Electric Charges & Fields

An intrinsic property of matter that gives rise to electric force is called CHARGE.

Origin

Atom : protons (+), electrons (-), neutrons (no charge). Charge of a body arises due to ~~excess~~ transfer of electrons.

Two kinds of charge

(i) Positive - glass rubbed with silk
(ii) Negative - ebonite rubbed with fur
Like charges repel ; unlike attract.

SI unit

1 coulomb (C) = charge that flows when 1 A current passes for 1 second.

$$\boxed{e = 1.6 \times 10^{-19}\ C}$$

<- smallest free
<- charge known

Mass of electron : 9.11×10^{-31} kg
Mass of proton : 1.67×10^{-27} kg

Dimensional formula of charge : [A T].

Properties of Electric Charge

1. Additivity

Total charge of a body = algebraic sum of all individual charges :

$$\boxed{Q = q_1 + q_2 + \dots + q_n}$$ <- with sign

2. Quantisation

Charge on any body is an integral multiple of e :

$$\boxed{q = n e, \quad n = +/- 1, +/- 2, \dots}$$ <- Millikan's
<- oil-drop expt.

Quarks have charge +/- ~~+/2~~ 1/3 e , 2/3 e but are not seen free - so e is the smallest free charge in nature.

3. Conservation

Total charge of an isolated system remains constant. Charge is neither created nor destroyed - only transferred.

4. Invariance

Charge does not depend on speed/frame (unlike mass) - it is a Lorentz invariant.

Methods of Charging

(a) By Friction

Induction sketch :

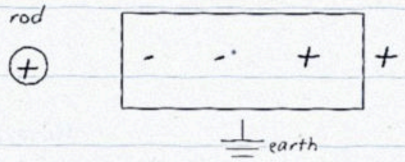
Two neutral bodies rubbed together :
electrons jump from one to the other.

Glass + silk -> glass (+) , silk (-)

Ebonite + fur -> ebonite (-) , fur (+)

(b) By Conduction

Touching a charged body with a neutral conductor - charge re-distributes till both reach the same potential.



(c) By Induction

Bring a charged rod near a conductor :

(i) nearer face gets opposite charge

(ii) farther face gets same charge

(iii) earth the conductor - far charge drains off ; remove earthing ; remove rod
-> body left with opposite charge.

Note : in induction the inducing body does not lose any of its own charge.
Body acquires charge without contact.

Gold-leaf electroscope - device used to

Coulomb's Law

Force between two point charges at rest in vacuum is :

(a) directly proportional to q1 q2 ,

(b) inversely proportional to r-squared,

(c) along the line joining the charges.

$$\boxed{F = \kappa\, q_1 q_2 / r^2}$$

<- scalar form
<- magnitude only

where $\kappa = 1 / (4 \pi \varepsilon_0) = 9 \times 10^{9} \cdot N\, m^2\, C^{-2}$

$\varepsilon_0 = 8.854 \times 10^{-12}\ C^2\, N^{-1}\, m^{-2}$

is the permittivity of free space.

Vector form

$\vec{F}_{12} = \kappa\, q_1 q_2 / r^2\ \hat{r}_{21}$

force on q1 due to q2 .

$\vec{F}_{12} = - \vec{F}_{21}$ (Newton's 3rd law)

In a medium

$F_{med} = q_1 q_2 / (4 \pi \varepsilon r^2)$

where $\varepsilon = \varepsilon_0 \varepsilon_r$; ε_r = dielectric const.

So $F_{med} / F_{vac} = 1 / \varepsilon_r < 1$ (always).

Medium weakens the electrostatic force.

Example 1 - Coulomb force

Two charges +2 uC and -3 uC are placed 0.30 m apart in air. Find the force between them.

Soln.

$q_1 = +2 \times 10^{-6}$ C

$q_2 = -3 \times 10^{-6}$ C

$r = 0.30$ m ; $k = 9 \times 10$ 9 N m 2 / C 2

$F = k\, q_1 q_2 / r^2$

$$= \frac{9 \times 10^9 \times (2 \times 10^{-6}) \times (3 \times 10^{-6})}{(0.30)^2}$$

$= 9 \times 10^9 \times 6 \times 10^{-12} \;/\; 0.09$

$= 54 \times 10^{-3} \;/\; 0.09$ N

$= 0.60$ N

$\boxed{F = 0.60 \text{ N (attractive)}}$ <- opposite
<- charges attract

If placed in water (eps_r = 80) :

$F_{water} = 0.60 / 80 = 7.5 \times 10^{-3}$ N

Force drops by a factor of 80.

Principle of Superposition

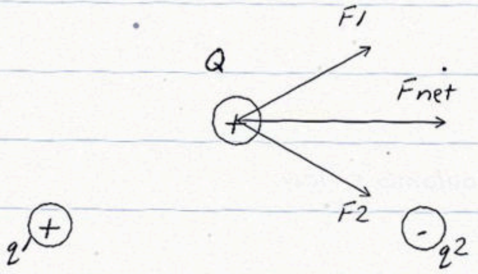
Force on one charge due to many charges
= vector sum of forces due to each one,
taken ~~seperately~~ separately - others having no effect.

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \dots + \vec{F}_{1n}$$

$$\vec{F}_1 = k\, q_1 \int_{j=2}^{n} (q_j / r_{1j}^2)\, \hat{r}_{j1}$$ <- summation <- form

Key idea

Coulomb's law is PAIR-WISE - the force between any two charges is unaffected by the presence of other charges.



Resultant by parallelogram law :

$$F_R = \mathrm{sqrt}(F_1^2 + F_2^2 + 2 F_1 F_2 \cos \Theta)$$

$$\tan \alpha = F_2 \sin \Theta / (F_1 + F_2 \cos \Theta)$$

Electric Field

Region around a charge in which it can exert a force on another charge.

Definition

Electric field at a point = force per unit POSITIVE test charge q0 placed at that point :

$$\boxed{\vec{E} = \vec{F} \,/\, q_0 \quad (q_0 \rightarrow 0)}$$

<- limit so q0
<- does not disturb

Unit : N / C = V / m

Dim'l : $[\, M \, L \, T^{-3} \, A^{-1} \,]$

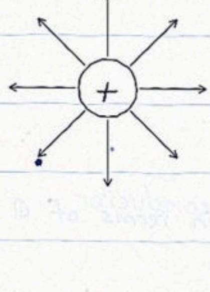
E is a VECTOR - same direction as F on +q0 .

Due to a point charge

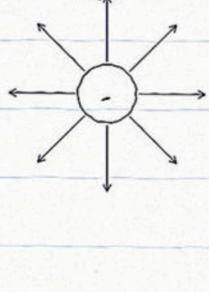
$\vec{E} = k\, Q \,/\, r^2 \;\hat{r} \;=\; Q \,/\, (4 \pi \varepsilon_0 r^2)\; \hat{r}$

(radially outward for +Q, inward for -Q)

field of +Q



field of -Q



Field of a System of Charges

$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots + \vec{E}_n$ (superposition)

$\vec{E}_P = \kappa \int_{i=1}^{n} (q_i / r_{iP}^2) \cdot \hat{r}_{iP}$

At a point P with position vector r, due to charge q__i at r__i :

$r_{iP} = \vec{r} - \vec{r}_i$

Electric Field Lines

Imaginary lines whose tangent at any point gives the direction of E there.

Properties :

(1) Start from +ve, end at -ve charges, or extend to infinity.

(2) Continuous curves - no breaks.

(3) Two lines ~~can~~ never intersect (else E would have 2 directions).

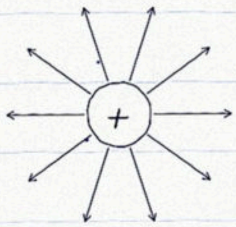
(4) Density (lines per unit area) is proportional to E at that point.

(5) Do not form closed loops in electrostatics (E is conservative).

(6) Perpendicular to the surface of a conductor in electrostatic equilibrium.

Patterns of Field Lines

(a) Isolated +Q

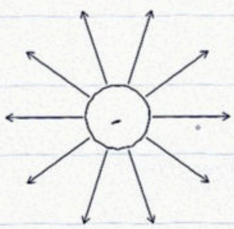


radially outward,

spherically symmetric.

density 1/r-squared.

(b) Isolated -Q

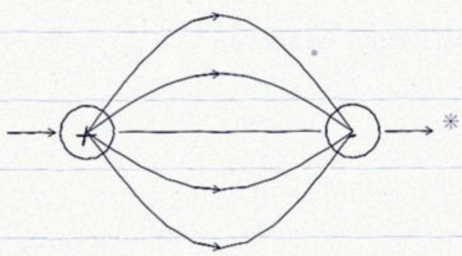


radially inward.

Same shape - reversed

direction.

(c) Dipole (+Q, -Q)



Continuous Charge Distribution

When charges are very close together we treat charge as continuous. 3 kinds :

(i) Linear charge density

$\lambda = dq \, / \, dl$ (C / m)

(charge on a thin wire)

(ii) Surface charge density

$\sigma = dq \, / \, dA$ ($C \, / \, m^2$)

(charge on a thin sheet / surface)

(iii) Volume charge density

$\rho = dq \, / \, dV$ ($C \, / \, m^3$)

(charge spread in a 3-D volume)

Field due to a continuous body

$dE = \kappa \, dq \, / \, r^2$ (each element)

$\vec{E} = \int \kappa \, dq \, / \, r^2 \; \hat{r}$ <- integrate over <- the whole body

For a wire : $dq = \lambda \, dl$

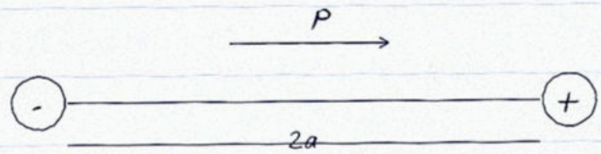
For surface: $dq = \sigma \, dA$

For volume : $dq = \rho \, dV$

Use symmetry to simplify the integral.

Electric Dipole

A pair of equal and opposite point charges $+q$, $-q$ separated by a small distance $2a$ is called an electric dipole.



Dipole moment

$\boxed{\vec{p} = q \cdot 2a \ \hat{p} \ \text{(from } -q \text{ to}}$ $+q$ ← vector qty,
<- unit C m

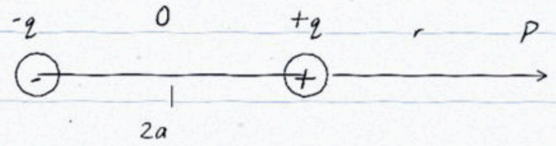
Magnitude $p = q(2a)$.
Direction : along the axis, from negative to positive charge.

Total charge of the dipole $= +q + (-q) = 0$
Net force on dipole in UNIFORM $E = 0$, but net TORQUE is non-zero (see later).

In a non-uniform field both ~~force~~ a net force AND a torque act on the dipole.

Field on Axis of Dipole

Consider point P at distance r from centre O on the axis (along p).



Distance from +q to P : r - a

Distance from -q to P : r + a

$E_+ = \kappa q / (r - a)^2$ (away from +q)

$E_- = \kappa q / (r + a)^2$ (towards -q)

Net field along axis (away from -q to +q):

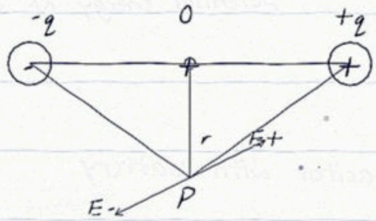
$$E_{ax} = E_+ - E_-$$
$$= \kappa q [1/(r-a)^2 - 1/(r+a)^2]$$
$$= \kappa q \cdot 4 a r / (r^2 - a^2)^2$$
$$= 2 \kappa p r / (r^2 - a^2)^2$$

For r >> a (short dipole) :

$$\boxed{E_{axial} = 2 \kappa p / r^3 \quad \text{(along } \vec{p}\text{)}}$$ <-1/r 3 <-falloff

Field on Equatorial Plane

Point P at distance r from O on the perpendicular bisector of the dipole.



$E+ = E- = \kappa q / (r \; 2 + a \; 2)$

Resolve into components - perpendicular to axis components cancel ; parallel ones add (antiparallel to p):

$E_{eq} = 2 \cdot \kappa q / (r^2 + a^2) \cdot \cos \Theta$

where $\cos \Theta = a / \mathrm{sqrt}(r^2 + a^2)$

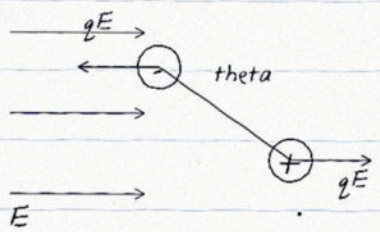
$E_{eq} = \kappa p / (r^2 + a^2)^{3/2}$

For short dipole ($r >> a$) :

$\boxed{E_{eq} = \kappa p / r^3 \quad (\text{opposite to } \vec{p}}$ <-} half of <- axial value

Dipole in Uniform Field

Dipole p in uniform field E - the two charges feel equal and opposite forces.



Force

$$\vec{F}_{net} = q\vec{E} + (-q)\,\vec{E} = 0$$

Net force on dipole in uniform E is zero.

Torque

Two equal opposite forces ; perpendicular distance between them = 2a sin theta :

$$\tau = q\,E \cdot 2a \sin\theta$$
$$= p\,E \sin\theta$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E} \quad (\text{tau})}$$

<- perpendicular
<- to both p, E

Torque is MAX when $\theta = 90$ deg : $\tau_{max} = p\,E$

Torque is ZERO when $\theta = 0$ or 180 deg (equilibrium).

Example 2 - Dipole torque

An electric dipole of moment 4×10^{-9} C m is placed in a uniform field $E = 5 \times 10^{4}$ N/C at an angle of 30 deg with the field. Find the torque on the dipole.

Soln.

$p = 4 \times 10^{-9}$ C m

$E = 5 \times 10^{4}$ N / C

$\theta = 30$ deg ; $\sin 30 = 1/2$

$$\begin{aligned}
tau &= p\,E \sin\theta \\
&= (4 \times 10^{-9})(5 \times 10^{4})(0.5) \\
&= 4 \times 5 \times 0.5 \times 10^{-5} \text{ N m} \\
&= 10 \times 10^{-5} \text{ N m} \\
&= 1.0 \times 10^{-4} \text{ N m}
\end{aligned}$$

$$\boxed{tau = 1.0 \times 10^{-4} \text{ N m}}$$

<- direction by
<- right-hand rule

Work done to rotate from theta1 to theta2:

$$W = p\,E\,(\cos\theta_1 - \cos\theta_2)$$

Potential energy of dipole in field :

$$U = -\vec{p} \cdot \vec{E} = -p\,E\cos\theta$$

Electric Flux

Number of electric field lines crossing a given area normally is called electric flux through that area.

For a small flat area

$$\boxed{d\phi = \vec{E} \cdot d\vec{A} = E\, dA \cos\theta}$$

<- scalar qty,
<- unit N m 2/C

$d\vec{A} = \hat{n}\, dA$; $\hat{n}$ is outward normal.

θ = angle between $\vec{E}$ and $\hat{n}$.

For a finite surface

$$\boxed{\phi = \int^{S} \vec{E} \cdot d\vec{A}}$$

<- surface
<- integral

Cases

(a) $\theta = 0 \rightarrow d\phi = E\, dA$ (max)

(b) $\theta = 90 \rightarrow d\phi = 0$ (E parallel to surface)

(c) $\theta = 180 \rightarrow d\phi = -E\, dA$ (entering)

ϕ is a SCALAR quantity,

dim'l : $[M L^3 T^{-3} A^{-1}]$

Gauss's Law

Total electric flux through ANY closed surface S = 1/eps0 times the total charge enclosed by S.

$$\boxed{\oint \vec{E} \cdot d\vec{A} = q_{enc} / \varepsilon_0}$$

<- closed
<- surface only

Important features

(1) Valid for ANY closed surface (Gaussian surface) of any shape.

(2) q_{enc} = algebraic sum of charges inside.

(3) Charges OUTSIDE S contribute to field on S but not to total flux.

(4) E on the surface in the integral is the TOTAL field (due to all charges).

(5) Direct consequence of inverse-square nature of Coulomb's law.

Choosing the Gaussian surface

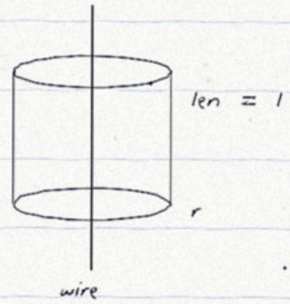
(i) $\vec{E}$ constant in magnitude on the surface

(ii) $\vec{E}$ parallel or perpendicular to $\hat{n}$

So we exploit the SYMMETRY of the charge distribution - spherical, cylindrical, or planar - to ~~guess~~ choose the surface.

Field of Infinite Line Charge

Straight wire of infinite length, uniform linear charge density lambda C/m.



Choose Gaussian surface

Coaxial cylinder of radius r and length 1.
By symmetry E is radial, same magnitude everywhere on the curved surface.

Flux

ϕ (curved) $= E \cdot (2 \Pi r 1)$
ϕ (flat ends) $= 0$ (E perp to $\hat{n}$)
Total $\phi = E \cdot 2 \Pi r 1$

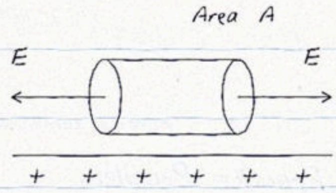
$q_{enc} = \lambda 1$
By Gauss : $E \cdot 2 \Pi r 1 = \lambda 1 / \varepsilon_0$

$$\boxed{E = \lambda / (2 \Pi \varepsilon_0 r)}$$

<-E   1 / r
<-radial

Field of Infinite Plane Sheet

Infinite plane sheet with uniform surface charge density sigma C/m 2 .



Gaussian pillbox

Take a cylinder (pillbox) of area A passing through the sheet, symmetrically.
By symmetry E is perpendicular to sheet.

Flux

$\Phi = E . A + E . A + 0 = 2 E A$

$q_{enc} = \sigma A$

Gauss : $2 E A = \sigma A / \varepsilon_0$

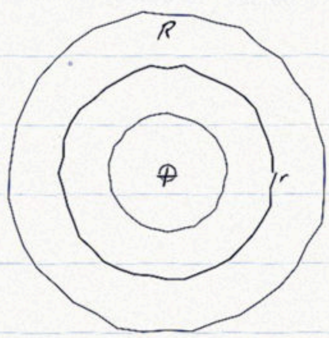
$$\boxed{E = \sigma / (2 \varepsilon_0)}$$

<- independent
<- of distance.

For a conducting sheet : E = sigma / eps0

Field of Charged Spherical Shell

Thin spherical shell of radius R carrying total charge Q (surface density sigma).



(a) Outside : $r > R$

$$\phi = E \cdot 4\pi r^2 = Q / \varepsilon_0$$

$$\boxed{E_{out} = \kappa Q / r^2}$$

<- same as a point
<- charge at. centre

(b) Surface : $r = R$

$$E_{surf} = \kappa Q / R^2 = \sigma / \varepsilon_0$$

(c) Inside : $r < R$

$q_{enc} = 0$ (no charge inside)

$$\boxed{E_{in} = 0}$$

<- shielding -
<- Faraday cage

Field jumps from 0 (just inside) to $\kappa Q/R$ 2.

Solid Sphere of Charge

Sphere of radius R with uniform volume charge density rho . Total charge $Q = \rho \cdot (4/3) \pi R^3$.

Outside ($r > R$)

$E \cdot 4 \pi r^2 = Q / \varepsilon_0$

$$\boxed{E_{out} = \kappa Q / r^2 .}$$ <- same as shell

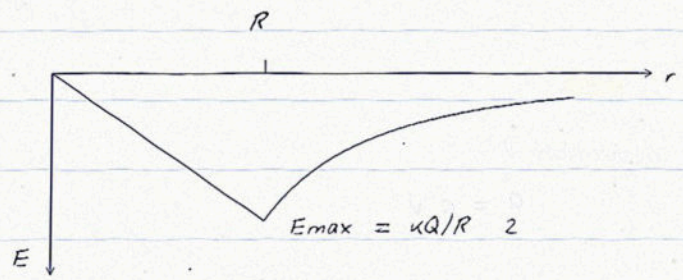
Inside ($r < R$)

$q_{enc} = \rho \cdot (4/3) \pi r^3$

$E \cdot 4 \pi r^2 = (4/3) \pi r^3 \rho / \varepsilon_0$

$$\boxed{E_{in} = \rho r / (3 \varepsilon_0) = \kappa Q r / R^3}$$ <- E linear in r

E vs r graph



Example 3 - Field of a line

An infinitely long wire carries a linear charge density of 5×10^{-6} C/m. Find E at a distance of 10 cm from the wire.

Soln.

$\lambda = 5 \times 10^{-6}$ C / m

$r = 10$ cm $= 0.10$ m

$$E = \lambda / (2 \pi \varepsilon_0 r)$$

$$= 2 k \lambda / r \qquad [\text{ since } 1/(2\pi\varepsilon_0) = 2k$$

$$= 2 (9 \times 10^{9})(5 \times 10^{-6}) / 0.10$$

$$= (9 \times 10^{4}) / 0.10$$

$$= 9.0 \times 10^{5} \text{ N / C}$$

$$\boxed{E = 9.0 \times 10^{5} \text{ N/C} \quad \text{(radial)}}$$

<- outward
<- for +ve line

Check : if r is doubled to 0.20 m -
E becomes 4.5×10^{5} N/C (halved).

E falls as $1/r$, slower than $1/r^2$ of a point charge. So a long wire has a long-range field.

Example 4 - Plane sheet

A large plane sheet has uniform surface charge density 8.85 x 10 -12 C/m 2. Find the electric field near the sheet.

Soln.

$\sigma = 8.85 \times 10^{-12}\ C / m^2$,

$\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/Nm^2$

$$
\begin{aligned}
E &= \sigma / (2\ \varepsilon_0) \\
&= (8.85 \times 10^{-12}) / [2 \cdot 8.85 \times 10^{-12}] \\
&= 1 / 2 \\
&= 0.5\ N / C
\end{aligned}
$$

$$\boxed{E = 0.5\ N / C}$$

<-uniform on
<-both sides

Two parallel sheets, +sigma and -sigma :

Between : $E = \sigma / \varepsilon_0$ (sum)

Outside : $E = 0$ (cancel)

Two sheets with same +sigma each :

Between : $E = 0$

Outside : $E = \sigma / \varepsilon_0$

Example 5 - Spherical Shell

A thin spherical shell of radius 10 cm carries a charge of 1×10^{-7} C. Find E at (a) 5 cm, (b) 10 cm, (c) 20 cm from the centre.

Soln.

$R = 0.10$ m ; $Q = 1 \times 10^{-7}$ C

(a) $r = 5$ cm $< R$ $\therefore$ $E = 0$ (inside).

(b) $r = R = 10$ cm :

$$E = \kappa Q / R^2$$
$$= (9 \times 10^9)(10^{-7}) / (0.10)^2$$
$$= 900 / 0.01$$
$$= 9.0 \times 10^4 \text{ N / C}$$

(c) $r = 20$ cm $> R$:

$$E = \kappa Q / r^2 = (9 \times 10^9)(10^{-7}) / 0.04$$
$$= 2.25 \times 10^4 \text{ N / C}$$

$$\boxed{E_a = 0 \,,\; E_b = 9.0e4 \,,\; E_c = 2.25e4 \text{ N/C}}$$

Notice : E jumps from 0 to 9.0e4 N/C at $r = R$ - the surface of the shell.

Example 6 - Flux from a cube

A charge of 8.85 nC is placed at the centre of a cube. Find the flux through (a)* the whole cube, (b) one face.

Soln.

$q = 8.85 \times 10^{-9}$ C

$\varepsilon_0 = 8.85 \times 10^{-12}$ SI

(a) By Gauss's law :

$$\Phi_{total} = q \;/\; \varepsilon_0$$
$$= (8.85 \times 10^{-9}) \;/\; (8.85 \times 10^{-12})$$
$$= 10^3 \text{ N m}^2 / \text{C}$$

(b) Cube has 6 identical faces ; by symmetry flux through each = Phi/6.

$$\Phi_{face} = 1000 \;/\; 6 = 166.7 \text{ N m}^2 / \text{C}$$

$$\boxed{\Phi_{face} = 1.67 \times 10^2 \text{ N m}^2 / \text{C}}$$

<- symmetry
<- argument

Trick : if the charge is at a CORNER of the cube, only 1/8 of the field lines go through the cube ; so the total flux through the cube becomes ~~q/eps0~~ q/(8 eps0).

Example 7 - Short dipole

A short electric dipole of moment $p = 6 \times 10^{-9}$ C m. Find E at distance 0.30 m on (a) axial, (b) equatorial line.

Soln.

$p = 6 \times 10^{-9}$ C m ; $r = 0.30$ m

(a) Axial line :

$$E_{ax} = 2 \kappa p / r^3$$
$$= 2 (9 \times 10^{9})(6 \times 10^{-9}) / (0.30)^3$$
$$= 108 / 0.027$$
$$= 4.0 \times 10^{3} \text{ N / C}$$

(direction : along $\vec{p}$)

(b) Equatorial line :

$$E_{eq} = \kappa p / r^3 = E_{ax} / 2$$
$$= 2.0 \times 10^{3} \text{ N / C}$$

(direction : antiparallel to $\vec{p}$)

$$\boxed{E_{ax} : E_{eq} = 2 : 1}$$

<- general
<- rule

Both fall off as $1/r^3$ - much faster than the $1/r^2$ of a point charge.

Example 8 - Quantisation

An object carries a charge of -3.2 uC.
How many excess electrons does it carry ?

Soln.

$q = -3.2 \times 10^{-6}$ C

$e = 1.6 \times 10^{-19}$ C

By quantisation : $q = n e$

$n = q / e$

$= (3.2 \times 10^{-6}) / (1.6 \times 10^{-19})$

$= 2 \times 10^{13}$ electrons

$$\boxed{n = 2 \times 10^{13} \text{ electrons (excess)}}$$

Mass of these electrons :

$m = n \cdot m_e = 2 \times 10^{13} \times 9.11 \times 10^{-31}$

$= 1.82 \times 10^{-17}$ kg (negligible !)

So macroscopic charging does NOT change the mass of a body in any noticeable way.

At macroscopic levels n is so huge that we treat charge as continuous - safely.

Summary - Key Formulas

Coulomb / Field

$F = \kappa\, q_1 q_2 / r^2$; $\kappa = 1/(4\pi\varepsilon_0)$

$E = F / q_0 = \kappa\, Q / r^2$ (point charge)

Dipole

$p = q\,(2a)$; $E_{ax} = 2\kappa\, p / r^3$; $E_{eq} = \kappa\, p / r^3$

tau $= \vec{p} \times \vec{E}$; $U = -\vec{p} \cdot \vec{E}$

Flux / Gauss

$\phi = \vec{E} \cdot \vec{A}$; $\oint \vec{E} \cdot d\vec{A} = q_{enc} / \varepsilon_0$

Applications of Gauss

Line : $E = \lambda / (2\pi\varepsilon_0 r)$

Sheet : $E = \sigma / (2\varepsilon_0)$ (insulator)

Conductor sheet : $E = \sigma / \varepsilon_0$

Shell : $E_{out} = \kappa\, Q / r^2$; $E_{in} = 0$

Solid sphere : $E_{in} = \kappa\, Q\, r / R^3$

Constants worth memorising

$e = 1.6 \times 10^{-19}$ C

$\kappa = 9.0 \times 10^{9}$ N m^2 / C^2

$\varepsilon_0 = 8.85 \times 10^{-12}$ C^2/ N m^2

$m_e = 9.11 \times 10^{-31}$ kg ; $m_p = 1.67 \times 10^{-27}$ kg

* * * End of Chapter 1 * * *