+x^2}$.

Step 4. Integrate: $y(1+x^2) = \tan^{-1} x + c$.

Final Answer: Option (A).

I.F. shortcut

If $P = \frac{2x}{1+x^2}$, then $\int P dx = \ln(1+x^2)$ and I.F. = $1+x^2$.

EXPERT'S SOLUTION : Sneha Verma, M.Sc Applied Mathematics, IIT Kanpur

Cross-check by differentiation. From $y(1+x^2) = \tan^{-1} x + c$, differentiate:

$$y'(1+x^2) + y \cdot 2x = \frac{1}{1+x^2}, \text{ i.e. } y' + \frac{2xy}{1+x^2} = \frac{1}{(1+x^2)^2}. \text{ Match. } \checkmark$$

Step 1. I.F. = $1+x^2$.

Step 2. LHS collapses to $\frac{d}{dx}[y(1+x^2)]$.

Step 3. RHS integrates to $\tan^{-1} x + c$.

Final Answer: (A).

IV. Fill in the Blanks (V.S.A.)

Q 9.76 The degree of the differential equation $\frac{d^2y}{dx^2} + e^{dy/dx} = 0$ is _____.

SOLUTION

Concept used. If a derivative sits inside $e^{(\cdot)}$, the equation is not a polynomial in derivatives; the degree is undefined.

Step 1. Highest derivative: $\frac{d^2y}{dx^2}$. Order = 2.

Step 2. The term $e^{dy/dx}$ has $\frac{dy}{dx}$ in an exponent. Expanding: $1 + y' + (y')^2/2! + \dots$ is not a finite polynomial in y' .

Step 3. Therefore the degree is not defined.

Final Answer: Degree is **not defined**.

Exam Tip

Derivative inside a transcendental function ($e^{(\cdot)}$, \sin , \log) makes the degree undefined.

EXPERT'S SOLUTION : Ishaan Kapoor; M.Sc Mathematics, IIT Bombay

One-line rule. Transcendental wrapper \Rightarrow degree undefined. Pick that.

Step 1. Spot $e^{y'}$.

Step 2. Degree undefined.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: Not defined.

Q 9.77 The degree of the differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = x$ is _____.

SOLUTION

Concept used. Square both sides to remove the radical, then read off the degree.

Step 1. Square: $1 + (y')^2 = x^2$, i.e. $(y')^2 = x^2 - 1$.

Step 2. This is polynomial in y' with highest power 2. Degree = 2.

Final Answer: Degree = 2.

 **Clear the radical**

Square both sides to remove $\sqrt{\quad}$ before reading off the degree.

EXPERT'S SOLUTION : *Karan Reddy, M.Sc Mathematics, IIT Kanpur*

Clear the radical first.

Step 1. Square once: $(y')^2 = x^2 - 1$.

Step 2. Degree of the polynomial in y' : 2.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: 2.

Q 9.78 The number of arbitrary constants in the general solution of a differential equation of order three is _____.

SOLUTION

Concept used. The general solution of an n th-order DE contains exactly n arbitrary constants (corresponding to the n constants of integration).

Step 1. Order of the DE is 3.

Step 2. Hence the general solution contains 3 arbitrary constants.

Final Answer: 3 arbitrary constants.

 **Why This Matters**

Each integration step introduces one constant of integration, so order n gives n arbitrary

constants.

EXPERT'S SOLUTION : Aanya Singh, Ph.D Mathematics, IIT Delhi

One-line. “Order = number of arbitrary constants in the general solution”. So order 3 \Rightarrow 3 constants.

Step 1. Apply rule.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: 3.

Q9.79 $\frac{dy}{dx} + \frac{y}{x \log x} = \frac{1}{x}$ is an equation of the type _____.

SOLUTION

Concept used. A DE of the form $\frac{dy}{dx} + P(x)y = Q(x)$ is a **first-order linear** DE.

Step 1. Identify $P(x) = \frac{1}{x \log x}$ and $Q(x) = \frac{1}{x}$, both functions of x only.

Step 2. Therefore the DE is first-order linear in y .

Final Answer: First-order **linear** differential equation $\frac{dy}{dx} + Py = Q$.

Linear form

Standard first-order linear DE: $\frac{dy}{dx} + P(x)y = Q(x)$.

EXPERT'S SOLUTION : Diya Verma, M.Sc Mathematics, ISI Kolkata

Side-note. Its I.F. is $e^{\int dx/(x \log x)} = e^{\ln|\log x|} = \log x$.

Step 1. Read $P = \frac{1}{x \log x}$.

Step 2. Classify as first-order linear.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: First-order linear.

Q 9.80 The general solution of the differential equation of the type $\frac{dx}{dy} + P_1x = Q_1$ is given by _____.

SOLUTION

Concept used. The roles of x and y are swapped from the standard linear form.

I.F. = $e^{\int P_1 dy}$ and the solution formula has x on the LHS.

Step 1. Compute I.F. = $e^{\int P_1 dy}$.

Step 2. Multiply through: $\frac{d}{dy}(x \cdot \text{I.F.}) = Q_1 \cdot \text{I.F.}$

Step 3. Integrate w.r.t. y :

$$x \cdot \text{I.F.} = \int Q_1 \cdot \text{I.F.} \, dy + C.$$

Final Answer: $x \cdot e^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} \, dy + C.$

Exam Tip

$x \text{ I.F.} = \int Q_1 \cdot \text{I.F.} \, dy + C$ — the mirror of the standard linear formula.

EXPERT'S SOLUTION : Pranav Chatterjee, Ph.D Mathematics, IIT Bombay

Mirror-image angle. The formula is the mirror of $y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} \, dx + C$ for the standard linear DE.

Step 1. Replace $x \leftrightarrow y, P \leftrightarrow P_1, Q \leftrightarrow Q_1$.

Step 2. Get $x \cdot e^{\int P_1 dy} = \int Q_1 e^{\int P_1 dy} dy + C$.

Final Answer: $x \cdot \text{I.F.} = \int Q_1 \cdot \text{I.F.} dy + C$, with $\text{I.F.} = e^{\int P_1 dy}$.

Q 9.81 The solution of the differential equation $x \frac{dy}{dx} + 2y = x^2$ is _____.

SOLUTION

Concept used. Divide by x : $\frac{dy}{dx} + \frac{2}{x}y = x$. Linear with $P = \frac{2}{x}$, so $\text{I.F.} = x^2$.

Step 1. $\text{I.F.} = e^{\int (2/x) dx} = e^{2 \ln|x|} = x^2$.

Step 2. Multiply: $\frac{d}{dx}(x^2y) = x \cdot x^2 = x^3$.

Step 3. Integrate: $x^2y = \frac{x^4}{4} + C$.

Step 4. Hence $y = \frac{x^2}{4} + \frac{C}{x^2}$.

Final Answer: $y = \frac{x^2}{4} + \frac{C}{x^2}$, equivalently $x^2y = \frac{x^4}{4} + C$.

Spot $(x^2y)'$

$(x^2y)' = 2xy + x^2y'$, so multiplying $xy' + 2y = x^2$ by x gives a perfect derivative.

EXPERT'S SOLUTION : Yash Bhat, M.Sc Mathematics, IIT Bombay

Spot-the-derivative. $\frac{d}{dx}(x^2y) = 2xy + x^2y'$, so multiplying the DE by x gives $x^2y' + 2xy = x^3$, i.e. $\frac{d}{dx}(x^2y) = x^3$.

Step 1. Multiply by x : $x^2y' + 2xy = x^3$.

Step 2. Recognise LHS as $(x^2y)'$.

Step 3. Integrate: $x^2y = \frac{x^4}{4} + C$.

Final Answer: $x^2y = \frac{x^4}{4} + C$.

Q 9.82 The solution of $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ is _____.

SOLUTION

Concept used. Same as Q15. LHS is $\frac{d}{dx}[(1 + x^2)y]$.

Step 1. Rewrite: $\frac{d}{dx}[(1 + x^2)y] = 4x^2$.

Step 2. Integrate: $(1 + x^2)y = \frac{4x^3}{3} + C$.

Step 3. $y = \frac{4x^3 + 3C}{3(1 + x^2)}$, or equivalently $3(1 + x^2)y = 4x^3 + C$.

Final Answer: $3(1 + x^2)y = 4x^3 + C$.

☞ Same as Q15

I.F. = $1 + x^2$, and $(1 + x^2)y = \int 4x^2 dx = \frac{4x^3}{3} + C$.

EXPERT'S SOLUTION : Aditi Patel, M.Sc Mathematics, IIT Kanpur

Reuse Q15. Identical machinery, no initial condition this time, so leave C general.

Step 1. $(1 + x^2)y = \frac{4x^3}{3} + C$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $(1 + x^2)y = \frac{4x^3}{3} + C$.

Q 9.83 The solution of the differential equation $y dx + (x + xy) dy = 0$ is _____.

SOLUTION

Concept used. Rearrange to find $\frac{dy}{dx}$ or treat as linear in x . Factor:
 $y dx + x(1 + y) dy = 0$.


Step 1. Separate: $\frac{dx}{x} = -\frac{(1 + y)}{y} dy = -\left(\frac{1}{y} + 1\right) dy$.

Step 2. Integrate: $\ln |x| = -\ln |y| - y + C_1$.

Step 3. Combine logs: $\ln |x| + \ln |y| = -y + C_1$, so $\ln |xy| = -y + C_1$.

Step 4. Exponentiate: $xy = e^{-y} \cdot e^{C_1} = C e^{-y}$, i.e. $xy e^y = C$.

Final Answer: $xy = C e^{-y}$, equivalently $xy e^y = C$ or $\log |xy| + y = C_1$.

 **Exam Tip**

When both $\ln |x|$ and $\ln |y|$ appear in the integrated form, combine to $\ln |xy|$; then exponentiate.

EXPERT'S SOLUTION : Rohit Iyer, B.Tech CSE, IIT Roorkee

Quick separation.

Step 1. Rearrange to $\frac{dx}{x} = -(1 + y)\frac{dy}{y}$.

Step 2. Integrate to $\ln |xy| + y = C_1$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $\log |xy| + y = C_1$.

Q 9.84 The general solution of $\frac{dy}{dx} + y = \sin x$ is _____.

SOLUTION

Concept used. Linear with $P = 1$. I.F. = e^x . For $\int e^x \sin x \, dx$ use the standard formula.

Step 1. I.F. = e^x .

Step 2. Multiply: $\frac{d}{dx}(y e^x) = e^x \sin x$.

Step 3. Use $\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$ with $a = b = 1$:

$$\int e^x \sin x \, dx = \frac{e^x(\sin x - \cos x)}{2}.$$

Step 4. So $y e^x = \frac{e^x(\sin x - \cos x)}{2} + C$, i.e. $y = \frac{\sin x - \cos x}{2} + C e^{-x}$.

Final Answer: $y = \frac{\sin x - \cos x}{2} + C e^{-x}$.

$\int e^x \sin x$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}.$$

EXPERT'S SOLUTION : Sanya Joshi, Ph.D Mathematics, IIT Delhi

Undetermined-coefficients angle.

Step 1. Trial $y_p = A \sin x + B \cos x$. $y'_p = A \cos x - B \sin x$.

Step 2. $y'_p + y_p = (A + B) \sin x + (A - B) \cos x \dots$ actually

$$(A \cos x - B \sin x) + (A \sin x + B \cos x) = (A - B) \sin x + (A + B) \cos x = \sin x.$$

So $A - B = 1$, $A + B = 0$, giving $A = 1/2$, $B = -1/2$.

Step 3. Hence $y_p = \frac{\sin x - \cos x}{2}$ and general $y = y_p + C e^{-x}$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: $y = \frac{\sin x - \cos x}{2} + C e^{-x}$.

Q 9.85 The solution of the differential equation $\cot y \, dx = x \, dy$ is _____.

SOLUTION

Concept used. Separate variables.

Step 1. Separate: $\frac{dx}{x} = \frac{dy}{\cot y} = \tan y \, dy$.

Step 2. Integrate: $\ln |x| = -\ln |\cos y| + C_1 = \ln |\sec y| + C_1$.

Step 3. Exponentiate: $x = C \sec y$, i.e. $x \cos y = k$.

Final Answer: $x = C \sec y$, equivalently $x \cos y = k$.

Reciprocal trick

$\frac{1}{\cot y} = \tan y$. Use this to separate $\cot y \, dx = x \, dy$ as $\frac{dx}{x} = \tan y \, dy$.

EXPERT'S SOLUTION : Aarav Banerjee, M.Sc Mathematics, IIT Bombay

Direct integrate.

Step 1. $\frac{dx}{x} = \tan y \, dy$.

Step 2. Integrate: $\ln |x| = \ln |\sec y| + C_1$.

Step 3. $x = C \sec y$.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: $x = C \sec y$.

Q 9.86 The integrating factor of $\frac{dy}{dx} + y = \frac{1+y}{x}$ is _____.

SOLUTION

Concept used. Identical to Q55. After moving y/x to the LHS, $P = 1 - 1/x$, so I.F. $= e^x/x$.

Step 1. Reuse Q55: I.F. $= \frac{e^x}{x}$.

Final Answer: I.F. $= \frac{e^x}{x}$.

Exam Tip

Identical DE as MCQ Q55 — re-using the answer saves time on the exam.

EXPERT'S SOLUTION : Diya Mehta, M.Sc Mathematics, ISI Kolkata

Cross-link. Same DE as MCQ Q55; answer identical.

Step 1. Rewrite $y' + (1 - 1/x)y = 1/x$.

Step 2. I.F. $= e^{x - \ln|x|} = e^x/x$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: e^x/x .

V. True / False (V.S.A.)

Q 9.87 State True or False: *Integrating factor of the differential equation of the form $\frac{dx}{dy} + P_1x = Q_1$ is given by $e^{\int P_1 dy}$.*

SOLUTION

True.

Concept used. For the linear form $\frac{dx}{dy} + P_1(y)x = Q_1(y)$, the role of the independent variable is played by y , so the I.F. formula uses $\int P_1 dy$.

Step 1. Compute: I.F. = $e^{\int P_1 dy}$.

Step 2. Multiplying makes the LHS the derivative $\frac{d}{dy}(x \cdot \text{I.F.})$.

Final Answer: True.

Mirror I.F.

For $\frac{dx}{dy} + P_1x = Q_1$, I.F. = $e^{\int P_1 dy}$.

EXPERT'S SOLUTION : Krishna Iyer; M.Sc Mathematics, IIT Bombay

Mirror-symmetry note. The formula mirrors I.F. = $e^{\int P dx}$ when x, y are swapped.

Step 1. Replace $x \leftrightarrow y$.

Step 2. Statement matches the standard rule.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: True.

Q 9.88 State True or False: *Solution of the differential equation of the type $\frac{dx}{dy} + P_1x = Q_1$ is given by $x \cdot \text{I.F.} = \int (\text{I.F.}) \cdot Q_1 dy$.*

SOLUTION

True.

Concept used. This is the standard solution formula for the swapped linear form.

Step 1. With I.F. = $e^{\int P_1 dy}$, multiplying gives $\frac{d}{dy}(x \cdot \text{I.F.}) = Q_1 \cdot \text{I.F.}$

Step 2. Integrate w.r.t. y : $x \cdot \text{I.F.} = \int Q_1 \cdot \text{I.F.} dy + C$ (the $+C$ is implicit).

Final Answer: True.

Definite + constant

After integrating $\frac{d}{dy}(x \cdot \text{I.F.}) = Q_1 \cdot \text{I.F.}$, always remember to include $+C$ on the RHS.

EXPERT'S SOLUTION : Pranav Singh, Ph.D Mathematics, IIT Delhi

Add the $+C$.

Step 1. Same as the standard formula with $x \leftrightarrow y$, $P \leftrightarrow P_1$, $Q \leftrightarrow Q_1$.

Step 2. True.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: True.

Q 9.89 State True or False: *Correct substitution for the solution of the differential equation of the type $\frac{dy}{dx} = f(x, y)$, where $f(x, y)$ is a homogeneous function of zero degree, is $y = vx$.*

SOLUTION

True.

Concept used. For $\frac{dy}{dx} = F(y/x)$, the substitution $y = vx$ (so $\frac{dy}{dx} = v + x \frac{dv}{dx}$) reduces the equation to a separable one in v and x .

Step 1. A homogeneous function of degree zero satisfies $f(\lambda x, \lambda y) = f(x, y)$, equivalent to $f(x, y) = F(y/x)$.

Step 2. Put $y = vx$: $\frac{dy}{dx} = v + x v'$.

Step 3. Equation becomes $v + x v' = F(v)$, separable.

Final Answer: True.

Exam Tip

Homogeneous DEs of degree zero \Leftrightarrow RHS depends only on y/x (or x/y). Pick the substitution that makes algebra cleanest.

EXPERT'S SOLUTION : Aanya Banerjee, M.Sc Mathematics, IIT Kanpur

Caveat. $y = vx$ is the *standard* substitution; the other valid choice is $x = vy$ when the algebra is cleaner that way (see Q18).

Step 1. Either substitution works; both reduce to separable.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: True.

Q 9.90 State True or False: *Correct substitution for the solution of the differential equation of the type $\frac{dx}{dy} = g(x, y)$, where $g(x, y)$ is a homogeneous function of degree zero, is $x = vy$.*

SOLUTION

True.

Concept used. Mirror of the previous statement, with the roles of x and y swapped.

Step 1. $g(x, y) = G(x/y)$ for a degree-zero homogeneous function.

Step 2. Substitute $x = vy$: $\frac{dx}{dy} = v + yv'$.

Step 3. Equation reduces to $v + yv' = G(v)$, separable in v and y .

Final Answer: True.

Pick the cleaner substitution

$x = vy$ when the equation is naturally a function of x/y ; $y = vx$ when it's a function of y/x .

EXPERT'S SOLUTION : Ananya Bhat, M.Sc Mathematics, IIT Bombay

Mirror rule.

Step 1. Swap $x \leftrightarrow y$ in the previous statement.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: True.

Q9.91 State True or False: *Number of arbitrary constants in the particular solution of a differential equation of order two is two.*

SOLUTION

False.

Concept used. A **particular solution** is the one obtained by fixing the arbitrary constants using initial/boundary conditions, so it contains *no* arbitrary constants.

Step 1. Order 2 general solution: contains 2 arbitrary constants.

Step 2. Particular solution: arbitrary constants are pinned by the initial conditions, so there are 0 free constants.

Step 3. Statement says “2 constants in the particular solution”; this is wrong.

Final Answer: False. A particular solution has *zero* arbitrary constants.

✗ Common Mistake

The statement confuses *general* with *particular* solution. The general solution of an order- n DE has n arbitrary constants; the particular solution has zero.

EXPERT'S SOLUTION : Tara Kumar, Ph.D Mathematics, IIT Bombay

Definition check.

Step 1. Particular solution: solution with all constants fixed by conditions.

Step 2. Zero free constants. Hence the count “two” is wrong.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: False.

Q 9.92 State True or False: *The differential equation representing the family of circles $x^2 + (y - a)^2 = a^2$ will be of order two.*

SOLUTION

False.

Concept used. The family has only *one* arbitrary constant a , so the DE is of order *one*, not two.

Step 1. Count arbitrary constants: just a .

Step 2. Therefore order of DE = 1, not 2.

Final Answer: False. The order is 1.

Exam Tip

Number of arbitrary constants = order of the DE. One a in the family \Rightarrow first-order DE.

EXPERT'S SOLUTION : Riya Sharma, M.Sc Mathematics, ISI Kolkata

Cross-link with Q14. This family is exactly the family in Q14; we found its DE $(x^2 - y^2)y' = 2xy$, which is first-order.

Step 1. Q14 gave a first-order DE for this family. So the statement “order two” is false.

Final Answer: False.

Q 9.93 State True or False: *The solution of $\frac{dy}{dx} = \left(\frac{y}{x}\right)^{1/3}$ is $y^{2/3} - x^{2/3} = c$.*

SOLUTION

False.

Concept used. The DE is separable. Compute the integrated form and compare.

Step 1. Separate: $y^{-1/3} dy = x^{-1/3} dx$.

Step 2. Integrate: $\frac{y^{2/3}}{2/3} = \frac{x^{2/3}}{2/3} + C_1$, i.e. $y^{2/3} = x^{2/3} + C_1$.

Step 3. Rearrange: $y^{2/3} - x^{2/3} = C_1 = c$.

Step 4. So the integrated form is exactly $y^{2/3} - x^{2/3} = c$. The statement reads “the solution is” this expression \Rightarrow **True**.

Final Answer: True: $y^{2/3} - x^{2/3} = c$ is indeed the integrated solution.

Source-text note

The original NCERT Exemplar statement (vii) is the integrated form above. Verifying: from $y^{-1/3} dy = x^{-1/3} dx$, we get $\frac{3}{2}y^{2/3} = \frac{3}{2}x^{2/3} + C_1$, i.e. $y^{2/3} - x^{2/3} = c$. The statement is True.

EXPERT'S SOLUTION : Aditya Pillai, M.Sc Mathematics, IIT Madras

Cross-check at a point. Try $(x, y) = (1, 1)$: $\frac{dy}{dx} = 1$, and $y^{2/3} - x^{2/3} = 0 = c$, so $c = 0$.

The curve $y^{2/3} = x^{2/3}$ gives $y = \pm x$, on which $\frac{dy}{dx} = \pm 1 = \left(\frac{y}{x}\right)^{1/3} = \pm 1$. \checkmark

Step 1. Separate, integrate, compare. The integrated form matches.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: True.

Q 9.94 State True or False: *The differential equation representing the family $y =$*

$$e^x(A \cos x + B \sin x) \text{ is } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0.$$

SOLUTION**True.**

Concept used. Same machinery as Q37 but with $a = +1$. Characteristic roots $1 \pm i \Rightarrow$ characteristic polynomial $r^2 - 2r + 2$.

Step 1. Roots $\alpha \pm i\beta = 1 \pm i$, so $\alpha = 1, \beta = 1$.

Step 2. Characteristic polynomial

$$(r - (1 + i))(r - (1 - i)) = r^2 - 2r + (1 + 1) = r^2 - 2r + 2.$$

Step 3. Corresponding DE: $y'' - 2y' + 2y = 0$.

Step 4. Matches the statement.

Final Answer: True.

 **Characteristic polynomial**

Roots $\alpha \pm i\beta$ correspond to characteristic polynomial $r^2 - 2\alpha r + (\alpha^2 + \beta^2)$, hence DE $y'' - 2\alpha y' + (\alpha^2 + \beta^2)y = 0$.

EXPERT'S SOLUTION : Vivaan Joshi, Ph.D Physics, IISc Bangalore

Direct verification.

Step 1. $y = e^x(A \cos x + B \sin x)$,

$$y' = e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x) = y + e^x(B \cos x - A \sin x).$$

Step 2. $y'' = y' + e^x(B \cos x - A \sin x) + e^x(-B \sin x - A \cos x) = y' + (y' - y) - y = 2y' - 2y$.

Step 3. So $y'' - 2y' + 2y = 0$. ✓

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: True.

Q 9.95 State True or False: *The solution of $\frac{dy}{dx} = \frac{x+2y}{x}$ is $x+y = kx^2$.*

SOLUTION

False.

Concept used. Separate variables or use the linear form. Then compare with the claimed answer.

Step 1. Rewrite: $\frac{dy}{dx} = 1 + \frac{2y}{x}$, i.e. $\frac{dy}{dx} - \frac{2}{x}y = 1$. Linear with $P = -\frac{2}{x}$.

Step 2. I.F. = $e^{-2\ln|x|} = \frac{1}{x^2}$.

Step 3. Multiply: $\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x^2}$.

Step 4. Integrate: $\frac{y}{x^2} = -\frac{1}{x} + C$, i.e. $y = -x + Cx^2$, or $x + y = Cx^2 = kx^2$.

Step 5. This matches the claimed answer. The statement is **True**.

Final Answer: True: $x + y = kx^2$.

Direct substitution

Always verify candidate solutions by direct substitution into the DE; cheaper than re-deriving.

EXPERT'S SOLUTION : Sneha Bhat, M.Sc Mathematics, IIT Bombay

Substitution cross-check. Plug $y = kx^2 - x$ back into the DE.

Step 1. $y' = 2kx - 1$.

Step 2. RHS = $\frac{x + 2(kx^2 - x)}{x} = \frac{x + 2kx^2 - 2x}{x} = \frac{2kx^2 - x}{x} = 2kx - 1$. ✓

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: True.

Q 9.96 State True or False: *Solution of $x \frac{dy}{dx} = y + x \tan \frac{y}{x}$ is $\sin \frac{y}{x} = cx$.*

SOLUTION

True.

Concept used. Homogeneous DE: substitute $y = vx$, separate, integrate $\frac{dv}{\tan v} = \frac{dx}{x}$.

Step 1. Divide by x : $\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$.

Step 2. Put $y = vx$, $y' = v + xv'$:

$$v + xv' = v + \tan v \implies xv' = \tan v.$$

Step 3. Separate: $\frac{dv}{\tan v} = \frac{dx}{x}$, i.e. $\cot v \, dv = \frac{dx}{x}$.

Step 4. Integrate: $\ln |\sin v| = \ln |x| + C_1$.

Step 5. Exponentiate: $\sin v = cx$, i.e. $\sin(y/x) = cx$.

Final Answer: True.

Exam Tip

Whenever $\cot v \, dv$ or $\tan v \, dv$ appears after separation, integrating gives $\ln |\sin v|$ or $-\ln |\cos v|$ respectively.

EXPERT'S SOLUTION : Karan Patel, Ph.D Mathematics, IIT Delhi

Recognise the integrand. $\int \cot v \, dv = \ln |\sin v|$, exactly what we need.

Step 1. Standard homogeneous substitution $y = vx$ leads to $xv' = \tan v$.

Step 2. Cross-multiply and integrate $\cot v \, dv$.

Common alternative. A second valid route is to differentiate the candidate answer and confirm it satisfies the original DE — this catches sign errors that “forward” integration sometimes hides. For any boxed answer $F(x, y) = C$, differentiate implicitly and rearrange; you must recover the given DE.

Marking-scheme angle. Examiners look for (i) the standard-form identification, (ii) the correct intermediate work, and (iii) the boxed final answer with an arbitrary constant. Skipping any of the three risks dropped marks even when the final answer is correct.

Final Answer: True.

Q 9.97 State True or False: *The differential equation of all non-horizontal lines in a*

plane is $\frac{d^2x}{dy^2} = 0$.

SOLUTION

True.

Concept used. A non-horizontal line can be written as $x = my + c$ (now treating x as a function of y , which is well-defined because the line is not horizontal \Rightarrow unique x for each y).

Step 1. $x = my + c$, two arbitrary constants m, c .

Step 2. Differentiate w.r.t. y : $\frac{dx}{dy} = m$.

Step 3. Differentiate again: $\frac{d^2x}{dy^2} = 0$.

Step 4. Both constants eliminated. DE is order 2.

Final Answer: True.

Exam Tip

This is the mirror of Q2 (non-vertical lines $\Rightarrow y'' = 0$). For non-horizontal lines, treat x as a function of y , giving $\frac{d^2x}{dy^2} = 0$.

EXPERT'S SOLUTION : Aarav Iyer, M.Sc Mathematics, IIT Bombay

Symmetry angle.

Step 1. Q2: non-vertical lines $y = mx + c \Rightarrow y'' = 0$.

Step 2. Mirror: non-horizontal lines $x = my + c \Rightarrow \frac{d^2x}{dy^2} = 0$.

Step 3. True.

Numerical sanity check. Substitute a simple specific value (or limit) into both the original DE and the integrated form to confirm consistency. Even a one-point check rules out the most common algebraic slips (sign errors, dropped factors of two, missing $+C$).

Reasoning recap. The strategy of identifying the standard form first (separable / linear / homogeneous), then applying the matching template, is the single biggest time-saver in any differential-equations exam. Marking schemes typically award separate credit for: (i) correct identification of the form, (ii) correct intermediate algebra, (iii) the final answer with an arbitrary constant present where required. Show each of these clearly.

Final Answer: True.

Key Takeaways

- The **order** of a differential equation is the highest derivative present; the **degree** is the power of that highest derivative once the equation is a polynomial in derivatives, and is undefined otherwise (e.g. when a derivative sits inside \sin , e^{\cdot} or under a radical that cannot be cleared).
- For a family of curves with n arbitrary constants, the differential equation obtained by eliminating those constants has order exactly n .
- **Variable separable** form $\frac{dy}{dx} = f(x)g(y)$ is solved by writing $\frac{dy}{g(y)} = f(x) dx$ and integrating both sides.
- A **homogeneous** DE $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ is solved by the substitution $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$, which makes the variables separable in v and x .
- A first-order **linear** DE $\frac{dy}{dx} + P(x)y = Q(x)$ is solved by the integrating factor I.F. = $e^{\int P dx}$, giving $y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dx + C$.

End of NCERT Exemplar Problems