Differential Equations

An equation involving derivatives of a dependent variable w.r.t. an independent variable is called a differential equation.

$$dy / dx = f(x, y)$$

<-ordinary DE
<-(one ind. var)

Examples

(i) $dy/dx + 3y = \sin x$

(ii) $d^2y/dx^2 + 5(dy/dx) + 6y = 0$

(iii) $(dy/dx)^3 + y^2 = e^x$

(iv) $y\,dx + x\,dy = 0$

Why study DE ?

DE describe how things change.
Population growth, radioactive decay, cooling, motion, current in circuits, mixture problems - all give ~~equation~~ DEs.

Two kinds

(a) Ordinary DE : one ind. variable.

(b) Partial DE : two or more ind. vars.

We study only ordinary DEs in class 12.

Order of a DE

Order = order of the highest order derivative that occurs in the DE.

Order = highest derivative's order

<-always a
<-positive integer

Examples (find order)

(i) $dy/dx + \sin x = 0$

highest is dy/dx

(ii) $d^2y/dx^2 + y = 0$ -> order = 2

(iii) $d^3y/dx^3 + x^2(d^2y/dx^2)^3 = 0$

highest order derivative = d 3 y/dx 3

therefore order = 3

Quick rule

Spot the ~~biggest~~ highest derivative present.
Its order (1, 2, 3 ...) is the order of DE.
Powers / coefficients do NOT affect order.

Note : (dy/dx) 5 has order 1, not 5.
Power on derivative != order of derivative.

Degree of a DE

Degree = power of the highest order derivative, AFTER the DE is made free from radicals & fractions in derivatives, and is polynomial in the derivatives.

Degree = power of highest deriv.

<-after clearing
<-radicals

Conditions for degree to exist

(i) DE must be polynomial in all derivatives.

(ii) No fractional or ~~irrational~~ negative powers on derivatives.

If sin(dy/dx) , e (dy/dx) , log(dy/dx) appears -> degree is not defined.

Example

$(d^2y/dx^2)^3 + (dy/dx)^2 + y = 0$

highest derivative : d 2 y/dx 2

its power : 3

order = 2, degree = 3

<-answer

Order & Degree - more examples

Eg 1

$(dy/dx)^3 + 2y(dy/dx) + 3 = 0$

polynomial in dy/dx, highest order = 1

order = 1 , degree = 3

Eg 2

$sqrt(1 + (dy/dx)^2) = d^2y/dx^2$

square both sides to remove sqrt :

$1 + (dy/dx)^2 = (d^2y/dx^2)^2$

order = 2 , degree = 2 <-clear radical <-first !

Eg 3

$dy/dx + \sin(dy/dx) = 0$

sin(dy/dx) is not polynomial in dy/dx;

so degree is NOT defined.

order = 1 , degree = N.D.

Eg 4

$(d^4y/dx^4) + 3\, d^3y/dx^3 + 2(dy/dx)^7 = 0$

highest deriv. d 4 y/dx 4, power = 1

order = 4 , degree = 1

Tip : Always reduce DE to polynomial form in derivatives BEFORE writing the degree.

General & Particular Solutions

Solution

A function $y = f(x)$ is a solution of the DE if it satisfies the DE identically.

General solution

Contains as many arbitrary constants as the order of the DE.

nth order DE -> n constants

<-primitive / <-complete soln

Particular solution

Obtained from the general solution by giving specific values to the constants. Usually constants are fixed using given initial / boundary conditions.

Eg.

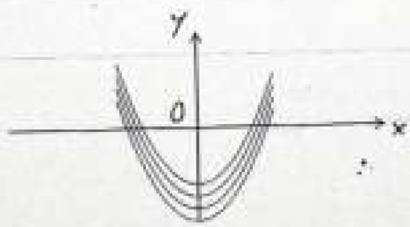
DE : $dy/dx = 2x$

general soln : $y = x^2 + c$

if $y(0) = 3$: $3 = 0 + c$ => $c = 3$

particular : $y = x \;\; 2 +$

3 <-I.V.P. <-answer



Formation of a DE

Given a family of curves containing n arbitrary constants, we can form an nth order DE by eliminating the constants.

Procedure

(1) Write the equation of the family.

(2) Differentiate w.r.t. x as many times as the number of constants.

(3) Eliminate the constants between the original eqn and its derivatives.

(4) The relation in x, y, y', y'' ... is the DE.

Eg. - family $y = mx$

family : $y = m x$ ...(1)

differentiate : $dy/dx = m$ ...(2)

from (2) put m in (1) :

$y = x\,(dy/dx)$

$$\boxed{x\, dy/dx - y = 0}$$

<- DE of all
<- lines thru 0

Note : the # of arbitrary constants equals the order of the resulting DE.

Formation - 2 constant example

Family

$y = a \sin(x + b)$ a, b arbitrary

Differentiate twice

$dy/dx = a \cos(x + b)$ ...(1)

$d^2y/dx^2 = - a \sin(x + b)$ ...(2)

Eliminate a, b

from (2) : $d^2y/dx^2 = - y$

(since $a \sin(x+b) = y$)

$$d^2y/dx^2 + y = 0$$

<-2nd order DE
<-of S.H.M.

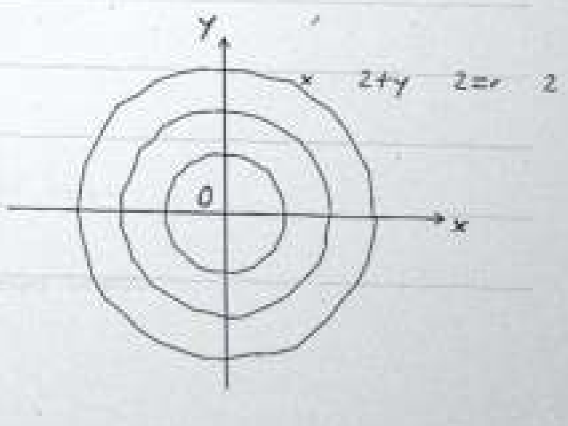
Eg - circle family

$x^2 + y^2 = r^2$ (r is constant)

differentiate : $2x + 2y(dy/dx) = 0$

$$x + y\,(dy/dx) = 0$$

<-DE of all
<-circles at O



n constants -> nth order DE

Methods of solving 1st order DE

•

We study three standard methods :

(I) Variable separable
(II) Homogeneous (substitution $y = vx$)
(III) Linear (use integrating factor)

Choose the right method

Look at the RHS $f(x, y)$ of $dy/dx = f(x, y)$:

* splits as $g(x) \cdot h(y)$ -> separable
* homogeneous in x, y -> put $y = vx$
* of form $dy/dx + Py = Q$ -> linear

$dy/dx = g(x)\, h(y)$ -> separable

$dy/dx = F(y/x)$ -> homogeneous

$dy/dx + P(x)\, y = Q(x)$ -> linear

Always check

1. separable ~~works~~ first.
2. if not, test homogeneous.
3. finally check linear form.

Always add + C after integrating.

Variable Separable Method

If a DE can be written in the form

$$dy/dx = g(x) \cdot h(y)$$

<-factor RHS
<-into x . y parts

Rearrange to put all y on one side and all x on the other :

$$dy \,/\, h(y) = g(x)\, dx$$

Integrate both sides (don't forget + C):

$$\int dy/h(y) = \int g(x)\, dx + C$$

<-implicit form
<-of soln

Steps

(1) Factorise RHS as $g(x)\, h(y)$.

(2) Separate : $dy/h(y) = g(x)\, dx$.

(3) Integrate both ~~ends~~ sides.

(4) Add a single constant of integration.

(5) Simplify, use initial condition if given.

Caution : never lose solns where $h(y) = 0$.

Separable - Example 1

Q.

solve $dy/dx = (1 + y^2) / (1 + x^2)$

Soln

separate : $dy / (1 + y^2) = dx / (1 + x^2)$

integrate both sides :

int $dy/(1+y^2)$ = int $dx/(1+x^2)$

$\tan^{-1} y = \tan^{-1} x + C$

$\boxed{\tan^{-1} y - \tan^{-1} x = C}$ <-general <-solution

Q. (with I.C.)

$dy/dx = 2 x y$, $y(0) = 1$

separate : $dy / y = 2 x \, dx$

integrate : $\ln y = x^2 + C$

$y = A e^{x^2}$ $(A = e^C)$

use $y(0)=1$: $1 = A.1 \Rightarrow A = 1$

$\boxed{y = e^{x^2}}$ <-particular soln

Separable - Example 2

Q.

solve $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Soln

divide by tan x . tan y :

$(\sec^2 x / \tan x) \, dx + (\sec^2 y / \tan y) \, dy = 0$

now each part depends on one variable.

integrate :

int $\sec^2 x/\tan x \, dx$ + int $\sec^2 y/\tan y \, dy = 0$

use d(tan u)/du = sec 2 u :

ln tan x + ln tan y = ln C

ln tan x . tan y = ln C

tan x . tan y = C	<-general <-solution

Tip

int f(x)/f(x) dx = ln f(x) + C

choose ~~=~~ ln C cleverly,

e.g. set C = ln K to remove log later.

It often simplifies the final form a lot.

Always check answer by differentiating.

Homogeneous Functions

A function $F(x, y)$ is said to be homogeneous of degree n in x, y if :

$$F(\lambda x, \lambda y) = \lambda^n F(x, y)$$

<- all terms
<- same degree

Test

$F(x, y) = x^2 + xy + y^2$

$F(\lambda x, \lambda y) = \lambda^2(x^2 + xy + y^2)$

$= \lambda^2 F(x, y)$

homogeneous of degree 2.

Alternate form

$F(x, y) = x^n \cdot g(y/x)$ or $y^n \cdot h(x/y)$

i.e. RHS depends on the ratio y/x (or x/y).

Examples

(i) $F = (x^2 - y^2) / xy$

homogeneous, deg 0

(ii) $F = x \sin(y/x) + y \cos(y/x)$

homogeneous, deg 1

(iii) $F = x^2 + \sin x$ - NOT homog.

Homogeneous DE - Method

A DE of the form

$$dy/dx = F(y/x)$$

<-or any homog. <-DE of degree 0

is solved by the substitution :

$$y = v x \Rightarrow dy/dx = v + x\, du/dx$$

<-key step !

After substitution

$v + x\, du/dx = F(v)$

$x\, du/dx = F(v) - v$

$dv / (F(v) - v) = dx / x$

now this IS variable separable in v, x.

Steps

(1) Check homogeneity of $f(x, y)$.

(2) Put $y = v x$, $dy/dx = v + x\, du/dx$.

(3) Separate variables, integrate.

(4) Substitute back ~~$v = x/y$~~ $v = y/x$.

(5) Apply initial condition if any.

Alt : if RHS = $F(x/y)$; put $x = v y$ instead.

Homogeneous - Example

Q.

solve $dy/dx = (x + y) / x$

Soln

RHS $= 1 + y/x$ function of y/x

homogeneous -> put $y = v x$

$dy/dx = v + x \, dv/dx$

$v + x \, dv/dx = 1 + v$

$x \, dv/dx = 1$

$dv = dx / x$

$v = \ln x + C$

put $v = y/x$:

$$y / x = \ln x + C$$

$$y = x (\ln x + C)$$ <-general <-soln

Check

$dy/dx = \ln x + C + x \cdot (1/x)$

$= \ln x + C + 1 = y/x + 1$

i.e. $dy/dx = (x + y)/x$, verified.

Homogeneous - 2nd Example

Q.

$(x^2 + y^2)\, dy = x\, y\, dx$

Rewrite

$dy/dx = x\, y \,/\, (x^2 + y^2)$

both numerator and denominator are homogeneous of degree 2 -> homogeneous DE.

Substitute $y = v\, x$

$v + x \,.\, du/dx = v \,/\, (1 + v^2)$

$x\, du/dx = v/(1 + v^2) - v$

$= - v^3 \,/\, (1 + v^2)$

separate :

$(1 + v^2)/v^3\, dv = - dx \,/\, x$

$\text{int } (v^{-3} + 1/v)\, dv = - \ln |x| + C$

$- 1/(2\, v^2) + \ln |v| = - \ln |x| + C$

put $v = y/x$:

$$- x^2/(2\, y^2) + \ln |y| = C$$

<- implicit
<- general soln

$\ln |x| + \ln |v| = \ln |xv| = \ln |y|$ used.

Linear DE - First Order

A DE of the form

$$dy/dx + P(x)\, y = Q(x)$$

<-standard
<-linear form

where P, Q are functions of x (or const.)
is called a linear DE in y.

Integrating Factor (I.F.)

$$I.F. = e^{\int P\, dx}$$

<-memorise !

Multiply the whole DE by I.F. Then
LHS becomes d/dx (y . I.F.) :

$$d/dx\, (y \cdot I.F.) = Q \cdot I.F.$$

Integrate both sides w.r.t. x :

$$y \cdot I.F. = \int (Q \cdot I.F.)\, dx + C$$

<-solution

Linear DE - Steps & Tips

Method

(1) Bring DE to form $dy/dx + P y = Q$.

(2) Identify $P(x)$ and $Q(x)$.

(3) Compute I.F. $= e^{\text{int } P \, dx}$.

(4) Multiply DE by I.F.

(5) LHS = $d/dx\,(y \cdot \text{I.F.})$.

(6) Integrate : $y \cdot \text{I.F.} = \text{int } Q \cdot \text{I.F.} \, dx + C$.

(7) Divide by I.F. to get y explicitly.

Watch for

* ~~forget~~ always rearrange $dy/dx + Py = Q$ first - coeff. of dy/dx must be 1.
* Inside e {int P dx} no constant of integration.
* C is added only after FINAL integration.
* If e {ln f} appears, simplify to f.

Useful identities

$e^{\ln u} = u$

$e^{-\ln u} = 1/u$

$e^{k \ln u} = u^k$

$e^{\ln u + \ln v} = u v$

These help simplify I.F. quickly.

Practice : compute I.F. mentally first.

Linear DE - Example 1

Q.

solve $dy/dx + y = e^{-x}$.

Soln

Compare with $dy/dx + P y = Q$:

$P(x) = 1$, $Q(x) = e^{-x}$

$I.F. = e^{\text{int } 1\, dx} = e^{x}$

multiply both sides by e . x :

$e^{x}\, dy/dx + e^{x} y = e^{x} . e^{-x} = 1$

$d/dx\, (y\, e^{x}) = 1$

integrate :

$y\, e^{x} = x + c$

$$\boxed{y = (x + C)\, e^{-x}}$$

<-general
<-solution

Quick check

$dy/dx = e^{-x} - (x+C)\, e^{-x}$

$dy/dx + y = e^{-x} - (x+C)e^{-x} + (x+C)e^{-x}$

$= e^{-x}$

checks out - always verify like this.

Linear DE - Example 2

Q.

solve $x\, dy/dx + 2y = x^2 \quad (x > 0)$

Make leading coeff = 1

divide by x : $dy/dx + (2/x)\, y = x$

$P(x) = 2/x$, $Q(x) = x$

Find I.F.

$\int P\, dx = \int 2/x\, dx = 2 \ln x$

$I.F. = e^{2 \ln x} = x^2$

Multiply & integrate

$x^2\, dy/dx + 2x\, y = x^3$

$d/dx\, (y\, x^2) = x^3$

$y\, x^2 = x^4/4 + C$

$$\boxed{y = x^2/4 + C / x^2}$$ <- general <- solution

If y(1) = 0

$0 = 1/4 + C \Rightarrow C = -1/4$

$$\boxed{y = x^2/4 - 1/(4x^2)}$$ <- particular <- soln

Linear DE - dx/dy form

Sometimes the DE is NOT linear in y but IS linear in x. Treat x as dependent var.

$$dx/dy + P(y)\, x = Q(y)$$ <- linear <- in x

$$I.F. = e^{\int P(y)\, dy}$$

Solution :

$$x \cdot I.F. = \int Q(y) \cdot I.F.\, dy + C$$

Eg.

$y\, dx - (x + 2y^2)\, dy = 0$

rewrite : $dx/dy - x/y = 2y$

$P(y) = -1/y$, $Q(y) = 2y$

$I.F. = e^{-\ln y} = 1/y$

$d/dy\,(x/y) = 2$

$x/y = 2y + C$

$$x = 2y^2 + Cy$$ <- answer

Applications - Growth & Decay

Law

Rate of change is proportional to the amount present at that instant.

$$dN / dt = \kappa N$$

<- $\kappa > 0$ growth
<- $\kappa < 0$ decay

Solve

separate : $dN / N = \kappa \, dt$

integrate : $\ln N = \kappa t + C$

$N = N_0 e^{\kappa t}$

$$N(t) = N_0 e^{\kappa t}$$

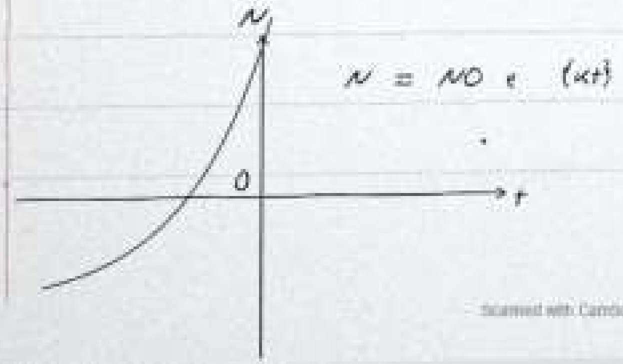
<- NO = initial
<- amount

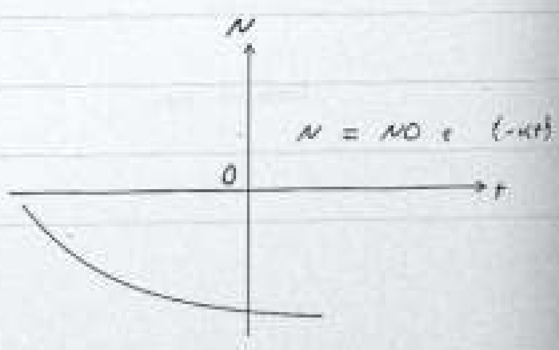
Half life (decay)

at half life $t = T$: $N = N_0/2$

$N_0/2 = N_0 e^{\kappa T}$

$T = \ln 2 / (-\kappa)$ $(\kappa < 0)$





Newton's Law of Cooling

Rate of change of temperature of a body is proportional to (T - T_s), where T_s is the temperature of surroundings.

$$dT/dt = -\kappa (T - T_s)$$ <- $\kappa > 0$ <-constant

Solve (separable)

$d(T - T_s) / (T - T_s) = -\kappa \, dt$

$\ln \quad T - T_s = -\kappa t + C$

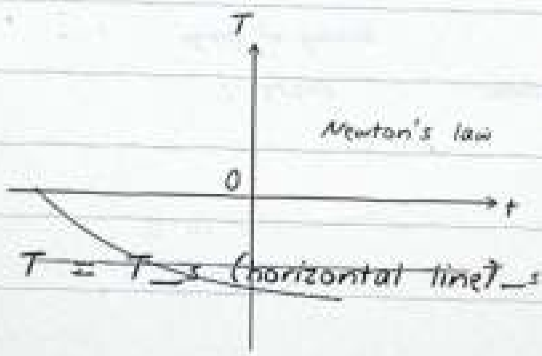
$T - T_s = A e^{-\kappa t}$

$$T(t) = T_s + (T_0 - T_s) e^{-\kappa t}$$ <-T0=init

Observation

as t -> inf, ~~T -> 0~~ T -> T_s

body cools to surrounding temperature.



asymptote : ~~T = T_s (horizontal line)~~ T_s

Mixture / Population problems

Eg - Mixture

Salt water flows into a tank at rate $r1$ L/min with conc. $c1$ g/L. Mixture leaves the tank at rate $r2$ L/min. Let $Q(t)$ be amount of salt at time t.

$$dQ/dt = c1 \cdot r1 - (Q / V)\, r2$$

<- in - out
<- balance

if $r1 = r2 = r$, V constant, the DE is linear in Q -> solve by integrating factor.

Eg - Population (logistic-free)

$dP/dt = \kappa P$ (small population)

$P(t) = P_0 e^{\kappa t}$

Bacteria doubling

P doubles in time T : $2 = e^{\kappa T}$

$\kappa = \ln 2 / T$

Money (compound)

$dA/dt = r A$ $A(t) = A_0 e^{rt}$

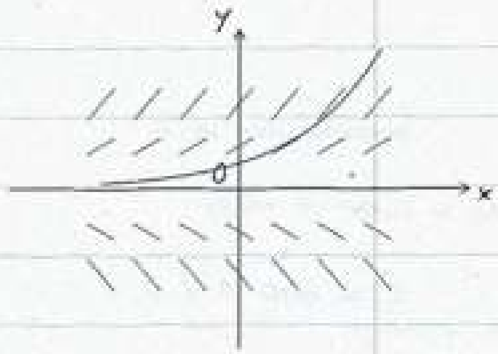
continuous compounding at rate r.

All such laws are special cases of growth.

Solution Curves & Slope Field

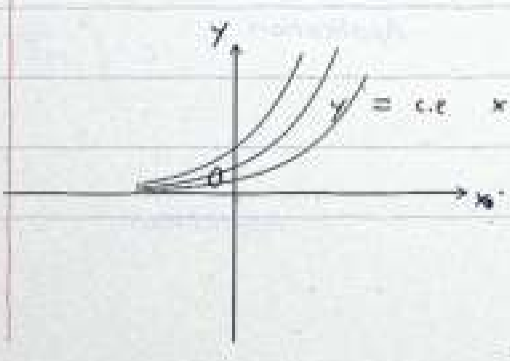
Geometric meaning of $dy/dx = f(x, y)$:
at every point (x, y) it gives the slope.

Drawing tiny dashes with these slopes gives a slope field. A solution curve is everywhere tangent to the dashes.



Example shown : $dy/dx = y$
solution : $y = C e^x$

Family of solutions



each value of C gives one solution curve.
I.C. picks one curve.
$y(0)=1 \Rightarrow C = 1.$

Initial Value Problems (I.V.P.)

An I.V.P. is a DE together with values of the unknown function (and possibly its derivatives) at a single point.

$$dy/dx = f(x, y), \quad y(x_0) = y_0$$

<-1st order <-I.V.P.

Steps

(1) Solve the DE for the general soln.

(2) Plug in the given values to find C.

(3) Write the particular solution.

Eg.

$dy/dx = 3x^2, \quad y(0) = 1$

general : $y = x^3 + C$

$y(0) = 1$: $1 = 0 + C$, $C = 1$.

$$y = x^3 + 1$$

Existence & uniqueness (idea)

If $f(x, y)$ and f/y are continuous in a region around $(x0, y0)$, then a unique soln of the I.V.P. exists near $(x0, y0)$.

Common Mistakes & Tips

Mistakes

1. forgetting + C after ~~derivative~~ integration.

2. not making the coeff. of dy/dx = 1 before finding P, Q (linear DE).

3. adding C inside the exponent of I.F. e {int P dx + C} is wrong.

4. confusing order with degree. $(dy/dx)^5$ has order 1, degree 5.

5. losing trivial solns when dividing by factors that can be zero.

Tips

* Always rewrite in standard form first.
* Test : homog. ? put y = vx.
* Linear ? compute I.F.
* Both sides of ln . stay positive.
* Replace e C by A (positive const.).
* Verify the answer by differentiation.
* Initial condition fixes the constant.

Always check whether degree exists.

Quick Formula Sheet - I

Standard forms

Separable : $dy/dx = g(x)\, h(y)$

Homogeneous : $dy/dx = F(y/x)$

Linear : $dy/dx + P\,y = Q$

Key substitutions

$y = v\,x \Rightarrow dy/dx = v + x\, dv/dx$

$x = v\,y \Rightarrow dx/dy = v + y\, dv/dy$

Integrating factor

$dy/dx + P(x)\, y = Q(x)$

$I.F. = e^{\int P\, dx}$

$dx/dy + P(y)\, x = Q(y)$

$I.F. = e^{\int P\, dy}$

Solutions

$y \cdot I.F. = \int Q \cdot I.F.\, dx + C$

$x \cdot I.F. = \int Q \cdot I.F.\, dy + C$

Both forms come from the same idea.

Quick Formula Sheet - II

Standard integrals (used a lot)

int dx/x = ln x + C

int e^{ax} dx = e^{ax} / a + C

int dx/(1 + x^2) = $\tan^{-1}$ x + C

int dx / sqrt(1 - x^2) = $\sin^{-1}$ x + C

int $\sec^2$ x dx = tan x + C

int f'(x)/f(x) dx = ln f(x) + C

Applications recap

Growth : $N = N_0 e^{kt}$

Decay : $N = N_0 e^{-kt}$

Cooling : T = T_s + (T_0 - T_s) e^{-kt}

Half-life

T = ln 2 / k (decay constant k)

Doubling time

T = ln 2 / k (growth rate k)

Remember : a DE describes the rate of change ; the soln gives the function itself.

Practice Problems

Order & degree

(1) $(d^2y/dx^2) + 3(dy/dx) + 5y = 0$

Ans : order 2, degree 1

(2) $(dy/dx)^2 + \sin(d^2y/dx^2) = 0$

Ans : order 2, degree N.D.

Separable

(3) $dy/dx = x\, e^{-y}$

Ans : $e^y = x^2/2 + C$

Homogeneous

(4) $x\, dy - y\, dx = \text{sqrt}(x^2 + y^2)\, dx$

Hint : put $y = v\, x$.

Linear

(5) $dy/dx + 2\, y = \sin x$

I.F. $= e^{2x}$

$y\, e^{2x} = \text{int}\ e^{2x} \sin x\, dx + C$

(6) $x\, dy/dx + y = x \ln x$, $x > 0$

I.F. $= x$ $\to$ $d/dx\, (x\, y) = x \ln x$

Application

Practice (contd.) & Summary

Worded problem

(7) Bacteria triple in 5 hours. Find time to multiply 10 times the original.

$dN/dt = \kappa N \Rightarrow N = N_0 e^{\kappa t}$

$3 N_0 = N_0 e^{5\kappa} \Rightarrow \kappa = (\ln 3)/5$

$10 = e^{\kappa T} \Rightarrow T = \ln 10 / \kappa$

$T = 5 \ln 10 / \ln 3$ 10.48 hr

Final summary

* Order = highest derivative's order.
* Degree = power of that derivative (after clearing radicals).
* 3 main methods of 1st order DE : separable / homog. / linear.
* Always add + C ; use I.C. for particular.
* I.F. converts linear DE to perfect deriv.
* Applications : growth, decay, cooling, mixtures, populations, finance.

Golden rule

Identify form -> Apply method -> Verify

<-always

End of Chapter 9.