Collegedunia NCERT Notes

The Ultimate NCERT Revision Guide for Class 12 Mathematics

Chapter 9: Differential Equations

What this chapter covers: the language of differential equations — order, degree, general and particular solutions — followed by the three workhorse methods for first-order first-degree DEs: *variables separable*, *homogeneous*, and *linear* (integrating factor). Growth/decay applications and Bernoulli/variation-of-parameters lie outside the rationalised NCERT; we cover the exam-relevant pieces as [JEE/NEET Extension] sections.

1 Introduction and Basic Concepts

A **differential equation** (DE) is any equation that relates an unknown function $y = f(x)$ to its derivatives. Where algebra gave us equations like $x^2 - 3x + 2 = 0$ whose solutions are *numbers*, calculus gives us equations like $\frac{dy}{dx} = 2x$ whose solutions are *functions* — here, $y = x^2 + C$.

Differential equations are the native language of change. The motion of a falling body, the discharge of a capacitor, radioactive decay, the spread of an epidemic, the cooling of a cup of tea — every quantitative law in physics, chemistry, biology, or economics that involves a rate of change is, at its core, a differential equation.

1.1 Ordinary vs Partial DEs

An **ordinary differential equation** (ODE) involves derivatives of y with respect to a *single* independent variable x. A **partial differential equation** (PDE) involves partial derivatives of a function of *two or more* independent variables (e.g. the heat equation $\partial u/\partial t = k\,\partial^2 u/\partial x^2$). The Class 12 syllabus deals exclusively with ordinary DEs.

Ordinary Differential Equation

An equation involving an unknown function $y(x)$ and one or more of its derivatives:

$$F\left(x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \dots,\ \frac{d^ny}{dx^n}\right) = 0.$$

We write $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$, and $y_n = \frac{d^ny}{dx^n}$ for compactness.

1.2 Order of a Differential Equation

The **order** of a DE is the order of the *highest* derivative that appears in it. Order is always a positive integer.

Order

Order $=$ highest n such that $\frac{d^ny}{dx^n}$ appears in the equation.

Examples.

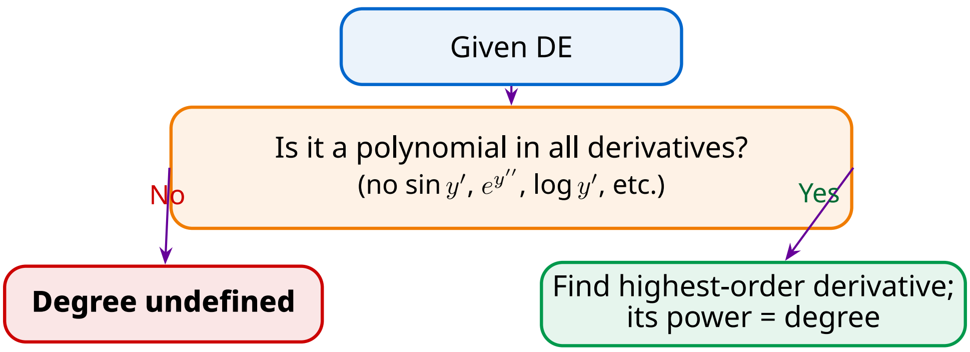
- $\frac{dy}{dx} = e^x$ — highest derivative is y', so order = 1.
- $\frac{d^2y}{dx^2} + y = 0$ — highest is y'', order = 2.
- $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$ — highest is y''', order = 3.

1.3 Degree of a Differential Equation

The **degree** is the highest power of the *highest-order* derivative, *provided* the DE is a polynomial in all derivatives. If any derivative appears inside a transcendental function (sin, cos, log, $e^{\cdot}$ etc.), the degree is **not defined**.

Degree

Step 1: Check that the DE can be written as a polynomial in $y', y'', \ldots, y^{(n)}$. If *any* derivative sits inside a non-polynomial function, degree is undefined.
Step 2: Identify the highest-order derivative.
Step 3: The power of that highest-order derivative (after clearing radicals/fractions) is the degree.



Worked examples.

- $\frac{dy}{dx} + \sin x \cdot y = 0$: polynomial in y' (no derivative inside a transcendental). Highest derivative y', power 1. **Order 1, Degree 1.**
- $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 - y = 0$: polynomial in y'', y'. Highest derivative y'', its power is 2. **Order 2, Degree 2.**
- $\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0$: y' sits inside sin, so the DE is not a polynomial in y'. **Order 1, Degree not defined.**
- $\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0$: polynomial in y''', y''. Highest is y''', power 1. **Order 3, Degree 1** (the cube on y'' does *not* count — only the highest derivative's power matters).

Degree is the power of the *highest-order* derivative, not the largest power

In $(y''')^2 + (y'')^3 + (y')^4 = 0$ the highest order is y''', raised to power **2**. So degree = 2 (not 4, not 3). The cube on y'' and the fourth power on y' are irrelevant — degree looks only at the *top* derivative.

Clear radicals first

If a DE contains $\sqrt{y''}$ or $(y')^{1/3}$, square or cube both sides until every derivative appears to an integer power. Then read off the degree. If after this rationalisation any transcendental function still wraps a derivative, the degree is undefined.

Order and Degree are always positive integers

Both are defined to be positive integers whenever they exist. If you compute a fractional or negative value, you have made an algebraic slip — rationalise before deciding.

Contents

2 General and Particular Solutions

A **solution** of a DE on an interval I is any function $y = \phi(x)$ that, together with its derivatives, satisfies the DE identically on I. Geometrically, the graph $y = \phi(x)$ is called a **solution curve** or **integral curve**.

2.1 Family of Solutions: the General Solution

Differentiating reduces information; integrating restores it — with one arbitrary constant per integration. So an n**th-order** DE typically has a solution involving n **arbitrary constants**, called the **general solution** (or primitive). Different choices of constants give different members of a one-parameter (or n-parameter) family of curves.

Counting Arbitrary Constants

$$\text{order of DE} = n \iff \text{general solution has exactly } n \text{ arbitrary constants.}$$

A **particular solution** is obtained by assigning specific numerical values to those constants (usually via initial/boundary conditions).

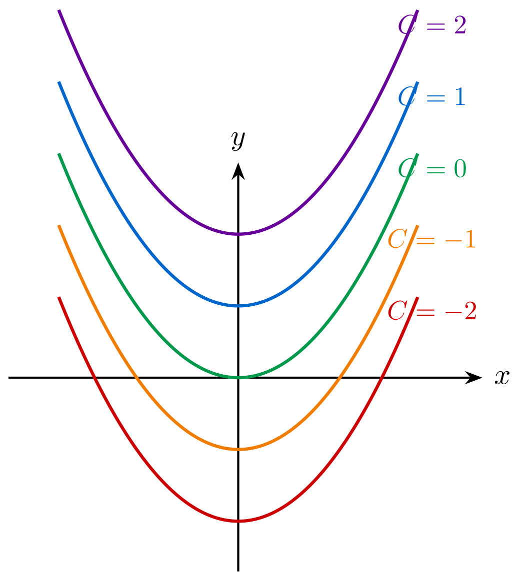
Example. The second-order DE $\dfrac{d^2y}{dx^2} + y = 0$ has general solution

$$y = A\cos x + B\sin x, \qquad A, B \in \mathbb{R}.$$

Two arbitrary constants (A and B) — matching the order 2. Setting $A = 2,\ B = -1$ gives the particular solution $y = 2\cos x - \sin x$.

2.2 Visualising a Family of Curves

The general solution defines an entire family of curves, one per choice of constant. Plotting several members together is the cleanest way to *see* what the DE encodes.



Family $y = \frac{1}{2}x^2 + C$ *— general solution of* $\frac{dy}{dx} = x$.

Every curve in the family has the same slope $\frac{dy}{dx} = x$ at each x — only the vertical offset C differs. An initial condition like $y(0) = 1$ picks out the single curve passing through $(0, 1)$, namely $C = 1$.

2.3 Verifying a Solution

Given a candidate function and a DE, you check the candidate by computing the relevant derivatives and substituting them into the DE. If the equation reduces to an identity (LHS $=$ RHS for all x in the domain), the candidate is a solution.

Verify $y = e^{-3x}$ *is a solution of* $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$.

Differentiate: $y' = -3e^{-3x}$, $y'' = 9e^{-3x}$. Substitute:

$$\text{LHS} = 9e^{-3x} + (-3e^{-3x}) - 6e^{-3x} = (9 - 3 - 6)e^{-3x} = 0 = \text{RHS}. \checkmark$$

Verify $y = a\cos x + b\sin x$ *is a solution of* $y'' + y = 0$.

$y'' = -a\cos x - b\sin x = -y$, so $y'' + y = 0$. ✓

Checklist before claiming a function "solves" a DE

1. Compute all derivatives that appear in the DE.
2. Substitute every derivative back in.
3. Simplify the LHS algebraically.
4. Confirm the simplified LHS equals the RHS **as an identity in** x — not just at one point.

Particular solution has *zero* arbitrary constants

A particular solution of a third-order DE has **0** arbitrary constants, not 3. The 3 constants of the general solution are precisely what an initial condition pins down. Common MCQ trap: "Number of arbitrary constants in the particular solution of a 4th-order DE" — the answer is 0, not 4.

The family $\to$ specific curve translation

A radioactive sample obeys $\frac{dN}{dt} = -\lambda N$; its general solution $N = N_0 e^{-\lambda t}$ describes *any* sample of *any* starting size. To match *your particular* sample you need one measurement — e.g. $N(0) = 10^{20}$ atoms — which sets the constant N_0. General solution = the physics; particular solution = the experiment.

3 Method 1: Variables Separable

The simplest and most common technique. If the right-hand side of $\frac{dy}{dx} = F(x, y)$ factors as a product of a function of x alone and a function of y alone, we can separate the variables and integrate.

3.1 Recognising a separable DE

A first-order DE is **variables-separable** if it can be brought to the form

$$\frac{dy}{dx} = g(x)\,h(y) \qquad \iff \qquad \frac{1}{h(y)}\,dy = g(x)\,dx.$$

The two variables sit on opposite sides; each side then integrates independently.

Separable DE — General Recipe

$$\frac{dy}{dx} = g(x)\,h(y) \implies \int \frac{dy}{h(y)} = \int g(x)\,dx + C$$

provided $h(y) \neq 0$.

3.2 Step-by-step solution flow

1. Write the DE as $\dfrac{dy}{dx} = g(x)\,h(y)$
2. Separate: $\dfrac{dy}{h(y)} = g(x)\,dx$
3. Integrate both sides
4. Add a single constant C; simplify

3.3 Worked example 1 — a polynomial RHS

Solve $\dfrac{dy}{dx} = \dfrac{x+1}{2-y}$ *for* $y \neq 2$.

Rearrange so each variable sits on its own side:

$$(2-y)\,dy = (x+1)\,dx.$$

Integrate:

$$\int (2-y)\,dy = \int (x+1)\,dx \implies 2y - \frac{y^2}{2} = \frac{x^2}{2} + x + C_1.$$

Multiply by 2 and rename the constant:

$$\boxed{x^2 + y^2 + 2x - 4y + C = 0} \quad (C = 2C_1).$$

3.4 Worked example 2 — separable on first look

Solve $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$.

Since $1 + y^2 \neq 0$, separate:

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2} \implies \tan^{-1} y = \tan^{-1} x + C.$$

3.5 Worked example 3 — a particular solution

Solve $\dfrac{dy}{dx} = -4xy^2$ *with* $y(0) = 1$.

Separate (assuming $y \neq 0$):

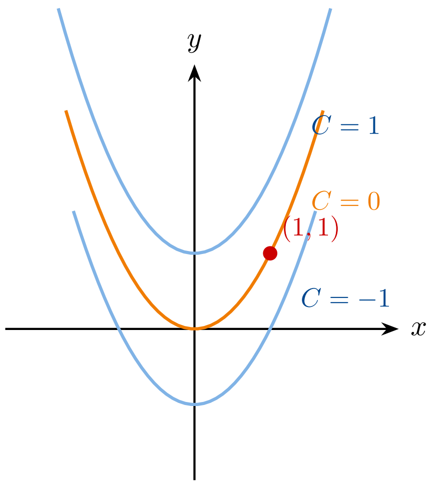
$$\frac{dy}{y^2} = -4x\,dx \implies -\frac{1}{y} = -2x^2 + C \implies y = \frac{1}{2x^2 - C}.$$

Use $y(0) = 1$: $1 = -1/C$, so $C = -1$. Particular solution:

$$\boxed{y = \frac{1}{2x^2 + 1}}.$$

3.6 The solution curve through a point

A separable DE plus an initial condition $y(x_0) = y_0$ picks out one curve from the family. The diagram below shows three members of $y = x^2 + C$ together with the unique member passing through $(1, 1)$.



Initial condition $y(1) = 1$ selects the curve $y = x^2$ (orange).

3.7 Worked example 4 — a curve through a given point

Find the curve passing through $(1, 1)$ satisfying $x\,dy = (2x^2 + 1)\,dx$, $x \neq 0$.

Divide by x:

$$dy = \left(2x + \frac{1}{x}\right) dx \implies y = x^2 + \log|x| + C.$$

Substitute $(1, 1)$: $1 = 1 + 0 + C$, so $C = 0$. Required curve: $\boxed{y = x^2 + \log|x|}$.

Three signs of separability

The RHS is separable if you can do *one* of the following:

(a) Factor it as $g(x) \cdot h(y)$.

(b) Move all y-terms (including dy) to one side, all x-terms (including dx) to the other.

(c) Recognise it as a quotient $\dfrac{P(x)}{Q(y)}$ or $\dfrac{P(x)\,Q(y)}{R(x)\,S(y)}$ that splits cleanly.

If none of these works, try the homogeneous or linear methods instead.

Don't forget the constant of integration

A solution of a first-order DE *must* carry one arbitrary constant. If you write "$\log y = x$" instead of "$\log y = x+C$", you have written down a single particular solution and called it the general solution. Always add C on the RHS before exponentiating or rearranging.

"Split, integrate, combine"

Separable DEs follow three beats: **Split** the variables, **Integrate** each side, **Combine** into the implicit relation $H(y) = G(x)+C$. Three steps, one constant.

4 Method 2: Homogeneous Differential Equations

When the variables can't be separated directly, a clever substitution may reduce the DE to a separable one. The first big example of this is the homogeneous DE.

4.1 Homogeneous functions

A function $F(x, y)$ is **homogeneous of degree** n if

$$F(\lambda x,\, \lambda y) = \lambda^n\, F(x, y) \quad \text{for every } \lambda \neq 0.$$

Examples.

- $F(x, y) = x^2 + xy$ has $F(\lambda x, \lambda y) = \lambda^2(x^2 + xy) = \lambda^2 F(x, y)$ — degree 2.
- $F(x, y) = \cos(y/x)$ has $F(\lambda x, \lambda y) = \cos(y/x) = \lambda^0 F(x, y)$ — degree 0.
- $F(x, y) = \sin x + \cos y$ is *not* homogeneous — no λ^n factor pops out.

4.2 Homogeneous DE: the standard form

A first-order DE $\dfrac{dy}{dx} = F(x, y)$ is called **homogeneous** when F is homogeneous of degree zero, i.e. $F(\lambda x, \lambda y) = F(x, y)$ for every $\lambda \neq 0$. Equivalently, F depends only on the ratio y/x:

$$F(x, y) = g\left(\frac{y}{x}\right).$$

Homogeneous DE

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right).$$

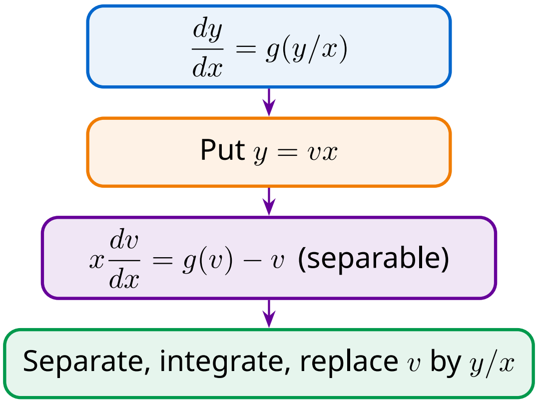
Substitute $y = vx$ (so $v = y/x$). Then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}, \qquad \text{and the DE becomes} \qquad x\frac{dv}{dx} = g(v) - v,$$

which is variables-separable in v and x.

4.3 The substitution $y = vx$

The substitution turns the unsolvable-looking ratio y/x into a single variable v, separating x from v.



4.4 Worked example 5 — a classic linear-fraction RHS

Show that $(x - y)\,\dfrac{dy}{dx} = x + 2y$ *is homogeneous and solve it.*

Rearrange: $\dfrac{dy}{dx} = \dfrac{x+2y}{x-y}$. Divide top and bottom by x:

$$\frac{dy}{dx} = \frac{1 + 2(y/x)}{1 - (y/x)} = g(y/x).$$

So the DE is homogeneous of degree zero. Substitute $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{1+2v}{1-v} \implies x\frac{dv}{dx} = \frac{1+2v}{1-v} - v = \frac{1+v+v^2}{1-v}.$$

Separate and integrate:

$$\int \frac{1-v}{v^2+v+1}\,dv = \int \frac{dx}{x}.$$

After standard partial-fraction/completing-square work and replacing $v = y/x$, the general solution is

$$\log(x^2 + xy + y^2) = 2\sqrt{3}\,\tan^{-1}\left(\frac{x+2y}{\sqrt{3}\,x}\right) + C.$$

4.5 The dual substitution $x = vy$

If the DE is naturally written as $\dfrac{dx}{dy} = h(x/y)$, use $x = vy$ instead. Same idea, just with x and y swapped in their roles.

How to spot a homogeneous DE in 5 seconds

Look at $\frac{dy}{dx} = F(x, y)$. If every term in F (numerator and denominator combined, before any cancellation) has the **same total degree** in x and y, the DE is homogeneous. E.g. in $\frac{x^2+y^2}{2xy}$ both top (x^2, y^2) and bottom (xy) are degree 2 — homogeneous.

Homogeneous DE $\neq$ homogeneous linear DE

Some textbooks (and older syllabi) also use "homogeneous" for a *linear* DE with zero right-hand side, like $y'' + 4y = 0$. That is a *different* use of the word. In Class 12 NCERT, "homogeneous DE" means $F(x, y)$ is a homogeneous function of degree zero, period.

4.6 Worked example 6 — a transcendental homogeneous DE

Solve $x\cos(y/x)\,\frac{dy}{dx} = y\cos(y/x) + x$.

Divide by $x\cos(y/x)$:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{1}{\cos(y/x)}.$$

Put $y = vx$:

$$v + x\frac{dv}{dx} = v + \sec v \implies \cos v\, dv = \frac{dx}{x} \implies \sin v = \log|x| + \log|C|.$$

Replacing $v = y/x$: $\sin(y/x) = \log|Cx|$.

The trick in one sentence

A homogeneous DE collapses to a separable one through the substitution $y = vx$ (or $x = vy$). After integrating, restore the original variables by writing $v = y/x$.

5 Method 3: Linear Differential Equations

The third method handles a different shape: DEs that are *linear* in y and y', but with x-dependent coefficients on y.

5.1 The standard linear form

A first-order DE is **linear** when it can be written as

$$\frac{dy}{dx} + P(x)\,y = Q(x),$$

where P and Q are functions of x alone (or constants). The defining feature: y and y' each appear to the first power, never as a product or inside a transcendental.

Linear DE — Standard Form

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

Integrating Factor (IF) $= e^{\int P(x)\,dx}$.
Solution: $y \cdot \text{IF} = \int Q(x) \cdot \text{IF}\, dx + C$.

5.2 Why the integrating factor works

Multiplying the DE by $\mu(x) = e^{\int P\,dx}$ makes the LHS a perfect derivative:

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y = \frac{d}{dx}\big[\mu(x)\,y\big].$$

This is because $\frac{d\mu}{dx} = \mu(x)P(x)$ by construction (that's the whole purpose of the choice). Integrating both sides then immediately gives the solution.

The whole trick in one line

The IF is engineered so that $\frac{d}{dx}(y \cdot \text{IF}) = Q \cdot \text{IF}$. Integrate once, divide by IF, done.

5.3 The four-step recipe

1. Bring the DE to the form $\frac{dy}{dx} + P(x)\,y = Q(x)$
2. Compute IF $= e^{\int P\,dx}$
3. Write $y \cdot \text{IF} = \int Q \cdot \text{IF}\, dx + C$
4. Solve the integral on the right, then isolate y

5.4 Worked example 7 — a basic linear DE

Solve $\frac{dy}{dx} - y = \cos x$.

Here $P = -1$, $Q = \cos x$. IF $= e^{\int -1\,dx} = e^{-x}$. Multiply through and integrate:

$$y\,e^{-x} = \int e^{-x}\cos x\,dx + C.$$

Using integration by parts twice (or the standard formula):

$$\int e^{-x}\cos x\,dx = \frac{e^{-x}(\sin x - \cos x)}{2}.$$

So $y = \dfrac{\sin x - \cos x}{2} + Ce^{x}$.

5.5 Worked example 8 — when P involves $1/x$

Solve $x\dfrac{dy}{dx} + 2y = x^2$ *for* $x \neq 0$.

Divide by x: $\dfrac{dy}{dx} + \dfrac{2}{x}y = x$. So $P = 2/x$, $Q = x$, and

$$\text{IF} = e^{\int 2/x\,dx} = e^{2\log|x|} = x^2.$$

Solution:

$$y\,x^2 = \int x\cdot x^2\,dx + C = \frac{x^4}{4} + C \implies y = \frac{x^2}{4} + \frac{C}{x^2}.$$

5.6 Worked example 9 — linear in x, not in y

Some DEs are linear when rewritten with x as the dependent variable. The dual standard form is

$$\frac{dx}{dy} + P_1(y)\,x = Q_1(y), \qquad \text{IF} = e^{\int P_1\,dy}.$$

Solve $y\,dx - (x + 2y^2)\,dy = 0$.

Divide by $y\,dy$ and rearrange:

$$\frac{dx}{dy} - \frac{x}{y} = 2y, \quad \text{so} \quad P_1 = -\frac{1}{y},\ Q_1 = 2y.$$

$\text{IF} = e^{\int -1/y\,dy} = e^{-\log y} = 1/y$. Then

$$x\cdot\frac{1}{y} = \int 2y\cdot\frac{1}{y}\,dy + C = 2y + C \implies x = 2y^2 + Cy.$$

5.7 Worked example 10 — particular solution with cot x

Solve $\dfrac{dy}{dx} + y\cot x = 2x + x^2\cot x$ *with* $y(\pi/2) = 0$.

$P = \cot x$, $Q = 2x + x^2\cot x$. $\text{IF} = e^{\int \cot x\,dx} = e^{\log|\sin x|} = \sin x$. Then

$$y\sin x = \int (2x + x^2\cot x)\sin x\,dx + C = \int 2x\sin x\,dx + \int x^2\cos x\,dx + C.$$

Use integration by parts ($\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$): the two integrals cancel out neatly, giving

$$y\sin x = x^2\sin x + C.$$

Use $y(\pi/2) = 0$: $0 = (\pi/2)^2 + C$, so $C = -\pi^2/4$. Particular solution:

$$y = x^2 - \frac{\pi^2}{4\sin x}.$$

Spotting linear vs separable

If the DE has the shape "$y' + (\text{stuff in } x)\, y = (\text{stuff in } x)$", reach for the integrating factor. If it factors as $g(x)h(y)$, separate variables. If neither, check homogeneity. These three methods together handle the vast majority of Class 12 / JEE first-order DEs.

Bring to standard form *before* reading off P

If you see $xy' + 2y = x^2$, don't read $P = 2$. Divide by x first: $y' + (2/x)y = x$, so $P = 2/x$. Forgetting this gives the wrong IF and a wrong answer.

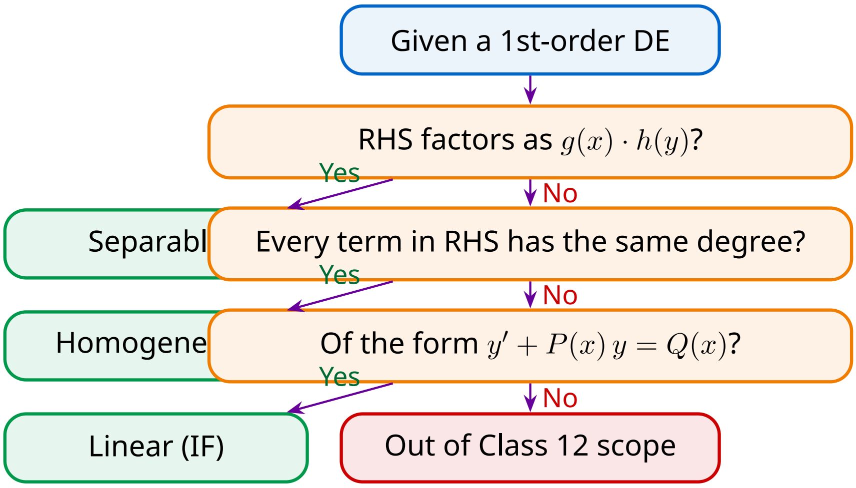
PIE: P, IF, Equation

For any linear DE, march in this order: **P**arameter (read off P and Q), **I**ntegrating **F**actor ($e^{\int P\,dx}$), then write the **E**quation $y \cdot \text{IF} = \int Q \cdot \text{IF}\, dx + C$. P-I-E. Three letters, three steps.

5.8 Comparison of the three methods

Method	Standard form	Key step	Final shape
Separable	$\frac{dy}{dx} = g(x)\,h(y)$	Split, then integrate both sides	$H(y) = G(x) + C$
Homogeneous	$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$	Put $y = vx$; becomes separable in v, x	implicit in y/x
Linear (1st order)	$\frac{dy}{dx} + P(x)\,y = Q(x)$	$\text{IF} = e^{\int P\,dx}$	$y\,\text{IF} = \int Q\,\text{IF}\,dx + C$

5.9 A decision flow for picking the method



Where linear DEs show up

An RL circuit driven by an EMF source obeys $L\,\frac{di}{dt} + R\,i = V(t)$ — exactly the linear DE with $P = R/L$ and $Q = V(t)/L$. The integrating factor $e^{Rt/L}$ predicts how the current builds up after the switch is closed. The same maths runs Newton's law of cooling, drug elimination from the bloodstream, and the charging of a capacitor.

6 Growth, Decay, and Applications [JEE/NEET Extension]

Population growth, radioactive decay, Newton's law of cooling, and continuously compounded interest were trimmed from the rationalised NCERT but remain favourite contexts for JEE and NEET word problems. Each reduces to a first-order DE solvable by separation of variables.

6.1 The exponential law of natural growth and decay

If a quantity $N(t)$ changes at a rate proportional to its current size, then

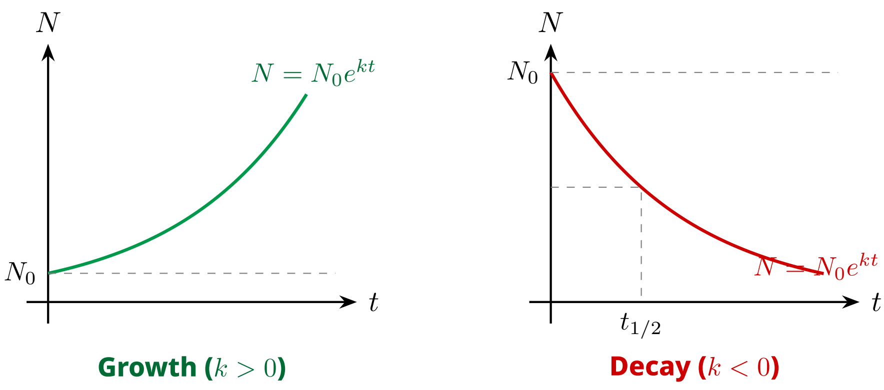
$$\frac{dN}{dt} = kN \quad \Longrightarrow \quad N(t) = N_0\,e^{kt},$$

where $N_0 = N(0)$. $k > 0$ gives *growth*; $k < 0$ gives *decay*.

Exponential Growth / Decay

$$\frac{dN}{dt} = kN, \qquad N(t) = N_0 e^{kt}.$$

Doubling time (when $k > 0$): $t_2 = \dfrac{\log 2}{k}$. **Half-life** (when $k < 0$): $t_{1/2} = \dfrac{\log 2}{|k|}$.



6.2 Worked example — bacterial growth

A culture has 10^5 *bacteria. After 2 hours, the count is* 1.1×10^5. *Find when the count reaches* 2×10^5*, given that growth is proportional to the present count.*

DE: $\dfrac{dN}{dt} = kN \Rightarrow N = N_0 e^{kt}$ with $N_0 = 10^5$.

At $t = 2$: $1.1 \times 10^5 = 10^5\, e^{2k} \Rightarrow k = \frac{1}{2}\log(1.1)$.

Set $N = 2 \times 10^5$: $e^{kt} = 2 \Rightarrow t = \dfrac{\log 2}{k} = \dfrac{2 \log 2}{\log 1.1} \approx 14.55$ hours.

6.3 Newton's Law of Cooling

The rate at which a body cools is proportional to the temperature difference between the body and its surroundings:

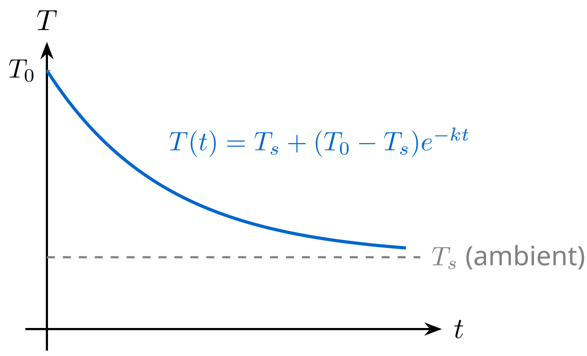
$$\frac{dT}{dt} = -k(T - T_s), \qquad T(0) = T_0.$$

This is linear (and separable). Solution:

$$T(t) = T_s + (T_0 - T_s)\, e^{-kt}.$$

The cooling curve

$T(t)$ approaches the ambient T_s exponentially. After a long time, the body and surroundings reach thermal equilibrium — the temperature difference decays with time constant $1/k$.

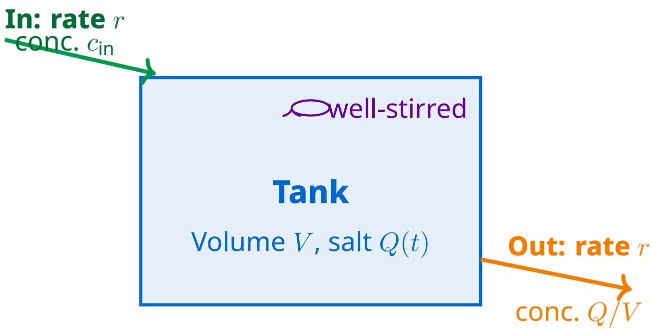


6.4 Mixing-tank problem

A tank holds V litres of brine with $Q(t)$ kg of salt. Fresh brine (concentration c_{in}) flows in at rate r litres/min; the well-stirred mixture flows out at the same rate. Conservation of salt gives

$$\frac{dQ}{dt} = r\,c_{\text{in}} - \frac{r}{V}\,Q,$$

a linear first-order DE in Q.



Salt balance: $\frac{dQ}{dt} = r\,c_{in} - \frac{r}{V}\,Q$ *(linear DE)*

6.5 Worked example — continuously compounded interest

If principal increases continuously at r% per year and Rs 100 doubles in 10 years, find r. (Use $\log 2 = 0.6931$.)

DE: $\frac{dP}{dt} = \frac{r}{100}P \Rightarrow P = P_0 e^{rt/100}$.

$2P_0 = P_0 e^{10r/100} \Rightarrow 10r/100 = \log 2 \Rightarrow r = 10\log 2 \approx 6.93$%.

"Rate proportional to amount" $\Rightarrow$ exponential

Whenever a problem says "the rate of change of X is proportional to X", the answer is an exponential. Skip the DE setup and jump to $X(t) = X_0 e^{kt}$ in your head — just figure out k from the data.

Carbon-14 dating

Living tissue maintains a steady ratio of ^{14}C to ^{12}C through the atmosphere. After death, no fresh ^{14}C is taken in, and the existing ^{14}C decays exponentially with half-life 5730 years. Measuring the remaining ^{14}C fraction N/N_0 in a sample and inverting $N = N_0 e^{-kt}$ gives the sample's age. Same DE; different timescale.

6.6 Beyond the syllabus: Bernoulli and variation of parameters

Two further first-order methods sit just outside Class 12 NCERT. We mention them for completeness; both are formally beyond the board scope, but *Bernoulli* occasionally shows up in JEE Main.

Bernoulli's equation: $\dfrac{dy}{dx} + P(x)y = Q(x)\,y^n$. The substitution $v = y^{1-n}$ converts it to a linear DE in v.

Variation of parameters: a technique for solving non-homogeneous *linear* DEs of higher order; it generalises the integrating-factor trick. The Class 12 syllabus does not require it; we list it only so the term doesn't surprise the student in a JEE prep book.

Don't use Bernoulli in board exams

The CBSE board only assumes the three methods (separable, homogeneous, linear) in the rationalised NCERT. Solving a board problem with Bernoulli or variation of parameters is not wrong, but graders may not award credit if your working assumes those tools. Stick to the three methods for boards; use Bernoulli for JEE only when the problem genuinely demands it.

7 Quick Reference Summary

A consolidated cheat sheet for the night before the exam.

7.1 Definitions at a glance

Term	Meaning
Order	Order of the highest derivative present in the DE
Degree	Power of the highest-order derivative, after rationalising radicals; undefined if any derivative sits inside a transcendental
General solution	Solution containing as many arbitrary constants as the order of the DE
Particular solution	Solution with no arbitrary constants (obtained from the general solution by imposing initial / boundary conditions)
Ordinary DE	DE with only one independent variable (the only kind in Class 12)
Homogeneous function of degree n	$F(\lambda x, \lambda y) = \lambda^n F(x, y)$
Homogeneous DE	$\frac{dy}{dx} = g(y/x)$ (i.e. F homogeneous of degree zero)
Linear DE (1st order)	$\frac{dy}{dx} + P(x)y = Q(x)$
Integrating Factor (IF)	$\mu(x) = e^{\int P\,dx}$

7.2 Method-selection flow

If the DE looks like...	Method	Final solution shape
$\frac{dy}{dx} = g(x)h(y)$	Variables separable	$\int \frac{dy}{h(y)} = \int g(x)\,dx + C$
$\frac{dy}{dx} = g(y/x)$	Homogeneous, $y = vx$	Separable in v, x; then $v \to y/x$
$\frac{dy}{dx} + P(x)y = Q(x)$	Linear, IF $= e^{\int P\,dx}$	$y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$
$\frac{dx}{dy} + P_1(y)x = Q_1(y)$	Linear (in x), IF $= e^{\int P_1\,dy}$	$x \cdot \text{IF} = \int Q_1 \cdot \text{IF}\,dy + C$
$\frac{dN}{dt} = kN$ *[JEE]*	Separable	$N = N_0\,e^{kt}$
$\frac{dT}{dt} = -k(T - T_s)$ *[JEE]*	Linear / separable	$T = T_s + (T_0 - T_s)\,e^{-kt}$

7.3 Key integrals to remember

The five integrals you'll use most

$$\int \frac{dx}{x} = \log|x| + C \qquad \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

$$\int e^{ax}\,dx = \frac{e^{ax}}{a} + C \qquad \int \sec^2 x\,dx = \tan x + C$$

$$\int \cot x\,dx = \log|\sin x| + C$$

7.4 Standard integrating factors

Coefficient $P(x)$	$\int P\,dx$	**IF** $= e^{\int P\,dx}$
k (constant)	kx	e^{kx}
$\frac{n}{x}$	$n\log\lvert x\rvert$	x^n
$\tan x$	$\log\lvert\sec x\rvert$	$\sec x$
$\cot x$	$\log\lvert\sin x\rvert$	$\sin x$
$\frac{1}{x\log x}$	$\log\lvert\log x\rvert$	$\log x$
$\frac{2x}{1+x^2}$	$\log(1+x^2)$	$1+x^2$

7.5 Twelve-second checklist

The 12-second exam checklist

1. Identify order and degree.
2. Choose: separable, homogeneous, or linear (in y or x).
3. For homogeneous: substitute $y = vx$ (or $x = vy$).
4. For linear: IF $= e^{\int P\,dx}$; solution $y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$.
5. Apply initial condition (if asked).
6. Always write $+C$ *before* exponentiating or rearranging.

"SHL" — the three-step exam strategy

Separable first — the cheapest method. If it doesn't split, try **H**omogeneous (look at degrees). If still stuck, force the form $y' + Py = Q$ and go **L**inear. Almost every Class 12 first-order DE falls into one of these three buckets.

Verify your final answer

Once you have a candidate solution, differentiate it and plug back into the original DE. If the LHS reduces to the RHS identically, you're done. This 30-second check has saved more marks than any other habit.