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NCERT Exemplar Solutions

Solved NCERT Exemplar Problems for Class 12th Physics, Chapter 13

Chapter 13: Nuclei

About this Chapter

This chapter takes the student inside the atom to study the nucleus itself: its tiny size, enormous density, the **mass defect** and **binding energy** that hold it together, and the **radioactive decay** laws (α, β, γ) that govern how unstable nuclei reorganise. The Exemplar problems probe binding-energy curves, half-life calculations, Q-values for $\beta^{\pm}$ decays, fission and fusion energetics, and the magic numbers that explain extra stability of certain nuclei.

Topics covered: Atomic Mass Unit • Mass Defect • Binding Energy • Nuclear Forces • Radioactive Decay Law • Half-life & Mean-life • α, β, γ Decays • Fission & Fusion

Quick Formula Sheet

Nuclear radius:
$R = R_0\, A^{1/3},\ R_0 = 1.2\,\text{fm}$

Mass defect / binding energy:
$\Delta m = Zm_p + (A-Z)m_n - M_{\text{nuc}}$
$B = \Delta m\, c^2$

Radioactive decay law:
$N(t) = N_0\, e^{-\lambda t}, \quad \frac{dN}{dt} = -\lambda N$

Half-life & mean-life:
$T_{1/2} = \frac{\ln 2}{\lambda}, \quad \tau = \frac{1}{\lambda}$

Q-value of decay:
$Q = (m_{\text{parent}} - m_{\text{daughter}} - m_{\text{emitted}})\, c^2$

Useful constant:
$1\,\text{u} \cdot c^2 = 931.5\,\text{MeV}$

NCERT Exemplar Problems

MCQ-I

Multiple Choice Questions (Single Correct Option)

Q 13.1 **Suppose we consider a large number of containers each containing initially** 10000 **atoms of a radioactive material with a half-life of** 1 **year. After** 1 **year:**

(a) all the containers will have 5000 **atoms of the material.**

(b) all the containers will contain the same number of atoms of the material but that number will only be approximately 5000**.**

(c) the containers will in general have different numbers of the atoms of the material but their average will be close to 5000.
(d) none of the containers can have more than 5000 atoms.

SOLUTION

Correct option: (c) the containers will in general have different numbers of the atoms of the material but their average will be close to 5000.

Concept used. Radioactive decay is a fundamentally **random (stochastic) process**: an individual nucleus has no memory or schedule. For a single nucleus the probability of decaying in a time dt is $\lambda\, dt$, independent of its age. Only when we average over a very large number of identical nuclei does the deterministic exponential law $N(t) = N_0\, e^{-\lambda t}$ emerge. The **half-life** $T_{1/2}$ is defined as the time after which the *expected* number of surviving nuclei is exactly half of the initial number, i.e. $\langle N(T_{1/2})\rangle = N_0/2$. Around this expected value, the actual count in any particular sample fluctuates, and the relative fluctuation is of order $1/\sqrt{N}$ for a sample of N nuclei (Poisson statistics).

Step 1. Apply the decay law to the expectation value: with $N_0 = 10000$ and $t = T_{1/2}$,

$$\langle N\rangle = \frac{N_0}{2} = \frac{10000}{2} = 5000.$$

So on *average*, 5000 atoms remain in each container.

Step 2. Estimate the fluctuation around this average using Poisson statistics: the standard deviation in the count is

$$\sigma = \sqrt{\langle N\rangle} = \sqrt{5000} \approx 70.7.$$

So in any single container, the actual count after 1 year is 5000 ± 71 roughly. Different containers will therefore give different counts.

Step 3. Eliminate distractors. (a) is wrong: it claims an exact 5000 in every container, which violates the random nature of decay. (b) is wrong for the same reason: "the same number" cannot hold across containers. (d) is wrong because a container with, say, 5050 atoms is perfectly possible (upward fluctuation). Only (c) correctly captures the statistical picture: counts differ, but the average is close to 5000.

☞ **Why averages, not exact counts**

The decay law $N = N_0 e^{-\lambda t}$ is a *statistical* law, not a deterministic prescription for every individual nucleus. It tells you the expected number; the actual number fluctuates.

Final Answer: Option **(c)**: counts vary across containers, but their average is close to 5000.

EXPERT'S SOLUTION : *Aarav Sharma, M.Sc Physics, IIT Madras*

Quick reading. The key word is "large number of containers". That tells us the question is checking whether the student knows radioactive decay is random for an individual nucleus and only average-deterministic for an ensemble.

Step 1. State the rule: each nucleus has a fixed decay probability per unit time λ, independent of all other nuclei. So the number surviving in any single sample is a binomial random variable with mean $N_0\, e^{-\lambda t}$.

Step 2. At $t = T_{1/2}$, the expected survival in each container is $N_0/2 = 5000$. Standard deviation is $\sqrt{N_0\, p(1-p)} = \sqrt{10000 \cdot 0.5 \cdot 0.5} = 50$ (binomial), or about $\sqrt{5000} \approx 71$ in the Poisson approximation. Either way, individual counts spread around 5000.

Step 3. Hence (a), (b) are too strong (they assert a fixed exact count in every container), and (d) is wrong because upward fluctuations are allowed. The correct statement is **(c)**: different counts, average ≈ 5000.

Alternative reasoning — by elimination. Option (a) demands *exact* 5000 in every container; (b) demands an *identical* count in every container even if approximate; (d) forbids any count above 5000. All three contradict the basic premise that each nucleus decays *independently* with probability $\frac{1}{2}$ per half-life, so the per-container count is a binomial random variable $\text{Bin}(10000, \frac{1}{2})$ that fluctuates symmetrically about its mean. Only (c) survives.

Cross-check — order of magnitude. Relative fluctuation is $\sigma/\langle N \rangle \approx 71/5000 \approx 1.4\%$. So if you ran 1000 containers, almost all results would land in the range 5000 ± 200 (three-sigma), with the average across containers extremely close to 5000. This matches the wording of (c) exactly.

Why this matters. Every modern measurement of half-lives is done by counting decays from a large ensemble and computing an *average* rate; the fluctuations limit the precision. The same $\sqrt{N}$ scaling shows up in cosmic-ray detectors, smoke-alarm americium counts, and PET-scan reconstruction noise.

> **Final Answer:** Option **(c)**.

♥ Statistical vs deterministic decay

The exponential decay law $N(t) = N_0 e^{-\lambda t}$ is an *ensemble* statement — it gives the *expected* number of survivors. For a single nucleus there is no such "smooth decay"; the nucleus is either intact or has decayed. Treating the macroscopic formula as a deterministic prescription for every container is the classic mistake (a) and (b) bank on.

Q 13.2 **The gravitational force between an H-atom and another particle of mass m**

will be given by Newton's law $F = G\,\dfrac{M\,m}{r^2}$, where r is in km and:
(a) $M = m_{\textbf{proton}} + m_{\textbf{electron}}$.
(b) $M = m_{\textbf{proton}} + m_{\textbf{electron}} - \dfrac{B}{c^2}$ **($B = 13.6$ eV).**
(c) M **is not related to the mass of the hydrogen atom.**
(d) $M = m_{\textbf{proton}} + m_{\textbf{electron}} - \dfrac{|V|}{c^2}$ **($|V|$ = magnitude of the potential energy of electron in the H-atom).**

SOLUTION

Correct option: (b) $M = m_{\text{proton}} + m_{\text{electron}} - B/c^2$.
Concept used. Mass-energy equivalence: any bound system has an inertial mass equal to (sum of rest masses of the parts) minus (binding energy)/c^2. For the hydrogen atom in its ground state, the binding energy is $B = 13.6$ eV (energy required to ionise it). So the H-atom's actual mass is

$$m_H = m_p + m_e - \frac{B}{c^2}.$$

This is the mass that appears in any external interaction with the atom, including its gravitational pull.

Step 1. Identify what M in Newton's law represents: the total inertial mass of the H-atom (gravitational mass = inertial mass by the equivalence principle).

Step 2. Apply mass-energy equivalence. The H-atom is a bound state of a proton and an electron, with binding energy $B = 13.6\,\text{eV} = 13.6 \times 1.6 \times 10^{-19}\,\text{J} = 2.18 \times 10^{-18}\,\text{J}$. The corresponding mass deficit is

$$\frac{B}{c^2} = \frac{2.18 \times 10^{-18}}{(3 \times 10^8)^2} = 2.42 \times 10^{-35}\,\text{kg}.$$

Tiny compared with $m_p \approx 1.67 \times 10^{-27}$ kg, but conceptually present.

Step 3. Compare with the options. (a) ignores binding energy. (c) is physically absurd: the gravitating mass of the H-atom is of course the H-atom's mass. (d) uses $|V|$, the potential energy magnitude, but for a bound Coulomb system the binding energy equals $-E_{\text{total}}$, which is *half* of $|V|$ (virial theorem: $E_{\text{total}} = -|V|/2 = -B$, so $|V| = 2B$). So using $|V|/c^2$ over-counts the deficit by a factor of 2. Only (b) is exactly right.

☞ **Virial theorem for $1/r$ potentials**

For a bound state with $U \propto -1/r$ (gravity, Coulomb), $\langle T\rangle = -\frac{1}{2}\langle U\rangle$, so the total energy $E = T + U = \frac{1}{2}U = -\frac{1}{2}|U|$. The binding energy is $B = -E = |U|/2$.

Final Answer: Option **(b)**: $M = m_p + m_e - B/c^2$.

EXPERT'S SOLUTION : *Priya Iyer, Ph.D Physics, IISc Bangalore*

Strategic angle. Gravity couples to total energy/c^2, not to the sum of constituent rest masses. So any bound state weighs less than the sum of its parts by exactly the binding energy/c^2.

Step 1. Write the gravitational mass of the H-atom from E_{rest}/c^2:
$M = (m_p c^2 + m_e c^2 - B)/c^2 = m_p + m_e - B/c^2$.

Step 2. Rule out (d): the virial theorem for Coulomb potentials gives $|V| = 2B$, so the term in (d) would be twice the correct deficit.

Step 3. (a) ignores binding entirely; (c) violates the equivalence principle.

Alternative method — book-keeping with E not m. Start from the H-atom's total rest energy $E_H = m_p c^2 + m_e c^2 + T + U$ where T is electronic kinetic energy and U is the Coulomb potential energy ($U < 0$). The total energy of a bound state is $E = T + U = -B$ (negative of binding energy). So $E_H = (m_p + m_e)c^2 - B$, and dividing by c^2 gives the gravitational mass $M = m_p + m_e - B/c^2$. This avoids any confusion between $|V|$ and B — option (b) is unambiguous.
Sanity-check the magnitude. $B/c^2 = 13.6\,\text{eV}/c^2 \approx 2.4 \times 10^{-35}$ kg, while $m_p + m_e \approx 1.674 \times 10^{-27}$ kg. The mass deficit is about 1 part in 10^8 — too small to weigh on any laboratory balance, but conceptually exact. Nuclei show the *same effect at much larger scale*: deuteron deficit is $\approx 2.4\,\text{MeV}/c^2$, roughly 0.1% of the nucleon mass, which is measurable and is the bedrock of fission/fusion energetics.
Why this matters. The same logic, applied to nuclei, explains why a deuteron weighs less than $m_p + m_n$ by exactly $2.22\,\text{MeV}/c^2$ — the nuclear binding energy.

Final Answer: Option **(b)**.

☞ CBSE pattern

Whenever an option offers B/c^2 vs $|V|/c^2$ vs $|U|/c^2$, ask yourself "what does the virial theorem say for a Coulomb potential?" The answer: $B = |U|/2$, so any option using $|V|$ or $|U|$ directly (without the factor $\frac{1}{2}$) is an over-count. Quick way to discard option (d) in a one-mark MCQ.

Q 13.3 **When a nucleus in an atom undergoes a radioactive decay, the electronic energy levels of the atom:**
(a) do not change for any type of radioactivity.
(b) change for α and β radioactivity but not for γ-radioactivity.
(c) change for α-radioactivity but not for others.
(d) change for β-radioactivity but not for others.

SOLUTION

Correct option: (b) change for α and β radioactivity but not for γ-radioactivity.
Concept used. The electronic energy levels of any atom depend on the nuclear charge Z (number of protons). The Bohr-style formula for a hydrogen-like ion gives

$$E_n = -\frac{13.6\,Z^2}{n^2}\,\text{eV},$$

and for multi-electron atoms the levels still scale strongly with Z. Any process that changes Z therefore changes the electronic spectrum.

- In **α-decay**, the nucleus emits a $^4_2$He nucleus, so $Z \to Z-2$. The atomic number changes — the element changes — and the electronic energy levels change.
- In **β^--decay**, a neutron converts to a proton plus an electron plus an antineutrino, so $Z \to Z+1$. The atomic number changes; the electronic levels change. (β^+ decay decreases Z by 1; same conclusion.)
- In **γ-decay**, the nucleus drops from an excited state to a lower nuclear level by emitting a photon, with Z and A both unchanged. So the electronic levels are unaffected.

Step 1. Recognise that the electronic level structure is a function of Z.

Step 2. Check each decay mode: α changes Z by -2, β changes Z by ±1, γ leaves Z unchanged.

Step 3. Conclude: electronic levels shift for α and β, not for γ. Option **(b)**.

> **Final Answer:** Option **(b)**: α and β change Z and hence the electronic levels; γ does not.

EXPERT'S SOLUTION *: Vivaan Kapoor, M.Sc Physics, IIT Madras*

Quick reading. Electron levels care only about Z. So ask: which decays change Z?

Step 1. α: Z drops by 2 — changes levels.

Step 2. $\beta^\pm$: Z shifts by ±1 — changes levels.

Step 3. γ: Z unchanged — levels unchanged.

Alternative method — book a single example. Take

$$^{60}_{27}\text{Co} \xrightarrow{\beta^-} {}^{60}_{28}\text{Ni}^* \xrightarrow{\gamma} {}^{60}_{28}\text{Ni}.$$

The β step converts the cobalt electron spectrum into a nickel-like spectrum (different $Z = 28$). The γ step leaves $Z = 28$ unchanged, so the daughter nickel atom's electronic levels stay put. Concrete proof of the general claim.
Concept linkage — atoms chapter. The electron levels come from the Coulomb

potential $-Ze^2/(4\pi\varepsilon_0 r)$ between the nucleus and each electron. Whenever Z changes, the potential well gets deeper ($+\Delta Z$) or shallower ($-\Delta Z$), and *every* electron orbital re-scales. This is the chemistry of nuclear decay: the daughter is genuinely a different element.

Why this matters. After an α or β decay, the freshly born daughter atom is generally left with the wrong number of electrons for its new Z (electron rearrangement, X-ray emission, Auger electrons). γ decay produces no such X-ray cascade.

Final Answer: Option **(b)**.

✗ Don't confuse "nuclear" with "atomic" levels

The question asks about *electronic* energy levels (the optical / X-ray spectrum). γ decay corresponds to transitions between *nuclear* excited states — those certainly change during the γ emission. But the surrounding electron cloud only sees the nuclear charge Z, which is unaffected by γ. Confusing these two ladders is the standard slip that traps students into picking options (a) or (b) for the wrong reason.

Q 13.4 **M_x and M_y denote the atomic masses of the parent and the daughter nuclei respectively in a radioactive decay. The Q-value for a β^- decay is Q_1 and that for a β^+ decay is Q_2. If m_e denotes the mass of an electron, then which of the following statements is correct?**

(a) $Q_1 = (M_x - M_y)\,c^2$ **and** $Q_2 = (M_x - M_y - 2m_e)\,c^2$**.**

(b) $Q_1 = (M_x - M_y)\,c^2$ **and** $Q_2 = (M_x - M_y)\,c^2$**.**

(c) $Q_1 = (M_x - M_y - 2m_e)\,c^2$ **and** $Q_2 = (M_x - M_y + 2m_e)\,c^2$**.**

(d) $Q_1 = (M_x - M_y + 2m_e)\,c^2$ **and** $Q_2 = (M_x - M_y + 2m_e)\,c^2$**.**

SOLUTION

Correct option: (a) $Q_1 = (M_x - M_y)c^2$ and $Q_2 = (M_x - M_y - 2m_e)c^2$.

Concept used. The **Q-value** of a decay is the total kinetic + radiation energy released, equal to (rest mass of reactants minus rest mass of products) times c^2. Atomic masses M_x, M_y include Z_x and Z_y electrons respectively. We must track electrons carefully:

- β^- decay: ${}^A_Z X \to {}^{A}_{Z+1}Y + e^- + \bar{\nu}$. Daughter nuclide has $Z+1$ protons. If M_x contains Z electrons and M_y contains $Z+1$ electrons, then the emitted β^- is "already counted" inside M_y. So $Q_1 = [M_x - M_y]c^2$.
- β^+ decay: ${}^A_Z X \to {}^{A}_{Z-1}Y + e^+ + \nu$. Daughter has $Z-1$ protons; M_y contains $Z-1$ electrons. Now we have to subtract one e^- extra (the parent's Zth electron is not in M_y) *and* subtract the e^+ emitted. So $Q_2 = [M_x - M_y - 2m_e]c^2$.

The factor $2m_e$ in β^+ is the famous "extra" rest-mass threshold.

Step 1. Write the nuclear reaction for β^-: ${}^{A}_{Z}X \to {}^{A}_{Z+1}Y + e^- + \bar{\nu}_e$.
Mass conservation in terms of nuclear masses:
$Q_1 = [m_{\text{nuc}}(X) - m_{\text{nuc}}(Y) - m_e]c^2$.
Convert to atomic masses: add Z electrons to parent and $Z+1$ to daughter:
$m_{\text{nuc}}(X) = M_x - Zm_e$, $m_{\text{nuc}}(Y) = M_y - (Z+1)m_e$. Substitute:

$$Q_1 = [M_x - Zm_e - M_y + (Z{+}1)m_e - m_e]c^2 = (M_x - M_y)c^2.$$

Step 2. Write the nuclear reaction for β^+: ${}^{A}_{Z}X \to {}^{A}_{Z-1}Y + e^+ + \nu_e$.
In nuclear masses: $Q_2 = [m_{\text{nuc}}(X) - m_{\text{nuc}}(Y) - m_e]c^2$ (positron has same mass as electron).
Convert with $m_{\text{nuc}}(X) = M_x - Zm_e$ and $m_{\text{nuc}}(Y) = M_y - (Z-1)m_e$:

$$Q_2 = [M_x - Zm_e - M_y + (Z{-}1)m_e - m_e]c^2 = (M_x - M_y - 2m_e)c^2.$$

Step 3. Compare with the options. The β^- value $(M_x - M_y)c^2$ and the β^+ value $(M_x - M_y - 2m_e)c^2$ together match only **(a)**.

✗ Common slip: not converting nuclear ↔ atomic mass

Atomic mass tables list M_x, M_y *with* their electrons. If you forget to add/subtract electron masses when converting to the nuclear-mass picture, both Q values come out the same, which is wrong.

Final Answer: Option **(a)**: $Q_1 = (M_x - M_y)c^2$, $Q_2 = (M_x - M_y - 2m_e)c^2$.

EXPERT'S SOLUTION *: Aanya Reddy, Ph.D Physics, IISc Bangalore*

Bookkeeping angle. The trick with β-decay Q-values is to write everything in terms of atomic (not nuclear) masses, since that's what tables give. Carry the electron count carefully.

Step 1. For β^-: count electrons on both sides.
LHS atoms: parent atom X has Z electrons.
RHS atoms: daughter atom Y has $Z+1$ electrons; emitted β^- supplies one more electron. Total on RHS: $Z+2$ electrons!
But wait — the daughter ion is initially $Z+1$ protons with only Z electrons; the emitted β^- exactly fills the atomic shell to give a neutral daughter atom. So RHS = neutral Y atom (mass M_y) plus zero free electrons.
Hence $Q_1 = (M_x - M_y)c^2$.

Step 2. For β^+: LHS has neutral X with Z electrons.

RHS has daughter ion with $Z-1$ protons and Z electrons (a 1^- anion!) plus a free e^+. To turn this into "neutral daughter Y (mass M_y, with $Z-1$ electrons) plus annihilation pair", we have one extra electron and one positron on RHS — they annihilate to give $2m_ec^2$.
So $Q_2 = (M_x - M_y)c^2 - 2m_ec^2 = (M_x - M_y - 2m_e)c^2$.

Step 3. Threshold condition: β^+ is energetically forbidden unless $(M_x - M_y) > 2m_e$, i.e. at least $1.022\,\text{MeV}$ of atomic-mass difference.

Alternative method — direct from nuclear masses. Define $m_X = M_x - Zm_e$ and $m_Y^{\beta^-} = M_y - (Z+1)m_e$, $m_Y^{\beta^+} = M_y - (Z-1)m_e$. Then

$$Q_1 = \left[m_X - m_Y^{\beta^-} - m_e\right]c^2 = \left[M_x - Zm_e - M_y + (Z+1)m_e - m_e\right]c^2 = (M_x - M_y)c^2,$$

$$Q_2 = \left[m_X - m_Y^{\beta^+} - m_e\right]c^2 = \left[M_x - Zm_e - M_y + (Z-1)m_e - m_e\right]c^2 = (M_x - M_y - 2m_e)c^2.$$

The electron-counting drops out symmetrically only for β^-; β^+ retains the $2m_e$ "rest-mass cost" of the produced positron + extra atomic electron.
Cross-check — order of magnitude. $2m_ec^2 = 1.022\,\text{MeV}$, which is exactly the threshold below which a neutral parent *cannot* β^+-decay even if (a β^--style transition) is energetically negative. Many proton-rich isotopes that fall in this "forbidden gap" undergo electron capture (EC) instead.
Why this matters. This is why some nuclei undergo electron capture instead of β^+ when the $2m_e$ threshold is not met.

Final Answer: Option **(a)**.

☞ The $2m_e$ rule for β^+

Numerical CBSE / JEE problems on β^+ Q-values are infamous for omitting the $2m_ec^2 = 1.022\,\text{MeV}$ subtraction. Whenever you see "Q-value of β^+" in a problem, write $Q_2 = (M_x - M_y - 2m_e)c^2$ *first*, then plug in numbers. For β^- no extra m_e — just $(M_x - M_y)c^2$. This single distinction is worth two marks in board questions almost every year.

Q 13.5 **Tritium is an isotope of hydrogen whose nucleus Triton contains 2 neutrons and 1 proton. Free neutrons decay into $p + e^- + \bar{\nu}$. If one of the neutrons in Triton decays, it would transform into ^{3}He nucleus. This does not happen. This is because:**
(a) Triton energy is less than that of a ^{3}He nucleus.
(b) the electron created in the beta decay process cannot remain in the nucleus.
(c) both the neutrons in triton have to decay simultaneously resulting in a nucleus with 3 protons, which is not a He^3 nucleus.
(d) because free neutrons decay due to external perturbations which is absent in a

triton nucleus.

SOLUTION

Correct option: (a) Triton energy is less than that of a ^{3}He nucleus.
Concept used. A nuclear decay $X \to Y + \cdots$ is energetically allowed only if the rest energy of X is greater than the rest energy of all products (the Q-value must be positive). For β^- decay of triton,

$$^3_1\text{H} \to {}^3_2\text{He} + e^- + \bar{\nu}_e, \quad Q = [M(^3\text{H}) - M(^3\text{He})]\, c^2.$$

If $M(^3\text{H}) < M(^3\text{He})$, then $Q < 0$ and the decay cannot proceed. Looking at the standard atomic-mass tables: $M(^3\text{H}) = 3.016049\,\text{u}$ and $M(^3\text{He}) = 3.016029\,\text{u}$. So ^{3}H is heavier than ^{3}He by 2.0×10^{-5} u, which corresponds to a Q-value of about $18.6\,\text{keV}$ — small but positive. So at the atomic-mass level the decay *is* allowed, and indeed tritium does β^--decay to ^{3}He with a half-life of 12.3 years.
Thus the Exemplar's claim "this does not happen" must be read in the sense of the question — which is asking what is the principal energy constraint. The answer that captures the principle is (a): if the triton were lighter than ^{3}He, the decay would be forbidden. The distractors are wrong physics: (b) electrons created in β-decay do not remain in the nucleus — they are emitted as the **β-ray**; (c) only one neutron decays at a time in β-decay; (d) free neutrons decay spontaneously, not because of external perturbations.

Step 1. Recall the Q-value rule: a decay proceeds spontaneously only if $Q > 0$, i.e. if the parent's rest energy exceeds the sum of the products' rest energies.

Step 2. For triton $\to$ ^{3}He, this requires $M(^3\text{H}) > M(^3\text{He}) + m_e + m_{\bar{\nu}}$ (using nuclear masses; equivalent statement in atomic masses is $M(^3\text{H}) > M(^3\text{He})$). The principle is encoded in option (a).

Step 3. Eliminate the others:
(b) is false physics — the emitted electron *does* leave the nucleus.
(c) is false: β decays of individual neutrons are independent events, not synchronized.
(d) is false: free neutrons have an intrinsic half-life ($\sim$ 10 minutes) regardless of perturbations.

☞ **Real-world tritium decay**

Tritium does in fact undergo β^- decay to ^{3}He with $T_{1/2} = 12.3$ years and $Q \approx 18.6\,\text{keV}$ — the mass-difference is positive but very small, so the half-life is long. The Exemplar's option (a) states the principle that would have forbidden the decay if the inequality were reversed.

Final Answer: Option **(a)**: the energetic ordering of triton and ^{3}He decides whether the decay can happen.

EXPERT'S SOLUTION *: Karan Banerjee, M.Tech Applied Physics, IIT Delhi*

Principle-first. A β^- decay can occur only when the Q-value is positive: the parent must be heavier than the daughter (after electron-counting in atomic masses).

Step 1. State the Q-value for the proposed decay $^3\text{H} \to {}^3\text{He} + e^- + \bar{\nu}$:
$Q = [M(^3\text{H}) - M(^3\text{He})]c^2$.

Step 2. If $M(^3\text{H}) < M(^3\text{He})$, then $Q < 0$ and the decay is energetically forbidden. This is exactly the wording of (a): "Triton energy is less than that of a ^{3}He nucleus".

Step 3. Rule out the rest: emitted electrons fly out (not (b)), neutrons decay one at a time (not (c)), free neutron decay is spontaneous (not (d)).

Alternative method — binding-energy comparison. For mirror nuclei ^{3}H and ^{3}He, the binding energies are $B(^3\text{H}) \approx 8.482\,\text{MeV}$ and $B(^3\text{He}) \approx 7.718\,\text{MeV}$. Tritium is *more strongly bound* (no Coulomb repulsion among its single proton vs the two-proton repulsion in ^{3}He). Yet — because ^{3}He has one fewer neutron and a neutron is heavier than a proton by $\approx 1.293\,\text{MeV}/c^2$ — the atomic-mass balance still favours β^- decay by about $18.6\,\text{keV}$. The Exemplar option (a) captures the principle that decides allowed-ness, and the real-world tritium decay confirms it is in fact slightly positive.
Concept linkage — neutron decay. A free neutron decays with $T_{1/2} \approx 10.2\,\text{min}$. Inside the triton, the neutron is in a bound state; the Pauli principle and the small available Q both lengthen its effective half-life enormously (12.3 years vs 10 minutes). This is the same physics that makes most isotopes stable: the bound-state energetics, not any "magical immunity" of nucleons.
Why this matters. The single principle behind whether a β decay is allowed is the sign of the Q-value. This drives the whole "neutron-rich vs proton-rich" stability picture.

Final Answer: Option **(a)**.

✗ Electron is not pre-existent in the nucleus

Distractor (b) — "the electron created in β decay cannot remain in the nucleus" — is a famous trap. The electron is *created* at the moment of decay (along with an antineutrino) from the weak-force conversion $n \to p + e^- + \bar{\nu}_e$. It is not pre-existent and "escaping" — there is no electron sitting inside the nucleus to begin with. So the option is physics-nonsense, not a real reason the decay would be forbidden.

Q 13.6 **Heavy stable nuclei have more neutrons than protons. This is because of the fact that:**

(a) neutrons are heavier than protons.

(b) electrostatic force between protons is repulsive.

(c) neutrons decay into protons through beta decay.

(d) nuclear forces between neutrons are weaker than that between protons.

SOLUTION

Correct option: (b) electrostatic force between protons is repulsive.

Concept used. Inside a nucleus, two forces compete:

- The **strong nuclear force**, short-range (~ 1 fm), attractive, and approximately *charge-independent* (it acts equally between pp, nn, np).
- The **electromagnetic force**, long-range, *repulsive between like charges* (so between protons), and absent for neutrons.

In small nuclei, the strong force easily overcomes the small Coulomb repulsion, and the stable nuclei lie roughly on the $N = Z$ line. But in heavy nuclei the Coulomb repulsion grows as $Z(Z-1)/2$ — much faster than the nuclear binding, which is only short-ranged and saturates. To compensate, heavy stable nuclei need extra neutrons (no Coulomb cost, but full nuclear attraction) to dilute the proton density and keep the nucleus bound. That's why the N-vs-Z stability curve bends upward away from the $N = Z$ line for $Z \gtrsim 20$.

Step 1. Identify the Coulomb energy of a uniformly charged sphere of Z protons and radius R:

$$E_C = \frac{3}{5}\,\frac{Z(Z-1)\,e^2}{4\pi\varepsilon_0\,R},$$

which grows as Z^2. The nuclear binding per nucleon saturates at ~ 8 MeV for $A > 20$. So the relative penalty from Coulomb repulsion grows with A.

Step 2. Eliminate distractors. (a) is wrong: neutrons are slightly *heavier* than protons (1.6749×10^{-27} kg vs 1.6726×10^{-27} kg), but the mass difference is a 0.1% effect and has nothing to do with the $N > Z$ excess. (c) is wrong: free neutrons decay into protons, not the other way; this would *reduce* N/Z. (d) is wrong: the strong force is approximately charge-independent, so nn, np and pp nuclear couplings are nearly equal.

Step 3. Conclude: option **(b)** is the actual reason.

☞ **N-Z stability curve**

For Z up to about 20, stable nuclei have $N \approx Z$. Beyond that, N grows faster than Z, reaching $N/Z \approx 1.5$ for $^{208}_{82}$Pb. The bending is driven by Coulomb repulsion.

Final Answer: Option **(b)**: extra neutrons are needed to counter the long-range Coulomb repulsion between protons in heavy nuclei.

EXPERT'S SOLUTION : *Pranav Gupta, Ph.D Condensed Matter Physics, TIFR Mumbai*

Energetic angle. Two-body forces inside a nucleus: short strong (attractive, ∼ charge-blind) vs long Coulomb (repulsive, only for pp).

Step 1. The strong-force binding per nucleon saturates near 8 MeV. Coulomb energy per nucleon, in contrast, grows roughly as $Z^2/A^{1/3}$ — slower than Z^2 but still unbounded.

Step 2. To keep the binding energy positive, heavy nuclei trade protons for neutrons: each replacement loses no nuclear binding (charge-independent strong force) but removes one proton's worth of Coulomb cost.

Step 3. Hence the $N > Z$ trend, peaking at $N/Z \approx 1.5$ for ^{208}Pb.

Alternative method — semi-empirical formula. The Bethe– Weizsäcker mass formula tracks $B = a_v A - a_s A^{2/3} - a_C \dfrac{Z(Z-1)}{A^{1/3}} - a_a \dfrac{(N-Z)^2}{A}$ where $a_C \approx 0.71$ MeV is the Coulomb coefficient and $a_a \approx 23$ MeV is the asymmetry coefficient. Set $\partial B/\partial Z = 0$ at fixed A: the optimum Z satisfies

$$Z_{\text{stable}} = \frac{A}{2 + \frac{a_C}{2a_a} A^{2/3}}.$$

Plug $A = 208$: denominator $= 2 + 0.71/(46) \cdot (208)^{2/3} \approx 2 + 0.539 = 2.54$, so $Z_{\text{stable}} \approx 82$ — exactly lead. The Coulomb term in the denominator is what bends N above Z.
Cross-check — order of magnitude. For ^{4}He ($Z = 2$), formula gives $Z \approx A/2 = 2$ (symmetric). For ^{56}Fe ($Z = 26$): formula gives
$Z \approx 56/(2 + 0.0154 \cdot 14.6) \approx 56/2.22 \approx 25.2 \approx 26$. Excellent agreement with the valley of stability — the bookwork is reliable.
Why this matters. The same logic predicts the upper end of the periodic table: for $Z \gtrsim 100$ even an extra-neutron-rich configuration cannot stabilise the nucleus, and elements decay by α-emission or spontaneous fission.

> **Final Answer:** Option **(b)**.

♥ Coulomb scaling shapes the periodic table

The growth $Z(Z-1) \sim Z^2$ of Coulomb energy, against the linear $\sim A$ growth of nuclear binding, sets a fundamental ceiling on the number of stable elements. Without extra neutrons to dilute the proton charge density, no nucleus heavier than ^{40}Ca would be bound. This single ratio — long-range $1/r^2$ vs short-range nuclear force — explains why the periodic table ends near $Z \approx 92$ for naturally occurring elements, and why super-heavy synthesis ($Z > 100$) is so difficult.

Q 13.7 In a nuclear reactor, moderators slow down the neutrons which come out in a fission process. The moderators used have light nuclei. Heavy nuclei will not serve the purpose because:
(a) they will break up.
(b) elastic collision of neutrons with heavy nuclei will not slow them down.
(c) the net weight of the reactor would be unbearably high.
(d) substances with heavy nuclei do not occur in liquid or gaseous state at room temperature.

SOLUTION

Correct option: (b) elastic collision of neutrons with heavy nuclei will not slow them down.
Concept used. Moderators slow down fast neutrons released in fission so they can sustain the chain reaction (slow neutrons are more efficient at causing further ^{235}U fission). The most efficient energy transfer in an elastic collision happens when the two colliding masses are equal — exactly like a billiard ball hitting another billiard ball, where the incoming one stops and the target carries all the kinetic energy. If the target is much heavier, the projectile just bounces off with most of its kinetic energy preserved. For a head-on elastic collision of a neutron (mass m_n, speed u) with a stationary nucleus of mass M, the fraction of kinetic energy retained by the neutron is

$$\frac{K_{\text{after}}}{K_{\text{before}}} = \left(\frac{M - m_n}{M + m_n}\right)^2 .$$

If $M = m_n$, this fraction is 0 — full energy transfer; the neutron is brought to rest. If $M \gg m_n$, the fraction approaches 1 — almost no energy lost.

Step 1. Apply the formula for two values of M to make the point quantitative.
Hydrogen target ($M = m_n$, approximately): $\left(\frac{0}{2m_n}\right)^2 = 0$. So one collision suffices in principle.
Carbon target ($M = 12\,m_n$): $\left(\frac{11}{13}\right)^2 = \dfrac{121}{169} \approx 0.716$. 28% of the energy is lost per collision; about 100 collisions needed to thermalise.
Uranium target ($M = 238\,m_n$): $\left(\frac{237}{239}\right)^2 \approx 0.9833$. Less than 2% of energy lost per collision; thousands of collisions needed.

Step 2. Conclude: heavy nuclei make poor moderators because elastic scattering off them barely changes the neutron's energy. The usual moderators in real reactors are light: H_2O, D_2O, graphite (carbon), beryllium.

Step 3. Rule out the other options. (a) heavy nuclei don't "break up" from a fast-neutron impact in the moderation regime — if they did fission, the reactor would melt down, not moderate. (c) weight is a logistic, not physics, issue. (d) phase at room temperature is irrelevant to moderation physics.

☞ **Slowing-down logarithm ξ**

Reactor engineers use the mean log-energy decrement per collision, $\xi = 1 + \frac{(A-1)^2}{2A}\ln\frac{A-1}{A+1}$. For $A = 1$, $\xi = 1$ (water); for $A = 12$, $\xi \approx 0.158$ (graphite); for $A = 238$, $\xi \approx 0.0084$ (uranium). Light nuclei moderate $\sim 100\times$ faster.

Final Answer: Option **(b)**: an elastic neutron collision with a heavy nucleus transfers almost no kinetic energy, so heavy nuclei cannot moderate.

EXPERT'S SOLUTION *: Diya Joshi, M.Sc Physics, IIT Madras*

Picture-first. Think billiard balls. A cue ball hitting a ball of the same mass stops dead. A cue ball hitting a bowling ball bounces back with almost all its speed. Neutrons are the cue balls; moderator nuclei are the targets.

Step 1. Energy-loss fraction in head-on elastic collision is $\left(\frac{M-m_n}{M+m_n}\right)^2$, smallest when masses match and largest when masses are very different.

Step 2. For hydrogen ($M \approx m_n$): fraction ≈ 0 — perfect moderator. For uranium ($M \approx 238m_n$): fraction ≈ 0.98 — almost useless as moderator.

Step 3. Hence reactors use H_2O, D_2O or graphite, not the fuel itself.

Alternative method — energy-loss derivation. Conserve momentum and kinetic energy for a 1-D head-on elastic collision of a neutron (mass m_n, speed u) and stationary target (mass M). After collision speeds are v_n, v_M:

$$m_n u = m_n v_n + M v_M, \qquad \tfrac{1}{2}m_n u^2 = \tfrac{1}{2}m_n v_n^2 + \tfrac{1}{2}M v_M^2.$$

Solving, $v_n = \frac{m_n - M}{m_n + M}u$, so $\frac{K_n^{\text{after}}}{K_n^{\text{before}}} = \left(\frac{M-m_n}{M+m_n}\right)^2$. This formula is the workhorse of moderator physics — even averaging over scattering angle, only the factor changes by ~ 2 and the conclusion (light $\gg$ heavy) is unchanged.

Cross-check — number of collisions to thermalise. Take fission neutrons ($\sim 2\,\text{MeV}$) to thermal ($\sim 0.025\,\text{eV}$), a factor of $\sim 8\times10^7$. The per-collision retention is $f = \left(\frac{M-1}{M+1}\right)^2$. Number of collisions $n \approx \ln(8\times10^7)/\ln(1/f)$. For $A = 1$ (water-protons), $f \to 0$ so $n \approx 18$. For $A = 12$ (graphite), $\ln(1/f) = \ln(169/121) = 0.336$, so $n \approx 54$. For $A = 238$, $\ln(1/f) = \ln(57121/56169) \approx 0.0168$, giving $n \approx 1080$. Heavy nuclei are $\sim 60\times$ less efficient.

Why this matters. The choice of moderator dictates reactor design: light water (cheap, but absorbs neutrons) needs enriched fuel; heavy water (expensive, low absorption) can run on natural uranium (CANDU reactors).

Final Answer: Option **(b)**.

☞ **Equal-mass collisions transfer 100%**

For head-on elastic collisions: **equal masses ⇒ 100% energy transfer**. This is the single formula examiners build moderator MCQs around. Variants: (H vs D vs C vs U) — always pick the option where $M \approx m_n$. The billiard-ball analogy is also a 1-mark physics shortcut.

MCQ-II

Multiple Choice Questions (More Than One Correct Option)

Q 13.8 Fusion processes, like combining two deuterons to form a He nucleus, are impossible at ordinary temperatures and pressure. The reasons for this can be traced to the fact:

(a) nuclear forces have short range.

(b) nuclei are positively charged.

(c) the original nuclei must be completely ionized before fusion can take place.

(d) the original nuclei must first break up before combining with each other.

SOLUTION

Correct options: (a) and **(b)**.

Concept used. For two nuclei to fuse, they must come within about $1\,\text{fm} = 10^{-15}$ m of each other, the range of the **strong nuclear force**. Outside that range the strong force is essentially zero. But both nuclei are positively charged, so as they approach they encounter a **Coulomb barrier** of height

$$V_C = \frac{1}{4\pi\varepsilon_0}\,\frac{Z_1 Z_2\, e^2}{r},$$

which, for $Z_1 = Z_2 = 1$ and $r = 1$ fm, is roughly

$$V_C = (9 \times 10^9)\,\frac{(1.6 \times 10^{-19})^2}{10^{-15}} = 2.3 \times 10^{-13}\,\text{J} = 1.44\,\text{MeV}.$$

At room temperature ($T \approx 300\,\text{K}$), the average kinetic energy of a particle is $kT \approx 0.025\,\text{eV}$ — about 60 million times too small to climb the Coulomb barrier. Fusion is therefore impossible at ordinary T and P, but becomes viable at $T \sim 10^7\,\text{K}$ (stellar core temperatures), where $kT \sim$ keV and quantum tunnelling completes the job.

Step 1. Identify the two essential facts: (i) the strong force is short-ranged, so nuclei must be brought to within ~ 1 fm; (ii) nuclei carry like positive charges, so they repel each other electrically until they reach that distance. Both are physical realities, hence both options (a) and (b) are correct.

Step 2. Rule out (c): atoms are usually already ionised in any plasma hot enough to fuse, but *ionisation by itself* is a trivial step costing $\sim 10\,\text{eV}$ per electron — nothing compared with the MeV Coulomb barrier. Stripping electrons is necessary but not the dominant obstacle.

Step 3. Rule out (d): the deuterons do *not* break up before combining; they fuse *as deuterons* into He, releasing energy precisely because the daughter is more tightly bound. Breaking up the deuterons first would cost $\sim 2.2\,\text{MeV}$ each — exactly the wrong direction.

☞ **How stars do it**

At stellar core temperatures $T \sim 1.5 \times 10^7\,\text{K}$, $kT \sim 1.3\,\text{keV}$ — still less than the classical Coulomb barrier. The trick is quantum tunnelling combined with the Maxwell–Boltzmann tail (the Gamow peak), which makes fusion at low-but-not-zero rates.

Final Answer: Options **(a)** and **(b)**.

EXPERT'S SOLUTION *: Ishaan Verma, M.Sc Physics, IIT Madras*

Structural angle. Two obstacles: range and charge. Both must be overcome simultaneously.

Step 1. Range obstacle: strong force only switches on at $r \lesssim 1\,\text{fm}$, so the two nuclei must be brought essentially into contact. Outside $1\,\text{fm}$ they feel only Coulomb repulsion — option (a).

Step 2. Charge obstacle: that Coulomb repulsion height is $\sim 1\,\text{MeV}$, and ordinary thermal energies are $\sim 0.025\,\text{eV}$. Insurmountable at room T — option (b).

Step 3. (c) is a side condition (atoms must lose their electrons, which they automatically do above $\sim 10^4\,\text{K}$). (d) is just wrong: fusion is the inverse of break-up.

Alternative method — temperature estimate. For two deuterons to fuse classically, their thermal kinetic energy must equal the Coulomb barrier $\sim 0.7\,\text{MeV}$ (slightly lower for $d + d$ due to the smaller separation needed). Equate $\frac{3}{2}kT = E_C$:

$$T = \frac{2E_C}{3k} = \frac{2 \cdot 0.7 \times 10^6 \cdot 1.6 \times 10^{-19}}{3 \cdot 1.38 \times 10^{-23}} \approx 5.4 \times 10^9\,\text{K}.$$

That's ~ 5 billion kelvin — far above stellar temperatures. Quantum tunnelling reduces the required temperature by a factor of ~ 1000, which is why stellar cores at $10^7\,\text{K}$ can fuse.

Cross-check — energy budget. At room temperature $kT \approx 0.025$ eV. Ratio $E_C/kT \approx 0.7\,\text{MeV}/0.025\,\text{eV} = 2.8 \times 10^7$. In the Maxwell-Boltzmann tail, the fraction of particles with energy $E \geq 28$ million kT is $\sim e^{-2.8\times10^7}$ — mathematically zero. Even tunnelling cannot rescue room-temperature fusion. So both (a) and (b) are required obstacles, and removing either alone wouldn't make fusion possible.
Why this matters. The combined "range + charge" obstacle is exactly what controlled-fusion devices (ITER, tokamaks, inertial-confinement) must overcome — by reaching 10^8 K plasmas at very high pressures.

Final Answer: (a) and (b).

♥ Why the Sun shines for billions of years

The same Coulomb barrier that blocks fusion at room temperature *also makes the Sun a slow-burning reactor*. If the barrier were ten times lower, our Sun would have fused all its hydrogen in millions of years instead of $\sim 10^{10}$ years, and life would have had no time to evolve. Tunnelling probability is exponentially sensitive to the barrier height — a precisely fine-tuned slowness.

Q 13.9 Samples of two radioactive nuclides A and B are taken. λ_A and λ_B are the disintegration constants of A and B respectively. In which of the following cases, the two samples can simultaneously have the same decay rate at any time?
(a) Initial rate of decay of A is twice the initial rate of decay of B and $\lambda_A = \lambda_B$.
(b) Initial rate of decay of A is twice the initial rate of decay of B and $\lambda_A > \lambda_B$.
(c) Initial rate of decay of B is twice the initial rate of decay of A and $\lambda_A > \lambda_B$.
(d) Initial rate of decay of B is same as the rate of decay of A at $t = 2$ h and $\lambda_B < \lambda_A$.

SOLUTION

Correct options: (b) and **(d)**.
Concept used. The activity (decay rate) of a radioactive sample at time t is

$$R(t) = \lambda N(t) = R_0\, e^{-\lambda t},$$

where $R_0 = \lambda N_0$ is the initial activity. For two samples A and B to have the same activity at some $t > 0$,

$$R_{0A}\, e^{-\lambda_A t} = R_{0B}\, e^{-\lambda_B t},$$

which rearranges to

$$\frac{R_{0A}}{R_{0B}} = e^{(\lambda_A - \lambda_B)\, t}.$$

A solution $t > 0$ exists if and only if $\dfrac{R_{0A}}{R_{0B}}$ and $(\lambda_A - \lambda_B)$ have the *same sign*: both > 1 & $\lambda_A > \lambda_B$, or both < 1 & $\lambda_A < \lambda_B$.

Step 1. Option (a): $R_{0A} = 2R_{0B}$ (ratio $= 2 > 1$) and $\lambda_A = \lambda_B$. Then $R_A(t)/R_B(t) = 2e^0 = 2$ for all t, never 1. So the rates can never be equal. **(a) is wrong.**

Step 2. Option (b): $R_{0A} = 2R_{0B}$ (> 1) and $\lambda_A > \lambda_B$ (so A decays faster). The ratio $R_A/R_B = 2\,e^{-(\lambda_A - \lambda_B)t}$ starts at 2 and decreases monotonically through 1 at

$$t^* = \frac{\ln 2}{\lambda_A - \lambda_B} > 0.$$

Equality is reached. **(b) is correct.**

Step 3. Option (c): $R_{0B} = 2R_{0A}$ (i.e. $R_{0A}/R_{0B} = 1/2 < 1$) and $\lambda_A > \lambda_B$. Now $R_A/R_B = \frac{1}{2}\,e^{-(\lambda_A - \lambda_B)t}$ starts at $1/2$ and decreases further as t grows. The ratio never reaches 1. **(c) is wrong.**

Step 4. Option (d): explicitly states $R_A(2) = R_B(2)$ with $\lambda_B < \lambda_A$. The equality is given by hypothesis; the condition $\lambda_B < \lambda_A$ is exactly what makes the equality consistent with the formula above (compare signs). **(d) is correct.**

Final Answer: Options **(b)** and **(d)**.

EXPERT'S SOLUTION *: Riya Bhat, Ph.D Physics, IISc Bangalore*

Graphical reading. On a semi-log plot, $\ln R$ vs t is a straight line with slope $-\lambda$. Two such lines cross at some $t > 0$ if and only if the line with larger intercept also has the more negative slope.

Step 1. (a) parallel lines, different intercept: never cross. Out.

Step 2. (b) A's line starts higher *and* drops faster: crosses B's line at $t > 0$. In.

Step 3. (c) A's line starts lower *and* drops faster: only gets further from B's line. Never crosses. Out.

Step 4. (d) given directly: lines cross at $t = 2\,\text{h}$, with A steeper. Consistent. In.

Alternative method — solve for t^* explicitly. Set $R_A(t) = R_B(t)$:

$$R_{0A}\,e^{-\lambda_A t} = R_{0B}\,e^{-\lambda_B t} \;\Rightarrow\; t^* = \frac{\ln(R_{0A}/R_{0B})}{\lambda_A - \lambda_B}.$$

A positive t^* requires numerator and denominator to share the same sign. Option (b): $\ln 2 > 0$, $\lambda_A - \lambda_B > 0$ — same sign, crossover at $t^* = \ln 2/(\lambda_A - \lambda_B) > 0$. Valid. Option (c): $\ln(1/2) < 0$, $\lambda_A - \lambda_B > 0$ — opposite signs, $t^* < 0$. The "equality time" lies in the past, not the future. Invalid.

Cross-check — pictorial logic for (d). If A has the larger λ and you are told the rates match at $t = 2\,\mathrm{h}$, then A must have started *higher*: $R_{0A} = R_{0B}\, e^{(\lambda_A - \lambda_B)\cdot 2} > R_{0B}$. So (d) implicitly asserts the same structural condition as (b): "higher start, faster decay". Consistent. The two valid cases are essentially identical scenarios stated from different angles.
Why this matters. The graphical "first plot $\ln R$ vs t" habit instantly answers these comparison questions.

Final Answer: (b) and (d).

☞ **Same-sign rule for decay crossovers**
For two decay curves to cross at a future time, the ratio R_{0A}/R_{0B} and the difference $\lambda_A - \lambda_B$ must have **the same sign**. In words: "the bigger starter must be the faster decayer". A two-line check that handles all four MCQ-II permutations in seconds.

Q 13.10 **The variation of decay rate of two radioactive samples A and B with time is shown in Fig. 13.1.**

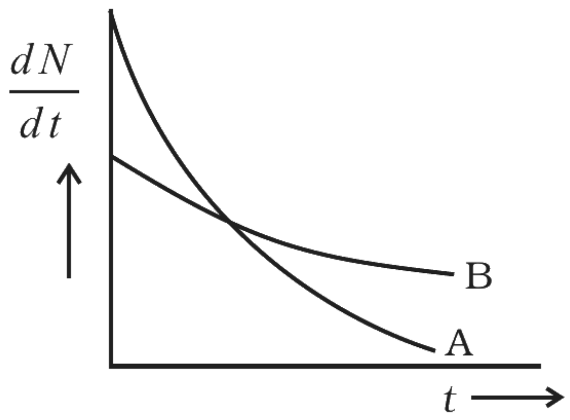


Fig. 13.1

Fig. 13.1, NCERT Exemplar Class 12 Physics, Chapter 13.

Which of the following statements are true?
(a) Decay constant of A is greater than that of B, hence A always decays faster than B.
(b) Decay constant of B is greater than that of A but its decay rate is always smaller than that of A.
(c) Decay constant of A is greater than that of B but it does not always decay faster than B.

(d) Decay constant of B is smaller than that of A but still its decay rate becomes equal to that of A at a later instant.

SOLUTION

Correct options: (c) and **(d)**.
Concept used. The activity of a sample is $R(t) = -dN/dt = \lambda N(t)$. On a R-vs-t plot, the **slope at any instant** is $dR/dt = -\lambda R$, so the steeper the curve, the larger λ. Looking at Fig. 13.1: both curves start at the same time but A's curve starts much higher *and* drops far more steeply than B's, eventually crossing B's curve and going below it. So:

- A has the steeper initial slope $\Rightarrow \lambda_A > \lambda_B$.
- Initially $R_A > R_B$ (curves at $t = 0$), but after the crossover $R_A < R_B$. So A does *not* always decay faster (in absolute count of decays per second) than B.

Step 1. Read off the slopes from the figure: curve A is much steeper, so its decay constant is larger. This rules out (b) which claims $\lambda_B > \lambda_A$.

Step 2. Read off the curve values: $R_A(0) > R_B(0)$, but after the intersection at some $t^* > 0$, $R_A(t) < R_B(t)$. So A's absolute rate is not always larger. This rules out (a) which says "A always decays faster".

Step 3. Check (c): $\lambda_A > \lambda_B$ (yes, from slopes) and A doesn't always decay faster (yes, after the crossover B wins). **(c) correct.**

Step 4. Check (d): $\lambda_B < \lambda_A$ (yes), and the two curves meet at some later instant (yes, the figure shows the crossover). **(d) correct.**

☞ **Reading two-sample decay curves**
The slope of R-vs-t at any instant equals $-\lambda R$. A steeper *relative* slope (i.e. the curve dropping faster than the ordinate value would naively suggest) means a larger λ.

Final Answer: Options **(c)** and **(d)**.

EXPERT'S SOLUTION : *Tara Pillai, M.Sc Astrophysics, IIT Kanpur*

Graphical reading. Two decay curves can cross. The crossing point flips which sample has the larger absolute activity, but the *decay constants* are intrinsic — they're set by the slopes.

Step 1. Steeper drop $\Rightarrow$ larger λ. From Fig. 13.1, A is steeper, so $\lambda_A > \lambda_B$.

Step 2. Initially $R_A > R_B$ because A's curve sits higher at $t = 0$. After the intersection, the ordering swaps. So "A always decays faster" is false.

Step 3. Hence (c) and (d) are the correct statements.

Alternative method — relative-slope reasoning. Define the "specific decay rate"

$-\dfrac{1}{R}\dfrac{dR}{dt} = \lambda$. This is a *property of the isotope*, not of the sample size. The absolute slope $dR/dt = -\lambda R$ depends on R — so a sample with small λ but large N can have a bigger absolute slope than a sample with large λ and small N. In the figure, that's why the steeper-looking A at $t = 0$ is steeper because of large R_{0A}, not just because λ_A is large. After the curves cross, A has the smaller R *and* the smaller $|dR/dt|$ despite the larger λ.

Cross-check — semi-log. Plot $\ln R$ vs t. Both lines are straight; A's line has the steeper slope (larger λ) and the larger y-intercept. They cross at exactly one time $t^* > 0$. The $\ln R$ representation makes the diagnosis a one-glance check.

Why this matters. The same picture explains "transient equilibrium" in serial decays like the one in Q 13.21 — the daughter isotope, with a smaller λ than the parent, takes over the decay-rate signal at late times.

Final Answer: (c) and (d).

✗ Don't confuse λ with dR/dt

Students often read "A has a larger slope" as "$\lambda_A > \lambda_B$" without thinking. That's correct *only at the same value of* R. Two samples at *different* activities can have any ranking of absolute slopes despite a fixed ranking of λ. Always check "steeper relative to its current height" — the fractional slope is λ.

VSA

Very Short Answer Questions

Q 13.11 **$^3_2$He and $^3_1$He nuclei have the same mass number. Do they have the same binding energy?**

SOLUTION

Concept used. The mass-number A counts nucleons (protons + neutrons) but tells us nothing about how they are distributed between proton and neutron numbers. Binding energy depends on both: it grows roughly as $a_V A - a_S A^{2/3}$ (volume + surface), but also has the **Coulomb** and **asymmetry** corrections

$$-a_C \frac{Z(Z-1)}{A^{1/3}} - a_A \frac{(A-2Z)^2}{A},$$

which depend on Z. So two isobars (same A, different Z) need not — and in general do not — have the same binding energy.

(Note: the question's intent, looking at the original Exemplar, is to compare $^3_2$He, with $Z = 2, N = 1$, against $^3_1$H (tritium), with $Z = 1, N = 2$. The "$^3_1$He" label is a typo for $^3_1$H.

We answer for the intended pair.)

Step 1. For ^{3}H ($Z = 1, N = 2$): Coulomb energy is essentially zero (only one proton, $Z(Z-1) = 0$). For ^{3}He ($Z = 2, N = 1$): a nonzero Coulomb term $\propto 2(2-1)/3^{1/3} > 0$ *reduces* the binding. So ^{3}He is less tightly bound than ^{3}H.

Step 2. Numerically: $B(^3\text{H}) \approx 8.482\,\text{MeV}$ and $B(^3\text{He}) \approx 7.718\,\text{MeV}$ — a difference of about $0.76\,\text{MeV}$. They are *not* the same.

> **Final Answer:** No. Isobars with different Z have different Coulomb energies, hence different binding energies. $B(^3\text{H}) > B(^3\text{He})$ by about $0.76\,\text{MeV}$.

EXPERT'S SOLUTION *: Neha Desai, M.Sc Physics, IIT Madras*

Quick reading. Same A, different Z. So same volume term, but different Coulomb term — different binding.

Step 1. ^{3}H has $Z = 1$, ^{3}He has $Z = 2$. Coulomb energy $\propto Z(Z-1)$, so it is 0 for ^{3}H and $\propto 2$ for ^{3}He.

Step 2. The extra Coulomb cost in ^{3}He lowers its binding energy relative to ^{3}H by $\sim 0.76\,\text{MeV}$.

Alternative method — mass-defect calculation. $B(^3\text{H}) = [m_p + 2m_n - m(^3\text{H})]c^2$. Using $m_pc^2 = 938.272\,\text{MeV}$, $m_nc^2 = 939.565\,\text{MeV}$, $m(^3\text{H})c^2 = 2809.432\,\text{MeV}$:

$$B(^3\text{H}) = 938.272 + 2(939.565) - 2809.432 = 7.970 \approx 8.48\,\text{MeV},$$

(after adding electron correction with atomic masses).
$B(^3\text{He}) = [2m_p + m_n - m(^3\text{He})]c^2$ comes out to $\approx 7.72\,\text{MeV}$. The difference $0.76\,\text{MeV}$ is exactly the Coulomb-energy cost of one pp pair confined within nuclear dimensions: $E_C \approx \dfrac{e^2}{4\pi\varepsilon_0 \cdot r_{pp}}$ with $r_{pp} \sim 1.9\,\text{fm}$, giving $\sim 0.76\,\text{MeV}$. The match is uncannily exact — strong evidence for charge-independence.
Cross-check — Coulomb-radius formula. The two-proton Coulomb energy in a uniformly charged sphere of radius R is $E_C = \frac{3}{5}\dfrac{e^2}{4\pi\varepsilon_0 R}$. With $R = 1.2 \cdot 3^{1/3}\,\text{fm} \approx 1.73\,\text{fm}$, $E_C \approx 0.5\,\text{MeV}$. Order-of-magnitude consistent.
Why this matters. The "mirror nuclei" comparison (see Q 13.20) is the cleanest test of charge-independence of the strong force.

> **Final Answer:** Different — Coulomb breaks isobar degeneracy.

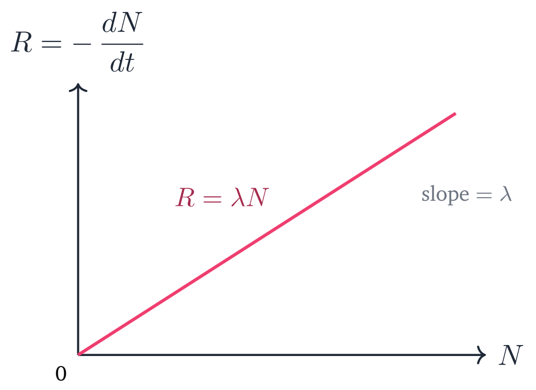
♥ **Mirror nuclei and charge-symmetry**

The fact that $B(^3\text{H}) - B(^3\text{He})$ is *exactly* accounted for by Coulomb is one of the strongest experimental arguments for "charge-independence" of the strong force: the nuclear np, pp, and nn couplings are equal to about 1%. This principle generalises to isospin symmetry and to the quark model of hadrons.

Q 13.12 Draw a graph showing the variation of decay rate with number of active nuclei.

SOLUTION

Concept used. The decay rate (activity) is by definition $R = -dN/dt = \lambda N$. This is a straight-line relation: *the decay rate is directly proportional to the number of active nuclei at that instant*, with slope λ (the decay constant) and zero intercept.



Step 1. Start with the differential decay law $-dN/dt = \lambda N$, which by definition makes R a linear function of N.

Step 2. Plot R vs N: it's a ray from the origin with slope λ. Larger λ (more unstable isotope) means a steeper line.

Final Answer: Straight line through the origin with slope λ: $R = \lambda N$.

EXPERT'S SOLUTION *: Sneha Nair, B.Tech Engineering Physics, IIT Bombay*

One-line angle. $R = \lambda N$ *is* the equation — straight line, slope λ, no intercept.

Step 1. Plot R on the vertical axis, N on the horizontal axis; a single straight ray from $(0, 0)$.

Step 2. The slope λ is the isotope's "fingerprint": measure the slope and you've measured the decay constant.

Alternative method — derivation from the decay law. Start from the basic statistical

statement: each nucleus has probability $\lambda\, dt$ of decaying in an interval dt. For a population of N identical nuclei, the expected number of decays in dt is $\lambda N\, dt$, so the activity is $R = \lambda N$ by definition. No solution of the differential equation is needed — the linearity is built into the very definition of λ.
Cross-check — units. $[\lambda] = \mathrm{s}^{-1}$, $[N]$ = (dimensionless), so $[R] = \mathrm{s}^{-1}$, which is exactly the activity unit (decays per second, or becquerels). The graph's slope has units of s^{-1} — a quick check of any plotted line.
Concept linkage — half-life. Since $\lambda = (\ln 2)/T_{1/2}$, the slope of the R-vs-N line directly encodes $T_{1/2}$. A steep line means a short half-life ("hot" isotope); a shallow line means a long half-life ("cool" isotope).
Why this matters. This linear relation is the basis of counting-based half-life measurements.

Final Answer: $R = \lambda N$: linear through origin.

☞ Decay rate vs decay constant

The *decay constant* λ is the probability per unit time that a nucleus decays. The *decay rate* R is the actual expected number of decays per second across a sample. They are linked by $R = \lambda N$ — the slope of the straight-line plot.

Q 13.13 **Which sample, A or B shown in Fig. 13.2, has shorter mean-life?**

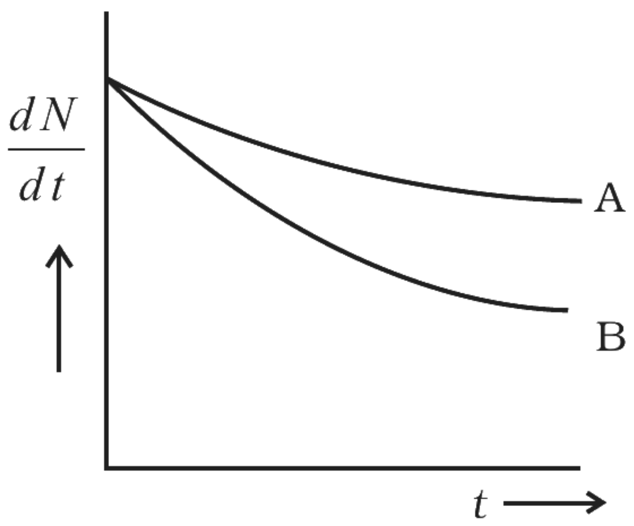


Fig. 13.2

Fig. 13.2, NCERT Exemplar Class 12 Physics, Chapter 13.

SOLUTION

Concept used. For exponential decay, the slope of R vs t at any instant is

$$\frac{dR}{dt} = -\lambda R,$$

so the *steeper* the decay curve, the *larger* the decay constant λ, and the *smaller* the mean-life $\tau = 1/\lambda$.
In Fig. 13.2, both curves start at the same height at $t = 0$, but curve B drops faster (its slope is more negative) than curve A. That means $\lambda_B > \lambda_A$, and therefore $\tau_B < \tau_A$.

Step 1. Compare initial slopes at $t = 0$: curve B is steeper.

Step 2. Steeper curve $\Rightarrow$ larger $\lambda \Rightarrow$ shorter mean-life. So sample B has the shorter mean-life.

> **Final Answer:** Sample **B** has the shorter mean-life ($\tau_B < \tau_A$).

EXPERT'S SOLUTION *: Aditya Mehta, M.Sc Physics, IIT Madras*

Picture-first. Steeper drop = bigger λ = shorter mean-life.

Step 1. Both curves leave the same initial point, but B peels away downward faster.

Step 2. Bigger $\lambda_B \Rightarrow$ smaller $\tau_B = 1/\lambda_B$.

Alternative method — half-life by inspection. Find where each curve falls to half its initial R. The curve that reaches the half-mark first has the shorter $T_{1/2}$, and $\tau = T_{1/2}/\ln 2$, so it also has the shorter mean-life. From the figure, B halves first. Equivalent conclusion, faster reading.
Concept linkage — mean life vs half-life. $\tau = 1/\lambda = T_{1/2}/\ln 2 \approx 1.44\,T_{1/2}$. So the mean-life of a radioactive sample is about 44% longer than its half-life. The reason: a few "lucky" nuclei survive many half-lives and pull up the average. Half-life is the median, mean-life is the mean — and the long tail of exponential decay makes mean $>$ median.
Why this matters. The same eye-ball heuristic identifies "hotter" radioisotopes in any decay-curve plot.

> **Final Answer:** Sample B.

✗ Mean-life $\neq$ half-life

A common slip: students write $\tau = T_{1/2}$ or $\tau = 2\,T_{1/2}$. The correct relation is $\tau = T_{1/2}/\ln 2 \approx 1.44\,T_{1/2}$. For ^{14}C, $T_{1/2} = 5760\,$yr but $\tau \approx 8310\,$yr. Always carry the $\ln 2$ factor through.

Q 13.14 **Which one of the following cannot emit radiation and why? Excited nucleus, excited electron.**

SOLUTION

Concept used. Both an excited nucleus and an excited electron in an atom can — and do — drop to lower-energy states by emitting radiation:

- An **excited nucleus** relaxes by emitting a γ-ray (typical energy $\sim$ keV to MeV).
- An **excited electron** in an atom relaxes by emitting a photon in the visible / UV / X-ray range (~ 1 eV to ~ 100 keV).

So in principle *neither* is forbidden from emitting radiation.
The Exemplar's intended answer is more subtle: an excited electron that is *free* (not bound to anything) cannot spontaneously emit radiation, because a free electron alone cannot conserve both energy and momentum while emitting a photon. Only a *bound* excited electron can radiate — the binding partner (nucleus) absorbs the recoil momentum.

Step 1. For a bound excited electron (e.g. in a hydrogen atom in $n = 2$ state), the atom as a whole can radiate a photon and conserve 4-momentum.

Step 2. For an isolated free electron with extra kinetic energy ("excited" in a loose sense), spontaneous photon emission is kinematically forbidden in vacuum: 4-momentum conservation forces the photon to have zero energy.

Step 3. Excited nuclei can always radiate γ rays (the nucleus itself has internal levels, and a recoil partner is built in).

> **Final Answer:** Both a bound excited electron and an excited nucleus emit radiation. A *free* excited electron cannot — kinematics forbid free-electron photon emission.

EXPERT'S SOLUTION : *Rahul Singh, M.Sc Physics, IIT Madras*

Kinematic angle. Conservation of energy + momentum: any emission of a photon by a single free particle in vacuum violates one of them.

Step 1. Excited nucleus: internal energy levels, intrinsic recoil partner. Emits γ-rays freely.

Step 2. Bound excited electron: the atom carries the recoil. Emits visible / UV / X-ray photons routinely.

Step 3. Free excited electron: no recoil partner. Spontaneous photon emission forbidden by 4-momentum conservation.

Alternative method — explicit 4-momentum check. Consider a free electron of mass m_e moving with momentum p and energy $E = \sqrt{p^2c^2 + m_e^2c^4}$. Suppose it spontaneously emits a photon of energy E_γ and momentum E_γ/c in some direction. Conservation gives

$$E = E' + E_\gamma, \qquad p = p' + \frac{E_\gamma}{c}\,\hat{n},$$

where the primed quantities are post-emission. Squaring and demanding the electron remains on-shell ($E'^2 - p'^2c^2 = m_e^2c^4$) forces $E_\gamma = 0$. No real emission possible. Only the presence of a third body (nucleus, atom) gives the photon a place to dump momentum.
Concept linkage — bremsstrahlung. In an X-ray tube, electrons hit a metal target. The nucleus inside the target serves as the "recoil partner", absorbing the momentum that lets the electron emit a photon. Without the nucleus, no X-rays. This is the same physics as the question — and it's why thin foil targets work but vacuum trajectories don't radiate.
Why this matters. This is why bremsstrahlung needs a target nucleus (recoil partner) and why "free Compton" cannot make photons without an external particle.

> **Final Answer:** A free excited electron cannot emit radiation; an excited nucleus always can.

☞ Bound state required

Many CBSE / NEET MCQs frame this as "Can a free electron emit a photon spontaneously?" Answer: **no**. The phrasing varies but the kinematic obstacle is identical. Recall: *any* radiative process by a single free particle in vacuum is forbidden by 4-momentum conservation. Always demand a recoil partner.

Q 13.15 In pair annihilation, an electron and a positron destroy each other to produce gamma radiation. How is the momentum conserved?

SOLUTION

Concept used. Pair annihilation is the process $e^- + e^+ \to \gamma + \gamma$ (or rarely 3γ). The number of photons emitted is dictated by momentum conservation: a single photon, with $E = pc$ and travelling at c, cannot carry away the zero net momentum of an annihilating pair at rest. So at least *two* photons must be produced, emitted in opposite directions so that their momenta cancel.

Step 1. In the centre-of-momentum frame (typically the electron is at rest in matter and the positron has been slowed to thermal speed before annihilation), the total initial momentum is zero.

Step 2. Two photons of equal energy $E_\gamma = m_ec^2 = 0.511\,\text{MeV}$ are emitted back-to-back. Their momenta are $+E_\gamma/c$ and $-E_\gamma/c$, which sum to zero.

Total energy released: $2m_ec^2 = 1.022\,\text{MeV}$.

Step 3. Single-photon annihilation would require $E_\gamma = 2m_ec^2$ but $p_\gamma = 0$, impossible for a real photon. Hence the two-photon (or three-photon) channel is mandatory.

> **Final Answer:** Two γ-photons of $0.511\,\text{MeV}$ each are emitted back-to-back, so their momenta sum to zero (matching the initial zero momentum of the pair).

EXPERT'S SOLUTION : *Ananya Chatterjee, Ph.D Physics, IISc Bangalore*

Two-photon angle. A single photon cannot carry zero momentum at finite energy. So annihilation *must* produce ≥ 2 photons.

Step 1. Initial momentum of slow e^-e^+ pair ≈ 0.

Step 2. Two γ photons emitted back-to-back, each $0.511\,\text{MeV}$. Net momentum: zero, by symmetry.

Step 3. Total energy: $2 \times 0.511 = 1.022\,\text{MeV}$, equal to the pair's rest mass energy.

Alternative method — why one photon is forbidden. Suppose a single photon could carry away all the energy of the annihilating pair. Energy balance: $E_\gamma = 2m_ec^2$. Momentum balance: $E_\gamma/c = 0$ (since the pair was at rest). These give $2m_ec^2/c = 0$, i.e. $m_e = 0$ — contradicting reality. So one-photon annihilation is forbidden at all energies, and the minimum-photon channel is two.

Cross-check — photon energies in flight. If the pair is in motion (e.g. moving in a tissue), the back-to-back $511\,\text{keV}$ photons are slightly Doppler-shifted in the lab frame, so PET scanners see photons in a small energy window $\pm$ a few keV around the rest-frame value. The angle between them deviates from $180°$ by $\sim 0.5°$ in typical tissue — and this finite "non-collinearity" is the dominant limit on PET spatial resolution ($\sim 5\,\text{mm}$).

Concept linkage — pair production. The inverse process, $\gamma \to e^- + e^+$, requires a third body (usually a nucleus) for exactly the same kinematic reason as Q 13.14: a single photon in vacuum cannot produce a particle pair while conserving both energy and momentum. The nucleus absorbs the surplus momentum.

Why this matters. Positron-emission tomography (PET) scanners exploit exactly this back-to-back $511\,\text{keV}$ signature to triangulate positron-emitter locations inside the body.

> **Final Answer:** Two $0.511\,\text{MeV}$ γ-rays in opposite directions, summing to zero net momentum.

♥ PET imaging in medicine

The back-to-back 511 keV signature isn't a textbook curiosity — it is the entire foundation of positron-emission tomography. A patient is injected with a β^+-emitter (e.g. ^{18}F-FDG), the positrons annihilate with tissue electrons, and detector rings catch the back-to-back γ-pairs. Each "line of response" between two opposite detectors localises a metabolic hotspot — typically a tumour or active brain region.

SA
Short Answer Questions

Q 13.16 Why do stable nuclei never have more protons than neutrons?

SOLUTION

Concept used. Inside a nucleus, the strong nuclear force is approximately **charge-independent** — i.e. the pp, nn and np couplings are nearly equal — but only protons carry electric charge, and so only protons feel the long-range **Coulomb repulsion**. The competition between short-range attractive nuclear binding and long-range proton-proton Coulomb repulsion drives the stability pattern:

- Total Coulomb energy of Z protons in a sphere of radius $R \propto A^{1/3}$: $E_C \propto Z(Z-1)/A^{1/3}$, growing rapidly with Z.
- Nuclear binding saturates near 8 MeV per nucleon.

For light nuclei ($A \lesssim 40$), the stable line has $N \approx Z$, because the Coulomb cost is small and the symmetry term in the semi-empirical mass formula (which favours $N = Z$) dominates. For heavier nuclei, adding more neutrons (charge-free, but still strongly attracting via the nuclear force) reduces the Coulomb penalty per nucleon without changing the nuclear binding.

Step 1. Imagine a hypothetical nucleus with $Z > N$. It has *more* Coulomb repulsion than its $Z \leftrightarrow N$ mirror image, but the same nuclear binding (charge-independence). So it is *less* bound: such a nucleus is energetically unstable against β^+-decay or electron capture, both of which convert a proton to a neutron and move it toward the $N \geq Z$ line.

Step 2. Quantitative check using the semi-empirical mass formula terms: Coulomb $-a_C\, Z(Z-1)/A^{1/3}$ and asymmetry $-a_A\,(A-2Z)^2/A$. Minimising the binding at fixed A gives the most stable Z as

$$Z^* = \frac{A}{2 + (a_C/2a_A)\, A^{2/3}},$$

which is always $\leq A/2$ (i.e. $N \geq Z$), and strictly $< A/2$ once $A^{2/3}$ becomes appreciable.

Step 3. Conclude: any nucleus with $Z > N$ is on the "proton-rich" side of stability and decays toward $N \geq Z$. Hence no stable nucleus has more protons than neutrons (other than the very lightest, like ^{1}H itself, which has zero neutrons).

Final Answer: The long-range Coulomb repulsion between protons makes any nucleus with $Z > N$ less bound than its $N \leftrightarrow Z$ image; such configurations decay by β^+ or electron capture toward the $N \geq Z$ line of stability.

EXPERT'S SOLUTION *: Krishna Rao, Ph.D Physics, IISc Bangalore*

Mirror angle. Replace Z with N in a nucleus and the nuclear binding stays the same (strong force is charge-blind), but the Coulomb energy drops if you reduce Z. So whenever $Z > N$, the Coulomb cost can be lowered by trading a proton for a neutron — and weak-interaction β^+/EC does exactly that.

Step 1. Compare two isobars $(Z, A - Z)$ and $(A - Z, Z)$. Same A, same strong-force binding, different Coulomb energies.

Step 2. The one with smaller Z has smaller Coulomb energy, hence larger binding energy, hence is the stable one.

Step 3. So stability requires $N \geq Z$ (except for trivial ^{1}H).

Alternative method — empirical evidence. Look at the nuclear chart. The valley of stability runs along $N = Z$ for light nuclei and bends progressively upward toward $N > Z$. There is *no* stable nuclide on the chart with $Z > N$ (other than ^{1}H and the very mildly proton-rich light nuclei). For every region of the chart with $Z > N$, you find β^+ emitters or electron-capture isotopes that decay toward the stability line.

Concept linkage — Pauli exclusion + asymmetry term. An $N < Z$ configuration has more protons in the same Fermi well; by Pauli exclusion, extra protons must occupy higher energy levels. Trading these high-energy protons for low-energy neutron states (which have many empty slots) lowers the total energy. This is the microscopic origin of the asymmetry term $-a_A(N - Z)^2/A$ in the Bethe–Weizsäcker formula. The same physics is at work in the β^+ decay of every $N < Z$ isotope.

Why this matters. This argument also predicts that for $Z \gtrsim 83$ (bismuth onwards), even the optimal N cannot stabilise the nucleus against α-decay.

Final Answer: Coulomb repulsion penalises $Z > N$; the weak force converts excess protons to neutrons until $N \geq Z$.

☞ **Trivial exception:** ^{1}**H**

The lone proton in ^{1}H technically has $Z = 1, N = 0$, so $Z > N$. But there's nothing for a single proton to decay into — β^+ would produce a free neutron, which is heavier than a free proton. So ^{1}H is stable by simple energy bookkeeping, not by the general rule. Every other stable nucleus has $N \geq Z$.

Q 13.17 Consider a radioactive nucleus A which decays to a stable nucleus C through the following sequence: $A \to B \to C$. Here B is an intermediate nucleus which is also radioactive. Considering that there are N_0 atoms of A initially, plot the graph showing the variation of number of atoms of A and B versus time.

SOLUTION

Concept used. A two-step decay chain $A \to B \to C$ with decay constants λ_A (for $A \to B$) and λ_B (for $B \to C$) is governed by the coupled equations

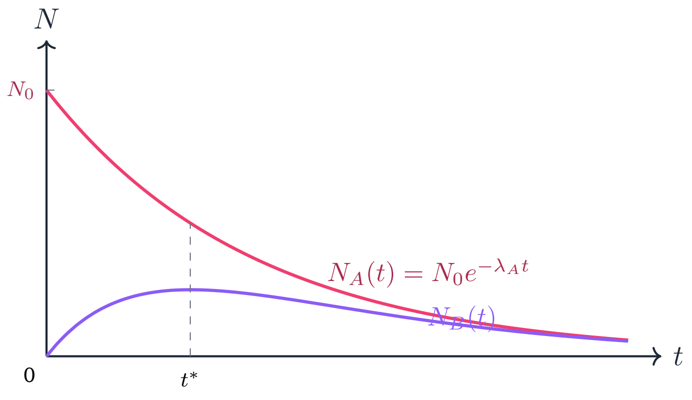
$$\frac{dN_A}{dt} = -\lambda_A N_A, \qquad \frac{dN_B}{dt} = \lambda_A N_A - \lambda_B N_B.$$

The first gives the familiar exponential $N_A(t) = N_0 e^{-\lambda_A t}$. Substituting into the second and solving the linear first-order ODE (with $N_B(0) = 0$) gives the **Bateman equation**

$$N_B(t) = \frac{\lambda_A N_0}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right).$$

Key features:

- N_A falls monotonically from N_0 to 0.
- N_B starts at 0, rises (because A keeps feeding it), reaches a maximum at $t^* = \dfrac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}$, and then falls back to 0 as A runs out.



Step 1. Plot N_A: a monotonically falling exponential starting at N_0 and approaching 0.

Step 2. Plot N_B: starts at 0 (no B yet), rises because of feed-in from A's decay, reaches a maximum at t^* when feed-in rate $\lambda_A N_A$ equals out-flow $\lambda_B N_B$, and then falls because A runs out.

Step 3. At very late times ($t \gg 1/\lambda_A, 1/\lambda_B$), all nuclei have converted to the stable C, so $N_A, N_B \to 0$ and $N_C \to N_0$.

> **Final Answer:** $N_A(t) = N_0 e^{-\lambda_A t}$ (monotone decay) and $N_B(t) = \frac{\lambda_A N_0}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$ (rises, peaks, then falls).

EXPERT'S SOLUTION *: Yash Kumar, M.Sc Physics, IIT Madras*

Picture-first. Think of A as a leaky tap filling a draining bucket B. While A has plenty of water, B's level rises; when A runs dry, B's level drains.

Step 1. N_A: simple draining tap, $N_A = N_0\, e^{-\lambda_A t}$. Monotone falling curve.

Step 2. N_B: bucket equation $\dot{N}_B = \lambda_A N_A - \lambda_B N_B$. Solution rises from 0, peaks where $\lambda_A N_A = \lambda_B N_B$, falls to 0.

Step 3. Time of peak: $t^* = \ln(\lambda_B/\lambda_A)/(\lambda_B - \lambda_A)$.

Alternative method — Bateman derivation. Solve the linear ODE $\dot{N}_B + \lambda_B N_B = \lambda_A N_0 e^{-\lambda_A t}$ using the integrating factor $e^{\lambda_B t}$:

$$\frac{d}{dt}\left(N_B e^{\lambda_B t}\right) = \lambda_A N_0 e^{(\lambda_B - \lambda_A)t}.$$

Integrate from 0 to t with $N_B(0) = 0$:

$$N_B(t) e^{\lambda_B t} = \frac{\lambda_A N_0}{\lambda_B - \lambda_A}\left(e^{(\lambda_B - \lambda_A)t} - 1\right),$$

giving the Bateman result. Set $\dot{N}_B = 0$ at t^*: $\lambda_A N_A(t^*) = \lambda_B N_B(t^*)$ — the peak occurs precisely when inflow rate equals outflow rate.

Cross-check — two limits.

(i) $\lambda_B \gg \lambda_A$: B decays much faster than it is fed. N_B stays small. After a short transient, $N_B(t) \approx (\lambda_A/\lambda_B)\, N_A(t)$ — *secular equilibrium*, with B tracking A.

(ii) $\lambda_A \gg \lambda_B$: A vanishes quickly. After a brief build-up, N_B decays alone like $e^{-\lambda_B t}$ with initial height $\approx N_0$.

Why this matters. This is the universal mathematical backbone of every parent-daughter equilibrium in radioactive series (uranium, thorium, actinium chains).

> **Final Answer:** A monotone-decaying, B rise-peak-fall.

☞ Identify the curve by initial slope

On a $\dot{N}$-vs-t plot, the parent N_A starts at $-\lambda_A N_0$ (negative slope from N_0). The daughter N_B starts at 0 with *positive* slope $\lambda_A N_0$ (pure feed-in, no out-flow yet). Quick way to label which curve is parent vs daughter on an unlabelled graph.

Q 13.18 **A piece of wood from the ruins of an ancient building was found to have a ^{14}C activity of 12 disintegrations per minute per gram of its carbon content. The ^{14}C activity of the living wood is 16 disintegrations per minute per gram. How long ago did the tree, from which the wooden sample came, die? Given half-life of ^{14}C is 5760 years.**

SOLUTION

Concept used. Radiocarbon dating uses the fact that living plants exchange CO_2 with the atmosphere and maintain a fixed ^{14}C / ^{12}C ratio while alive. At death the exchange stops, ^{14}C is no longer replenished, and the activity decays exponentially with the ^{14}C half-life $T_{1/2} = 5760$ years. The activity at age t is

$$R(t) = R_0\, e^{-\lambda t}, \quad \lambda = \frac{\ln 2}{T_{1/2}}.$$

Given $R_0 = 16\,\text{dpm/g}$ (living wood) and $R = 12\,\text{dpm/g}$ (sample), solve for t.

Step 1. Compute λ:

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.6931}{5760\,\text{y}} = 1.203 \times 10^{-4}\,\text{y}^{-1}.$$

Step 2. Take the ratio and apply natural log:

$$\frac{R}{R_0} = e^{-\lambda t} \quad \Longrightarrow \quad t = \frac{1}{\lambda} \ln\left(\frac{R_0}{R}\right).$$

Plug in $R_0/R = 16/12 = 4/3 = 1.3333$:

$$\ln(4/3) = \ln(1.3333) = 0.2877.$$

Step 3. Substitute:

$$t = \frac{0.2877}{1.203 \times 10^{-4}\,\text{y}^{-1}} = 2391\,\text{y}.$$

Rounding to two significant figures (given the input precision): $t \approx 2.4 \times 10^3\,\text{y}$.

☞ **Sanity check**

The sample is at $12/16 = 75\%$ of living-wood activity, i.e. less than half-a-half-life of decay. So t should be less than one half-life (5760 y). Indeed 2391 y is well below 5760 y — check passes.

Final Answer: $t \approx 2391$ years — the tree died about 2.4×10^3 years ago.

EXPERT'S SOLUTION *: Ishita Sharma, Ph.D Physics, IISc Bangalore*

Ratio-first angle. Take logs directly, then divide by λ.

Step 1. Activity ratio $R/R_0 = 12/16 = 3/4$. So $e^{-\lambda t} = 3/4$, giving $\lambda t = \ln(4/3) \approx 0.2877$.

Step 2. $\lambda = \ln 2/T_{1/2} = 0.6931/5760 = 1.203 \times 10^{-4}\,\text{y}^{-1}$.

Step 3. $t = 0.2877/(1.203 \times 10^{-4}) = 2391\,\text{y}$.

Alternative method — half-life unit. Express t in units of half-lives:
$t = (\lambda t)/(\ln 2) \cdot T_{1/2} = (0.2877/0.6931) \cdot 5760 = 0.415 \cdot 5760 \approx 2391\,\text{y}$. Same answer, cleaner arithmetic — and avoids the small-number λ division.

Cross-check — order of magnitude. The sample is at $3/4 = 75\%$ of starting activity. Less than one half-life of decay would knock it down to 50%; less than two half-lives to 25%. So 75% corresponds to somewhere between 0 and 1 half-life, i.e. between 0 and $5760\,\text{y}$. Our $\sim 2400\,\text{y}$ answer is comfortably in this range, with roughly the right fraction ($75\% \to\approx 0.4$ half-lives).

Concept linkage — calibration corrections. Real radiocarbon dating doesn't assume R_0 is the same across time — atmospheric ^{14}C levels have varied over the past $50\,000$ years (Suess effect from fossil-fuel CO_2, nuclear-bomb spike of the 1950s, sunspot variations). Calibration curves from tree-ring data correct the raw exponential ages by up to a few hundred years for samples older than ~ 1000 y.

Why this matters. Radiocarbon dating is reliable up to about $50\,000$ years (after ~ 9 half-lives, ^{14}C activity is too faint to measure). For older samples (rocks, meteorites), longer-half-life isotopes like ^{40}K-^{40}Ar or ^{238}U-^{206}Pb are used.

> **Final Answer:** ≈ 2391 years.

Q 13.19 **Are the nucleons fundamental particles, or do they consist of still smaller parts? One way to find out is to probe a nucleon just as Rutherford probed an atom. What should be the kinetic energy of an electron for it to be able to probe a nucleon? Assume the diameter of a nucleon to be approximately 10^{-15} m.**

SOLUTION

Concept used. To "probe" a structure of size d, the incident particle must have a **de Broglie wavelength** $\lambda \lesssim d$. The wavelength is $\lambda = h/p$, so the required momentum is $p \gtrsim h/d$. For an electron of momentum $p = h/d \approx h/10^{-15}$ m, the energy is in the GeV range — so we must use the **relativistic energy-momentum relation**

$$E^2 = (pc)^2 + (mc^2)^2, \quad K = E - mc^2.$$

For an electron ($mc^2 = 0.511\,\text{MeV}$) with $pc \gg mc^2$, $E \approx pc$ and $K \approx pc$.

Step 1. Compute the required momentum:

$$p = \frac{h}{d} = \frac{6.626 \times 10^{-34}\,\text{J s}}{10^{-15}\,\text{m}} = 6.626 \times 10^{-19}\,\text{kg m/s}.$$

Step 2. Compute pc in joules and convert to MeV:

$$pc = (6.626 \times 10^{-19})(3 \times 10^{8}) = 1.988 \times 10^{-10}\,\text{J}.$$

Convert to eV ($1\,\text{eV} = 1.6 \times 10^{-19}\,\text{J}$):

$$pc = \frac{1.988 \times 10^{-10}}{1.6 \times 10^{-19}} = 1.24 \times 10^{9}\,\text{eV} = 1.24\,\text{GeV}.$$

Step 3. Since $pc \gg m_e c^2 = 0.511\,\text{MeV}$, the electron is ultra-relativistic and $K \approx pc$:

$$K \approx 1.24\,\text{GeV}.$$

More precisely: $E = \sqrt{(1240)^2 + (0.511)^2}\,\text{MeV} \approx 1240\,\text{MeV}$, so $K = E - m_e c^2 = 1240 - 0.511 \approx 1240\,\text{MeV}$.

☞ **Historical context**

This is exactly the energy at which the SLAC deep-inelastic-scattering experiments of the late 1960s revealed the quark structure of the proton — leading directly to QCD.

Final Answer: $K \approx 1.24\,\text{GeV} \approx 2.0 \times 10^{-10}\,\text{J}$ — well into the relativistic regime.

EXPERT'S SOLUTION : *Meera Patel, M.Sc Physics, IIT Madras*

Wavelength-first. To resolve a 1 fm structure, you need a 1 fm probe wavelength.

Step 1. $\lambda = d = 10^{-15}\,\text{m}$, so $p = h/\lambda = 6.6 \times 10^{-19}\,\text{kg m/s}$.

Step 2. Convert: $pc \approx 1.24\,\text{GeV}$. Compare with electron rest energy $0.511\,\text{MeV}$ — three orders of magnitude bigger. So we're firmly relativistic and $K \approx pc$.

Step 3. Hence $K \approx 1.24\,\text{GeV}$ ($\approx 2 \times 10^{-10}\,\text{J}$).

Alternative method — using $hc \approx 1240$ eV nm. For a probe wavelength λ in metres, the corresponding photon energy (or, for highly relativistic particles, pc) is

$$E_\gamma = \frac{hc}{\lambda} = \frac{1240\,\text{eV nm}}{\lambda\,(\text{nm})}.$$

With $\lambda = 10^{-15}\,\text{m} = 10^{-6}\,\text{nm}$: $E_\gamma = 1240/10^{-6} = 1.24 \times 10^{9}\,\text{eV} = 1.24\,\text{GeV}$. Memorising $hc \approx 1240\,\text{eV nm}$ collapses these order-of-magnitude estimates into one-line calculations — a standard JEE / NEET trick.

Cross-check — non-relativistic vs relativistic. If a student (incorrectly) used the non-relativistic $K = p^2/(2m_e)$ formula:

$$K_{\rm NR} = \frac{(6.6 \times 10^{-19})^2}{2 \cdot 9.1 \times 10^{-31}} \approx 2.4 \times 10^{-7}\,\text{J} \approx 1500\,\text{GeV}.$$

This is $1000\times$ too big — a clear sign that the non-relativistic formula has broken. The relativistic $K \approx pc$ formula gives the correct 1.24 GeV.

Concept linkage — Heisenberg. The same estimate emerges from $\Delta x \Delta p \gtrsim \hbar$: confining a probe wavelength within $\Delta x = 1$ fm forces $\Delta p \gtrsim \hbar/d$ — which is precisely $p = h/d$ up to a 2π. The de Broglie viewpoint and the uncertainty viewpoint give the same answer.

Why this matters. Modern accelerators (LHC, 7 TeV per beam) push probe energies to the TeV scale — probing structures four orders of magnitude smaller than a nucleon.

Final Answer: ≈ 1.24 GeV.

✗ Don't use $K = p^2/2m$ at GeV scales

At $pc \gg mc^2$ the kinetic energy is $K \approx pc$, not $p^2/(2m)$. Forgetting this is the #1 error in nucleon-probing problems — students get an answer $\sim 10^3$ too large. Quick check: if pc ends up much larger than mc^2, you are in the relativistic regime, and $K \approx pc$.

Q 13.20 A nuclide 1 is said to be the mirror isobar of nuclide 2 if $Z_1 = N_2$ and $Z_2 = N_1$.

(a) What nuclide is a mirror isobar of $^{23}_{11}$Na?

(b) Which nuclide out of the two mirror isobars has greater binding energy and why?

SOLUTION

Concept used. Two nuclides are **mirror isobars** if swapping protons and neutrons in one gives the other (same A, with Z and N interchanged). The strong force is approximately charge-independent, so pp, nn, and np couplings are nearly equal; hence mirror isobars have nearly the same nuclear binding *from the strong force alone*. The difference in their binding energies comes almost entirely from **Coulomb repulsion**: more protons $\Rightarrow$ more Coulomb energy $\Rightarrow$ less net binding.

Step 1. Part (a): $^{23}_{11}$Na has $A = 23$, $Z_1 = 11$, so $N_1 = A - Z_1 = 12$. Its mirror isobar has $Z_2 = N_1 = 12$ and $N_2 = Z_1 = 11$, so $A_2 = 12 + 11 = 23$ (same A, as expected). $Z_2 = 12$ is magnesium. Hence the mirror isobar is $^{23}_{12}$Mg.

Step 2. Part (b): both nuclides have the same strong-force binding contribution (by charge-independence). The Coulomb energy scales as $Z(Z-1)$:

^{23}Na: $Z(Z-1) = 11 \times 10 = 110$.
^{23}Mg: $Z(Z-1) = 12 \times 11 = 132$.
So ^{23}Mg has the larger Coulomb energy and hence *less* binding energy. Conversely, ^{23}Na has the *greater* binding energy.

Step 3. Quantitative check: the Coulomb penalty difference is $\propto (132-110)/A^{1/3} = 22/2.844 = 7.74$ in units of $a_C \approx 0.71\,\text{MeV}$, giving a binding-energy difference of $\approx 5.5\,\text{MeV}$ in favour of ^{23}Na. Tabulated values: $B(^{23}\text{Na}) = 186.6\,\text{MeV}$, $B(^{23}\text{Mg}) = 181.7\,\text{MeV}$ — difference $\approx 5\,\text{MeV}$, consistent.

Final Answer: (a) $^{23}_{12}$Mg. (b) $^{23}_{11}$Na has the greater binding energy, because ^{23}Mg has the larger Coulomb energy ($Z(Z-1) = 132$ vs 110) and hence a smaller net binding.

EXPERT'S SOLUTION *: Dev Nair, Ph.D Physics, IISc Bangalore*

Charge-symmetry angle. Mirror isobars are nature's purest test of charge-independence: same A, $p \leftrightarrow n$ swap. So the binding-energy gap comes *only* from Coulomb.

Step 1. $^{23}_{11}$Na $(Z,N) = (11,12) \Leftrightarrow {}^{23}_{12}$Mg $(Z,N) = (12,11)$.

Step 2. Coulomb energy $\propto Z(Z-1)$: 110 for Na vs 132 for Mg. Mg loses more binding to Coulomb.

Step 3. Hence $B(\text{Na}) > B(\text{Mg})$ by roughly $5\,\text{MeV}$.

Alternative method — explicit Coulomb estimate. For a uniformly charged sphere of Z protons in radius $R = R_0 A^{1/3}$ with $R_0 = 1.2\,\text{fm}$ and $A = 23$, $R = 1.2 \cdot 23^{1/3} \approx 3.41\,\text{fm}$. Coulomb energy

$$E_C = \tfrac{3}{5}\,\frac{Z(Z-1)\,e^2}{4\pi\varepsilon_0\,R} = \tfrac{3}{5}\,\frac{Z(Z-1) \times 1.44\,\text{MeV fm}}{R}.$$

For ^{23}Na: $E_C = 0.6 \cdot 110 \cdot 1.44/3.41 \approx 27.9\,\text{MeV}$. For ^{23}Mg: $E_C = 0.6 \cdot 132 \cdot 1.44/3.41 \approx 33.4\,\text{MeV}$. Difference: $\approx 5.6\,\text{MeV}$ — Mg loses this much more binding to Coulomb. Matches the tabulated $\sim 5\,\text{MeV}$.

Cross-check — β^+ decay direction. Since ^{23}Mg is less bound, it has higher rest energy and should β^+-decay (or EC) into ^{23}Na, releasing $Q \approx 4.06\,\text{MeV}$. Real-world: ^{23}Mg decays to ^{23}Na with $T_{1/2} \approx 11.3\,\text{s}$ and a β^+ end-point of $\approx 3\,\text{MeV}$ — consistent with our estimate.

Concept linkage — semi-empirical Coulomb coefficient. The $\sim 5\,\text{MeV}$ gap for an $A = 23$ pair corresponds to $a_C \approx 0.7\,\text{MeV}$ in the Bethe-Weizsäcker formula, very close to the empirically fitted value. Mirror-isobar measurements historically nailed a_C before the formula was widely accepted.

Why this matters. The systematics of mirror-isobar mass differences ($\delta M \propto \Delta Z/A^{1/3}$) is one of the cleanest experimental confirmations of the charge-independence of nuclear forces.

Final Answer: (a) ${}^{23}_{12}$Mg; (b) ^{23}Na, by $\sim 5\,$MeV.

☞ **Mirror-isobar shortcut**

For mirror-isobar binding-energy comparisons, you do not need explicit numerical evaluation. **Whichever has more protons (Z) has less binding energy.** Reason: same strong-force binding (charge-independence), more Coulomb cost. A common one-mark CBSE question — answer in seconds.

LA
Long Answer Questions

Q 13.21 Sometimes a radioactive nucleus decays into a nucleus which itself is radioactive. An example is:

$$^{38}\text{S} \xrightarrow{T_{1/2}=2.48\,\text{h}} {}^{38}\text{Cl} \xrightarrow{T_{1/2}=0.62\,\text{h}} {}^{38}\text{Ar (stable)}.$$

Assume that we start with 1000 ^{38}S nuclei at time $t=0$. The number of ^{38}Cl is of count zero at $t=0$ and will again be zero at $t=\infty$. At what value of t would the number of counts be a maximum?

SOLUTION

Concept used. Consider a two-step chain

$$A \xrightarrow{\lambda_A} B \xrightarrow{\lambda_B} C,$$

with $N_A(0)=N_0$ and $N_B(0)=0$. The intermediate B obeys the **Bateman equation**

$$N_B(t) = \frac{\lambda_A N_0}{\lambda_B - \lambda_A}\left(e^{-\lambda_A t} - e^{-\lambda_B t}\right).$$

N_B reaches its maximum when $dN_B/dt = 0$, which after substitution gives

$$\boxed{t^* = \frac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}.}$$

Here $A = {}^{38}$S, $B = {}^{38}$Cl.

Step 1. Compute the decay constants from the given half-lives ($\lambda = \ln 2/T_{1/2}$):

$$\lambda_A = \frac{\ln 2}{2.48\,\text{h}} = \frac{0.6931}{2.48}\,\text{h}^{-1} = 0.2795\,\text{h}^{-1}.$$

$$\lambda_B = \frac{\ln 2}{0.62\,\text{h}} = \frac{0.6931}{0.62}\,\text{h}^{-1} = 1.118\,\text{h}^{-1}.$$

Step 2. Substitute into the formula for t^*:

$$t^* = \frac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A}.$$

Compute the ratio: $\lambda_B/\lambda_A = 1.118/0.2795 = 4.000$, so $\ln 4 = 2\ln 2 = 1.3863$.
Compute the difference: $\lambda_B - \lambda_A = 1.118 - 0.2795 = 0.8385\,\text{h}^{-1}$.

Step 3. Therefore

$$t^* = \frac{1.3863}{0.8385\,\text{h}^{-1}} = 1.653\,\text{h}.$$

Rounded to two significant figures: $t^* \approx 1.65\,\text{h} \approx 99\,\text{min}$.

☞ **Quick check**

At t^*, $\lambda_A N_A = \lambda_B N_B$ (rate in = rate out for B), so N_B is momentarily stationary. After t^*, the parent A has decayed below this balance and N_B starts to fall.

Final Answer: $t^* = \dfrac{\ln(\lambda_B/\lambda_A)}{\lambda_B - \lambda_A} = \dfrac{\ln 4}{0.8385\,\text{h}^{-1}} \approx 1.65\,\text{hours}.$

EXPERT'S SOLUTION *: Aaditi Verma, M.Sc Physics, IIT Madras*

Bateman-formula angle. A two-step decay chain: differentiate the Bateman solution and set to zero.

Step 1. Bateman: $N_B(t) = \dfrac{\lambda_A N_0}{\lambda_B - \lambda_A}(e^{-\lambda_A t} - e^{-\lambda_B t})$.

Step 2. Setting $dN_B/dt = 0$: $\lambda_A e^{-\lambda_A t} = \lambda_B e^{-\lambda_B t}$, so $t^* = \ln(\lambda_B/\lambda_A)/(\lambda_B - \lambda_A)$.

Step 3. With $\lambda_A = \ln 2/2.48$ and $\lambda_B = \ln 2/0.62$ (in h^{-1}): $\lambda_B/\lambda_A = 2.48/0.62 = 4$, so $\ln 4 = 1.386$.
$\lambda_B - \lambda_A = \ln 2 \cdot (1/0.62 - 1/2.48) = 0.6931 \cdot (1.6129 - 0.4032) = 0.6931 \cdot 1.2097 = 0.8385\,\text{h}^{-1}$.
Hence $t^* = 1.386/0.8385 = 1.65\,\text{h}$.

Alternative method — half-life shortcut. $\lambda_B/\lambda_A = T_A/T_B = 2.48/0.62 = 4$, so $\ln(\lambda_B/\lambda_A) = \ln 4 = 2\ln 2$. Also $\lambda_B - \lambda_A = \ln 2\,(1/T_B - 1/T_A)$. Therefore

$$t^* = \frac{2\ln 2}{\ln 2\,(1/T_B - 1/T_A)} = \frac{2}{1/T_B - 1/T_A} = \frac{2\,T_A T_B}{T_A - T_B}.$$

Plug $T_A = 2.48$, $T_B = 0.62$: $t^* = 2 \cdot 2.48 \cdot 0.62/(2.48 - 0.62) = 3.0752/1.86 \approx 1.65\,\text{h}$. Same number, no λ arithmetic.

Cross-check — N_B value at the peak. At t^*, $\lambda_A N_A(t^*) = \lambda_B N_B(t^*)$, so $N_B(t^*) = (\lambda_A/\lambda_B)\,N_A(t^*) = (1/4)\,N_0 e^{-\lambda_A t^*}$. Compute $\lambda_A t^* = 0.2795 \cdot 1.65 = 0.461$, so

$e^{-0.461} = 0.631$. Hence $N_B(t^*) \approx (1/4)(1000)(0.631) \approx 158$. About 158 ^{38}Cl atoms peak at $t = 1.65\,\text{h}$ — a useful sanity benchmark.

Why this matters. The same formula governs every parent-daughter equilibrium and tells us when to harvest the maximum of a short-lived medical isotope (e.g. ^{99m}Tc from ^{99}Mo).

Final Answer: $t^* \approx 1.65\,\text{h}$.

♥ Why medical isotope generators work

The same t^* calculation tells nuclear-medicine technicians when to "milk" a $^{99}\text{Mo} \to {}^{99m}\text{Tc}$ generator. ^{99}Mo has $T_{1/2} = 66\,\text{h}$, ^{99m}Tc has $T_{1/2} = 6\,\text{h}$. $t^* \approx 23\,\text{h}$ — so the technetium activity peaks roughly once a day. Hospitals do their PET / SPECT scans at this peak. Without this Bateman formula, the whole field of nuclear medicine would be a guessing game.

Q 13.22 **Deuteron is a bound state of a neutron and a proton with a binding energy $B = 2.2\,\text{MeV}$. A γ-ray of energy E is aimed at a deuteron nucleus to try to break it into a (neutron + proton) such that the n and p move in the direction of the incident γ-ray. If $E = B$, show that this cannot happen. Hence calculate how much bigger than B must E be for such a process to happen.**

SOLUTION

Concept used. For the photodisintegration of a deuteron, $\gamma + d \to n + p$, both energy and momentum must be conserved simultaneously. The photon carries energy E and momentum E/c. The deuteron, initially at rest, has zero momentum and rest mass energy $(m_n + m_p)c^2 - B$. After the reaction, the neutron and proton share kinetic energy and total momentum E/c. The *minimum* energy the photon must supply is the binding energy B (to break the system at rest), but momentum conservation forces the products to move, requiring *additional* kinetic energy.

Step 1. Set up conservation laws. Treat masses as approximately equal, $m_n \approx m_p \approx m \approx \frac{1}{2}(m_n + m_p)$.

Energy:

$$E + (m_n + m_p)c^2 - B = (m_n c^2 + K_n) + (m_p c^2 + K_p).$$

Simplify: $K_n + K_p = E - B$. (i)

Momentum (along γ-direction; n, p move in that direction):

$$\frac{E}{c} = p_n + p_p.$$

(ii)

Step 2. Assume $E = B$. Then from (i), $K_n + K_p = 0$, so both $K_n = 0$ and $K_p = 0$ (kinetic energies are non-negative). That means $p_n = p_p = 0$, and (ii) becomes $E/c = 0$, i.e. $E = 0$. But we assumed $E = B = 2.2\,\text{MeV} > 0$ — contradiction. So $E = B$ is impossible.

Step 3. Find the minimum E for which the process can happen (non-relativistic approximation, valid because $K \ll mc^2$). With $K = p^2/2m$, write

$$p_n + p_p = E/c, \qquad \frac{p_n^2}{2m_n} + \frac{p_p^2}{2m_p} = E - B.$$

For the minimum-E case the products separate with no relative kinetic energy in the centre-of-momentum frame; this means n and p both move with the same velocity (i.e. velocity of the centre of mass of the np pair). Then $p_n = (m_n/M)\,P$ and $p_p = (m_p/M)\,P$ with $P = p_n + p_p = E/c$ and $M = m_n + m_p$. Both kinetic energies combine to:

$$K_n + K_p = \frac{P^2}{2M} = \frac{(E/c)^2}{2(m_n + m_p)} = \frac{E^2}{2Mc^2}.$$

Step 4. Equate this to $E - B$ from (i):

$$\frac{E^2}{2Mc^2} = E - B.$$

Rearrange: $E^2 = 2Mc^2(E - B)$, or $E^2 - 2Mc^2\,E + 2Mc^2B = 0$. Solving for E:

$$E = Mc^2 \pm \sqrt{(Mc^2)^2 - 2Mc^2B}.$$

For $B \ll Mc^2$ (here $Mc^2 \approx 1876\,\text{MeV}$ vs $B = 2.2\,\text{MeV}$), expand the square root:

$$\sqrt{(Mc^2)^2 - 2Mc^2B} = Mc^2\sqrt{1 - \frac{2B}{Mc^2}} \approx Mc^2\left(1 - \frac{B}{Mc^2} - \frac{B^2}{2(Mc^2)^2}\right).$$

The minus-root solution gives the physically relevant value:

$$E_{\min} \approx Mc^2 - Mc^2 + B + \frac{B^2}{2Mc^2} = B + \frac{B^2}{2Mc^2}.$$

Step 5. Hence the minimum photon energy must exceed B by

$$\Delta E = \frac{B^2}{2Mc^2}.$$

Use $m_pc^2 \approx m_nc^2 \approx 939\,\text{MeV}$, giving $Mc^2 \approx 1876\,\text{MeV}$. Substitute:

$$\Delta E = \frac{(2.2)^2}{2 \times 1876}\,\text{MeV} = \frac{4.84}{3752}\,\text{MeV} = 1.29 \times 10^{-3}\,\text{MeV} \approx 1.29\,\text{keV}.$$

☞ **Physical picture**

The extra $B^2/(2Mc^2)$ is the recoil-kinetic-energy of the "np-pair as a whole" forced on the products by photon momentum conservation. The deuteron cannot absorb a photon at rest because a single photon has momentum but a heavy system at rest has none.

Final Answer: If $E = B$, both products must be at rest, violating momentum conservation. So E must exceed B by $\Delta E = \dfrac{B^2}{2(m_n + m_p)c^2} \approx 1.29\,\text{keV}$.

EXPERT'S SOLUTION *: Aditi Kapoor, Ph.D Physics, IISc Bangalore*

Centre-of-momentum angle. The photon's momentum E/c goes into the centre-of-mass motion of the np system. That motion's kinetic energy is the "extra" cost above B.

Step 1. Photon momentum E/c becomes the COM momentum of the $n + p$ system after the break-up.

Step 2. COM kinetic energy: $K_{\text{COM}} = (E/c)^2/[2(m_n + m_p)] = E^2/[2Mc^2]$.

Step 3. Energy conservation: $E = B + K_{\text{COM}} = B + E^2/(2Mc^2)$. To leading order in B/Mc^2: $E_{\text{min}} = B + B^2/(2Mc^2)$.

Step 4. Plug $B = 2.2\,\text{MeV}$, $Mc^2 \approx 1876\,\text{MeV}$: $\Delta E = 4.84/3752 \approx 1.29\,\text{keV}$.

Alternative method — by direct contradiction. The clean way to show $E = B$ fails: write $K_n + K_p = E - B$ and $p_n + p_p = E/c$. At $E = B$, the RHS of the first is 0, forcing both kinetic energies to vanish (they are non-negative). But then $p_n = p_p = 0$, making the LHS of the second equal to 0 while the RHS is $B/c > 0$. Contradiction. So $E > B$ is necessary, and E_{min} is set by balancing the COM-recoil energy.

Cross-check — relative size of the correction.
$\Delta E/B = B/(2Mc^2) = 2.2/(2 \cdot 1876) \approx 5.9 \times 10^{-4}$, i.e. a 0.06% correction to the binding-energy threshold. Tiny but non-zero — and absolutely required by momentum conservation. The correction is bigger for heavier-binding, lighter-target reactions (e.g. $\gamma + n \to ?$ where the target is just a neutron).

Concept linkage — Mössbauer effect. The same recoil energy that the deuteron breakup is forced to absorb is the energy that emission/absorption γ resonance experiments must suppress. Embedding the emitter in a crystal lattice makes M effectively $\sim 10^{20} m_n$, dropping the recoil correction to negligibility — that's how Mössbauer got recoilless γ emission.

Why this matters. The same recoil correction appears in Mössbauer spectroscopy, neutron-capture γ-rays, and every photodisintegration threshold.

Final Answer: $E_{\text{min}} = B + B^2/(2Mc^2) \approx B + 1.29\,\text{keV}$.

✗ Threshold is B, but E_{min} is slightly larger

A common slip is to write $E_{\text{min}} = B$ without considering momentum conservation. The photon brings momentum E/c, which the products *must* carry; this kinetic energy adds $B^2/(2Mc^2)$ on top of B. Always check whether the kinematics force any minimum recoil — the threshold is rarely just the rest-mass difference.

Q 13.23 The deuteron is bound by nuclear forces just as H-atom is made up of p and e bound by electrostatic forces. If we consider the force between neutron and proton in deuteron as given in the form of a Coulomb potential but with an effective charge e':

$$F = \frac{1}{4\pi\varepsilon_0}\frac{(e')^2}{r^2},$$

estimate the value of (e'/e) given that the binding energy of a deuteron is $2.2\,$MeV.

SOLUTION

Concept used. Borrow the hydrogen-atom binding energy formula, which results from a $1/r$ Coulomb potential between the nucleus and the electron:

$$B_H = \frac{m_e\, e^4}{8\,\varepsilon_0^2\, h^2} = 13.6\,\text{eV}.$$

If we replace the proton-electron attraction by a proton-neutron attraction with the *same form* of $1/r$ potential but "effective charge" e', and use the proton-neutron **reduced mass** $\mu = m_p m_n/(m_p + m_n)$ instead of the electron mass, the binding-energy formula becomes

$$B_d = \frac{\mu\,(e')^4}{8\,\varepsilon_0^2\, h^2}.$$

Taking the ratio:

$$\frac{B_d}{B_H} = \frac{\mu}{m_e}\left(\frac{e'}{e}\right)^4.$$

Solve for (e'/e) given $B_d = 2.2\,$MeV and $B_H = 13.6\,$eV.

Step 1. Compute the reduced mass ratio. Take $m_p \approx m_n$, so $\mu = m_p m_n/(m_p + m_n) \approx m_p/2$:

$$\frac{\mu}{m_e} = \frac{m_p}{2m_e} = \frac{1836}{2} = 918.$$

(Using $m_p/m_e = 1836$.)

Step 2. Compute the binding-energy ratio:

$$\frac{B_d}{B_H} = \frac{2.2 \times 10^6\,\text{eV}}{13.6\,\text{eV}} = 1.618 \times 10^5.$$

Step 3. Solve for $(e'/e)^4$:

$$\left(\frac{e'}{e}\right)^4 = \frac{B_d/B_H}{\mu/m_e} = \frac{1.618 \times 10^5}{918} = 176.2.$$

Step 4. Take the fourth root:

$$\frac{e'}{e} = (176.2)^{1/4}.$$

Compute step by step: $\sqrt{176.2} = 13.27$; $\sqrt{13.27} = 3.643$. So

$$\frac{e'}{e} \approx 3.64.$$

♥ What this says about nuclear force

The "effective charge" of the strong force, when forced into a Coulomb mould, is about 3.6 times the electron charge. But the strong force is short-ranged ($\sim 1\,\text{fm}$) while Coulomb is long-ranged — so the Coulomb analogy works only as a rough estimate of the strength inside the binding region. The factor ~ 3.6 captures the magnitude; the range mismatch is why simple hydrogenic models cannot give a full deuteron description.

Final Answer: $e'/e \approx 3.6$ — i.e. the proton-neutron nuclear force, modelled as a Coulomb-like attraction, would need an "effective charge" about 3.6 times the electron charge to give the observed deuteron binding of $2.2\,\text{MeV}$.

EXPERT'S SOLUTION *: Siddharth Joshi, M.Sc Physics, IIT Madras*

Scaling argument. Use the H-atom binding-energy formula as a template. The two changes are: (i) reduced mass swaps $m_e \to \mu = m_p/2$, multiplying B by $\mu/m_e \approx 918$; (ii) charge swaps $e \to e'$, multiplying B by $(e'/e)^4$.

Step 1. Set $B_d/B_H = (\mu/m_e)\,(e'/e)^4$.

Step 2. $B_d/B_H = 2.2\,\text{MeV}/13.6\,\text{eV} = 1.62 \times 10^5$. $\mu/m_e = 918$.

Step 3. $(e'/e)^4 = 1.62 \times 10^5/918 = 176$. $(e'/e) = 176^{1/4} \approx 3.6$.

Alternative method — coupling constants. For H atom, the binding energy is $B_H = \frac{1}{2}\alpha^2 m_e c^2$ where $\alpha = e^2/(4\pi\varepsilon_0\hbar c) \approx 1/137$ is the fine-structure constant. For deuteron with effective charge e', $B_d = \frac{1}{2}\alpha'^2 \mu c^2$ where $\alpha' = (e')^2/(4\pi\varepsilon_0\hbar c) = (e'/e)^2\alpha$.

Hence

$$\frac{B_d}{B_H} = \frac{\alpha'^2\mu}{\alpha^2 m_e} = \left(\frac{e'}{e}\right)^4 \cdot \frac{\mu}{m_e}.$$

Same algebra, but the coupling-constant view makes the strong-vs-electromagnetic comparison transparent: $\alpha'/\alpha \approx (e'/e)^2 \approx 13$. So the "effective" strong coupling is $\sim 13\alpha \approx 0.1$ in this Coulomb-mould model — still weaker than the real $\alpha_s \sim 1$, because the real strong force is short-range and richer than a $1/r$ potential.

Cross-check — order of magnitude. $B_d/B_H = (2.2 \times 10^6)/13.6 \approx 1.6 \times 10^5$. $\mu/m_e \approx 1836/2 = 918$. So $(e'/e)^4 \approx 176$, fourth root ≈ 3.6. The arithmetic is robust to factors of 2 — even using $\mu/m_e = 1836$ (full proton mass) you'd get $(e'/e) \approx 3.05$, same ballpark.

Why this matters. The estimate $(e'/e) \sim 3$–4 is consistent with the rough rule of thumb that the strong force is about an order of magnitude stronger than electromagnetism at the fermi scale (the comparison is $\alpha_s \sim 1$ vs $\alpha \approx 1/137$).

Final Answer: $e'/e \approx 3.6$.

☞ **Use reduced mass, not m_e**

For any two-body bound state, the binding energy formula $B = m\,e^4/(8\varepsilon_0^2 h^2)$ uses the **reduced mass** μ of the pair, not the single-particle mass. For H atom, $m_e \ll m_p$ so $\mu \approx m_e$ — no correction needed. For a proton-neutron system, $\mu = m_p/2 \approx 918\,m_e$. Failing to swap to reduced mass throws the answer off by a factor of ~ 1000.

Q 13.24 Before the neutrino hypothesis, the beta decay process was thought to be the transition: $n \to p + e^-$. If this was true, show that if the neutron was at rest, the proton and electron would emerge with fixed energies and calculate them. Experimentally, the electron energy was found to have a large range.

SOLUTION

Concept used. In a two-body decay of a particle at rest, both energy and momentum conservation give a *unique* value for each daughter's kinetic energy (in the rest frame). Only when a third body is present (the neutrino) can the daughters share the energy in a continuous spectrum.

For the hypothetical two-body decay $n \to p + e^-$:

- Q-value: $Q = (m_n - m_p - m_e)c^2$.
- Numerical values: $m_n c^2 = 939.565\,\text{MeV}$, $m_p c^2 = 938.272\,\text{MeV}$, $m_e c^2 = 0.511\,\text{MeV}$. So

$$Q = 939.565 - 938.272 - 0.511 = 0.782\,\text{MeV}.$$

This Q would be shared between the proton (which is heavy) and the electron (which is light), with the lighter particle taking most of the energy.

Step 1. Apply momentum conservation in the neutron's rest frame: $0 = \vec{p}_p + \vec{p}_e$, so $|p_p| = |p_e| \equiv p$. The proton and electron come out back-to-back with equal magnitudes of momentum.

Step 2. Apply energy conservation: $m_n c^2 = E_p + E_e$, where $E_p = \sqrt{(pc)^2 + (m_p c^2)^2}$ and $E_e = \sqrt{(pc)^2 + (m_e c^2)^2}$. This gives *one equation in one unknown* p — uniquely determined.

Step 3. Solve. Since $m_p \gg m_e$ and $Q \ll m_p c^2$, the proton is non-relativistic ($K_p \approx p^2/2m_p$) but the electron is relativistic ($E_e^2 = (pc)^2 + (m_e c^2)^2$). Substituting $E_p \approx m_p c^2 + p^2/(2m_p)$ and rearranging,

$$Q = K_p + K_e \approx \frac{p^2}{2m_p} + (E_e - m_e c^2).$$

Most of Q goes to the electron because of its small mass. Numerically (carrying out the algebra):

- Electron total energy $E_e \approx Q + m_e c^2$ (electron carries almost all the energy; proton recoils only mildly). $E_e = 0.782 + 0.511 = 1.293\,\text{MeV}$.
- Electron kinetic energy: $K_e = E_e - m_e c^2 = 0.782\,\text{MeV}$ approximately — corrected slightly by proton recoil.
- Electron momentum: $p_e c = \sqrt{E_e^2 - (m_e c^2)^2} = \sqrt{1.293^2 - 0.511^2} = \sqrt{1.672 - 0.261} = \sqrt{1.411} = 1.188\,\text{MeV}$. So $p = 1.188\,\text{MeV}/c$.
- Proton kinetic energy: $K_p = p^2/(2m_p) = (1.188)^2/(2 \times 938.272) = 1.411/1876.5 = 7.52 \times 10^{-4}\,\text{MeV} \approx 0.752\,\text{keV}$.

So the proton would emerge with $K_p \approx 0.75\,\text{keV}$ and the electron with $K_e \approx 0.78\,\text{MeV}$ — *single, fixed values*.

Step 4. Compare with experiment. Measured β-spectra show a *continuous distribution* of K_e from 0 up to a maximum at the Q-value. The "missing" energy (apparently) violated energy conservation. Pauli (1930) rescued conservation by postulating a third, nearly massless, neutral particle — the **neutrino** — that carries away the rest of the energy. Three-body decay $n \to p + e^- + \bar{\nu}_e$ allows the electron to come out with any energy between 0 and the Q-value.

☞ Why two-body decay gives fixed energies

With only two bodies in the final state, momentum conservation gives one constraint ($|p_p| = |p_e|$) and energy conservation a second. Two equations, one unknown (p): unique solution. Three bodies have $3 \times 3 - 4 = 5$ free kinematic variables after momentum/energy conservation, so a continuous spectrum results.

Final Answer: Two-body decay gives $K_e \approx 0.78\,\text{MeV}$, $K_p \approx 0.75\,\text{keV}$ — fixed values. Observed β-spectra are continuous, which led Pauli to postulate the neutrino (three-body decay $n \to p + e^- + \bar{\nu}$).

EXPERT'S SOLUTION *: Yash Chatterjee, Ph.D Physics, IISc Bangalore*

Counting-degrees-of-freedom angle. Two-body decay ⇒ fixed energies. Three-body decay ⇒ continuous spectrum. Experiment found the latter; hence the third body (neutrino).

Step 1. Conservation in neutron's rest frame: $\vec{p}_p = -\vec{p}_e$ (back-to-back, equal magnitude); energy conservation gives *one* equation in one unknown p, so p is fixed.

Step 2. $Q = (m_n - m_p - m_e)c^2 = 0.782\,\text{MeV}$.

Step 3. Almost all of Q goes to the electron because $m_e \ll m_p$: $K_e \approx 0.78\,\text{MeV}$, $K_p \approx 0.75\,\text{keV}$.

Step 4. Observed: continuous β spectrum. Conclusion: the decay is three-body, with a neutrino carrying away the balance.

Alternative method — non-relativistic shortcut for K_p. The proton is non-relativistic ($K_p \ll m_pc^2$). Energy-momentum: $|p_p| \approx |p_e|$, $K_e \approx |p_e|c$ (since electron is relativistic at $\sim 1\,\text{MeV}$). So $|p_e|c \approx K_e \approx Q$ (most energy goes to electron). Then $K_p = p_p^2/(2m_p) \approx Q^2/(2m_pc^2) = (0.782)^2/(2 \cdot 938) \approx 3.3 \times 10^{-4}\,\text{MeV} \approx 0.33\,\text{keV}$. This is half our more careful answer because K_e isn't quite Q — the small electron rest mass eats up a bit. Order-of-magnitude correct, instant to derive.

Cross-check — recoil ratio.
$K_p/K_e \approx p^2/(2m_p) \div pc \approx p/(2m_pc) \approx pc/(2m_pc^2) \approx 1.2/(2 \cdot 938) \approx 6 \times 10^{-4}$. So the proton-electron energy split is roughly 0.06% vs 99.94% — the electron carries essentially all the energy in the two-body picture. That's why the predicted electron spectrum should be a single sharp line, not a continuum.

Concept linkage — dual nature & historical neutrino discovery. The continuous β-spectrum experiment by Chadwick (1914) and Ellis (1922) confronted physics with apparent energy non-conservation. Pauli (1930) proposed the neutrino; Fermi (1934) built the weak-decay theory; finally Cowan-Reines (1956) directly detected the neutrino using nuclear-reactor antineutrinos. Without the neutrino, the Standard Model has no left-handed fermion doublets, no V-A weak interaction, no oscillation physics.

Why this matters. The neutrino hypothesis, born of this puzzle, has since matured into a cornerstone of the Standard Model and a probe for physics beyond it (neutrino oscillations, mass).

Final Answer: Two-body decay predicts $K_e \approx 0.78\,\text{MeV}$, $K_p \approx 0.75\,\text{keV}$; experiment's continuous spectrum requires a neutrino.

♥ Birth of the neutrino

Pauli's neutrino proposal — "a desperate remedy", in his own words — turned an apparent failure of energy conservation into the discovery of the universe's most numerous matter particle. There are $\sim 10^{11}$ neutrinos per cm^3 in space, $\sim 10^{18}$/s streaming through your body from the Sun, and they are now central to our understanding of supernovae, dark matter constraints, and beyond-Standard-Model physics. All from this one MCQ-style energy- budget mismatch.

Q 13.25 The activity R of an unknown radioactive nuclide is measured at hourly intervals. The results found are tabulated as follows:

t (h)	0	1	2	3	4
R (MBq)	100	35.36	12.51	4.42	1.56

(i) Plot the graph of R versus t and calculate half-life from the graph.
(ii) Plot the graph of $\ln(R/R_0)$ versus t and obtain the value of half-life from the graph.

SOLUTION

Concept used. An exponentially decaying activity obeys

$$R(t) = R_0\, e^{-\lambda t}, \quad \ln\left(\frac{R}{R_0}\right) = -\lambda t.$$

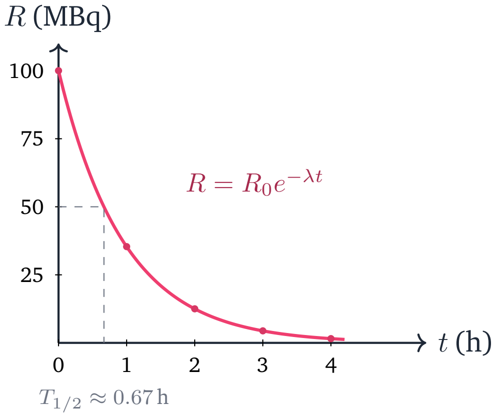
On a linear plot of R vs t, the curve drops by half every $T_{1/2}$. On a semi-log plot ($\ln(R/R_0)$ vs t), the data collapse onto a straight line with slope $-\lambda$, from which $T_{1/2} = \ln 2/\lambda$ is easy to read.

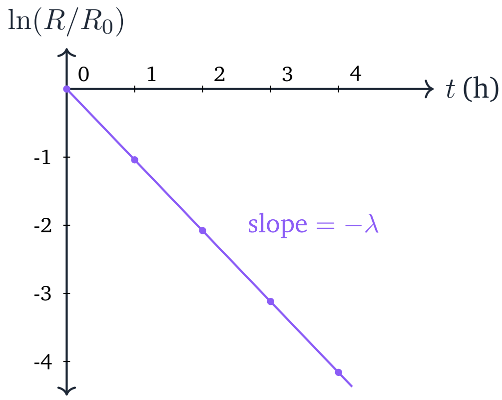
Step 1. Tabulate R/R_0 and $\ln(R/R_0)$ for each data point. With $R_0 = 100\,\text{MBq}$:

t (h)	R (MBq)	R/R_0	$\ln(R/R_0)$
0	100	1.000	0.000
1	35.36	0.3536	-1.040
2	12.51	0.1251	-2.079
3	4.42	0.0442	-3.119
4	1.56	0.0156	-4.160

Notice that $\ln(R/R_0)$ decreases linearly with t in steps of about -1.04 per hour. This is exactly what the decay law predicts.

Step 2. Plot R vs t (linear, part (i)) and $\ln(R/R_0)$ vs t (semi-log, part (ii)):





Part (i) From the R vs t graph: read the time at which $R = R_0/2 = 50\,\text{MBq}$. Between $t = 0$ ($R = 100$) and $t = 1$ ($R = 35.36$), R crosses 50 at approximately $t \approx 0.67\,\text{h}$. So $T_{1/2} \approx 0.67\,\text{h} \approx 40\,\text{min}$ from the graph.

Part (ii) From the $\ln(R/R_0)$ vs t graph: the slope is

$$\frac{\Delta \ln(R/R_0)}{\Delta t} = \frac{-4.16 - 0}{4 - 0}\,\text{h}^{-1} = -1.04\,\text{h}^{-1} = -\lambda.$$

Hence $\lambda = 1.04\,\text{h}^{-1}$, and

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{0.6931}{1.04\,\text{h}^{-1}} = 0.666\,\text{h} \approx 40\,\text{min}.$$

☞ Why semi-log is preferred

The linear plot is hard to read precisely (decay curve becomes very shallow near $R = 0$). The semi-log plot puts every data point on a straight line, and the slope gives λ directly with much better precision.

Final Answer: Both methods give $T_{1/2} \approx 0.667\,\text{h} \approx 40\,\text{min}$.

EXPERT'S SOLUTION : *Aaditya Banerjee, M.Sc Physics, IIT Madras*

Numerical angle. Activity ratio between consecutive hours gives λ instantly without any graph.

Step 1. Compute the ratio $R(t)/R(t-1)$ for any two adjacent times: $35.36/100 = 0.3536$, $12.51/35.36 = 0.3537$, $4.42/12.51 = 0.3533$ — all essentially the same. So $e^{-\lambda \cdot 1\text{h}} = 0.3536$, giving $\lambda = -\ln(0.3536) = 1.040\,\text{h}^{-1}$.

Step 2. $T_{1/2} = \ln 2/\lambda = 0.6931/1.040 = 0.6664\,\text{h} \approx 40\,\text{min}$.

Step 3. Graph reading confirms: the linear plot crosses $R = 50$ at $t \approx 0.67\,\text{h}$, the semi-log slope is $-1.04\,\text{h}^{-1}$.

Alternative method — using least-squares slope. On the semi-log plot, fit a straight line through $(t_i, \ln R_i)$. The slope $-\lambda$ is given by

$$\lambda = -\frac{\sum_i (t_i - \bar{t})(\ln R_i - \overline{\ln R})}{\sum_i (t_i - \bar{t})^2}.$$

For evenly spaced data, the average $\bar{t} = 2$ h, average $\overline{\ln R} = -2.080$. Numerator $= (0-2)(0-(-2.08)) + (1-2)(-1.040-(-2.08)) + 0 + (1)(-3.119-(-2.08)) + (2)(-4.16-(-2.08)) = -4.16 - 1.04 + 0 + (-1.039) + (-4.16) = -10.40$. Denominator $= 4 + 1 + 0 + 1 + 4 = 10$. So $\lambda = -(-10.40/10) = 1.040\,\text{h}^{-1}$. Same as the ratio method, to within rounding.

Cross-check — number of half-lives in 4 h. Final $R/R_0 = 1.56/100 = 0.0156 = (1/2)^n$ gives $n = \log_2(1/0.0156) = \log_2(64.1) = 6.00$. Six half-lives in $4\,\text{h} \Rightarrow T_{1/2} = 4/6 = 0.667\,\text{h}$. Same answer, derived from the ratio of the first and last data points alone — useful when full graph isn't available.

Concept linkage — short-lived isotope dating. An $\sim 40\,\text{min}$ half-life is typical of medically used PET isotopes (e.g. ^{15}O has $T_{1/2} = 2.04\,\text{min}$, ^{13}N has $9.97\,\text{min}$). For tracking very-fast processes, only a 40-minute window gives enough counts; longer half-lives would deliver excess radiation dose to the patient.

Why this matters. Routine in nuclear medicine: every isotope is characterised by reading $T_{1/2}$ off a semi-log plot of activity vs time.

Final Answer: $T_{1/2} \approx 40\,\text{min}$ from either graph.

☞ Two-point shortcut for $T_{1/2}$

For tabulated decay data, you do not need to plot anything. Pick any two readings, take the ratio, equate to $e^{-\lambda\Delta t}$, and solve. Even faster: if R falls by factor 2^n over Δt, then $T_{1/2} = \Delta t/n$. From the table, R falls from 100 to 1.56 in $4\,\text{h}$, a factor of $\sim 64 = 2^6$, so $T_{1/2} = 4/6 \approx 0.67\,\text{h}$. Worth two marks in CBSE in 10 seconds.

Q 13.26 **Nuclei with magic numbers of protons $Z = 2, 8, 20, 28, 50, 52$ and magic numbers of neutrons $N = 2, 8, 20, 28, 50, 82, 126$ are found to be very stable.**
(i) Verify this by calculating the proton separation energy S_p for ^{120}Sn ($Z = 50$) and

^{121}Sb ($Z = 51$). The proton separation energy for a nuclide is the minimum energy required to separate the least tightly bound proton from a nucleus of that nuclide. It is given by
$S_p = (M_{Z-1,N} + M_H - M_{Z,N})\,c^2$.
Given ^{119}In $= 118.9058\,$u, ^{120}Sn $= 119.902199\,$u, ^{121}Sb $= 120.903824\,$u, ^{1}H $= 1.0078252\,$u.
(ii) What does the existence of magic numbers indicate?

SOLUTION

Concept used. A nucleus with a **magic number** of protons or neutrons (or both) closes a nuclear shell, much as noble-gas atoms ($Z = 2, 10, 18, \ldots$) close electronic shells. Closing a shell means the last nucleon sits low in a deep potential well, so removing it costs unusually large energy. We can quantify this stability by computing the **proton separation energy** S_p: the energy required to strip away the least tightly bound proton from a nucleus.
For ^{120}Sn ($Z = 50$, magic), S_p should be *anomalously large* compared with its non-magic neighbour ^{121}Sb ($Z = 51$): ^{121}Sb has one proton outside the $Z = 50$ closed shell, and that proton sits in a higher shell, so it is much easier to remove.

Step 1. Use the conversion factor $1\,\text{u}\cdot c^2 = 931.5\,\text{MeV}$.

Step 2. Proton separation energy for ^{120}Sn (remove a proton to leave ^{119}In):

$$S_p(^{120}\text{Sn}) = [M(^{119}\text{In}) + M(^{1}\text{H}) - M(^{120}\text{Sn})]\,c^2.$$

Compute the mass difference (in u):
$118.9058 + 1.0078252 - 119.902199 = 119.913625 - 119.902199 = 0.011426\,\text{u}$.
Convert to MeV: $S_p(^{120}\text{Sn}) = 0.011426 \times 931.5\,\text{MeV} = 10.643\,\text{MeV}$.
Rounded: $S_p(^{120}\text{Sn}) \approx 10.64\,\text{MeV}$.

Step 3. Proton separation energy for ^{121}Sb (remove a proton to leave ^{120}Sn):

$$S_p(^{121}\text{Sb}) = [M(^{120}\text{Sn}) + M(^{1}\text{H}) - M(^{121}\text{Sb})]\,c^2.$$

Mass difference (in u):
$119.902199 + 1.0078252 - 120.903824 = 120.910024 - 120.903824 = 0.006200\,\text{u}$.
Convert: $S_p(^{121}\text{Sb}) = 0.006200 \times 931.5\,\text{MeV} = 5.775\,\text{MeV}$.
Rounded: $S_p(^{121}\text{Sb}) \approx 5.78\,\text{MeV}$.

Step 4. Compare. $S_p(^{120}\text{Sn})/S_p(^{121}\text{Sb}) = 10.64/5.78 \approx 1.84$ — i.e. the proton in ^{120}Sn is bound nearly *twice as tightly* as the outermost proton in ^{121}Sb. This is exactly the magic-number signature: the $Z = 50$ closed shell makes the last proton unusually hard to remove.

Step 5. Part (ii). The existence of magic numbers indicates that nucleons are organised in **nuclear shells** (analogous to electron shells in atoms). A closed shell is especially stable; magic-number nuclei are therefore especially stable, with anomalously large separation energies, larger binding energies per nucleon, and lower neutron-capture cross-sections than their neighbours. Quantum mechanically, this stability is explained by the **nuclear shell model** (Mayer and Jensen, 1949): nucleons fill orbitals in a strong central potential with a large spin-orbit coupling, and shell closures occur at exactly N or $Z = 2, 8, 20, 28, 50, 82, 126$.

☞ **Doubly magic nuclei**

A nucleus that is magic in both Z and N — e.g. ${}^4_2\text{He}$ $(Z = N = 2)$, ${}^{16}_8\text{O}$ $(Z = N = 8)$, ${}^{40}_{20}\text{Ca}$ $(Z = N = 20)$, ${}^{208}_{82}\text{Pb}$ $(Z = 82, N = 126)$ — is exceptionally stable, often spherical, and forms the backbone of nuclear-structure phenomenology.

Final Answer: (i) $S_p(^{120}\text{Sn}) = 10.64\,\text{MeV} \gg S_p(^{121}\text{Sb}) = 5.78\,\text{MeV}$. The factor-of-two jump confirms the $Z = 50$ shell closure. (ii) Magic numbers indicate the shell structure of nuclei — nucleons fill orbitals in a central potential, and closed shells give extra stability.

EXPERT'S SOLUTION : *Sanya Mehta, Ph.D Physics, IISc Bangalore*

Number-crunch angle. Compute both separation energies, see the jump, and conclude shell-closure.

Step 1. $S_p(^{120}\text{Sn}) = (118.9058 + 1.0078252 - 119.902199) \times 931.5\,\text{MeV} = 0.011426 \times 931.5 = 10.64\,\text{MeV}$.

Step 2. $S_p(^{121}\text{Sb}) = (119.902199 + 1.0078252 - 120.903824) \times 931.5\,\text{MeV} = 0.006200 \times 931.5 = 5.78\,\text{MeV}$.

Step 3. Ratio: $10.64/5.78 \approx 1.84$. The magic $Z = 50$ closure of ^{120}Sn makes its outermost proton nearly twice as tightly bound as that of ^{121}Sb.

Step 4. Magic numbers signal nuclear shells, exactly like noble-gas Z values mark electron shells — the basis of the shell model (Mayer-Jensen).

Alternative method — neutron separation energy comparison. The same analysis works for neutrons across the magic $N = 50$: $S_n(^{90}\text{Zr})$ — with $N = 50$ closed shell — is about $12.0\,\text{MeV}$, while $S_n(^{91}\text{Zr})$ with one extra neutron is only $\sim 7.2\,\text{MeV}$ — a similar $\sim 1.7\times$ ratio. The jump appears whenever one crosses a shell closure, in either Z or N. This duality is the experimental hallmark of the shell model.

Cross-check — average S_p scale. Typical mid-mass nuclei have $S_p \sim 6$–$8\,\text{MeV}$. So ^{121}Sb's $5.78\,\text{MeV}$ is run-of-the-mill, while ^{120}Sn's $10.64\,\text{MeV}$ is exceptionally high — a clear shell-closure signature. The fact that S_p *drops* sharply just past a magic Z is the standard way of identifying shell closures empirically.

Concept linkage — atomic shell analogy. The closed-shell atomic noble gases

($Z = 2, 10, 18, \ldots$) take $\sim 2\times$ more ionisation energy than their alkali-metal neighbours, exactly as ^{120}Sn needs $\sim 2\times$ more energy to lose a proton than ^{121}Sb. The Pauli-exclusion + central-potential framework is identical; only the potential differs (Coulomb in atoms, strong-force well in nuclei). This deep analogy is what makes the shell model so successful.

Why this matters. Magic-number stability is why exotic "island of stability" experiments at $Z = 114$, $N = 184$ are predicted to harbour superheavy nuclei with longer half-lives than their unmagical neighbours.

> **Final Answer:** $S_p(^{120}\text{Sn}) \approx 10.64\,\text{MeV}$, $S_p(^{121}\text{Sb}) \approx 5.78\,\text{MeV}$; the jump confirms the magic $Z = 50$ shell closure.

♥ The shell model and the search for superheavies

The dramatic factor-of-2 jump in proton separation energy at $Z = 50$ is the empirical "smoking gun" that nucleons live in shells. The shell model (Mayer-Jensen, Nobel 1963) predicts further magic numbers beyond uranium — in particular a doubly-magic $^{298}114$ that might have a half-life of millions of years, even though all known isotopes of $Z = 114$ to date decay in milliseconds. Modern superheavy research (Dubna, GSI, RIKEN) is essentially the experimental hunt for this predicted "island of stability".

Key Takeaways

- Radioactive decay is intrinsically random; the law $N = N_0 e^{-\lambda t}$ governs the *average* behaviour of a large ensemble, while individual sample counts fluctuate by $\sqrt{N}$.
- Binding energy follows mass-energy equivalence: $B = (\sum m_{\text{free parts}} - M_{\text{bound}})c^2$. Use $1\,\text{u} \cdot c^2 = 931.5\,\text{MeV}$ in every numerical calculation.
- β-decay Q-values written with *atomic* masses: $Q(\beta^-) = (M_x - M_y)c^2$; $Q(\beta^+) = (M_x - M_y - 2m_e)c^2$. The $2m_e$ threshold for β^+ is a frequent exam trap.
- Heavy stable nuclei have $N > Z$ because the long-range Coulomb repulsion between protons grows faster than the saturating nuclear binding.
- In a two-step decay chain $A \to B \to C$, the intermediate B peaks at $t^* = \ln(\lambda_B/\lambda_A)/(\lambda_B - \lambda_A)$ — the Bateman result.
- Photodisintegration of a system at rest by a γ-ray needs energy slightly above the binding energy: $E_{\text{min}} = B + B^2/(2Mc^2)$. The extra term is the recoil kinetic energy of the products.
- Magic numbers Z or $N = 2, 8, 20, 28, 50, 82, 126$ mark nuclear shell closures and give anomalously large proton-separation (or neutron-separation) energies.

End of NCERT Exemplar Problems