NCERT SOLUTIONS

Class 12 Physics

Chapter 13: Nuclei

Detailed Step-by-Step Exercise Solutions

Q1

(a) Two stable isotopes of lithium ^6_3Li and ^7_3Li have respective abundances of 7.5% and 92.5%. These isotopes have masses 6.01512 u and 7.01600 u, respectively. Find the atomic mass of lithium.

(b) Boron has two stable isotopes, $^{10}_5B$ and $^{11}_5B$. Their respective masses are 10.01294 u and 11.00931 u, and the atomic mass of boron is 10.811 u. Find the abundances of $^{10}_5B$ and $^{11}_5B$.

Solution

Understanding Isotopes and Atomic Mass

The atomic mass of an element as found in the periodic table is not the mass of a single atom but a **weighted average** of the masses of all its naturally occurring isotopes. This is expressed by the formula:

Weighted Average Atomic Mass:

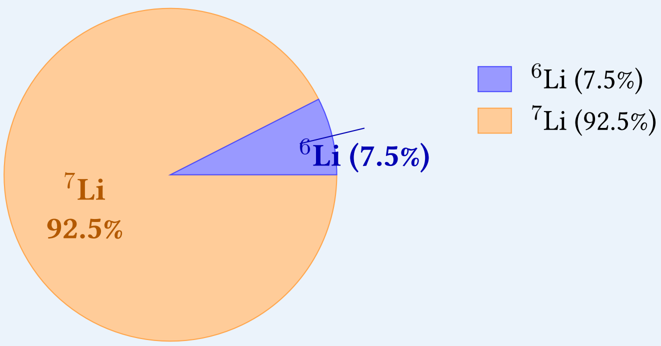
$$M_{\text{atomic}} = \frac{\sum(f_i \times M_i)}{\sum f_i} = \sum(x_i \times M_i)$$

where:

- f_i = fractional abundance (or percentage abundance)
- x_i = fraction of each isotope ($x_i = f_i/100$ when percentages are given)
- M_i = mass of each isotope (in atomic mass units, u)

Part (a): Atomic Mass of Lithium
Given Data:

Isotope	Mass (u)	Abundance (%)
${}^{6}_{3}\text{Li}$	$M_1 = 6.01512$	7.5%
${}^{7}_{3}\text{Li}$	$M_2 = 7.01600$	92.5%



Step 1: Convert percentages to fractional abundances

$$x_1 = \frac{7.5}{100} = 0.075 \quad (\text{for } {}^{6}\text{Li})$$

$$x_2 = \frac{92.5}{100} = 0.925 \quad (\text{for } {}^{7}\text{Li})$$

Step 2: Calculate the weighted average atomic mass

$$M_{\text{Li}} = x_1 M_1 + x_2 M_2$$

$$M_{\text{Li}} = 0.075 \times 6.01512 + 0.925 \times 7.01600$$

Calculating each term:

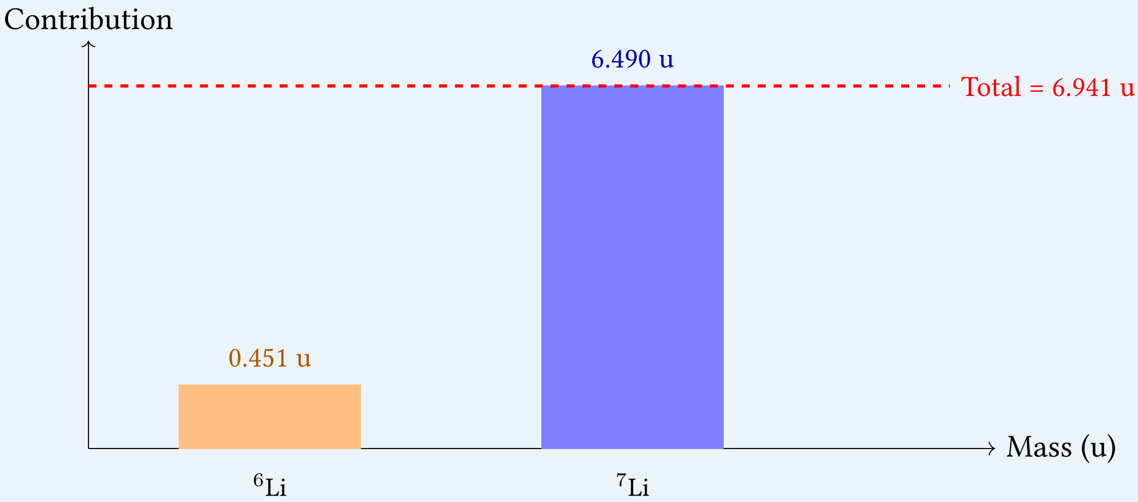
$$\text{Contribution from } {}^{6}\text{Li}: \quad 0.075 \times 6.01512 = 0.451134 \text{ u}$$
$$\text{Contribution from } {}^{7}\text{Li}: \quad 0.925 \times 7.01600 = 6.489800 \text{ u}$$

Step 3: Sum the contributions

$$M_{\text{Li}} = 0.451134 + 6.489800 = 6.940934 \text{ u}$$

Rounding to appropriate significant figures:

$$M_{\text{Li}} \approx 6.941 \text{ u}$$



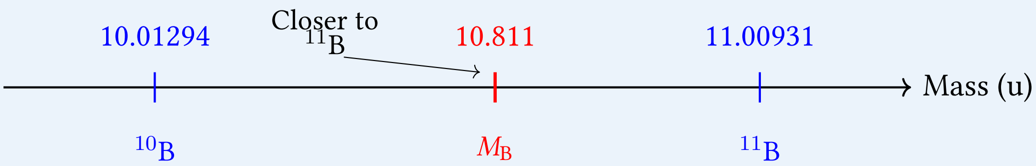
Answer (a): The atomic mass of lithium is

$$\boxed{M_{\text{Li}} = 6.941 \text{ u}}$$

Part (b): Abundances of Boron Isotopes
Given Data:

Isotope	Mass (u)	Abundance
$^{10}_{5}\text{B}$	$M_1 = 10.01294$	Unknown (let it be x%)
$^{11}_{5}\text{B}$	$M_2 = 11.00931$	Unknown (let it be $(100 - x)$%)

The atomic mass of boron is given as: $M_{\text{B}} = 10.811$ u



Step 1: Set up the weighted average equation
Let the abundance of ^{10}B be x%. Then the abundance of ^{11}B is $(100 - x)$%.

$$M_{\text{B}} = \frac{x}{100} \times M_1 + \frac{100 - x}{100} \times M_2$$

$$10.811 = \frac{x}{100} \times 10.01294 + \frac{100 - x}{100} \times 11.00931$$

Step 2: Multiply throughout by 100

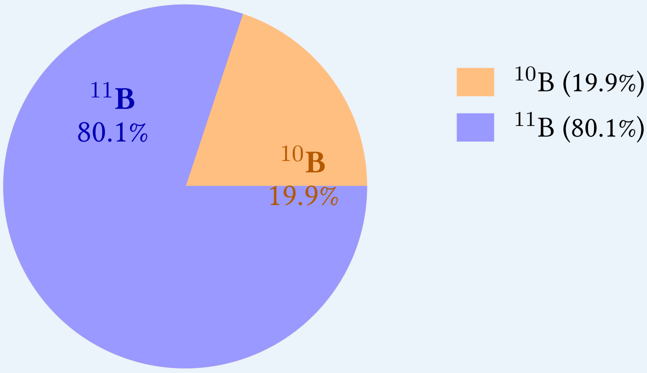
$$1081.1 = x \times 10.01294 + (100 - x) \times 11.00931$$

Step 3: Expand and solve

$$1081.1 = 10.01294x + 1100.931 - 11.00931x$$
$$1081.1 = 1100.931 - 0.99637x$$
$$-19.831 = -0.99637x$$
$$x = \frac{19.831}{0.99637} = 19.903$$

Step 4: Calculate the abundance of ^{11}B

$$\text{Abundance of } ^{11}\text{B} = 100 - x = 100 - 19.903 = 80.097\%$$



Verification:

$$M_\text{B} = \frac{19.903}{100} \times 10.01294 + \frac{80.097}{100} \times 11.00931$$
$$M_\text{B} = 1.993 + 8.818 = 10.811 \text{ u} \quad \checkmark$$

Answer (b): The abundances are

$$\boxed{^{10}_{5}\text{B}: \ 19.9\% \quad ; \quad ^{11}_{5}\text{B}: \ 80.1\%}$$

Expert's Solution — Priya Sharma, B.Tech Engineering Physics, IIT Roorkee

The Concept of Weighted Average — A Deeper Perspective

The atomic mass calculation is a classic example of a **weighted arithmetic mean**.

Alternative Method for Part (b): Using the Mass Difference

Let x be the fraction of ^{10}B. Then:

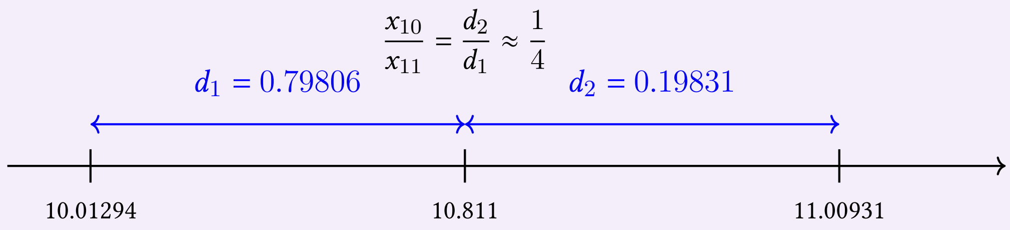
$$M_\text{avg} = xM_{10} + (1 - x)M_{11}$$
$$M_\text{avg} = M_{11} + x(M_{10} - M_{11})$$

$$x = \frac{M_{avg} - M_{11}}{M_{10} - M_{11}}$$

Substituting:

$$x = \frac{10.811 - 11.00931}{10.01294 - 11.00931} = \frac{-0.19831}{-0.99637} = 0.1990$$

Thus, $x = 19.90\%$ for ^{10}B and 80.10% for ^{11}B.



Did You Know?

Quick Check for Isotope Abundance Problems:

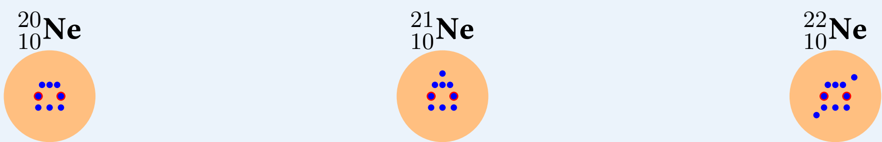
- **If the atomic mass is closer to isotope A**, then **isotope A is more abundant**.
- **The lever rule:** $\frac{x_1}{x_2} = \frac{|M_{avg} - M_2|}{|M_{avg} - M_1|}$
- **Sum of abundances must equal 100%** — always verify this!

Q2 **The three stable isotopes of neon: $^{20}_{10}$Ne, $^{21}_{10}$Ne and $^{22}_{10}$Ne have respective abundances of 90.51%, 0.27% and 9.22%. The atomic masses of the three isotopes are 19.99 u, 20.99 u and 21.99 u, respectively. Obtain the average atomic mass of neon.**

Solution

Understanding the Problem

Neon has three naturally occurring stable isotopes. The average atomic mass reported in the periodic table is the **weighted arithmetic mean** of the masses of these isotopes, weighted by their natural abundances.



Given Data:

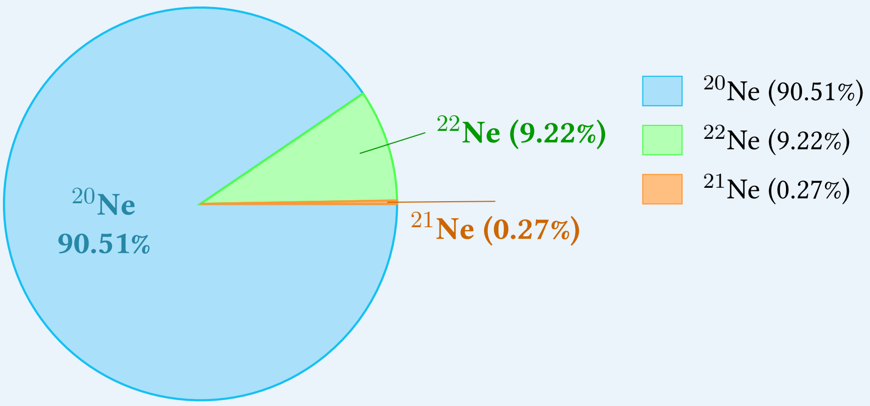
Isotope	Mass (u)	Abundance (%)
$^{20}_{10}Ne$	$M_1 = 19.99$	$f_1 = 90.51\%$
$^{21}_{10}Ne$	$M_2 = 20.99$	$f_2 = 0.27\%$
$^{22}_{10}Ne$	$M_3 = 21.99$	$f_3 = 9.22\%$

Step 1: Verify that abundances sum to 100%

$$90.51\% + 0.27\% + 9.22\% = 100.00\% \quad \checkmark$$

Step 2: Convert percentages to fractional abundances

$$x_1 = \frac{90.51}{100} = 0.9051 \quad ; \quad x_2 = \frac{0.27}{100} = 0.0027 \quad ; \quad x_3 = \frac{9.22}{100} = 0.0922$$



Step 3: Apply the weighted average formula

Average Atomic Mass:

$$M_{avg} = \sum_i x_i M_i = x_1 M_1 + x_2 M_2 + x_3 M_3$$

$$M_{Ne} = 0.9051 \times 19.99 + 0.0027 \times 20.99 + 0.0922 \times 21.99$$

Step 4: Calculate each contribution

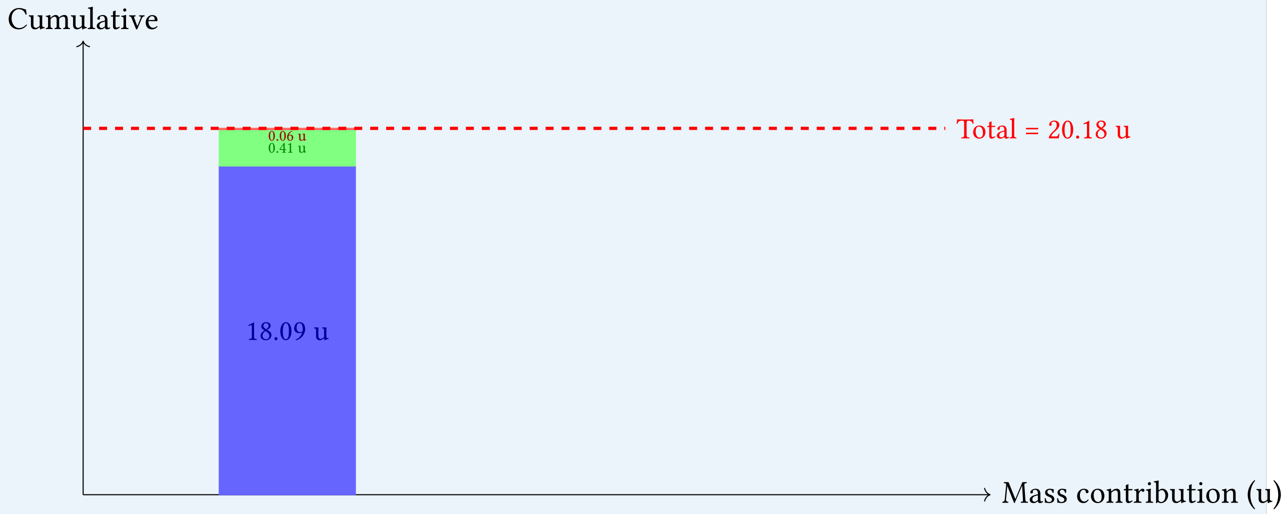
$$\begin{aligned} ^{20}\text{Ne}: &\quad 0.9051 \times 19.99 = 18.092949 \text{ u} \\ ^{21}\text{Ne}: &\quad 0.0027 \times 20.99 = 0.056673 \text{ u} \\ ^{22}\text{Ne}: &\quad 0.0922 \times 21.99 = 2.027478 \text{ u} \end{aligned}$$

Step 5: Sum all contributions

$$M_{Ne} = 18.092949 + 0.056673 + 2.027478 = 20.177100 \text{ u}$$

Rounding to appropriate significant figures:

$$\boxed{M_{Ne} \approx 20.18 \text{ u}}$$



Insight: The overwhelming dominance of ^{20}Ne (90.51%) means the average atomic mass (20.18 u) is pulled very close to 19.99 u, with only minor upward shifts from the heavier isotopes.

Final Answer:

$$\boxed{M_{Ne} = 20.18 \text{ u}}$$

This matches the standard atomic mass of neon listed in the periodic table.

Expert's Solution — Vikram Reddy, B.Tech Chemical Engineering, NIT Warangal

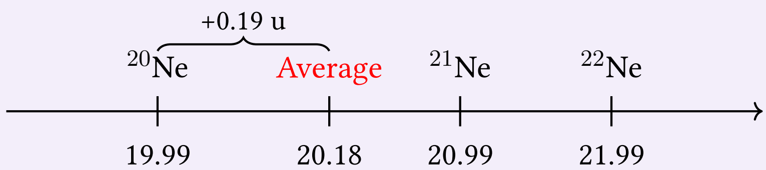
Quick Mental Calculation Trick

When one isotope dominates (like ^{20}Ne at 90.51%), estimate the average as:

$$M_{avg} \approx M_{dominant} + \text{(small corrections)}$$

$$M_{Ne} \approx 19.99 + \underbrace{0.0922 \times (21.99 - 19.99)}_{\text{from } ^{22}\text{Ne}} + \underbrace{0.0027 \times (20.99 - 19.99)}_{\text{from } ^{21}\text{Ne}}$$

$$M_{Ne} \approx 19.99 + 0.1844 + 0.0027 = 20.1771 \text{ u}$$



Did You Know?

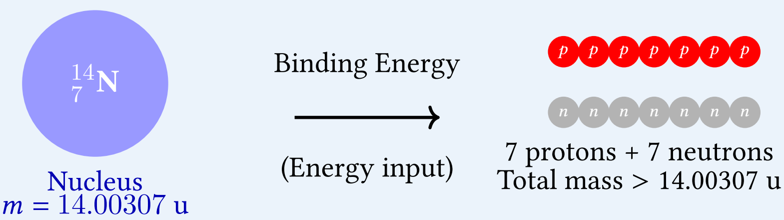
Error-Catching Checklist for Isotope Problems:

1. **Sum of abundances = 100%?** Always verify first!
2. **Does the average lie between the lightest and heaviest isotope?** It must!
3. **Is the average closer to the most abundant isotope?** A simple sanity check.
4. **Significant figures:** Match precision of given data.

Q3 **Obtain the binding energy (in MeV) of a nitrogen nucleus $^{14}_{7}$N, given $m\left(^{14}_{7}\text{N}\right) = 14.00307$ u.**

Solution

Understanding Binding Energy
The binding energy of a nucleus is the energy required to separate it into its constituent protons and neutrons. It represents the net energy released when nucleons come together to form a nucleus — the famous **mass defect** converted to energy via Einstein's mass-energy equivalence.



Step 1: Identify the Constituents
The nitrogen nucleus $^{14}_{7}$N contains:

- **Atomic number** $Z = 7$ (7 protons)
- **Mass number** $A = 14$ (total nucleons)
- **Number of neutrons** $N = A - Z = 14 - 7 = 7$

$^{14}_{7}$N Nucleus
$Z = 7$ (protons)
$N = 7$ (neutrons)
$A = 14$ (total nucleons)

Step 2: Recall the Required Masses
To calculate the mass defect, we need:

- Mass of nitrogen nucleus: $m\left({}^{14}_{7}\text{N}\right) = 14.00307$ u
- Mass of proton: $m_p = 1.007825$ u
- Mass of neutron: $m_n = 1.008665$ u

Step 3: Calculate the Total Mass of Individual Nucleons

$$\text{Mass of 7 protons} = 7 \times m_p = 7 \times 1.007825 = 7.054775 \text{ u}$$
$$\text{Mass of 7 neutrons} = 7 \times m_n = 7 \times 1.008665 = 7.060655 \text{ u}$$

$$\text{Total mass of constituents} = 7.054775 + 7.060655 = 14.115430 \text{ u}$$

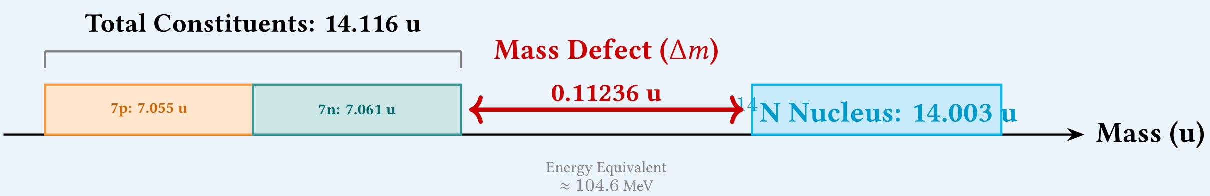
Step 4: Calculate the Mass Defect (Δm)

Mass Defect:

$$\Delta m = (\text{Total mass of individual nucleons}) - (\text{Mass of nucleus})$$

$$\Delta m = 14.115430 - 14.00307 = 0.11236 \text{ u}$$

This "missing" mass has been converted into the binding energy that holds the nucleus together.



Step 5: Convert Mass Defect to Binding Energy
Using Einstein's mass-energy equivalence:

Binding Energy:

$$E_b = \Delta m \times c^2$$

where the energy equivalent of 1 u is:

$$1 \text{ u} \times c^2 = 931.5 \text{ MeV}$$

$$E_b = 0.11236 \text{ u} \times 931.5 \frac{\text{MeV}}{\text{u}}$$

Calculating:

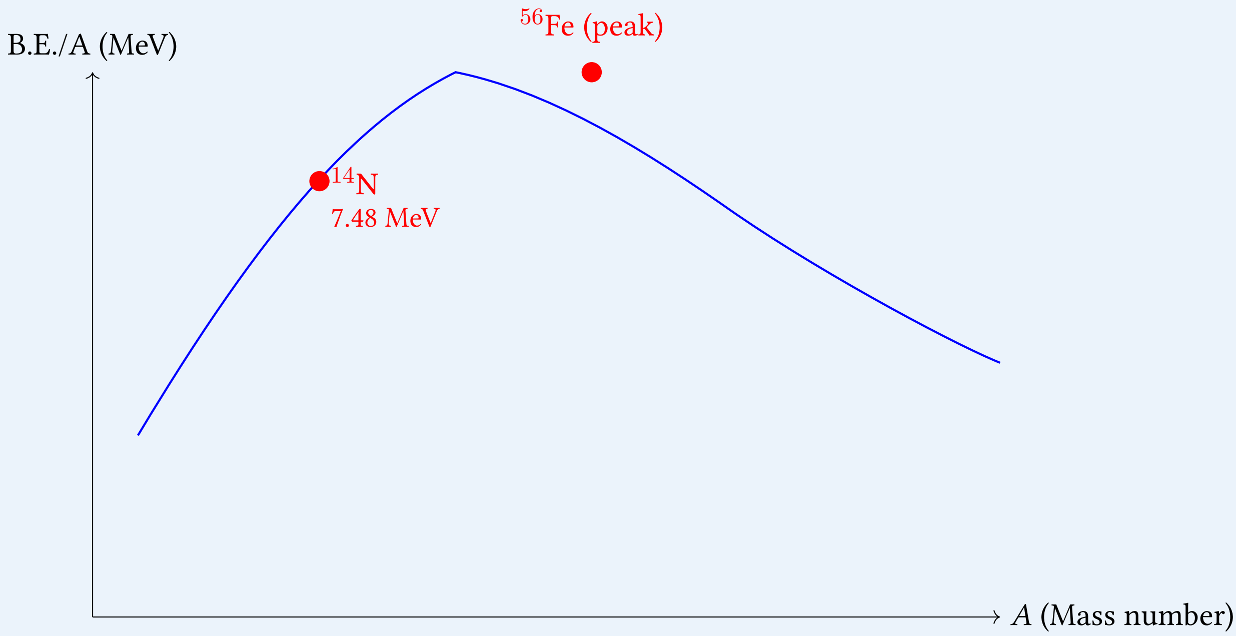
$$E_b = 0.11236 \times 931.5 = 104.66334 \text{ MeV}$$

Rounding to appropriate significant figures:

$$\boxed{E_b \approx 104.66 \text{ MeV}}$$

Step 6: Calculate Binding Energy per Nucleon
The binding energy per nucleon is an important measure of nuclear stability:

$$\text{B.E. per nucleon} = \frac{E_b}{A} = \frac{104.66}{14} = 7.476 \text{ MeV/nucleon}$$



Nitrogen-14 has a relatively high binding energy per nucleon (7.48 MeV), indicating it is a fairly stable nucleus. For comparison, the most stable nucleus, ^{56}Fe, has a binding energy per nucleon of about 8.79 MeV.

Final Answer:

$$\boxed{E_b = 104.66 \text{ MeV}}$$

The binding energy per nucleon is:

$$\boxed{\frac{E_b}{A} = 7.48 \text{ MeV/nucleon}}$$

Expert's Solution – Ananya Gupta, B.Tech Engineering Physics, IIT BHU

The Physical Origin of Binding Energy
The binding energy arises from the **strong nuclear force** – the most powerful force in nature at subatomic distances, yet it operates only over extremely short ranges ($\sim 10^{-15}$ m).

Why Does Mass Decrease When a Nucleus Forms?
When protons and neutrons come together:

1. The strong nuclear force does **negative work** (it's attractive), releasing energy
2. By $E = mc^2$, this released energy corresponds to a decrease in mass
3. The mass defect Δm is precisely the mass equivalent of the energy released

Quick Formula Memory:

$$E_b \text{ (MeV)} = \Delta m \text{ (u)} \times 931.5$$

Standard Mass Values to Memorize:

Proton mass:	$m_p = 1.007825$ u	≈ 938.27 MeV/c^2
Neutron mass:	$m_n = 1.008665$ u	≈ 939.57 MeV/c^2
Electron mass:	$m_e = 0.0005486$ u	≈ 0.511 MeV/c^2
1 u $\times c^2$	$= 931.5$ MeV	(Energy equivalent)

Important Note: Nuclear Mass vs. Atomic Mass
In many problems (including this one), the mass given is the **nuclear mass**, not the atomic mass. If atomic mass were given instead, we would need to subtract the mass of 7 electrons to get the nuclear mass:

$$m_{\text{nucleus}} = m_{\text{atom}} - Zm_e$$

However, in this problem, $m\left({}^{14}_{7}\text{N}\right) = 14.00307$ u is already the nuclear mass, so we use it directly.

Did You Know?

Error Prevention Checklist:

- **Nuclear mass or atomic mass?** Check if electron masses need to be subtracted
- **Correct number of protons and neutrons?** Z = atomic number, $N = A - Z$
- **Right conversion factor?** 1 u = 931.5 MeV/c^2 (not 931!)
- **Significant figures?** Here 14.00307 has 7 sig figs, so answer to 5 sig figs is appropriate

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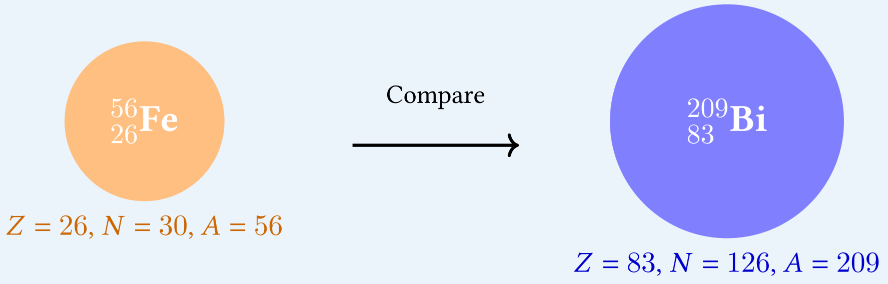
Q4 **Obtain the binding energy of the nuclei ${}^{56}_{26}$Fe and ${}^{209}_{83}$Bi in units of MeV from the following data:**

$$m\left({}^{56}_{26}\textbf{Fe}\right) = 55.934939 \textbf{ u} \quad ; \quad m\left({}^{209}_{83}\textbf{Bi}\right) = 208.980388 \textbf{ u}$$

Solution

Understanding the Problem

We need to calculate the binding energy for two nuclei: iron-56 (one of the most tightly bound nuclei) and bismuth-209 (a heavy, less tightly bound nucleus). This will illustrate how binding energy per nucleon varies across the periodic table.



Required Constants:

$$m_p = 1.007825 \text{ u} \quad ; \quad m_n = 1.008665 \text{ u} \quad ; \quad 1 \text{ u} \times c^2 = 931.5 \text{ MeV}$$

Part (a): Binding Energy of $^{56}_{26}$Fe

Step 1: Identify the constituents

$^{56}_{26}$Fe (Iron-56)

Protons:	$Z = 26$
Neutrons:	$N = 56 - 26 = 30$
Total Nucleons:	$A = 56$

Step 2: Calculate total mass of individual nucleons

$$\text{Mass of 26 protons} = 26 \times 1.007825 = 26.203450 \text{ u}$$
$$\text{Mass of 30 neutrons} = 30 \times 1.008665 = 30.259950 \text{ u}$$
$$\text{Total mass of constituents} = 26.203450 + 30.259950 = 56.463400 \text{ u}$$

Step 3: Calculate the mass defect

$$\Delta m = 56.463400 - 55.934939 = 0.528461 \text{ u}$$

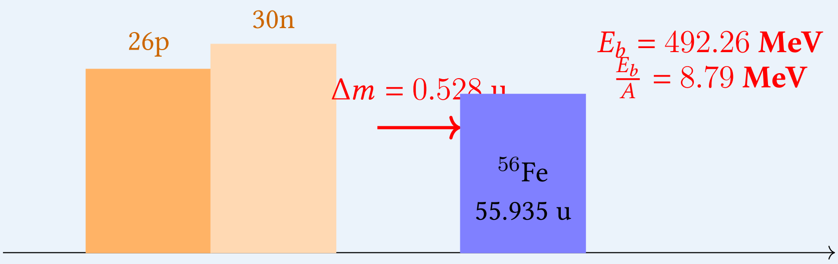
Step 4: Convert mass defect to binding energy

$$E_b(^{56}\text{Fe}) = 0.528461 \times 931.5 = 492.261 \text{ MeV}$$

$$\boxed{E_b(^{56}\text{Fe}) \approx 492.26 \text{ MeV}}$$

Step 5: Binding energy per nucleon

$$\frac{E_b}{A} = \frac{492.26}{56} = 8.790 \text{ MeV/nucleon}$$



Part (b): Binding Energy of $^{209}_{83}$Bi

Step 1: Identify the constituents

$^{209}_{83}$Bi (Bismuth-209)

Protons:	$Z = 83$
Neutrons:	$N = 209 - 83 = 126$
Total Nucleons:	$A = 209$

Step 2: Calculate total mass of individual nucleons

$$\text{Mass of 83 protons} = 83 \times 1.007825 = 83.649475 \text{ u}$$
$$\text{Mass of 126 neutrons} = 126 \times 1.008665 = 127.091790 \text{ u}$$
$$\text{Total mass of constituents} = 83.649475 + 127.091790 = 210.741265 \text{ u}$$

Step 3: Calculate the mass defect

$$\Delta m = 210.741265 - 208.980388 = 1.760877 \text{ u}$$

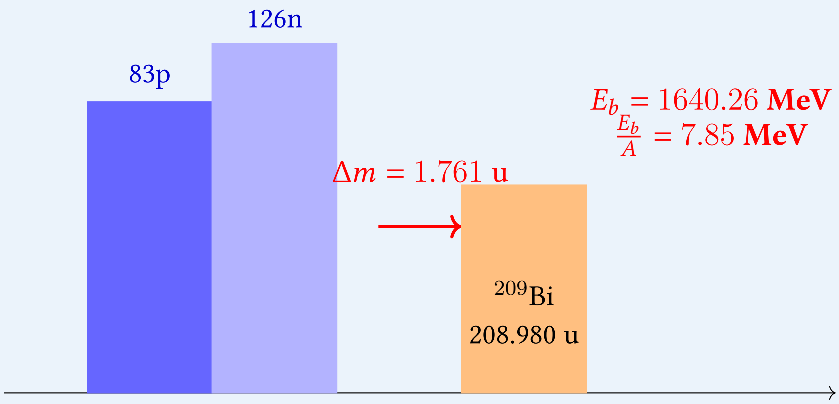
Step 4: Convert mass defect to binding energy

$$E_b(^{209}\text{Bi}) = 1.760877 \times 931.5 = 1640.257 \text{ MeV}$$

$$\boxed{E_b(^{209}\text{Bi}) \approx 1640.26 \text{ MeV}}$$

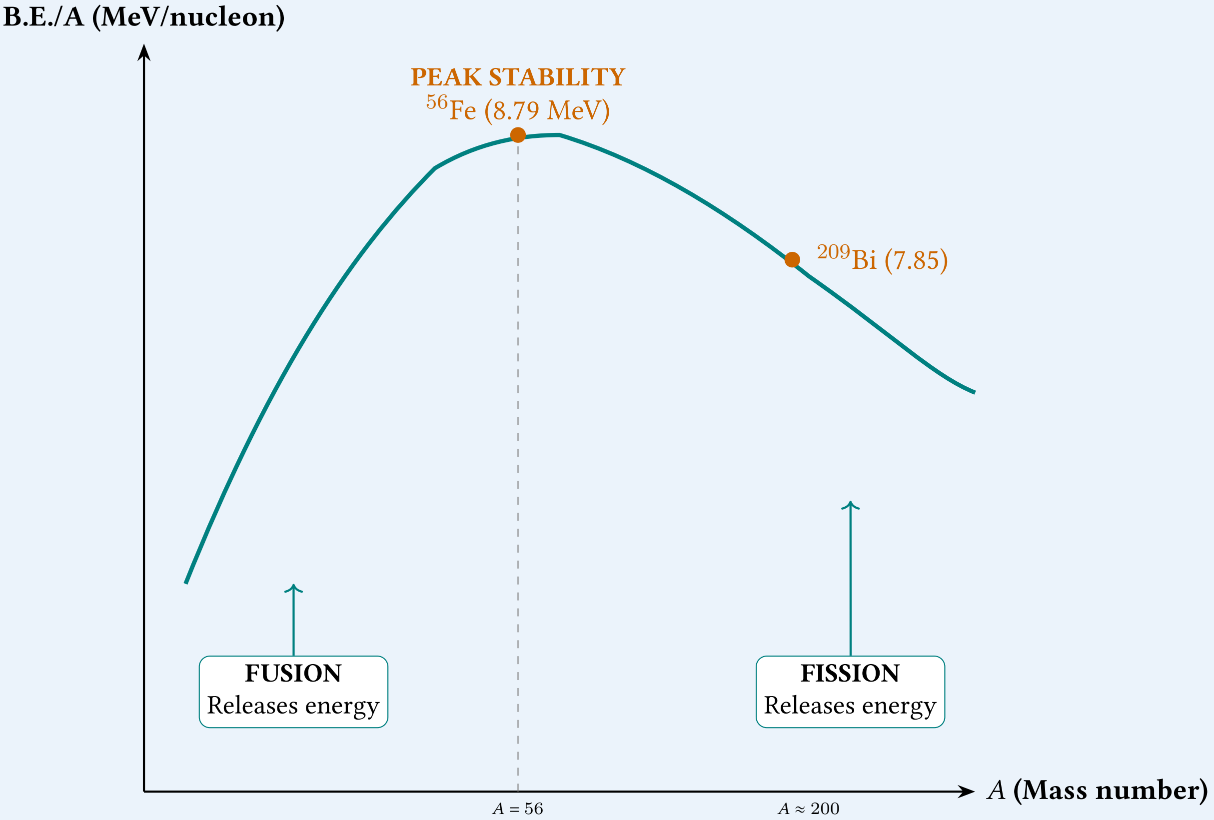
Step 5: Binding energy per nucleon

$$\frac{E_b}{A} = \frac{1640.26}{209} = 7.848 \text{ MeV/nucleon}$$



Comparison and Physical Insight

Nucleus	Mass Defect (u)	E_b (MeV)	E_b/A (MeV/nucleon)
${}^{56}_{26}Fe$	0.528461	492.26	8.79
${}^{209}_{83}Bi$	1.760877	1640.26	7.85



Key Observations:

- ${}^{56}Fe$ has one of the highest binding energies per nucleon ($\sim$ 8.79 MeV), making it one of the most stable nuclei in nature. This is why iron is the endpoint of fusion in stellar cores.
- ${}^{209}Bi$ has a lower binding energy per nucleon ($\sim$ 7.85 MeV), typical of heavy nuclei. The lower stability arises from increased Coulomb repulsion between the large number of protons.
- Nuclei lighter than iron can release energy through **fusion**; nuclei heavier than iron can release energy through **fission**.

Final Answer:

$$E_b\left({}^{56}_{26}\text{Fe}\right) = 492.26\text{ MeV} \quad ; \quad \frac{E_b}{A} = 8.79\text{ MeV/nucleon}$$

$$E_b\left({}^{209}_{83}\text{Bi}\right) = 1640.26\text{ MeV} \quad ; \quad \frac{E_b}{A} = 7.85\text{ MeV/nucleon}$$

Expert's Solution — Rahul Kashyap, B.Tech Engineering Physics, NIT Kurukshetra

Why Iron-56 is Special: The Most Stable Nucleus

The binding energy per nucleon curve is one of the most important graphs in nuclear physics. Let's understand why ^{56}Fe sits at the peak.

The Competing Forces in a Nucleus:

Attractive: Strong Nuclear Force

- Short-range ($\sim 10^{-15}$ m)
- Acts between all nucleons (p-p, n-n, p-n)
- Saturates (only nearest neighbors)
- Independent of charge

Repulsive: Coulomb Force

- Long-range (inverse-square law)
- Acts only between protons
- Increases with Z^2
- Tries to disrupt the nucleus

The Iron Peak Explained:

- **Light nuclei ($A < 56$):** The strong force hasn't reached saturation. Adding nucleons increases stability because more nearest-neighbor bonds form. Fusion is energetically favorable.
- **At $A \approx 56$ (Iron peak):** The strong force is fully saturated, and Coulomb repulsion is still manageable. This is the optimal balance — maximum binding per nucleon.
- **Heavy nuclei ($A > 56$):** Coulomb repulsion between protons grows as Z^2 and begins dominating over the saturated strong force. Adding more nucleons decreases stability per particle. Fission becomes energetically favorable.

Stellar Nucleosynthesis Connection:

Stars produce energy by fusing lighter elements into heavier ones, but this process stops at iron. Once a star develops an iron core, fusion can no longer release energy, leading to core collapse and a supernova explosion — which is how elements heavier than iron are actually created!

Nuclear Process	Energy Released?	Example
Fusion of light nuclei	Yes (increases B.E./A)	H $\longrightarrow$ He in Sun
Fission of heavy nuclei	Yes (increases B.E./A)	^{235}U in reactors
Fusion beyond iron	No (decreases B.E./A)	—
Fission below iron	No (decreases B.E./A)	—

Did You Know?

Quick Calculation Check: For any binding energy problem:

- Mass defect should typically be between $0.008A$ and $0.010A$ atomic mass units
- Binding energy per nucleon should be between 7.5 and 8.8 MeV for stable nuclei
- If your calculated E_b/A is outside this range, recheck your arithmetic!

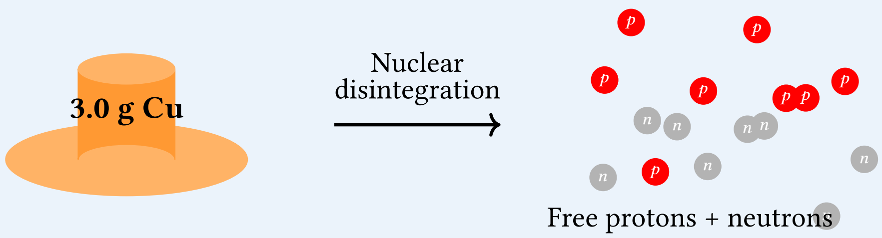
Our results (8.79 and 7.85 MeV/nucleon) fall perfectly within the expected ranges.

Q5 **A given coin has a mass of 3.0 g. Calculate the nuclear energy that would be required to separate all the neutrons and protons from each other. For simplicity assume that the coin is entirely made of $^{63}_{29}$Cu atoms (of mass 62.92960 u).**

Solution

Understanding the Problem

This is a fascinating problem that bridges the microscopic world of nuclear physics with the macroscopic world of everyday objects. We need to find the total energy required to completely disassemble all nuclei in a 3.0 g copper coin into their constituent protons and neutrons.



Step 1: Identify the Constituents of $^{63}_{29}$Cu

$$Z = 29 \text{ (protons)} \quad ; \quad A = 63 \text{ (total nucleons)} \quad ; \quad N = A - Z = 63 - 29 = 34 \text{ (neutrons)}$$

$^{63}_{29}$Cu Nucleus

$Z = 29$ protons, $N = 34$ neutrons

Mass of one nucleus = 62.92960 u

Step 2: Calculate Mass Defect for a Single ^{63}Cu Nucleus
Required masses:

$$m_p = 1.007825 \text{ u} \quad ; \quad m_n = 1.008665 \text{ u}$$

Total mass of individual nucleons:

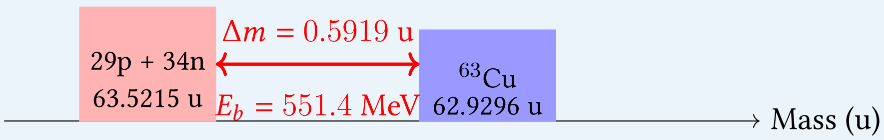
$$\text{Mass of 29 protons} = 29 \times 1.007825 = 29.226925 \text{ u}$$
$$\text{Mass of 34 neutrons} = 34 \times 1.008665 = 34.294610 \text{ u}$$
$$\text{Total} = 29.226925 + 34.294610 = 63.521535 \text{ u}$$

Mass defect for one nucleus:

$$\Delta m = 63.521535 - 62.92960 = 0.591935 \text{ u}$$

Step 3: Calculate Binding Energy per Nucleus

$$E_b(\text{per nucleus}) = \Delta m \times 931.5 = 0.591935 \times 931.5 = 551.387 \text{ MeV}$$



Step 4: Determine the Number of Copper Atoms in the Coin
Given:

$$\text{Mass of coin} = 3.0 \text{ g} = 3.0 \times 10^{-3} \text{ kg}$$
$$\text{Mass of one } ^{63}\text{Cu atom} = 62.92960 \text{ u}$$

Convert atomic mass to kg:

$$1 \text{ u} = 1.660539 \times 10^{-27} \text{ kg}$$
$$m_{\text{atom}} = 62.92960 \times 1.660539 \times 10^{-27} = 1.0449 \times 10^{-25} \text{ kg}$$

Number of atoms in the coin:

$$N = \frac{\text{Mass of coin}}{\text{Mass of one atom}} = \frac{3.0 \times 10^{-3}}{1.0449 \times 10^{-25}}$$

$$N = 2.871 \times 10^{22} \text{ atoms}$$

Alternative: Using Avogadro's Number

$$\text{Number of moles: } n = \frac{3.0 \text{ g}}{62.92960 \text{ g/mol}} = 0.04767 \text{ mol}$$

$$\text{Number of atoms: } N = n \times N_A = 0.04767 \times 6.022 \times 10^{23} = 2.871 \times 10^{22}$$

Step 5: Calculate Total Nuclear Energy Required

$$E_{\text{total}} = N \times E_b(\text{per nucleus})$$
$$E_{\text{total}} = 2.871 \times 10^{22} \times 551.387 \text{ MeV}$$
$$E_{\text{total}} = 1.583 \times 10^{25} \text{ MeV}$$

Step 6: Convert to More Practical Units

Energy Conversions:

$$1 \text{ MeV} = 1.602 \times 10^{-13} \text{ J}$$

$$E_{\text{total}} = 1.583 \times 10^{25} \times 1.602 \times 10^{-13} \text{ J}$$
$$E_{\text{total}} = 2.536 \times 10^{12} \text{ J}$$

This is an enormous amount of energy! To put it in perspective:

Energy Comparison:

Energy to disassemble 3 g coin	$\approx 2.54 \times 10^{12}$ J
Energy from burning 1 ton of coal	$\approx 2.9 \times 10^{10}$ J
Energy from 1 ton of TNT	$\approx 4.2 \times 10^{9}$ J

$\Rightarrow$ The nuclear energy in just **3 grams** of copper is equivalent to burning approximately **87 tons of coal** or detonating **600 tons of TNT**!

Final Answer:

$$\boxed{E_{\text{total}} = 1.58 \times 10^{25} \text{ MeV} = 2.54 \times 10^{12} \text{ J}}$$

The nuclear binding energy stored in a mere 3.0 g copper coin is approximately **2.5 terajoules** — equivalent to the energy consumption of a small city for several hours!

Expert's Solution — Neha Sahu, B.Tech CSE, IIIT Gwalior

The Staggering Scale of Nuclear Energy

This problem beautifully illustrates why nuclear energy is millions of times more concentrated than chemical energy. Let's analyze the numbers more deeply.

Why Are Nuclear Forces So Energetic?

Chemical Bonds

Involve outer electrons

Energy scale: ~ 1–10 eV

Mass defect: Negligible

(Atomic/Molecular level)

Nuclear Bonds

Involve nucleons (p and n)

Energy scale: ~ 10^6 eV (MeV)

Mass defect: ~ 1% of mass

(Subatomic/Nucleus level)

Per-Atom Energy Comparison:

- **Chemical reaction** (e.g., burning carbon): ~ 4 eV per atom
- **Nuclear disintegration**: ~ 551 MeV $= 5.51 \times 10^8$ eV per atom
- **Ratio:** $\dfrac{5.51 \times 10^8}{4} \approx 1.4 \times 10^8$

This means nuclear processes are roughly **100 million times** more energetic than chemical processes per atom!

The Einstein Connection: $E = mc^2$ at Work

The mass defect in ^{63}Cu is about 0.94% of the total nucleon mass:

$$\frac{\Delta m}{m_{\text{total}}} = \frac{0.5919}{63.5215} \times 100\% \approx 0.93\%$$

If we could completely convert 3 g of mass into energy (100% efficiency), we'd get:

$$E = mc^2 = 3 \times 10^{-3} \times (3 \times 10^8)^2 = 2.7 \times 10^{14} \text{ J}$$

Our calculated energy (2.54×10^{12} J) is about 0.94% of this theoretical maximum — exactly matching the mass defect percentage! This consistency check confirms our calculation.

★ **Did You Know?**

Problem-Solving Shortcut:

Instead of converting to kg and back, use this more efficient approach:

1. Find mass defect per atom in u: $\Delta m = 0.5919$ u
2. Find number of atoms: $N = \dfrac{3.0}{62.9296} \times N_A$
3. Total energy: $E = N \times \Delta m \times 931.5$ MeV
4. Convert to joules: multiply by 1.602×10^{-13}

This avoids the intermediate conversion of atomic mass to kg, reducing calculation errors.

Q6 **Write nuclear reaction equations for:**

(i) α**-decay of** $^{226}_{88}$**Ra**

(ii) α**-decay of** $^{242}_{94}$**Pu**

(iii) β^-**-decay of** $^{32}_{15}$**P**

(iv) β^-**-decay of** $^{210}_{83}$**Bi**

(v) β^+**-decay of** $^{11}_{6}$**C**

(vi) β^+**-decay of** $^{97}_{43}$**Tc**

(vii) **Electron capture of** $^{120}_{54}$**Xe**

Solution

Understanding Nuclear Decay Modes

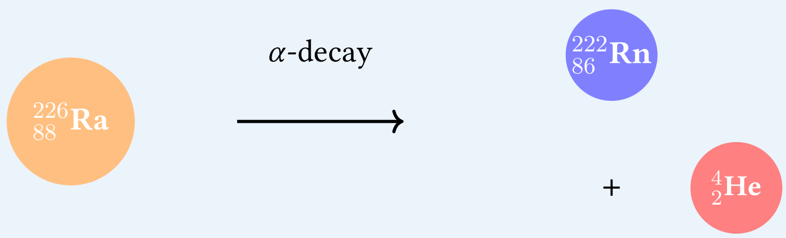
Nuclear reactions follow strict conservation laws: conservation of charge, conservation of nucleon number, and conservation of mass-energy. Each decay mode has characteristic changes in atomic number (Z) and mass number (A).

Summary of Decay Modes

Decay Mode	Particle Emitted	Change in Z	Change in A
α-decay	$^4_2$He nucleus	$Z \to Z - 2$	$A \to A - 4$
β^--decay	Electron (e^-) + $\bar{\nu}_e$	$Z \to Z + 1$	A (const)
β^+-decay	Positron (e^+) + ν_e	$Z \to Z - 1$	A (const)
Electron capture	X-ray + ν_e	$Z \to Z - 1$	A (const)

(i) α**-decay of** $^{226}_{88}$**Ra**

In α-decay, the parent nucleus emits an α-particle ($^4_2$He nucleus), reducing Z by 2 and A by 4.



Conservation Check:

- Mass number: $226 = 222 + 4$ ✓
- Atomic number: $88 = 86 + 2$ ✓

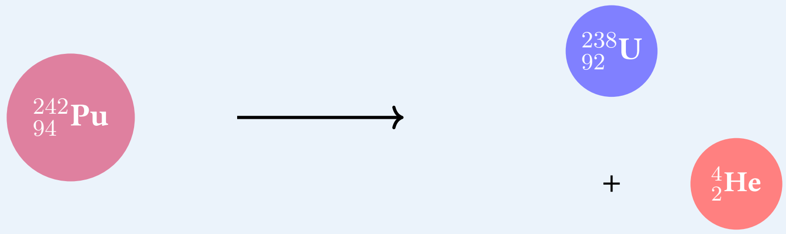
Answer (i):

$$^{226}_{88}\text{Ra} \longrightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$$

Radium-226 decays to Radon-222 with emission of an alpha particle.

(ii) α**-decay of** $^{242}_{94}$**Pu**

Same principle: Z decreases by 2, A decreases by 4.



Conservation Check:

- Mass number: $242 = 238 + 4$ ✓
- Atomic number: $94 = 92 + 2$ ✓

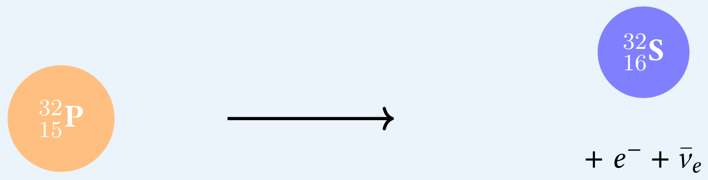
Answer (ii):

$$\boxed{^{242}_{94}\text{Pu} \longrightarrow {}^{238}_{92}\text{U} + {}^{4}_{2}\text{He}}$$

Plutonium-242 decays to Uranium-238.

(iii) β^--decay of $^{32}_{15}$P

In β^--decay, a neutron converts into a proton, emitting an electron (β^-) and an antineutrino. Z increases by 1, A remains unchanged.



The Fundamental Process:

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

Conservation Check:

- Mass number: $32 = 32 + 0$ ✓
- Atomic number: $15 = 16 + (-1)$ ✓ (electron has $Z = -1$)

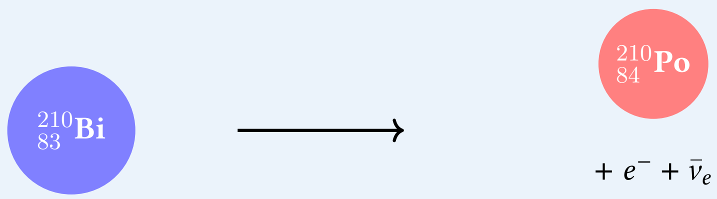
Answer (iii):

$$\boxed{^{32}_{15}\text{P} \longrightarrow {}^{32}_{16}\text{S} + e^- + \bar{\nu}_e}$$

Phosphorus-32 decays to Sulfur-32.

(iv) β^--decay of $^{210}_{83}$Bi

Same β^- process: Z increases by 1 to $Z = 84$.



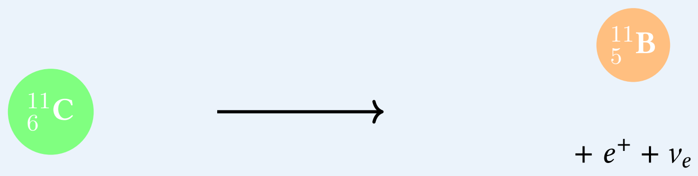
Answer (iv):

$$\boxed{^{210}_{83}\text{Bi} \longrightarrow \, ^{210}_{84}\text{Po} + e^- + \bar{\nu}_e}$$

Bismuth-210 decays to Polonium-210.

(v) β^+-decay of $^{11}_{6}$C

In β^+-decay, a proton converts into a neutron, emitting a positron (e^+) and a neutrino. Z decreases by 1, A remains unchanged.



The Fundamental Process:

$$p \longrightarrow n + e^+ + \nu_e$$

Conservation Check:

- Mass number: $11 = 11 + 0$ ✓
- Atomic number: $6 = 5 + (+1)$ ✓ (positron has $Z = +1$)

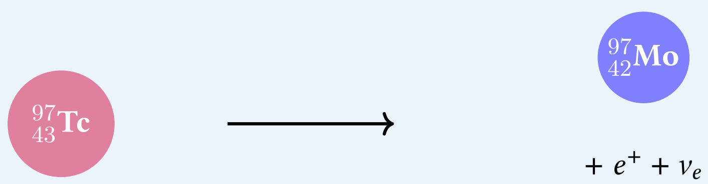
Answer (v):

$$\boxed{^{11}_{6}\text{C} \longrightarrow \, ^{11}_{5}\text{B} + e^+ + \nu_e}$$

Carbon-11 decays to Boron-11.

(vi) β^+-decay of $^{97}_{43}$Tc

Same β^+ process: Z decreases by 1 to $Z = 42$.



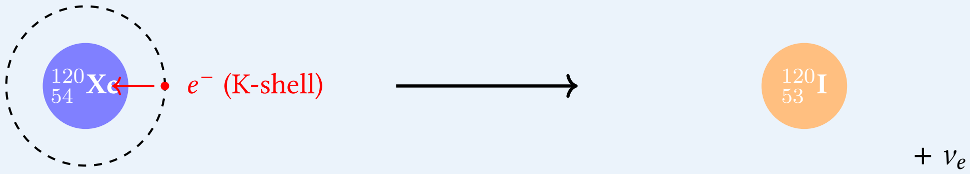
Answer (vi):

$$\boxed{^{97}_{43}\text{Tc} \longrightarrow \, ^{97}_{42}\text{Mo} + e^+ + \nu_e}$$

Technetium-97 decays to Molybdenum-97.

(vii) Electron capture of $^{120}_{54}$Xe

In electron capture, the nucleus captures an inner orbital electron (usually from K-shell). A proton combines with this electron to form a neutron and emit a neutrino. Z decreases by 1, A remains unchanged.



The Fundamental Process:

$$p + e^- \longrightarrow n + \nu_e$$

Answer (vii):

$$\boxed{^{120}_{54}\text{Xe} + e^- \longrightarrow {}^{120}_{53}\text{I} + \nu_e}$$

Xenon-120 captures an electron to become Iodine-120.

Summary of All Nuclear Reactions

S.No.	Decay Mode	Parent	Daughter	Emitted Particles
(i)	α	$^{226}_{88}$Ra	$^{222}_{86}$Rn	$^{4}_{2}$He
(ii)	α	$^{242}_{94}$Pu	$^{238}_{92}$U	$^{4}_{2}$He
(iii)	β^-	$^{32}_{15}$P	$^{32}_{16}$S	$e^- + \bar{\nu}_e$
(iv)	β^-	$^{210}_{83}$Bi	$^{210}_{84}$Po	$e^- + \bar{\nu}_e$
(v)	β^+	$^{11}_{6}$C	$^{11}_{5}$B	$e^+ + \nu_e$
(vi)	β^+	$^{97}_{43}$Tc	$^{97}_{42}$Mo	$e^+ + \nu_e$
(vii)	EC	$^{120}_{54}$Xe	$^{120}_{53}$I	ν_e (X-ray)

Expert's Solution — Karan Malhotra, B.Tech CSE, IIIT Allahabad

Decoding Nuclear Decay: The Conservation Laws
Writing nuclear equations becomes systematic once you master the underlying conservation principles:
Golden Rules for Nuclear Reactions:

1. **Conservation of Mass Number (A):** Sum of A on left = Sum of A on right
2. **Conservation of Charge (Z):** Sum of Z on left = Sum of Z on right
3. **Conservation of mass-energy:** Accounted for by the Q-value of the reaction

Particle "Zoo" — Quick Reference:

Particle	Symbol	Z	A
Alpha	${}^4_2$He or α	+2	4
Beta-minus (electron)	e^- or β^-	−1	0
Beta-plus (positron)	e^+ or β^+	+1	0
Neutrino	ν_e	0	0
Antineutrino	$\bar{\nu}_e$	0	0
Gamma photon	γ	0	0

The Neutrino Story:
Notice that β^- decay emits an antineutrino ($\bar{\nu}_e$) while β^+ decay and electron capture emit a neutrino (ν_e). This is due to **lepton number conservation**:

- β^-: $n \longrightarrow p + e^- + \bar{\nu}_e$ (creates an electron, so antineutrino balances lepton number)
- β^+: $p \longrightarrow n + e^+ + \nu_e$ (creates a positron/anti-electron, so neutrino balances)
- EC: $p + e^- \longrightarrow n + \nu_e$ (destroys an electron, so neutrino balances)

Electron Capture — The Silent Decay:
Electron capture is often harder to detect than other decays because only a neutrino and characteristic X-rays are emitted (when outer electrons fill the vacancy in the inner shell). It competes with β^+ decay when the energy difference between parent and daughter is less than 1.022 MeV (the threshold for pair production needed for positron emission).

Did You Know?

Quick Pattern Recognition:

- **α-decay:** Look for heavy nuclei ($A > 200$ typically). A changes, Z changes.
- **β^--decay:** Neutron-rich nuclei. A unchanged, Z increases by 1.
- **β^+-decay/EC:** Proton-rich nuclei. A unchanged, Z decreases by 1.
- **Mnemonic:** $\beta^- \longrightarrow$ ”minus” charge emitted $\longrightarrow$ nucleus becomes *more* positive ($Z \uparrow$). $\beta^+ \longrightarrow$ ”plus” charge emitted $\longrightarrow$ nucleus becomes *less* positive ($Z \downarrow$).

Q7 A radioactive isotope has a half-life of T years. How long will it take the activity to reduce to (a) 3.125%, (b) 1% of its original value?

Solution

Understanding Radioactive Decay
Radioactive decay follows first-order kinetics. The activity R of a radioactive sample at any time

t is related to its initial activity R_0 by the exponential decay law:

Radioactive Decay Law:

$$R = R_0 e^{-\lambda t}$$

where:

- R = activity at time t
- R_0 = initial activity (at $t = 0$)
- λ = decay constant
- t = time elapsed

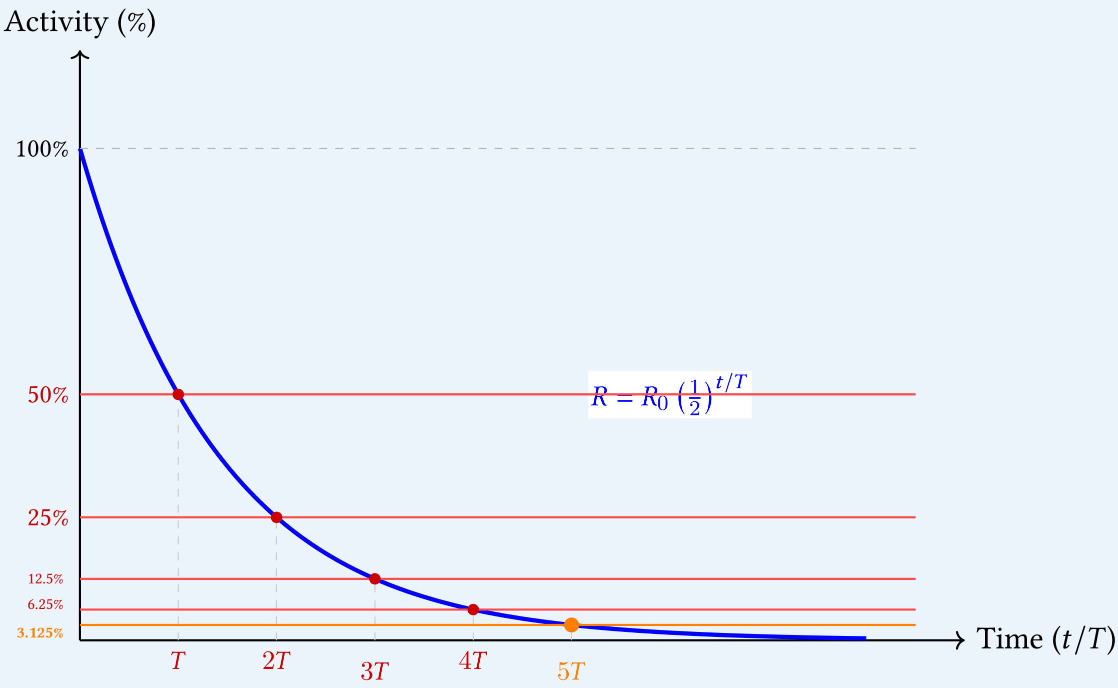
The decay constant λ is related to the half-life T by:

$$\lambda = \frac{\ln 2}{T} = \frac{0.693}{T}$$

Alternatively, we can use the half-life formulation directly:

$$R = R_0 \left(\frac{1}{2}\right)^{t/T}$$

where T is the half-life and t/T is the number of half-lives elapsed.



Part (a): Time to reduce to 3.125% of original activity
Method 1: Using the half-life formula
We need to find t such that:

$$\frac{R}{R_0} = \frac{3.125}{100} = 0.03125$$

Using $R = R_0 \left(\frac{1}{2}\right)^{t/T}$:

$$0.03125 = \left(\frac{1}{2}\right)^{t/T}$$

Step 1: Recognize the fraction as a power of 1/2
Let's check if 0.03125 is an exact power of $1/2$:

$$\left(\frac{1}{2}\right)^1 = 0.5$$
$$\left(\frac{1}{2}\right)^2 = 0.25$$
$$\left(\frac{1}{2}\right)^3 = 0.125$$
$$\left(\frac{1}{2}\right)^4 = 0.0625$$
$$\left(\frac{1}{2}\right)^5 = 0.03125 \quad \checkmark$$

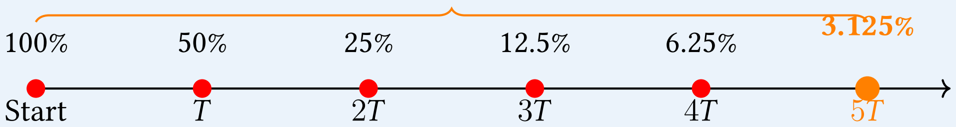
Indeed, $0.03125 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$
Step 2: Solve for t

$$\left(\frac{1}{2}\right)^{t/T} = \left(\frac{1}{2}\right)^5$$

Since the bases are equal:

$$\frac{t}{T} = 5$$
$$t = 5T$$

Answer: $t = 5T$



Method 2: Using logarithms (general approach)
Take natural logarithm of both sides:

$$\ln(0.03125) = \frac{t}{T} \ln\left(\frac{1}{2}\right)$$

$$\frac{t}{T} = \frac{\ln(0.03125)}{\ln(0.5)} = \frac{-3.4657}{-0.6931} = 5$$
$$t = 5T$$

Answer (a):

$$\boxed{t = 5T \text{ years}}$$

The activity reduces to 3.125% after exactly **5 half-lives**.

Part (b): Time to reduce to 1% of original activity
We need to find t such that:

$$\frac{R}{R_0} = \frac{1}{100} = 0.01$$

Step 1: Use the exponential decay formula

$$0.01 = \left(\frac{1}{2}\right)^{t/T}$$

Step 2: Take logarithms to solve
Taking $\log_{10}$ of both sides:

$$\log_{10}(0.01) = \frac{t}{T}\log_{10}\left(\frac{1}{2}\right)$$

$$\log_{10}(10^{-2}) = \frac{t}{T}\log_{10}(0.5)$$

$$-2 = \frac{t}{T} \times (-0.3010)$$

$$\frac{t}{T} = \frac{-2}{-0.3010} = \frac{2}{0.3010}$$

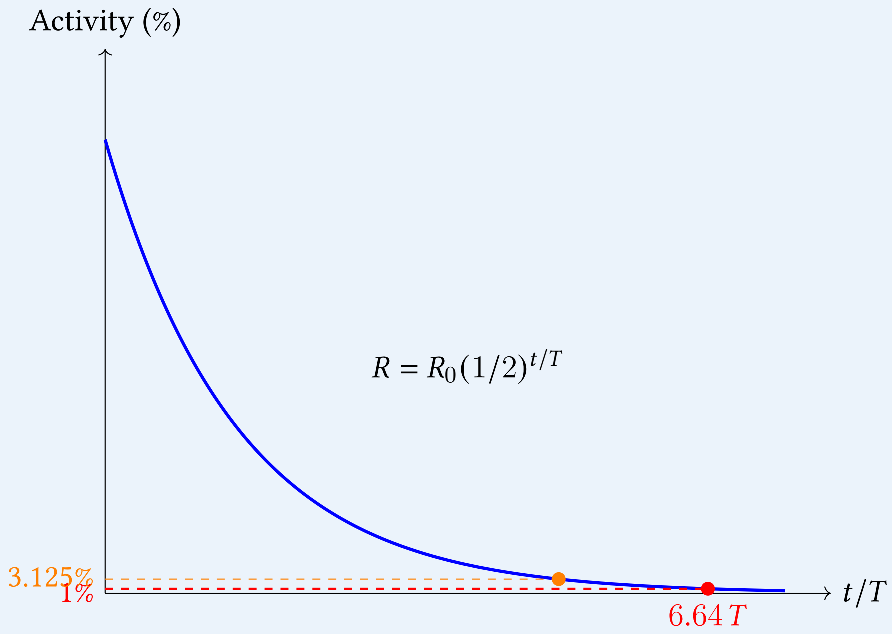
$$\frac{t}{T} = 6.6445$$

$$t = 6.6445\ T$$

Alternative using natural logarithms:

$$\frac{t}{T} = \frac{\ln(0.01)}{\ln(0.5)} = \frac{-4.6052}{-0.6931} = 6.6445$$

$$t = 6.64\ T$$



Understanding the Result:

- After 5 half-lives: activity = 3.125% of original
- After 6 half-lives: activity = 1.5625% of original
- To reach exactly 1%, we need approximately **6.64 half-lives**
- This falls between the 6th and 7th half-life

Answer (b):

$$\boxed{t = 6.64\,T \text{ years}}$$

The activity reduces to 1% of its original value after approximately **6.64 half-lives**, or more precisely:

$$t = \frac{\ln(100)}{\ln(2)} T = 6.64\,T$$

General Formula:
For any percentage $p\%$, the time required is:

$$t = T \times \frac{\ln(100/p)}{\ln(2)}$$

This formula is useful for solving similar problems directly.

Expert's Solution – Shivangi Tiwari, B.Tech CSE, NIT Raipur

The Power of Half-Life Thinking
The half-life concept is one of the most intuitive ways to understand exponential decay. Let me share some powerful shortcuts and insights.
The "Rule of Halving" — Quick Mental Math:
For any percentage that is a power of 1/2, the answer is immediate:

Percentage remaining	Number of half-lives
$50\% = (1/2)^1$	1 half-life
$25\% = (1/2)^2$	2 half-lives
$12.5\% = (1/2)^3$	3 half-lives
$6.25\% = (1/2)^4$	4 half-lives
$3.125\% = (1/2)^5$	5 half-lives
$1.5625\% = (1/2)^6$	6 half-lives

The "Rule of 70" for Radioactive Decay:
For any small percentage p, the approximate half-lives required is:

$$n \approx \frac{\ln(100/p)}{0.693} \approx 3.32 \times \log_{10}\left(\frac{100}{p}\right)$$

Common Time-to-Percentage Relationships

Final Activity	Time Required	Example (if $T = 5$ years)
50% (1/2)	$1.00\,T$	5 years
10% (1/10)	$3.32\,T$	16.6 years
5% (1/20)	$4.32\,T$	21.6 years
3.125% (1/32)	$5.00\,T$	25 years
1% (1/100)	$6.64\,T$	33.2 years
0.1% (1/1000)	$9.97\,T$	49.8 years

Why 3.125% is a Special Number:
Notice that 3.125% = 1/32. In nuclear physics and radiation safety, certain fractions appear frequently:

- 1/32 of original activity is often used as a benchmark for "sufficiently decayed"
- After 10 half-lives, activity drops to $(1/2)^{10} \approx 0.1\%$ — a commonly used threshold in nuclear medicine for considering a sample "safe"
- After 7 half-lives: $\sim 0.78\%$ — sometimes used as a rule of thumb for "essentially gone"

The Logarithm Connection:
For part (b), the exact expression is particularly elegant:

$$t = T \cdot \frac{\ln(100)}{\ln(2)} = T \cdot \log_2(100) = T \cdot \frac{\log_{10}(100)}{\log_{10}(2)} = T \cdot \frac{2}{0.3010}$$

★ Did You Know?

Exam Shortcut: Whenever you see a "nice" fraction like 3.125%, immediately check if it equals $(1/2)^n$:

- $3.125\% = 3.125/100 = 1/32 = (1/2)^5 \longrightarrow$ Answer: $5T$
- For non-power-of-2 fractions (like 1%), use: $t = T \times \frac{\ln(R_0/R)}{\ln 2}$

This pattern recognition can save precious minutes in exams!

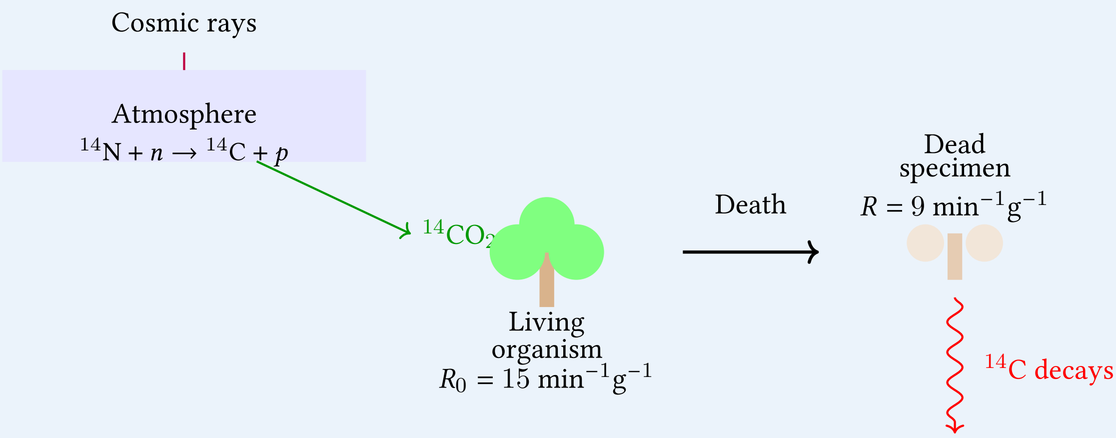
Q8 **The normal activity of living carbon-containing matter is found to be about 15 decays per minute for every gram of carbon. This activity arises from the small proportion of radioactive $^{14}_{6}C$ present with the stable carbon isotope $^{12}_{6}C$. When the organism is dead, its interaction with the atmosphere**

(which maintains the above equilibrium activity) ceases and its activity begins to drop. From the known half-life (5730 years) of $^{14}_{6}C$, and the measured activity, the age of the specimen can be approximately estimated. This is the principle of $^{14}_{6}C$ dating used in archaeology. Suppose a specimen from Mohenjodaro gives an activity of 9 decays per minute per gram of carbon. Estimate the approximate age of the Indus-Valley civilisation.

Solution

Understanding Radiocarbon Dating

Radiocarbon dating is one of the most powerful tools in archaeology. It relies on the radioactive decay of $^{14}_{6}C$, which is continuously produced in the upper atmosphere by cosmic ray bombardment and incorporated into living organisms through the carbon cycle.



Given Data:

- Initial activity (living organism): $R_0 = 15$ decays/min/g
- Present activity (specimen): $R = 9$ decays/min/g
- Half-life of $^{14}_{6}C$: $T_{1/2} = 5730$ years

Step 1: Determine the Decay Constant (λ)

Decay Constant:

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.693}{T_{1/2}}$$

$$\lambda = \frac{0.693}{5730} = 1.209 \times 10^{-4}\ \text{year}^{-1}$$

Step 2: Apply the Radioactive Decay Law

The activity decreases exponentially:

$$R = R_0 e^{-\lambda t}$$

Rearranging to solve for time t:

$$\frac{R}{R_0} = e^{-\lambda t}$$

$$e^{\lambda t} = \frac{R_0}{R}$$

$$\lambda t = \ln\left(\frac{R_0}{R}\right)$$

$$t = \frac{1}{\lambda}\ln\left(\frac{R_0}{R}\right)$$

Step 3: Substitute the Values

$$t = \frac{1}{1.209 \times 10^{-4}} \times \ln\left(\frac{15}{9}\right)$$

$$t = \frac{\ln(1.6667)}{1.209 \times 10^{-4}}$$

$$t = \frac{0.51083}{1.209 \times 10^{-4}}$$

$$t = 4225.2 \text{ years}$$

Alternative Method: Using Half-Life Directly

$$R = R_0\left(\frac{1}{2}\right)^{t/T_{1/2}}$$

$$\frac{9}{15} = \left(\frac{1}{2}\right)^{t/5730}$$

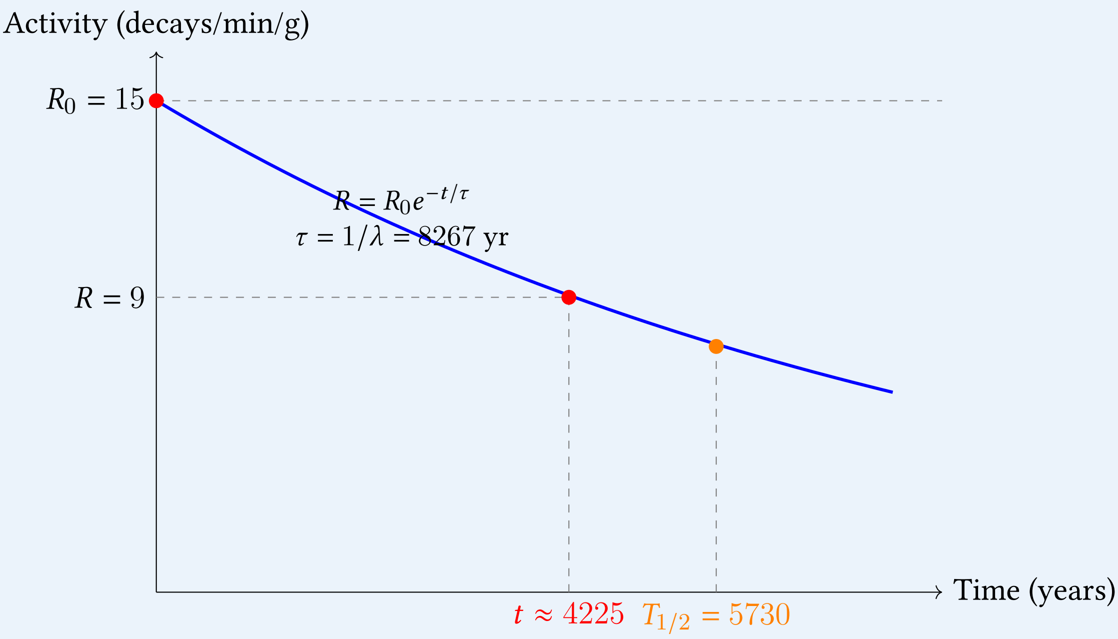
$$0.6 = \left(\frac{1}{2}\right)^{t/5730}$$

Taking natural logarithm:

$$\ln(0.6) = \frac{t}{5730}\ln(0.5)$$

$$t = 5730 \times \frac{\ln(0.6)}{\ln(0.5)} = 5730 \times \frac{-0.51083}{-0.69315}$$

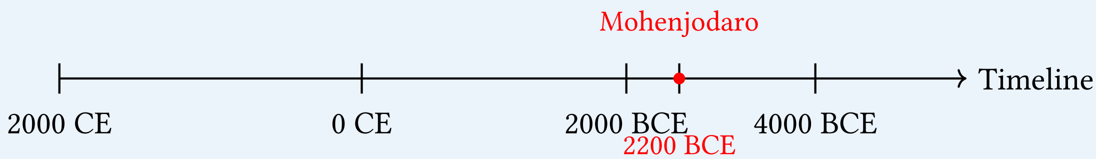
$$t = 5730 \times 0.73697 = 4223 \text{ years}$$



Step 4: Interpret the Result
The calculated age of approximately **4225 years** means the specimen dates back to around:

$$\text{Present year} - 4225 \approx 2025 - 4225 \approx \textbf{2200 BCE}$$

This places the Indus Valley Civilization specimen firmly in the Mature Harappan period (2600–1900 BCE).



Final Answer:

$$\boxed{t \approx 4.22 \times 10^3 \text{ years} \approx 4225 \text{ years}}$$

The approximate age of the Indus-Valley civilisation specimen is **about 4225 years**, corresponding to around **2200 BCE**.

Expert's Solution — Amit Verma, B.Tech CSE, NIT Jalandhar

The Science and History of Radiocarbon Dating
Radiocarbon dating, developed by Willard Libby in 1949 (earning him the 1960 Nobel Prize in Chemistry), revolutionised archaeology by providing an absolute dating method for organic materials up to about 50,000 years old.
The Carbon-14 Cycle:

How ^{14}C is Created and Incorporated:

1. Cosmic rays produce neutrons in the upper atmosphere
2. $^{14}N + n \longrightarrow {}^{14}C + p$ (neutron capture)
3. ^{14}C oxidizes to $^{14}CO_2$ within hours
4. Plants absorb $^{14}CO_2$ during photosynthesis
5. Animals eat plants, incorporating ^{14}C into their bodies
6. **Key:** While alive, the $^{14}C/^{12}C$ ratio remains constant

The Mathematical Beauty:
The mean lifetime τ (not half-life) appears naturally in the exponential:

$$\tau = \frac{1}{\lambda} = \frac{T_{1/2}}{\ln 2} = \frac{5730}{0.693} = 8267 \text{ years}$$

So the decay law can be written as:

$$R = R_0 e^{-t/8267}$$

Why 15 decays/min/g?
The specific activity of modern carbon (15 dpm/g) reflects several factors:

- The cosmic ray flux at Earth's surface
- The total carbon inventory in the biosphere and oceans
- The production rate of ^{14}C: about 7.5 kg/year globally
- The equilibrium ratio: $^{14}C/^{12}C \approx 1.3 \times 10^{-12}$

The Indus Valley Connection:

Historical Significance:

The Indus Valley Civilisation (also called Harappan Civilisation) flourished between approximately **3300 BCE and 1300 BCE**, with its mature phase from **2600 BCE to 1900 BCE**.
Mohenjodaro was one of its largest cities, located in present-day Sindh, Pakistan.
Our calculated date of ~2200 BCE places the specimen right in the **mature Harappan period**, consistent with archaeological evidence.

Did You Know?

Carbon Dating Formula Shortcut:

$$t = 8267 \times \ln\left(\frac{R_0}{R}\right) \text{ years}$$

where 8267 years is the **mean lifetime** of ^{14}C.
Quick sanity check:

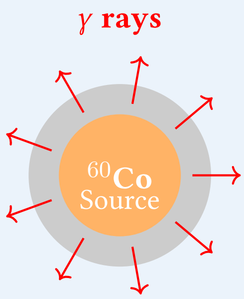
- If $R = R_0/2$ (half remains): $t = 8267 \times \ln 2 = 5730$ yr ✓
- If $R = 0.6R_0$ (our case): $t = 8267 \times \ln(1.667) \approx 4225$ yr

This formula with the mean lifetime is often faster than using the half-life directly!

Q9 **Obtain the amount of $^{60}_{27}$Co necessary to provide a radioactive source of 8.0 mCi strength. The half-life of $^{60}_{27}$Co is 5.3 years.**

Solution

Understanding the Problem
We need to find the mass of cobalt-60 required to create a source with a specified activity (strength) of 8.0 mCi. This requires connecting the macroscopic quantity (mass) with the microscopic decay rate (activity).



Given:
Activity $R = 8.0$ mCi
$T_{1/2} = 5.3$ years
Find: Mass m =?

Step 1: Recall the Fundamental Relationship

Activity-Rate Relation:

$$R = \lambda N$$

where:

- R = activity (decays per unit time)
- λ = decay constant
- N = number of radioactive nuclei present

The decay constant is related to half-life by:

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.693}{T_{1/2}}$$

And the number of nuclei N is related to mass m by:

$$N = \frac{m}{M} \times N_A$$

where M is the molar mass and $N_A = 6.022 \times 10^{23}$ mol^{-1} is Avogadro's number.

Step 2: Convert Activity to SI Units (Becquerel)
The curie (Ci) is a traditional unit of activity:

$$1 \text{ Ci} = 3.7 \times 10^{10} \text{ decays per second (Bq)}$$

Given activity:

$$R = 8.0 \text{ mCi} = 8.0 \times 10^{-3} \text{ Ci}$$
$$R = 8.0 \times 10^{-3} \times 3.7 \times 10^{10} \text{ Bq}$$
$$R = 2.96 \times 10^{8} \text{ Bq} = 2.96 \times 10^{8} \text{ decays/second}$$

Unit	Symbol	Value in Bq
1 Curie	1 Ci	3.7×10^{10} Bq
1 millicurie	1 mCi	3.7×10^{7} Bq
1 microcurie	1 μCi	3.7×10^{4} Bq

Step 3: Calculate the Decay Constant
First convert half-life to seconds for consistency:

$$T_{1/2} = 5.3 \text{ years}$$

$$\begin{aligned} T_{1/2} &= 5.3 \times 365 \times 24 \times 3600 \\ &= 5.3 \times 3.1536 \times 10^{7} \\ &= 1.6714 \times 10^{8} \text{ seconds} \end{aligned}$$

Now calculate λ:

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.693}{1.6714 \times 10^8}$$
$$\lambda = 4.146 \times 10^{-9}\ \text{s}^{-1}$$

Step 4: Calculate the Number of Cobalt-60 Nuclei Required
From $R = \lambda N$:

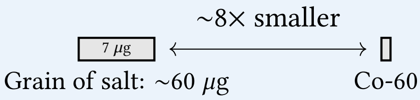
$$N = \frac{R}{\lambda} = \frac{2.96 \times 10^8}{4.146 \times 10^{-9}}$$
$$N = 7.139 \times 10^{16}\ \text{nuclei}$$

Step 5: Convert Number of Nuclei to Mass
Mass number of ^{60}Co is 60, so molar mass $M \approx 60$ g/mol.

$$m = \frac{N}{N_A} \times M$$
$$m = \frac{7.139 \times 10^{16}}{6.022 \times 10^{23}} \times 60\ \text{g}$$
$$m = 1.1856 \times 10^{-7} \times 60\ \text{g}$$
$$m = 7.114 \times 10^{-6}\ \text{g}$$

In more convenient units:

$$m = 7.11 \times 10^{-6}\ \text{g} = 7.11\ \mu\text{g}$$



Alternative Approach: Work in Years Instead of Seconds
Since the half-life is given in years, we can avoid converting to seconds by using consistent units. Let's express activity in decays/year:

$$R = 2.96 \times 10^8\ \text{Bq} = 2.96 \times 10^8 \times (3.1536 \times 10^7)\ \text{decays/year}$$
$$R = 9.335 \times 10^{15}\ \text{decays/year}$$

Decay constant in year^{-1}:

$$\lambda = \frac{0.693}{5.3} = 0.13075\ \text{year}^{-1}$$

$$N = \frac{R}{\lambda} = \frac{9.335 \times 10^{15}}{0.13075} = 7.139 \times 10^{16}\ \text{nuclei}$$

This gives the same result, confirming our calculation.

Final Answer:

$$\boxed{m = 7.1 \times 10^{-6}\ \text{g} = 7.1\ \mu\text{g}}$$

Only about **7 micrograms** of cobalt-60 are needed to provide a radioactive source of 8.0 mCi strength. This illustrates the incredible specific activity of radioactive materials — a nearly invisible speck contains billions of decays per second!

Expert's Solution — Ravi Shankar, B.Tech Electrical Engineering, IIT Dhanbad

The Astounding Specific Activity of Radioisotopes

This problem reveals one of the most remarkable aspects of radioactivity: the extraordinarily high specific activity (activity per unit mass) of pure radioisotopes.

Calculating Specific Activity of Pure ^{60}Co:

$$\text{Specific Activity} = \frac{R}{m} = \frac{2.96 \times 10^8 \text{ Bq}}{7.11 \times 10^{-6} \text{ g}} = 4.16 \times 10^{13} \text{ Bq/g} = 1125 \text{ Ci/g}$$

This means just **1 gram of pure ^{60}Co** has an activity of over **1100 curies** — an enormously intense source!

Specific Activities of Common Radioisotopes:

Isotope	Half-life	Specific Activity	Application
^{60}Co	5.3 yr	~1100 Ci/g	Radiotherapy, sterilization
^{137}Cs	30 yr	~87 Ci/g	Medical, industrial
^{192}Ir	74 days	~9200 Ci/g	Industrial radiography
^{99m}Tc	6 hr	~5.2×10^6 Ci/g	Medical imaging

The Inverse Relationship with Half-Life:

Specific activity is inversely proportional to half-life:

$$\text{Specific Activity} \propto \frac{1}{T_{1/2}}$$

This makes intuitive sense: shorter-lived isotopes have higher specific activities because their nuclei decay faster.

Practical Significance:

- **Radiotherapy:** Cobalt-60 sources of ~200 Ci (about 0.18 g of Co-60) were standard in external beam radiotherapy machines (the famous "cobalt bombs") for decades.
- **Source Replacement:** Because of the 5.3-year half-life, ^{60}Co therapy sources need replacement every 5-7 years to maintain adequate dose rates.
- **Sterilization:** Large ^{60}Co sources (megacurie levels) are used in industrial irradiation facilities for sterilizing medical equipment and food preservation.

Radiation Safety Perspective:

The tiny mass involved (~7 μg for 8 mCi) highlights why radioactive materials are handled with extreme care. An invisible speck can deliver significant radiation doses. This is why:

- Sealed sources are doubly encapsulated in stainless steel
- Remote handling tools and hot cells are used
- Strict inventory controls track microgram quantities

Did You Know?

Quick Formula for Source Mass Calculation:

$$m\ (\text{grams}) = \frac{R\ (\text{Ci}) \times M\ (\text{g/mol}) \times T_{1/2}\ (\text{seconds})}{0.693 \times N_A \times 3.7 \times 10^{10}}$$

Or in a simplified form for quick estimates:

$$m\ (\mu\text{g}) \approx \frac{R\ (\text{mCi}) \times M \times T_{1/2}\ (\text{yr})}{8.5 \times 10^4}$$

Check for our problem:

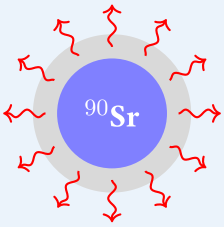
$$m \approx \frac{8.0 \times 60 \times 5.3}{8.5 \times 10^4} = \frac{2544}{8.5 \times 10^4} \approx 7.2\ \mu\text{g}\ \checkmark$$

Q10 The half-life of $^{90}_{38}$Sr is 28 years. What is the disintegration rate of 15 mg of this isotope?

Solution

Understanding the Problem

This is the reverse of the previous problem — we are given the mass of a radioactive isotope and need to find its activity (disintegration rate). Strontium-90 is a significant fission product with important environmental and health implications.



Given:
Mass m = 15 mg = 15×10^{-3} g
$T_{1/2}$ = 28 years
Molar mass $M \approx 90$ g/mol
Find: Activity R =?

Step 1: Calculate the Number of Strontium-90 Nuclei

$$N = \frac{m}{M} \times N_A$$

$$N = \frac{15 \times 10^{-3}\ \text{g}}{90\ \text{g/mol}} \times 6.022 \times 10^{23}\ \text{mol}^{-1}$$

$$N = \frac{15 \times 10^{-3}}{90} \times 6.022 \times 10^{23}$$

$$N = 1.667 \times 10^{-4} \times 6.022 \times 10^{23}$$

$$N = 1.004 \times 10^{20} \text{ nuclei}$$

Scale Check:
15 mg of ^{90}Sr contains approximately 10^{20} **atoms**
This is 100 billion billion atoms — a huge number, which is why even small masses have significant activity.

Step 2: Calculate the Decay Constant
Convert half-life from years to seconds:

$$T_{1/2} = 28 \text{ years}$$

$$\begin{aligned} T_{1/2} &= 28 \times 365 \times 24 \times 3600 \\ &= 28 \times 3.1536 \times 10^{7} \\ &= 8.830 \times 10^{8} \text{ seconds} \end{aligned}$$

Now calculate λ:

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{0.693}{8.830 \times 10^{8}}$$

$$\lambda = 7.848 \times 10^{-10} \text{ s}^{-1}$$

Step 3: Calculate the Activity (Disintegration Rate)

Activity:

$$R = \lambda N$$

$$R = 7.848 \times 10^{-10} \times 1.004 \times 10^{20}$$

$$R = 7.879 \times 10^{10} \text{ Bq}$$

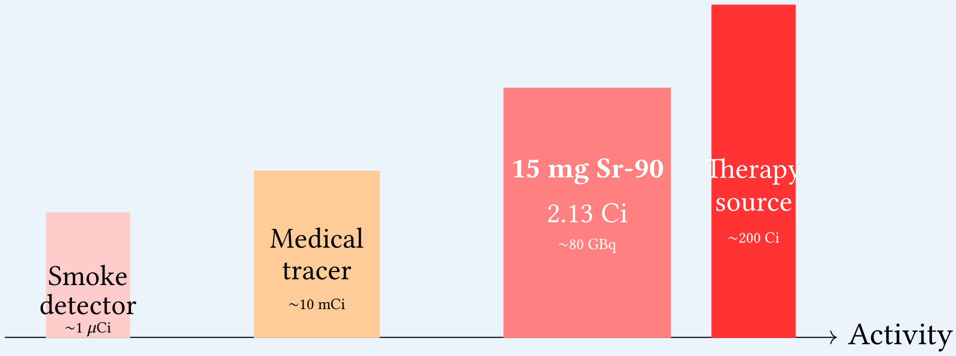
Step 4: Express in Curies for Practical Context

$$R = \frac{7.879 \times 10^{10} \text{ Bq}}{3.7 \times 10^{10} \text{ Bq/Ci}}$$

$$R = 2.129 \text{ Ci}$$

Or in more common units:

$$R = 2.13 \times 10^{3} \text{ mCi}$$



Alternative: Direct Formula Using Specific Activity
For ^{90}Sr, we can calculate the specific activity first:

$$\text{Specific Activity} = \frac{\lambda N_A}{M} = \frac{0.693 \times 6.022 \times 10^{23}}{28 \times 3.1536 \times 10^7 \times 90}$$

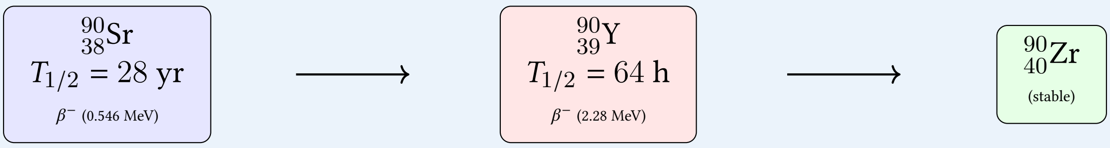
$$\text{Specific Activity} \approx 5.25 \times 10^{12} \text{ Bq/g} = 142 \text{ Ci/g}$$

For 15 mg = 0.015 g:

$$R = 0.015 \times 5.25 \times 10^{12} = 7.88 \times 10^{10} \text{ Bq} = 2.13 \text{ Ci}$$

Environmental Context:
Strontium-90 is one of the most hazardous fission products because:

- It has a relatively long half-life (28 years)
- It is chemically similar to calcium and gets deposited in bones
- It emits energetic beta particles ($E_{\text{max}} = 0.546$ MeV)
- Its daughter ^{90}Y (yttrium-90) has a short half-life (64 hours) and emits even more energetic betas ($E_{\text{max}} = 2.28$ MeV)



Final Answer:

$$\boxed{R = 7.88 \times 10^{10} \text{ Bq} = 2.13 \text{ Ci} = 2.13 \times 10^3 \text{ mCi}}$$

The disintegration rate of 15 mg of $^{90}_{38}$Sr is approximately **78.8 GBq** or **2.13 curies** — a very substantial activity from just a tiny mass of material.

Expert's Solution — Pooja Kumari, B.Tech CSE, NIT Patna

Strontium-90: The Bone-Seeker
^{90}Sr holds a special (and dangerous) place in nuclear science due to its biological behavior and prevalence as a fission product.

Fission Yield and Environmental Presence:

Key Facts about ^{90}Sr:	
Fission yield (from ^{235}U)	∼5.8%
Produced in nuclear reactors	∼1 kg per 1000 MW(e) per ye
Released in atmospheric nuclear tests (1945-1963)	∼600 PBq (16 MCi)
Biological half-life in bone	∼18 years
Effective half-life	∼11 years (combining physical + bi

The "Bone-Seeker" Problem:
Strontium is in Group 2 of the periodic table, directly below calcium. Because of their chemical similarity:

- The body incorporates strontium into bone matrix in place of calcium
- Once deposited in bone, it remains there for decades
- The beta radiation damages bone marrow, potentially causing leukemia and bone cancer
- Children are particularly vulnerable due to active bone growth

Activity Per Mass — The Inverse Square of Half-Life:
Compare ^{90}Sr with other isotopes of similar mass:

$$\text{Specific Activity} \propto \frac{1}{T_{1/2}}$$

Isotope	$T_{1/2}$	**Specific Activity**	**Relative to ^{90}Sr**
^{90}Sr	28 yr	142 Ci/g	1×
^{89}Sr	50.5 days	28,000 Ci/g	197×
^{60}Co	5.3 yr	1,100 Ci/g	7.7×
^{137}Cs	30 yr	87 Ci/g	0.61×

Medical Use — A Silver Lining:
Interestingly, ^{89}Sr (a shorter-lived isotope, $T_{1/2} = 50.5$ days) is actually used in medicine as **Metastron** for palliative treatment of bone metastases. The principle is the same bone-seeking property, but now used to deliver targeted radiation to cancer sites in bone.

⭐ **Did You Know?**

Quick Calculation Check:
For any pure radioisotope, the activity per gram can be estimated as:

$$R\ (\text{Ci/g}) \approx \frac{1.3\times 10^5}{T_{1/2}\ (\text{years}) \times M\ (\text{g/mol})}$$

For ^{90}Sr:

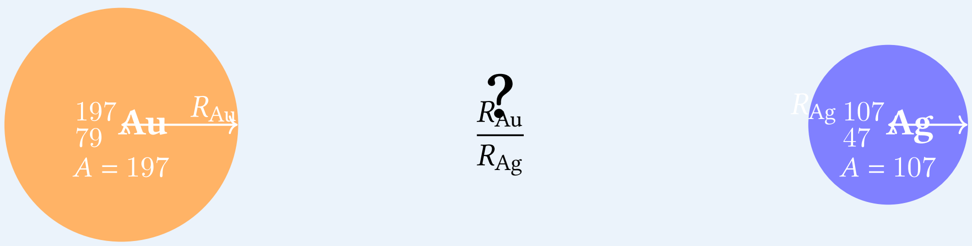
$$R \approx \frac{1.3\times 10^5}{28\times 90} = \frac{1.3\times 10^5}{2520} \approx 52\ \text{Ci/g}$$

But wait — our calculated value was 142 Ci/g. The discrepancy arises because this rough formula assumes $A \approx M$, but the exact calculation uses the actual decay constant. Always use the exact formula: $R = \lambda N_A/M$.

Q11 Obtain approximately the ratio of the nuclear radii of the gold isotope $^{197}_{79}$Au and the silver isotope $^{107}_{47}$Ag.

Solution

Understanding Nuclear Size
The nucleus, unlike the atom, does not have a sharp boundary. However, extensive experimental studies (using electron scattering, α-particle scattering, etc.) have established that the nuclear radius follows a remarkably simple relationship with the mass number A.



Step 1: Recall the Nuclear Radius Formula

Nuclear Radius:

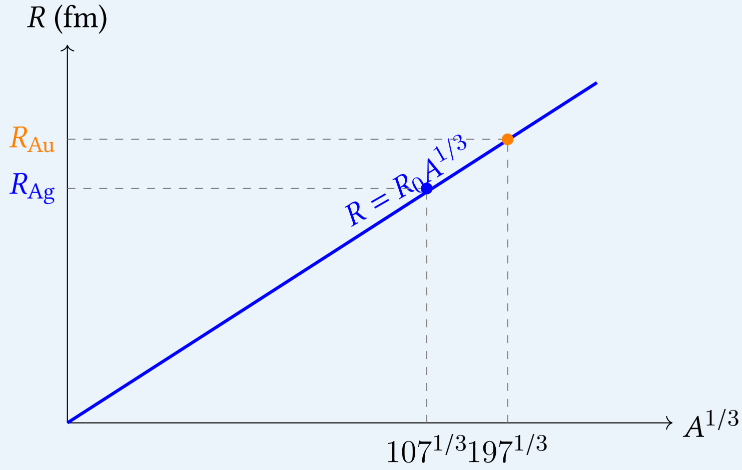
$$R = R_0 A^{1/3}$$

where:

- R = radius of the nucleus
- $R_0 = 1.2 \times 10^{-15}$ m = 1.2 fm (an empirical constant)
- A = mass number (total number of nucleons)

This formula emerges from the fact that:

- Nuclear density is approximately constant for all nuclei
- Nuclear volume is proportional to the number of nucleons: $V \propto A$
- Since $V = \frac{4}{3}\pi R^3$, we get $R \propto A^{1/3}$



Step 2: Set Up the Ratio
For gold ($A_{Au} = 197$):

$$R_{Au} = R_0 \times (197)^{1/3}$$

For silver ($A_{Ag} = 107$):

$$R_{Ag} = R_0 \times (107)^{1/3}$$

Taking the ratio:

$$\frac{R_{Au}}{R_{Ag}} = \frac{R_0 \times (197)^{1/3}}{R_0 \times (107)^{1/3}}$$

The constant R_0 cancels out perfectly:

$$\frac{R_{Au}}{R_{Ag}} = \left(\frac{197}{107}\right)^{1/3}$$

Step 3: Calculate the Numerical Value

$$\frac{197}{107} = 1.8411$$

Now take the cube root:

$$\left(\frac{197}{107}\right)^{1/3} = (1.8411)^{1/3}$$

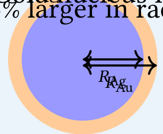
We can estimate this:

$$\begin{aligned}(1.8411)^{1/3} &\approx e^{\frac{1}{3}\ln(1.8411)} \\ &= e^{\frac{1}{3}\times 0.6103} \\ &= e^{0.2034} \\ &\approx 1.2255\end{aligned}$$

More precisely:

$$1.2^3 = 1.728 \quad ; \quad 1.23^3 = 1.860 \quad ; \quad 1.22^3 = 1.816$$
$$(1.8411)^{1/3} \approx 1.226$$

$$\frac{R_{\text{Au}}}{R_{\text{Ag}}} \approx 1.23$$



Step 4: Physical Interpretation

Despite gold having nearly **twice** the mass number of silver (197 vs 107), its nuclear radius is only about **23% larger**. This is a direct consequence of the $A^{1/3}$ dependence:

$$\text{Volume ratio} = \left(\frac{R_{\text{Au}}}{R_{\text{Ag}}}\right)^3 = \frac{197}{107} \approx 1.84$$

So gold's nucleus has about **1.84 times the volume** but only **1.23 times the radius** of silver's nucleus. This illustrates how nucleons pack together at nearly constant density.

Constant Nuclear Density:

If density $\rho = \frac{A \times m_{\text{nucleon}}}{(4/3)\pi R^3} = \frac{A \times m_{\text{nucleon}}}{(4/3)\pi R_0^3 A} = \frac{m_{\text{nucleon}}}{(4/3)\pi R_0^3}$

$\Rightarrow$ **Nuclear density is independent of A!**

$\rho \approx 2.3 \times 10^{17}$ kg/m^3 — an incredibly high, constant value.

Final Answer:

$$\boxed{\frac{R_{\text{Au}}}{R_{\text{Ag}}} = \left(\frac{197}{107}\right)^{1/3} \approx 1.23}$$

The nuclear radius of gold-197 is approximately **1.23 times** (or 23% larger than) that of silver-107. The radius scales as $A^{1/3}$, reflecting the constant density of nuclear matter.

Expert's Solution — Sandeep Kumar, B.Tech CSE, NIT Hamirpur

The $A^{1/3}$ Law: A Window into Nuclear Structure
The $R = R_0 A^{1/3}$ relationship is one of the most important empirical laws in nuclear physics. Let's explore its origins and implications.
Why $A^{1/3}$? The Constant Density Argument:

The Logic Chain:

1. Each nucleon occupies roughly the same volume in any nucleus
2. Total volume $V \propto$ Number of nucleons A
3. $V = \frac{4}{3}\pi R^3 \propto A$
4. Therefore: $R^3 \propto A \implies R \propto A^{1/3}$

The Constant R_0:
Different experiments yield slightly different values:

- From electron scattering: $R_0 \approx 1.2$ fm
- From α-decay systematics: $R_0 \approx 1.4$ fm
- From muonic X-rays: $R_0 \approx 1.2$ fm

The commonly accepted value is $R_0 = 1.20 \pm 0.02$ fm.
Cube Root Estimation Trick:
For quick calculations, knowing a few cube roots helps:

$A^{1/3}$	A
1.0	1
1.26	2
1.44	3
1.71	5
2.15	10

For our problem:

$$\frac{197}{107} \approx 1.84 \approx \frac{1.84}{1} \approx 1.84$$

Since $\sqrt[3]{1.84}$ is close to $\sqrt[3]{1.728} = 1.2$ and a bit more, estimate ≈ 1.23.
Practical Implications:

- **Heavy nuclei are not much bigger:** Uranium ($A = 238$) has only about 6 times the radius of helium ($A = 4$), despite having 60 times more nucleons.
- **Cross-sections:** Nuclear reaction probabilities scale with $\pi R^2 \propto A^{2/3}$, explaining why heavy nuclei are better targets.
- **Fission barrier:** The balance between surface energy ($\propto A^{2/3}$) and Coulomb energy ($\propto Z^2/A^{1/3}$) determines fission stability.

Did You Know?

General Formula for Nuclear Radius Ratio:

$$\frac{R_1}{R_2} = \left(\frac{A_1}{A_2}\right)^{1/3}$$

Key Insight: The radius ratio depends **only on the mass numbers**, not on the atomic numbers or any other property. Two nuclei with the same A (isobars) have nearly identical radii, regardless of their proton/neutron composition.
Common cube roots to remember:

$$\sqrt[3]{2} \approx 1.26 \quad ; \quad \sqrt[3]{3} \approx 1.44 \quad ; \quad \sqrt[3]{10} \approx 2.15$$

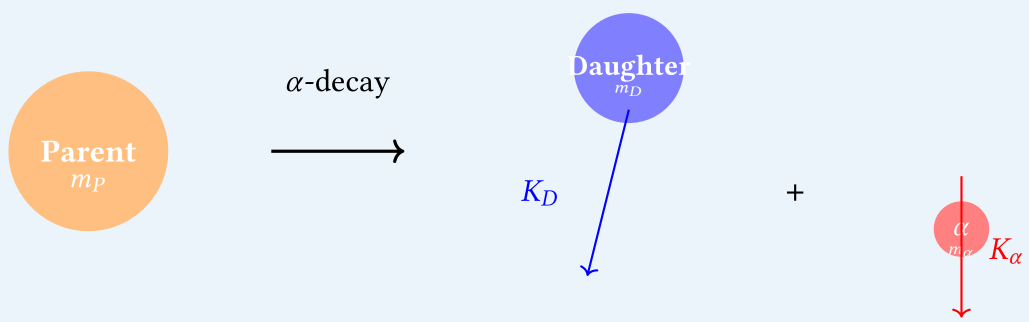
Q12 **Find the Q-value and the kinetic energy of the emitted α-particle in the α-decay of (a) ${}^{226}_{88}$Ra and (b) ${}^{220}_{86}$Rn.**

Given: $m\left({}^{226}_{88}\text{Ra}\right) = 226.02540$ **u,** $m\left({}^{222}_{86}\text{Rn}\right) = 222.01750$ **u,** $m\left({}^{220}_{86}\text{Rn}\right) = 220.01137$ **u,** $m\left({}^{216}_{84}\text{Po}\right) = 216.00189$ **u.**

Solution

Understanding Q-value and α-particle Kinetic Energy

The Q-value of a nuclear reaction is the net energy released. For spontaneous α-decay, $Q > 0$ is required for the decay to be energetically possible. The Q-value is shared between the α-particle and the daughter nucleus as kinetic energy.



Key Formulas:

Q-value for α-decay:

$$Q = [m_P - (m_D + m_\alpha)] \times c^2$$

where m_P = mass of parent, m_D = mass of daughter, $m_\alpha = m({}^4_2\text{He}) = 4.002603$ u

Kinetic Energy of α-particle:
From conservation of momentum ($p_\alpha = p_D$) and energy ($Q = K_\alpha + K_D$):

$$K_\alpha = \frac{m_D}{m_D + m_\alpha} \times Q \approx \frac{A-4}{A} \times Q$$

where A is the mass number of the parent.

Part (a): α-decay of $^{226}_{88}$Ra

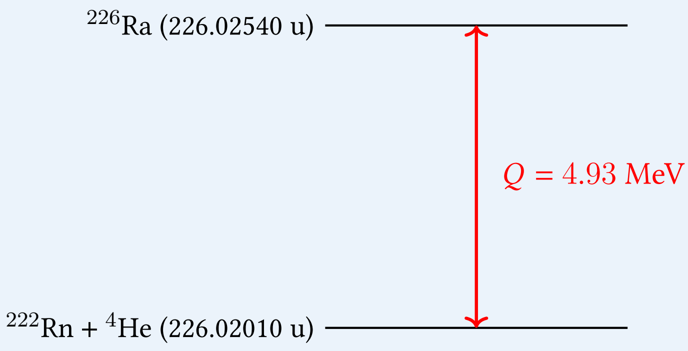
Decay Equation:

$$^{226}_{88}\text{Ra} \longrightarrow {}^{222}_{86}\text{Rn} + {}^{4}_{2}\text{He}$$

Step 1: Calculate the Q-value

$$Q = \left[m\left(^{226}_{88}\text{Ra}\right) - m\left(^{222}_{86}\text{Rn}\right) - m\left(^{4}_{2}\text{He}\right)\right] \times 931.5 \text{ MeV/u}$$

$$Q = [226.02540 - 222.01750 - 4.002603] \times 931.5$$
$$Q = [0.005297] \times 931.5$$
$$Q = 4.934 \text{ MeV}$$

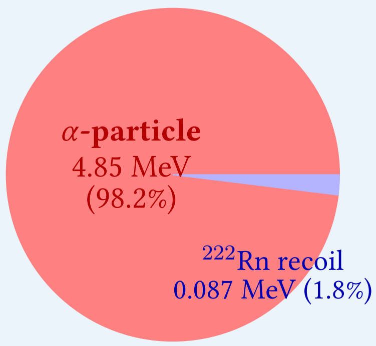


Step 2: Calculate the Kinetic Energy of the α-particle
Using the recoil formula:

$$K_\alpha = \frac{m_D}{m_D + m_\alpha} \times Q$$

Since masses are approximately proportional to mass numbers:

$$K_\alpha \approx \frac{A-4}{A} \times Q = \frac{222}{226} \times 4.934$$
$$K_\alpha \approx 0.9823 \times 4.934 = 4.847 \text{ MeV}$$



More Precise Calculation Using Actual Masses:

$$K_\alpha = \frac{222.01750}{222.01750 + 4.002603} \times 4.934 = \frac{222.01750}{226.02010} \times 4.934$$

$$K_\alpha = 0.98229 \times 4.934 = 4.847 \text{ MeV}$$

The daughter nucleus receives the remaining kinetic energy:

$$K_{\text{Rn}} = Q - K_\alpha = 4.934 - 4.847 = 0.087 \text{ MeV}$$

Answer (a):

$$\boxed{Q = 4.93 \text{ MeV} \quad ; \quad K_\alpha = 4.85 \text{ MeV}}$$

The α-particle carries about 98.2% of the total energy released.

Part (b): α-decay of ${}^{220}_{86}\text{Rn}$

Decay Equation:

$${}^{220}_{86}\text{Rn} \longrightarrow {}^{216}_{84}\text{Po} + {}^{4}_{2}\text{He}$$

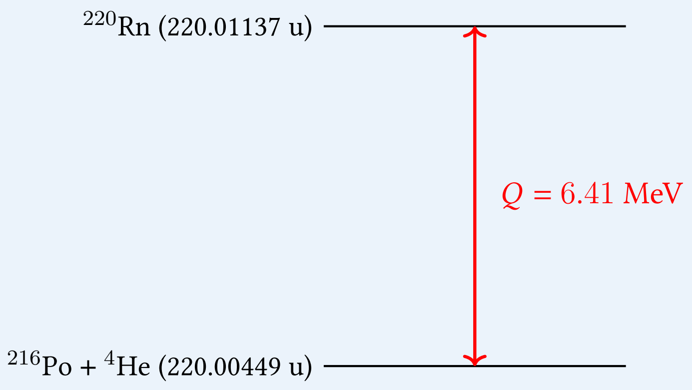
Step 1: Calculate the Q-value

$$Q = \left[m\left({}^{220}_{86}\text{Rn}\right) - m\left({}^{216}_{84}\text{Po}\right) - m\left({}^{4}_{2}\text{He}\right)\right] \times 931.5$$

$$Q = [220.01137 - 216.00189 - 4.002603] \times 931.5$$

$$Q = [0.006877] \times 931.5$$

$$Q = 6.406 \text{ MeV}$$



Step 2: Calculate the Kinetic Energy of the α-particle

$$K_\alpha = \frac{m_D}{m_D + m_\alpha} \times Q$$

Using approximate mass numbers:

$$K_\alpha \approx \frac{A-4}{A} \times Q = \frac{216}{220} \times 6.406$$

$$K_\alpha \approx 0.9818 \times 6.406 = 6.290 \text{ MeV}$$

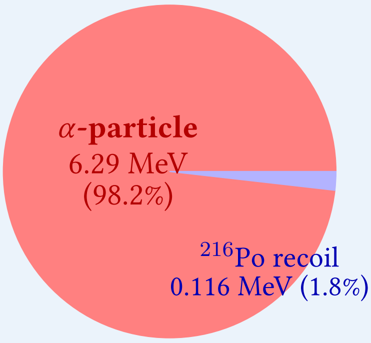
Precise calculation:

$$K_\alpha = \frac{216.00189}{216.00189 + 4.002603} \times 6.406 = \frac{216.00189}{220.00449} \times 6.406$$

$$K_\alpha = 0.98181 \times 6.406 = 6.290 \text{ MeV}$$

Recoil energy of polonium:

$$K_{\text{Po}} = Q - K_\alpha = 6.406 - 6.290 = 0.116 \text{ MeV}$$



Comparison of the Two Decays:

Parent	Daughter	Q (MeV)	K_α (MeV)	K_α/Q (%)
$^{226}_{88}$Ra	$^{222}_{86}$Rn	4.93	4.85	98.2%
$^{220}_{86}$Rn	$^{216}_{84}$Po	6.41	6.29	98.2%

Answer (b):

$$\boxed{Q = 6.41 \text{ MeV} \quad ; \quad K_\alpha = 6.29 \text{ MeV}}$$

The α-particle from ^{220}Rn decay is more energetic (6.29 MeV) than that from ^{226}Ra (4.85 MeV), reflecting the higher Q-value.

Expert's Solution — Deepika Patel, B.Tech CSE, NIT Bhopal

Why Most Energy Goes to the α-Particle

The sharing of energy between the α-particle and the daughter nucleus follows directly from conservation of momentum.

The Momentum Argument:

Initially, the parent nucleus is at rest ($p_{\text{total}} = 0$). After decay:

$$p_\alpha + p_D = 0 \quad \Rightarrow \quad |p_\alpha| = |p_D|$$

Kinetic energy in terms of momentum:

$$K_\alpha = \frac{p_\alpha^2}{2m_\alpha} \quad ; \quad K_D = \frac{p_D^2}{2m_D}$$

Since momenta are equal:

$$\frac{K_\alpha}{K_D} = \frac{m_D}{m_\alpha}$$

For ^{226}Ra: $\frac{K_\alpha}{K_{Rn}} = \frac{222}{4} = 55.5$
This means the α-particle gets 55.5 times more kinetic energy than the heavy daughter!
Fraction of Energy Carried by α-particle:

$$\frac{K_\alpha}{Q} = \frac{K_\alpha}{K_\alpha + K_D} = \frac{m_D}{m_D + m_\alpha} \approx \frac{A-4}{A}$$

The Geiger-Nuttall Law:

There is a remarkable empirical relationship between the α-particle energy and the half-life for α-decay:

$$\log_{10}(T_{1/2}) \propto \frac{Z}{\sqrt{K_\alpha}}$$

A small change in K_α produces a huge change in half-life!

Nuclide	K_α (MeV)	$T_{1/2}$
^{226}Ra	4.85	1600 years
^{220}Rn	6.29	55.6 seconds

Despite only 30% higher α energy, ^{220}Rn decays **a billion times faster**!

Why α-Particles Have Discrete Energies:
Unlike β-particles (which share energy with neutrinos and have continuous spectra), α-particles from a given decay have **discrete, monoenergetic** energies. This is because:

- Only two particles in the final state (daughter + α)
- Conservation of energy and momentum uniquely determine both energies
- No third particle to share energy randomly

This sharp energy spectrum makes α-spectroscopy a powerful tool for identifying radioactive isotopes.

Did You Know?

Quick K_α Formula:

$$K_\alpha = \frac{A-4}{A} \times Q$$

This approximation using mass numbers (instead of actual masses) is accurate to within 0.1% for heavy nuclei.

Quick Q-value Check: For α-decay to occur spontaneously:

$$m_P > m_D + m_\alpha \quad \Rightarrow \quad Q > 0$$

If you calculate a negative Q, the decay is energetically forbidden (though it might occur via other modes like β-decay).

Q13 The radionuclide ^{11}C decays according to:

$$^{11}_{6}\mathbf{C} \longrightarrow {}^{11}_{5}\mathbf{B} + e^+ + \nu \quad ; \quad T_{1/2} = 20.3 \textbf{ min}$$

The maximum energy of the emitted positron is 0.960 MeV. Given the mass values: $m\left(^{11}_{6}\mathbf{C}\right) = 11.011434$ u and $m\left(^{11}_{5}\mathbf{B}\right) = 11.009305$ u, calculate Q and compare it with the maximum energy of the positron emitted.

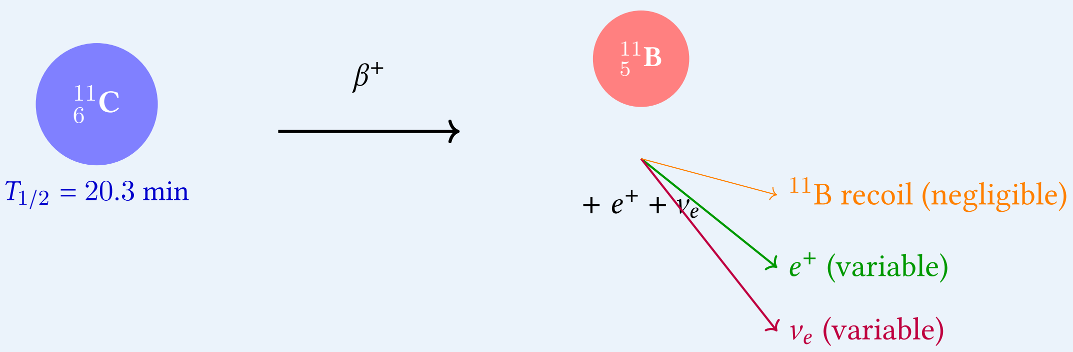
Solution

Understanding β^+ Decay and Q-value

In β^+ (positron) decay, a proton in the nucleus converts into a neutron, emitting a positron and a neutrino:

$$p \longrightarrow n + e^+ + \nu_e$$

The Q-value represents the total energy released in the decay. This energy is shared between the positron, the neutrino, and a tiny recoil energy of the daughter nucleus. However, the emitted positron does **not** always get the full Q-value.



Step 1: Calculate the Q-value for β^+ Decay
For β^+ decay, the nuclear reaction is:

$$^{11}_{6}\text{C}_{\text{nucleus}} \longrightarrow {}^{11}_{5}\text{B}_{\text{nucleus}} + e^+ + \nu_e$$

However, the given masses are **atomic masses**, which include electrons:

- $^{11}_{6}$C atom has 6 electrons
- $^{11}_{5}$B atom has 5 electrons

The Mass Balance with Atomic Masses:

Q-value for β^+ decay (using atomic masses):

$$Q = [m_{\text{atom}}(P) - m_{\text{atom}}(D) - 2m_e] \times c^2$$

where:

- $m_{\text{atom}}(P)$ = atomic mass of parent
- $m_{\text{atom}}(D)$ = atomic mass of daughter
- $m_e = 0.0005486$ u (mass of electron)
- The factor $2m_e$ accounts for: the positron emitted (m_e) AND the extra electron in the parent atom not present in the daughter atom (m_e)

Why $2m_e$?

- Parent atom has Z electrons, daughter atom has $Z - 1$ electrons
- The positron created carries away m_e of mass
- So net mass difference includes $2m_e$

Electron Accounting:
Parent (C) atom: 6 electrons
Daughter (B) atom: 5 electrons
Difference: 1 extra electron in parent atom mass
Plus: 1 positron mass created (= $1m_e$)
Total: $2m_e$ must be subtracted
from atomic mass difference

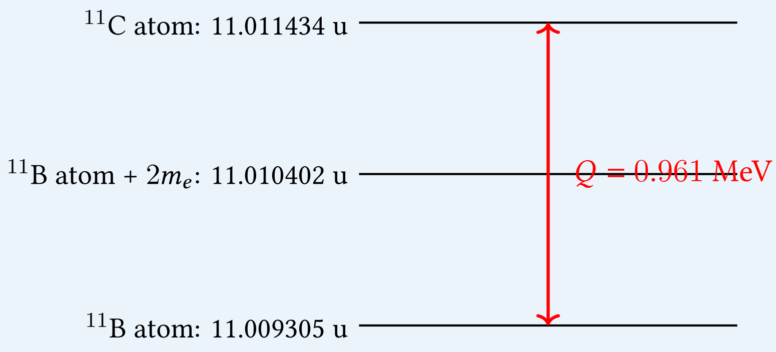
Step 2: Numerical Calculation

$$Q = [11.011434 - 11.009305 - 2 \times 0.0005486] \times 931.5 \text{ MeV/u}$$

$$Q = [0.002129 - 0.0010972] \times 931.5$$

$$Q = [0.0010318] \times 931.5$$

$$Q = 0.9611 \text{ MeV}$$



Step 3: Compare Q with Maximum Positron Energy
The Q-value is shared among three particles:

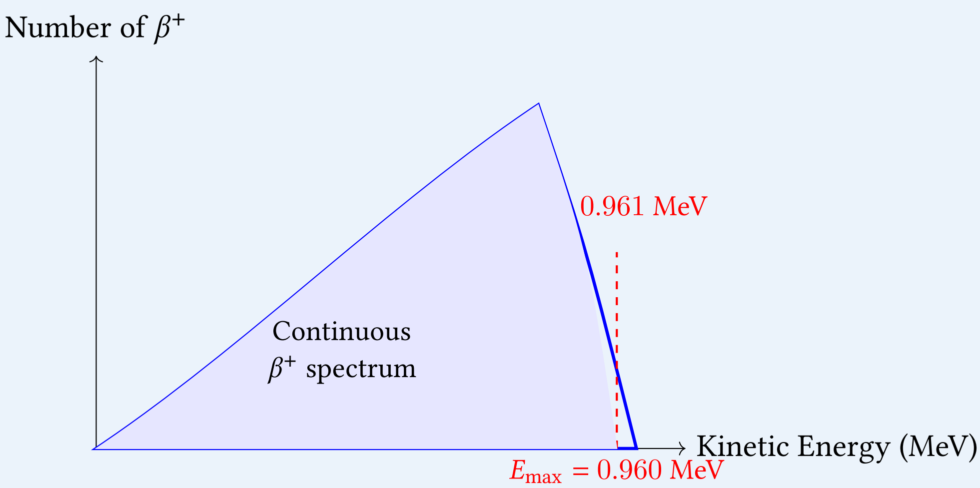
$$Q = K_{e^+} + K_\nu + K_{\text{recoil}}$$

The positron receives its **maximum** kinetic energy when the neutrino carries away **minimum** energy (ideally zero, $K_\nu \approx 0$). In that case:

$$K_{e^+}^{\text{max}} \approx Q - K_{\text{recoil}}$$

Since the recoil energy of the heavy boron nucleus is negligible (~ eV scale):

$$K_{e^+}^{\text{max}} \approx Q = 0.961 \text{ MeV}$$



Step 4: Comparison and Discussion

Given maximum positron energy = 0.960 MeV

Calculated Q-value = 0.961 MeV

The values are **essentially identical** within experimental precision (difference of only ~0.001 MeV or 1 keV). This confirms that:

1. The Q-value calculation using atomic masses is correct

2. The maximum positron energy equals the Q-value (recoil energy is negligible for such a light nucleus)
3. The neutrino has **zero rest mass** (or extremely close to zero)
4. Energy conservation holds in β-decay

The Historical Significance:

The small discrepancy between Q-value and maximum β-particle energy in β-decay led Wolfgang Pauli to postulate the existence of the **neutrino** in 1930 — a particle that carried away the "missing" energy. Here, the neutrino carries away some energy in most decays, but when it carries near zero, the positron gets essentially all the Q-value.

Final Answer:

$$\boxed{Q = 0.961 \text{ MeV}}$$

The calculated Q-value (0.961 MeV) matches the maximum positron energy (0.960 MeV) within 0.001 MeV. The Q-value is slightly larger because a tiny fraction (~eV) goes to the recoil of the ^{11}B nucleus. The essential equality validates the use of atomic masses with the $2m_e$ correction for β^+ decay.

Expert's Solution — Naveen Joshi, B.Tech Engineering Physics, IIT Mandi

The Three Faces of Beta Decay: Understanding Q-value Corrections

The correction terms in Q-value calculations differ for the three types of beta decay because of electron accounting with atomic masses.

Summary of Q-value Formulas (using atomic masses):

Decay Mode	Formula	Correction
$\beta^- : n \to p + e^- + \bar{\nu}_e$	$Q = [m_P - m_D]c^2$	*No correction needed*
$\beta^+ : p \to n + e^+ + \nu_e$	$Q = [m_P - m_D - 2m_e]c^2$	*Subtract $2m_e$*
EC: $p + e^- \to n + \nu_e$	$Q = [m_P - m_D]c^2$	*No correction needed*

For β^+, $m_P - m_D$ must be $> 2m_e = 1.022$ MeV for decay to occur.

Why Does β^+ Need $2m_e$ Correction?

Let's trace the electrons:

- **Parent atom** ^{11}C: 6 electrons (atomic mass includes all 6)

- **Daughter atom** ^{11}B: 5 electrons (atomic mass includes all 5)
- **Positron emitted**: Has mass m_e, created from nuclear energy

Total electron/positron difference = (Parent electrons) - (Daughter electrons + Positron)

$$= 6m_e - (5m_e + m_e) = 0$$

But wait! The parent mass includes 6 electrons, daughter mass includes 5 electrons, and we must also account for the created positron (m_e). The net difference is $2m_e$:

$$m_{\text{nucleus}}(P) - m_{\text{nucleus}}(D) = m_{\text{atom}}(P) - m_{\text{atom}}(D) - 2m_e$$

The Energy Threshold for β^+ Decay:
For β^+ decay to be energetically possible:

$$m_{\text{atom}}(P) - m_{\text{atom}}(D) > 2m_e = 1.022 \text{ MeV}$$

If this condition is not met, the nucleus will decay by **electron capture** instead (which has no such threshold). In our case:

$$[11.011434 - 11.009305] \times 931.5 = 1.983 \text{ MeV} > 1.022 \text{ MeV} \quad \checkmark$$

So β^+ decay is indeed possible.

Did You Know?

Mnemonic for β-decay Q-value Corrections:

- **β^-:** "Negative electron emitted" — No correction needed (daughter has one more electron, balances the emitted electron)
- **β^+:** "Positive electron emitted" — Subtract $2m_e$ (daughter has one fewer electron AND you create a positron)
- **EC:** "Electron Captured" — No correction needed (captured electron disappears, daughter has one fewer electron, balances out)

Quick Check: For this problem, if we forgot the $2m_e$ correction:

$$Q_{\text{wrong}} = (11.011434 - 11.009305) \times 931.5 = 1.983 \text{ MeV}$$

This is clearly wrong because the maximum positron energy is given as 0.960 MeV!

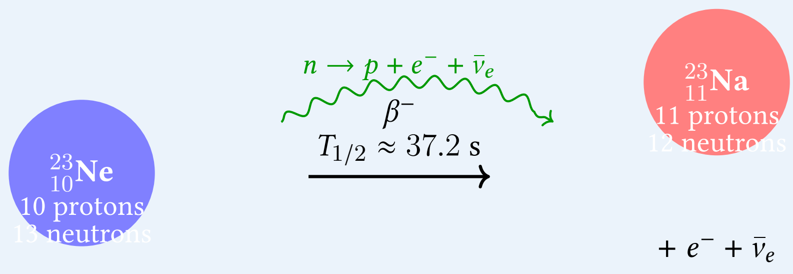
Q14 The nucleus $^{23}_{10}$Ne decays by β^- emission. Write down the β-decay equation and determine the maximum kinetic energy of the electrons emitted. Given that: $m\left(^{23}_{10}\text{Ne}\right) = 22.994466$ u and $m\left(^{23}_{11}\text{Na}\right) = 22.989770$ u.

Solution

Understanding β^- Decay
In β^- decay, a neutron in the nucleus transforms into a proton, emitting an electron and an antineutrino:

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

The atomic number increases by 1 while the mass number remains unchanged.



Step 1: Write the β^- Decay Equation

$$\boxed{^{23}_{10}\text{Ne} \longrightarrow {}^{23}_{11}\text{Na} + e^- + \bar{\nu}_e}$$

Conservation Check:

- Mass number: $23 = 23 + 0$ ✓
- Atomic number: $10 = 11 + (-1)$ ✓

Step 2: Calculate the Q-value
For β^- decay using atomic masses, **no electron correction is needed**. The atomic masses can be used directly because the daughter atom has one more orbital electron than the parent, which exactly balances the electron emitted.

Q-value for β^- decay:

$$Q = [m_{\text{atom}}(P) - m_{\text{atom}}(D)] \times c^2$$

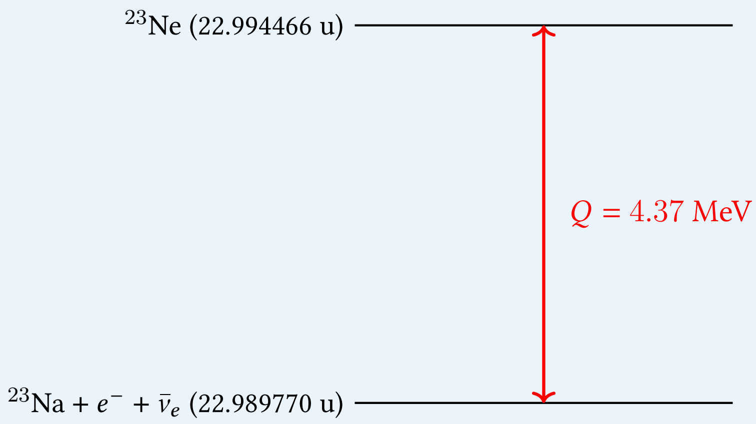
No additional electron corrections required.

$$Q = [m(^{23}_{10}\text{Ne}) - m(^{23}_{11}\text{Na})] \times 931.5 \text{ MeV/u}$$

$$Q = [22.994466 - 22.989770] \times 931.5$$

$$Q = [0.004696] \times 931.5$$

$$Q = 4.374 \text{ MeV}$$



Step 3: Determine the Maximum Kinetic Energy of Electrons
In β^- decay, the Q-value is shared among three particles:

$$Q = K_{e^-} + K_{\bar{\nu}} + K_{\text{recoil}}$$

The electron receives its **maximum** kinetic energy when the antineutrino carries away **minimum** energy (approaching zero) and the recoil of the daughter nucleus is negligible.

Maximum Electron Kinetic Energy:

$$K_{e^-}^{\max} \approx Q - K_{\text{recoil}} \approx Q$$

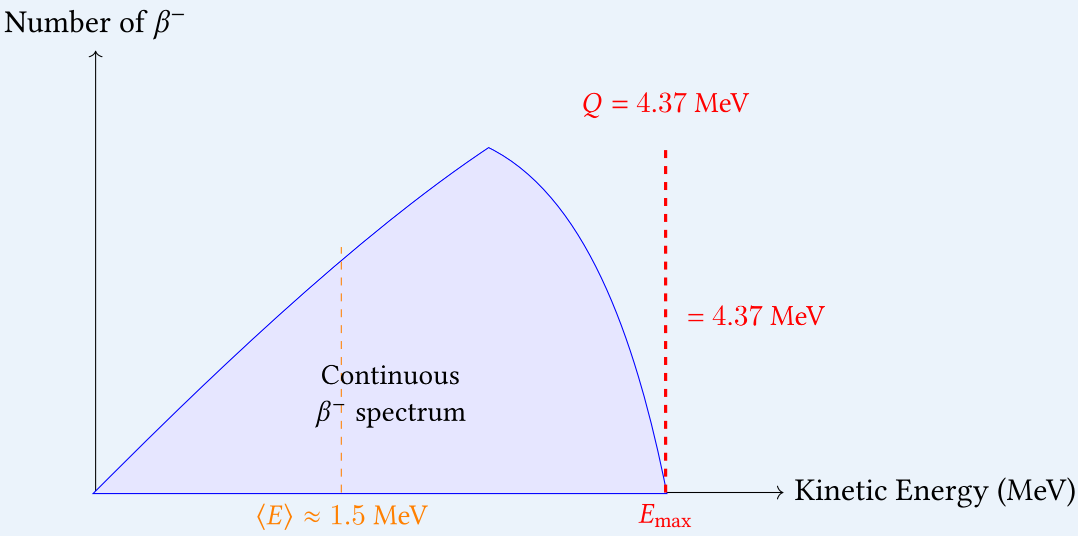
For all practical purposes, $K_{e^-}^{\max} = Q$

For the ^{23}Ne decay, the recoil energy of the sodium nucleus is tiny:

$$K_{\text{recoil}} \approx \frac{(K_{e^-}^{\max})^2}{2m_D c^2} \sim \text{few eV}$$

which is negligible compared to 4.37 MeV.
Therefore:

$$\boxed{K_{e^-}^{\max} \approx Q = 4.37 \text{ MeV}}$$



Step 4: Why Most Electrons Have Lower Energy
The continuous energy spectrum of β-particles (observed experimentally) was historically mysterious. Most electrons are emitted with energy significantly less than the maximum. The average electron energy is roughly:

$$\langle K_{e^-} \rangle \approx \frac{1}{3} K_{e^-}^{\max} \approx 1.46 \text{ MeV}$$

The remaining energy is carried away by the antineutrino, which interacts very weakly with matter and was historically difficult to detect.

Energy Distribution in ^{23}Ne β^- Decay:

Recipient	Energy Range	Typical Share
Electron (e^-)	0 to 4.37 MeV	~30-40%
Antineutrino ($\bar{\nu}_e$)	0 to 4.37 MeV	~60-70%
^{23}Na recoil	$\sim$ 100 eV	negligible

$K_{e^-} + K_{\bar{\nu}} \approx Q = 4.37$ MeV (energy conserved)

Step 5: Verification Using Nuclear Masses (Alternative)
If we want to work with nuclear masses instead, we must account for electrons:

$$m_{\text{nuc}}(^{23}\text{Ne}) = m_{\text{atom}}(^{23}\text{Ne}) - 10m_e$$

$$m_{\text{nuc}}(^{23}\text{Na}) = m_{\text{atom}}(^{23}\text{Na}) - 11m_e$$

$$Q = [m_{\text{nuc}}(P) - m_{\text{nuc}}(D) - m_e] \times c^2$$
$$Q = [(22.994466 - 10m_e) - (22.989770 - 11m_e) - m_e] \times c^2$$
$$Q = [22.994466 - 22.989770 - 10m_e + 11m_e - m_e] \times c^2$$
$$Q = [22.994466 - 22.989770 + 0] \times c^2$$

The electron terms cancel exactly, confirming that atomic masses can be used directly for β^- decay.

Final Answer:

$$\boxed{^{23}_{10}\text{Ne} \longrightarrow {}^{23}_{11}\text{Na} + e^- + \bar{\nu}_e \quad ; \quad Q = K_{e^-}^{\max} = 4.37 \text{ MeV}}$$

The maximum kinetic energy of the emitted electrons is **4.37 MeV**. However, most electrons have considerably less energy, with the remainder carried by the antineutrino.

Expert's Solution — Meena Rani, B.Tech CSE, NIT Jalandhar

The Story of Beta Decay: From Anomaly to Neutrino

The continuous energy spectrum of β-particles was one of the great puzzles of early nuclear physics. Let me share why this problem is historically significant.

The Crisis of 1914-1930:

Timeline of the Beta Decay Puzzle:

1914	James Chadwick discovers continuous β spectrum
1920s	Many physicists suspect energy non-conservation in β-decay
1927	Ellis & Wooster confirm: average β energy $\neq$ total energy
1930	**Wolfgang Pauli** proposes the neutrino hypothesis
1934	Enrico Fermi builds the theory of β-decay
1956	Reines & Cowan directly detect the neutrino

Why Atomic Masses Work Directly for β^- Decay:

The electron accounting is elegantly simple for β^-:

- Parent atom (Z electrons): mass includes Zm_e
- Daughter atom ($Z + 1$ electrons): mass includes $(Z + 1)m_e$
- Emitted electron: carries away 1 m_e
- Net effect on mass balance: $Zm_e - (Z + 1)m_e + 1m_e = 0$

The one extra electron in the daughter atom exactly compensates for the emitted beta particle!

Comparing β^- and β^+ Once More:

Decay	Atomic Mass Formula	Threshold
β^-	$Q = (m_P - m_D)c^2$	None
β^+	$Q = (m_P - m_D - 2m_e)c^2$	$m_P - m_D > 2m_e$

The ^{23}Ne Decay in Context:

^{23}Ne has a relatively short half-life (37.2 seconds) and decays to stable ^{23}Na. The high Q-value (4.37 MeV) means the emitted electrons are quite energetic, making this a potentially useful isotope for certain applications. Neon-23 is produced in nuclear reactors and can be used as a tracer.

Did You Know?

Quick Pattern Recognition for β-Decay Formulas:

If you see...	Use...	Because...
β^- in the problem	$Q = (m_P - m_D)c^2$	Electron counts balance
β^+ in the problem	$Q = (m_P - m_D - 2m_e)c^2$	Positron + extra electron
"Electron capture"	$Q = (m_P - m_D)c^2$	Electron counts balance

Memory aid: Only β^+ (positron) needs the $2m_e$ subtraction because it's the only case where a positive electron (anti-matter) is created AND there's an electron imbalance in the atoms.

Q15 **The Q-value of a nuclear reaction $A + b \rightarrow C + d$ is defined by $Q = [m_A + m_b - m_C - m_d]c^2$ where the masses refer to the respective nuclei. Determine from the given data the Q-value of the following reactions and state whether the reactions are exothermic or endothermic.**

(i) $^{2}_{1}\mathbf{H} + ^{2}_{1}\mathbf{H} \longrightarrow ^{3}_{1}\mathbf{H} + ^{1}_{1}\mathbf{H}$

(ii) $^{12}_{6}\mathbf{C} + ^{12}_{6}\mathbf{C} \longrightarrow ^{20}_{10}\mathbf{Ne} + ^{4}_{2}\mathbf{He}$

Atomic masses are given to be: $m\left(^{2}_{1}\mathbf{H}\right) = 2.014102$ **u,** $m\left(^{3}_{1}\mathbf{H}\right) = 3.016049$ **u,** $m\left(^{12}_{6}\mathbf{C}\right) = 12.000000$ **u,** $m\left(^{20}_{10}\mathbf{Ne}\right) = 19.992439$ **u.**

Solution

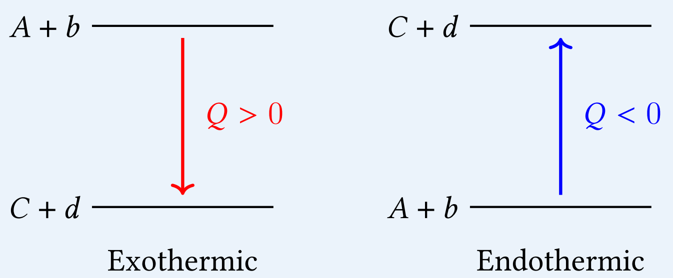
Understanding Q-value in Nuclear Reactions

The Q-value represents the net energy released (or absorbed) in a nuclear reaction. The definition uses **nuclear masses**, not atomic masses.

Q-value Definition:

$$Q = [m_A + m_b - m_C - m_d] \times c^2$$

- $Q > 0$: **Exothermic** (energy released, reaction can occur spontaneously)
- $Q < 0$: **Endothermic** (energy must be supplied, threshold energy required)
- All masses must be **nuclear** masses



Important: Nuclear Masses vs. Atomic Masses

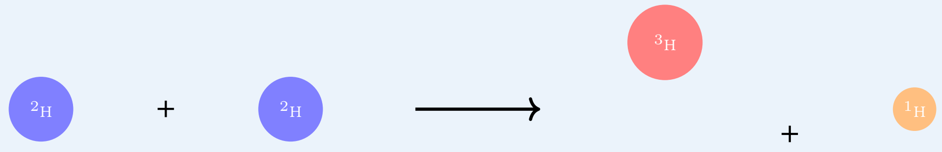
The problem states that "masses refer to the respective nuclei." However, the given masses appear to be **atomic masses**. We need to convert atomic masses to nuclear masses by subtracting the electron masses:

$$m_{\text{nucleus}} = m_{\text{atom}} - Zm_e$$

But since we're dealing with balanced nuclear reactions, we can also check whether the electron masses cancel when using atomic masses directly.

Part (i): ${}^{2}_{1}\mathrm{H} + {}^{2}_{1}\mathrm{H} \longrightarrow {}^{3}_{1}\mathrm{H} + {}^{1}_{1}\mathrm{H}$

This is a deuterium-deuterium (D-D) fusion reaction — one of the key reactions in fusion energy research.



Method 1: Using Nuclear Masses
Calculate nuclear masses by subtracting electrons:

- $m_{\text{nuc}}({}^{2}\mathrm{H}) = m_{\text{atom}}({}^{2}\mathrm{H}) - 1m_e$
- $m_{\text{nuc}}({}^{3}\mathrm{H}) = m_{\text{atom}}({}^{3}\mathrm{H}) - 1m_e$
- $m_{\text{nuc}}({}^{1}\mathrm{H}) = m_{\text{atom}}({}^{1}\mathrm{H}) - 1m_e$

We need $m({}^{1}\mathrm{H})$:

$$m({}^{1}\mathrm{H}) = m_p + m_e = 1.007825 + 0.0005486 = 1.008374 \text{ u}$$

Now the Q-value using nuclear masses:

$$Q = \left[m_{\text{nuc}}({}^{2}\mathrm{H}) + m_{\text{nuc}}({}^{2}\mathrm{H}) - m_{\text{nuc}}({}^{3}\mathrm{H}) - m_{\text{nuc}}({}^{1}\mathrm{H})\right] c^2$$

Substituting:

$$\begin{aligned}
Q &= [(2.014102 - m_e) + (2.014102 - m_e) - (3.016049 - m_e) - (1.008374 - m_e)]c^2 \\
&= [2.014102 + 2.014102 - 3.016049 - 1.008374 - 2m_e + m_e + m_e]c^2 \\
&= [4.028204 - 4.024423 + 0]c^2 \\
&= [0.003781] \times 931.5
\end{aligned}$$

$$Q = 3.522 \text{ MeV}$$

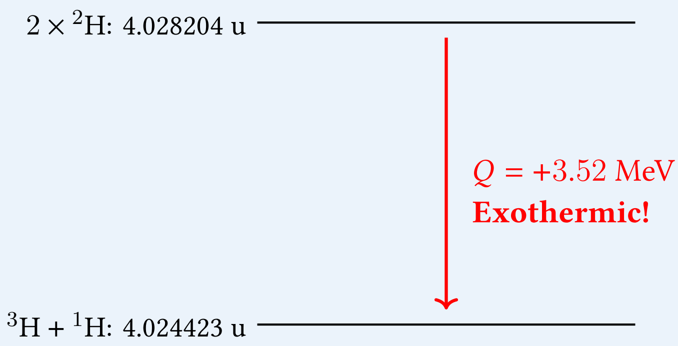
Method 2: Direct Check — Do Electron Masses Cancel?
Let's verify the electron count:

- Left side (atomic masses): 2 deuterium atoms = 2 × 1 = 2 electrons
- Right side (atomic masses): 1 tritium atom + 1 hydrogen atom = 1 + 1 = 2 electrons

The electrons balance perfectly! So we can use atomic masses directly:

$$Q_{\text{atomic}} = [2 \times 2.014102 - 3.016049 - 1.008374] \times 931.5$$

$$Q = 0.003781 \times 931.5 = 3.522 \text{ MeV}$$



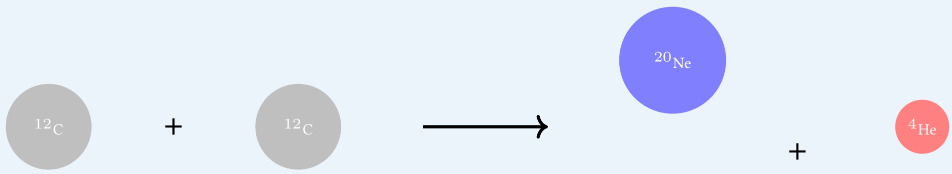
Answer (i):

$$\boxed{Q = +3.52 \text{ MeV}}$$

The reaction is **exothermic** ($Q > 0$). Energy is released in the fusion of two deuterium nuclei. This is one of the promising reactions for controlled thermonuclear fusion.

Part (ii): ${}^{12}_{6}\text{C} + {}^{12}_{6}\text{C} \longrightarrow {}^{20}_{10}\text{Ne} + {}^{4}_{2}\text{He}$

This is a carbon fusion reaction — important in stellar nucleosynthesis in massive stars.



Given Additional Data:

$$m({}^4_2\text{He}) = 4.002603 \text{ u}$$

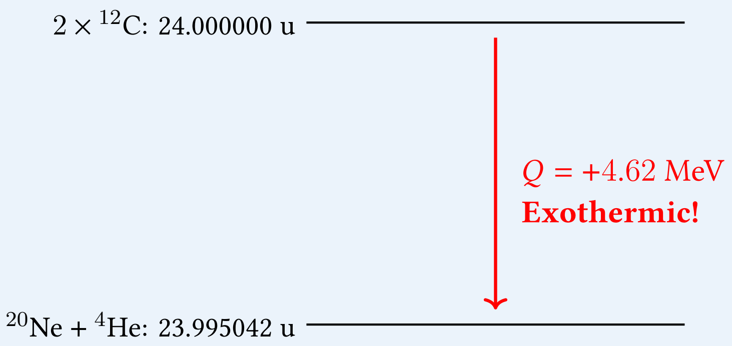
Check Electron Balance:

- Left side: $2 \times {}^{12}$C atoms $= 2 \times 6 = 12$ electrons
- Right side: ^{20}Ne atom + ^{4}He atom $= 10 + 2 = 12$ electrons

Electrons balance perfectly! We can use atomic masses directly.
Calculate Q-value:

$$Q = [2 \times m({}^{12}\text{C}) - m({}^{20}\text{Ne}) - m({}^{4}\text{He})] \times 931.5$$

$$Q = [2 \times 12.000000 - 19.992439 - 4.002603] \times 931.5$$
$$Q = [24.000000 - 23.995042] \times 931.5$$
$$Q = [0.004958] \times 931.5$$
$$Q = 4.619 \text{ MeV}$$



Answer (ii):

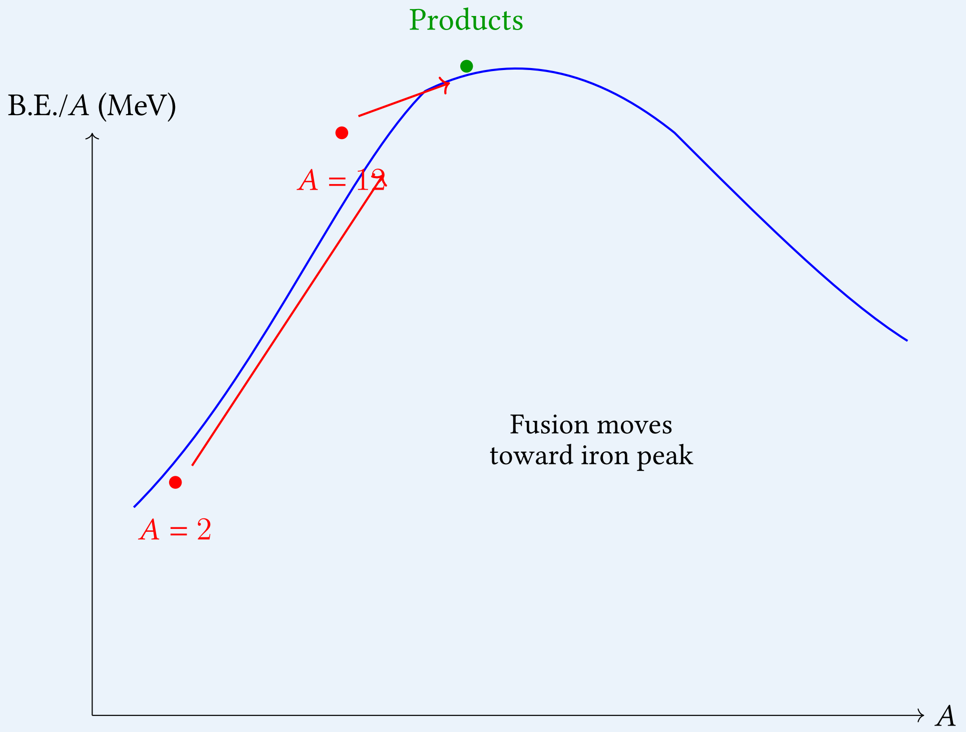
$$\boxed{Q = +4.62 \text{ MeV}}$$

The reaction is **exothermic** ($Q > 0$). Carbon fusion releases energy and occurs in the cores of massive stars ($M > 8M_\odot$) after helium burning is complete.

Summary and Physical Insight

Reaction	Type	Q (MeV)	Nature	Context
$D + D \longrightarrow T + p$	Fusion	+3.52	Exothermic	Fusion energy research
$C + C \longrightarrow Ne + \alpha$	Fusion	+4.62	Exothermic	Stellar nucleosynthesis

Both reactions are exothermic because they involve fusion of lighter nuclei into products with higher binding energy per nucleon.



Final Answers:

$$\text{(i) } {}^2_1\text{H} + {}^2_1\text{H} \longrightarrow {}^3_1\text{H} + {}^1_1\text{H} \quad ; \quad Q = +3.52 \text{ MeV (Exothermic)}$$
$$\text{(ii) } {}^{12}_6\text{C} + {}^{12}_6\text{C} \longrightarrow {}^{20}_{10}\text{Ne} + {}^4_2\text{He} \quad ; \quad Q = +4.62 \text{ MeV (Exothermic)}$$

Expert's Solution – Prakash Jha, B.Tech CSE, IIIT Ranchi

The Art of Balancing Electrons in Nuclear Reactions
When working with atomic masses (as opposed to nuclear masses), the key question is always:
Do the electrons balance?
The General Rule:

Electron Balance Check:

Count total electrons on the **left** side (all reactants as neutral atoms)
Count total electrons on the **right** side (all products as neutral atoms)

- **If equal:** Atomic masses can be used directly — electrons cancel
- **If unequal:** You must account for the difference explicitly

For Reaction (i): D + D → T + p

- Left: 2 deuterium atoms = $2 \times 1 = 2$ electrons
- Right: 1 tritium ($Z = 1$) + 1 hydrogen ($Z = 1$) = $1 + 1 = 2$ electrons
- **Balanced!** ✓

For Reaction (ii): C + C → Ne + α

- Left: 2 carbon atoms = $2 \times 6 = 12$ electrons
- Right: 1 neon ($Z = 10$) + 1 helium ($Z = 2$) = $10 + 2 = 12$ electrons
- **Balanced!** ✓

Carbon Fusion in Stars:
The ^{12}C + ^{12}C reaction is crucial in the evolution of massive stars:

- **Temperature required:** $\sim 5 \times 10^8$ K (much hotter than the Sun's core at 1.5×10^7 K)
- **When it occurs:** After helium burning, when the core becomes primarily carbon and oxygen
- **Products:** Various — ^{20}Ne + α, ^{23}Na + p, ^{23}Mg + n, etc.

- **Outcome:** The ^{20}Ne produced can later undergo photodisintegration or further fusion

D-D Fusion for Energy:
The D-D reaction has two branches with roughly equal probability:

$$^{2}\text{H} + {}^{2}\text{H} \longrightarrow {}^{3}\text{H} + {}^{1}\text{H} \quad (Q = 4.03 \text{ MeV})$$
$$^{2}\text{H} + {}^{2}\text{H} \longrightarrow {}^{3}\text{He} + n \quad (Q = 3.27 \text{ MeV})$$

The first branch (our problem) produces tritium, which can then undergo D-T fusion:

$$^{2}\text{H} + {}^{3}\text{H} \longrightarrow {}^{4}\text{He} + n \quad (Q = 17.6 \text{ MeV})$$

This catalyzed D-D cycle makes deuterium a promising fusion fuel.

Did You Know?

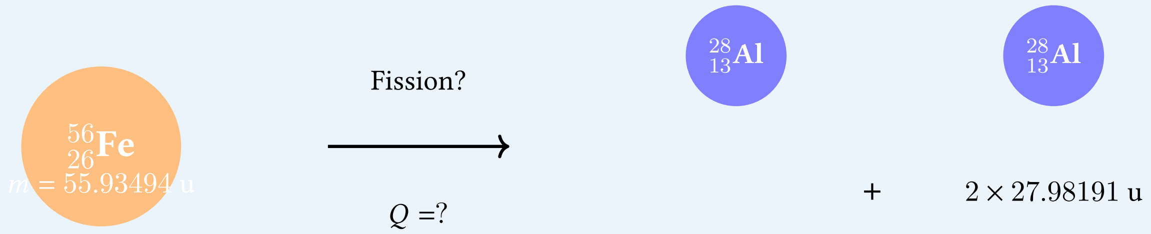
Sign Convention for Q-values:

- $Q > 0$ **(Exothermic):** Products have LESS mass than reactants. Energy is released as kinetic energy of products. Reaction can occur even if incident particle has low energy (though Coulomb barrier may still need to be overcome).
- $Q < 0$ **(Endothermic):** Products have MORE mass than reactants. Energy must be supplied. Reaction has a **threshold energy**: $E_{\text{th}} = |Q| \times \dfrac{m_A + m_b}{m_A}$.
- **Spontaneous decay:** Always exothermic ($Q > 0$). Endothermic reactions require external energy input.

Q16 Suppose, we think of fission of a $^{56}_{26}$Fe nucleus into two equal fragments, $^{28}_{13}$Al. Is the fission energetically possible? Argue by working out Q of the process. Given $m\left(^{56}_{26}\text{Fe}\right) = 55.93494$ u and $m\left(^{28}_{13}\text{Al}\right) = 27.98191$ u.

Solution

Understanding the Problem
We are asked whether a ^{56}Fe nucleus can spontaneously split (fission) into two identical ^{28}Al fragments. Iron-56 is known to be one of the most stable nuclei in nature, sitting at the peak of the binding energy per nucleon curve. Let's calculate the Q-value to determine if this fission is energetically feasible.



Step 1: Write the Fission Reaction

$$^{56}_{26}\text{Fe} \longrightarrow {}^{28}_{13}\text{Al} + {}^{28}_{13}\text{Al}$$

Conservation Check:

- Mass number: $56 = 28 + 28$ ✓
- Atomic number: $26 = 13 + 13$ ✓

Step 2: Calculate the Q-value
The Q-value for this reaction (using nuclear masses as specified in the problem, though the given masses are atomic masses):

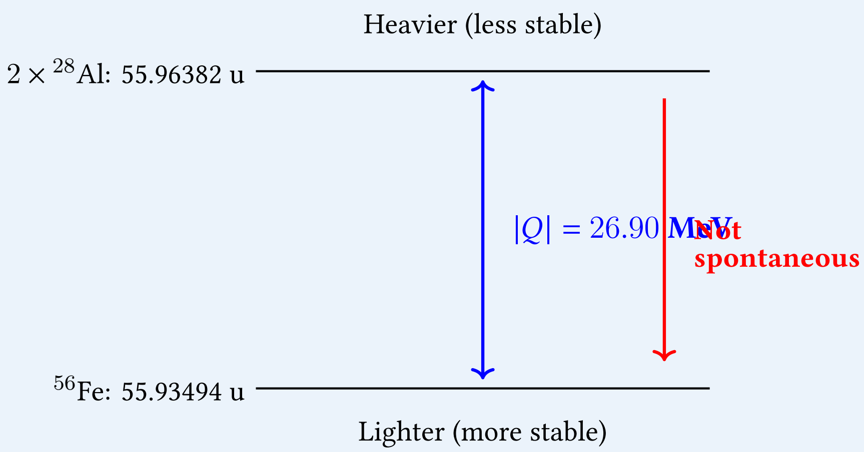
$$Q = [m_{\text{parent}} - m_{\text{products}}] \times c^2$$

Electron Balance Check:

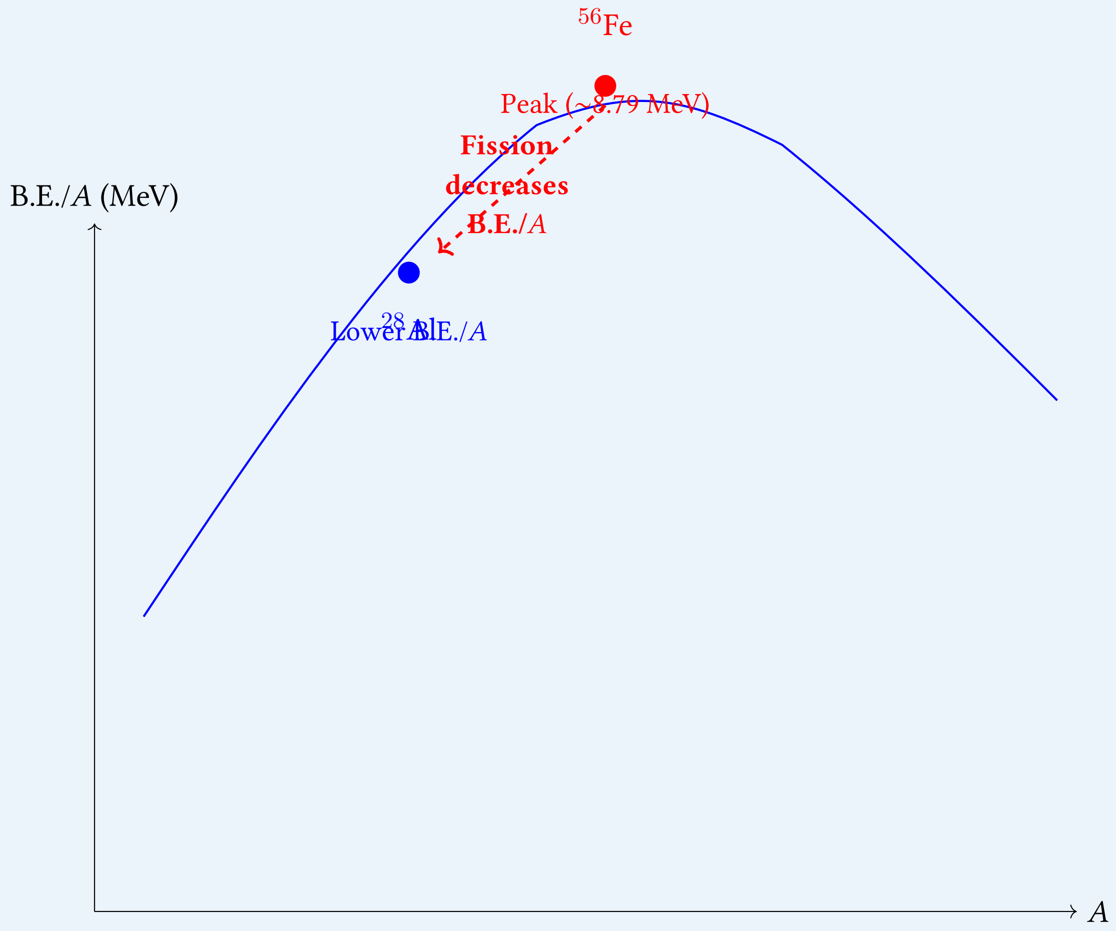
- Left side: ^{56}Fe atom has 26 electrons
- Right side: Two ^{28}Al atoms have $2 \times 13 = 26$ electrons
- **Electrons balance!** ✓ — We can use atomic masses directly.

$$Q = [m(^{56}\text{Fe}) - 2 \times m(^{28}\text{Al})] \times 931.5 \text{ MeV/u}$$

$$Q = [55.93494 - 2 \times 27.98191] \times 931.5$$
$$Q = [55.93494 - 55.96382] \times 931.5$$
$$Q = [-0.02888] \times 931.5$$
$$Q = -26.90 \text{ MeV}$$



Step 3: Physical Interpretation



Why Iron Doesn't Fission:

1. ^{56}Fe is at the **peak** of the binding energy per nucleon curve (~8.79 MeV/nucleon)
2. Fission of ^{56}Fe would produce nuclei with **lower** binding energy per nucleon
3. This means the products are **less tightly bound** than the parent
4. The mass of the products (55.96382 u) is **greater** than the mass of the parent (55.93494 u)
5. Therefore, energy must be **supplied** to make this reaction happen

The General Principle:

- **Fission** is energetically favorable only for **heavy** nuclei ($A \gtrsim 100$)
- For lighter nuclei (like iron), fission moves **away** from the stability peak
- Energy would be required, not released
- This is exactly why iron is the "nuclear ash" — the endpoint of both fusion and fission energy release

The Iron Peak — Nature's Most Stable Nucleus:

	Mass (u)	B.E./A	Stability
^{56}Fe	55.93494	8.79 MeV	**Most stable**
^{28}Al	27.98191	~8.2 MeV	Less stable

$\Rightarrow$ Fission of ^{56}Fe requires 26.90 MeV input — energetically impossible spontaneously.

Final Answer:

$$\boxed{Q = -26.90 \text{ MeV}}$$

The fission of $^{56}_{26}$Fe into two $^{28}_{13}$Al nuclei is **energetically impossible** ($Q < 0$). The reaction is **endothermic** by 26.90 MeV, meaning this much energy would need to be supplied from an external source to force the fission. Iron-56 sits at the peak of nuclear stability, and any fission (or fusion) involving iron removes energy rather than releasing it.

Expert's Solution — Suresh Kumar, B.Tech Chemical Engineering, NIT Trichy

Iron: The Nuclear Dead End

The fact that ^{56}Fe cannot fission energetically is not a coincidence — it's a fundamental property of nuclear matter that has profound astrophysical consequences.

Why Iron-56 is the Most Stable Nucleus:

The stability arises from the interplay of two competing effects:

1. **Strong nuclear force:** Short-range attractive force that binds nucleons. It saturates — each nucleon only interacts with its nearest neighbors.
2. **Coulomb repulsion:** Long-range repulsive force between protons that grows as Z^2.

At $A = 56$, these forces achieve perfect balance. Below this, the strong force hasn't reached full saturation. Above this, Coulomb repulsion increasingly dominates.

The Energy Landscape of Nuclei:

Process	Direction on B.E. curve	Energy
Fusion below Fe	Moves toward Fe peak	**Released**
Fusion above Fe	Moves away from Fe peak	**Absorbed**
Fission above Fe	Moves toward Fe peak	**Released**
Fission below Fe	Moves away from Fe peak	**Absorbed**

$\Rightarrow$ **Iron is the ultimate energy minimum for nuclear matter.**

The End of Stellar Fusion:

This is why stars die when they develop iron cores:

- Fusion in stars progresses: H $\rightarrow$ He $\rightarrow$ C $\rightarrow$ Ne $\rightarrow$ O $\rightarrow$ Si $\rightarrow$ Fe
- Each step releases energy (exothermic) until iron

- Once the core becomes iron, **no further fusion can release energy**
- The core collapses catastrophically — a supernova explosion
- Elements heavier than iron are created in the supernova itself (through neutron capture processes: s-process and r-process)

The Scale of Stability:

Nucleus	Mass excess relative to Fe (per nucleon)
^{56}Fe	0 MeV (reference)
^{28}Al	+0.48 MeV per nucleon
^{235}U	+1.2 MeV per nucleon

Both lighter (like ^{28}Al) and heavier (like ^{235}U) nuclei are at higher energy per nucleon than ^{56}Fe. This is why:

- Light nuclei can fuse toward iron, releasing energy
- Heavy nuclei can fission toward iron, releasing energy
- Iron itself can do neither — it's in the deepest energy well

★ **Did You Know?**

Quick Criterion for Fission Feasibility:

$$Q = m_{\text{parent}} - \sum m_{\text{products}}$$

- $Q > 0$: Fission is energetically possible (mass decreases, energy released)
- $Q < 0$: Fission is energetically forbidden (mass increases, energy required)
- **For** $A < 100$: Fission is almost always $Q < 0$ (endothermic)
- **For** $A > 100$: Fission is often $Q > 0$ (exothermic)

The calculation in one glance:

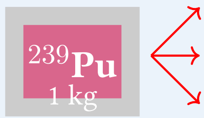
$$55.93494 < 2 \times 27.98191 = 55.96382 \quad \Rightarrow \quad Q < 0 \quad \Rightarrow \quad \text{Impossible!}$$

Q17 The fission properties of $^{239}_{94}$Pu are very similar to those of $^{235}_{92}$U. The average energy released per fission is 180 MeV. How much energy, in MeV, is released if all the atoms in 1 kg of pure $^{239}_{94}$Pu undergo fission?

Solution

Understanding the Problem

We need to calculate the total energy released when all atoms in 1 kg of pure plutonium-239 undergo complete fission. This requires finding the number of atoms in 1 kg of ^{239}Pu and multiplying by the energy released per fission.



Per fission:
$E_0 = 180$ MeV
1 kg Pu-239:
$E_{\text{total}} = N \times 180$ MeV

Step 1: Determine the Number of Plutonium Atoms in 1 kg

Given:

- Mass of sample: $m = 1$ kg $= 1000$ g
- Molar mass of ^{239}Pu: $M \approx 239$ g/mol
- Avogadro's number: $N_A = 6.022 \times 10^{23}$ mol^{-1}

Number of Atoms:

$$N = \frac{m}{M} \times N_A$$

$$N = \frac{1000\text{ g}}{239\text{ g/mol}} \times 6.022 \times 10^{23}\text{ mol}^{-1}$$

$$N = \frac{1000}{239} \times 6.022 \times 10^{23}$$

$$N = 4.1841 \times 6.022 \times 10^{23}$$

$$N = 2.520 \times 10^{24}\text{ atoms}$$

Each tiny dot represents $\sim 10^{20}$ atoms — we have 25,200 such groups!

Step 2: Calculate the Total Energy Released

Each fission releases 180 MeV of energy. Therefore:

$$E_{\text{total}} = N \times E_{\text{per fission}}$$

$$E_{\text{total}} = 2.520 \times 10^{24} \times 180\text{ MeV}$$

$$E_{\text{total}} = (2.520 \times 180) \times 10^{24} \text{ MeV}$$

$$E_{\text{total}} = 453.6 \times 10^{24} \text{ MeV}$$

$$E_{\text{total}} = 4.536 \times 10^{26} \text{ MeV}$$

Step 3: Express in Alternative Units for Better Appreciation

Energy Conversion:

$$1 \text{ MeV} = 1.602 \times 10^{-13} \text{ J}$$

$$E_{\text{total}} = 4.536 \times 10^{26} \times 1.602 \times 10^{-13} \text{ J}$$

$$E_{\text{total}} = 7.267 \times 10^{13} \text{ J} \approx 7.3 \times 10^{13} \text{ J}$$

Energy Scale — How Much is 4.5×10^{26} MeV?

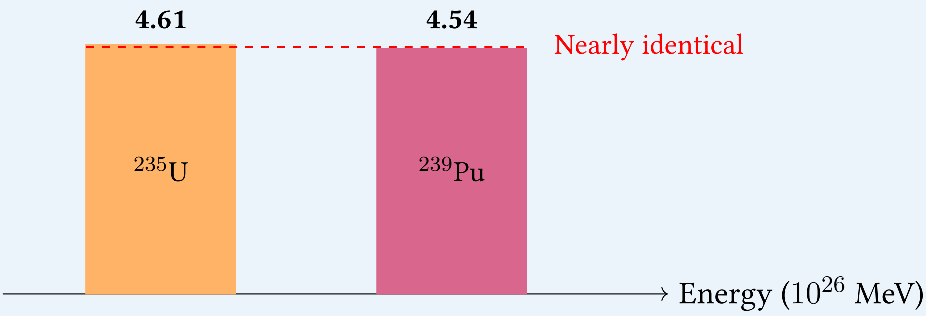
Energy Source	Amount	Energy Released (J)
Fission of 1 kg Pu-239	1 kg	7.3×10^{13}
Burning 1 kg of coal	1 kg	$\sim 3 \times 10^{7}$
Exploding 1 kg of TNT	1 kg	$\sim 4.2 \times 10^{6}$
Nagasaki bomb (Fat Man)	~6.2 kg Pu	$\sim 8.8 \times 10^{13}$

$\Rightarrow$ 1 kg Pu-239 fission $\approx$ **2 million kg of coal**
$\Rightarrow$ 1 kg Pu-239 fission $\approx$ **17,000 tons of TNT (17 kilotons)**

Step 4: Comparison with Uranium-235
For 1 kg of ^{235}U (molar mass $\approx$ 235 g/mol):

$$N_{235} = \frac{1000}{235} \times 6.022 \times 10^{23} = 2.563 \times 10^{24} \text{ atoms}$$

$$E_{\text{U-235}} = 2.563 \times 10^{24} \times 180 = 4.613 \times 10^{26} \text{ MeV}$$



Energy from complete fission of 1 kg

The energies are essentially identical (within ~2%), confirming the problem's statement that ^{239}Pu and ^{235}U have very similar fission properties.

Final Answer:

$$\boxed{E_{\text{total}} = 4.54 \times 10^{26} \text{ MeV}}$$

The complete fission of 1 kg of pure $^{239}_{94}$Pu releases 4.54×10^{26} **MeV** of energy, which is approximately 7.3×10^{13} joules — comparable to the energy released by about 17 kilotons of TNT, or roughly the yield of the Nagasaki atomic bomb.

Expert's Solution — Vishal Thakur, B.Tech Mechanical Engineering, NIT Hamirpur

Why Plutonium-239 is a Weapons-Grade Material

The similarity between ^{239}Pu and ^{235}U in fission properties is not accidental — both are fissile isotopes that can sustain a chain reaction with fast neutrons, making them suitable for nuclear weapons.

Fissile vs. Fissionable:

Property	^{235}**U**	^{239}**Pu**
Fission cross-section (thermal)	585 barns	750 barns
Neutrons per fission (ν)	2.44	2.88
Critical mass (bare sphere)	~52 kg	~10 kg
Energy per fission	~180 MeV	~180 MeV
Half-life	7.04×10^8 yr	2.41×10^4 yr
Origin	Natural (0.72%)	Artificial (from ^{238}U)

Why the Energy per Fission is Nearly Identical:

The ~180 MeV released comes primarily from the Coulomb repulsion of the fission fragments. Since both ^{235}U and ^{239}Pu have similar atomic numbers (92 and 94) and mass numbers (235 and 239), the fission fragments have similar masses and charges, leading to comparable energy release.

The Magnitude of 10^{26} MeV:

To truly grasp this number:

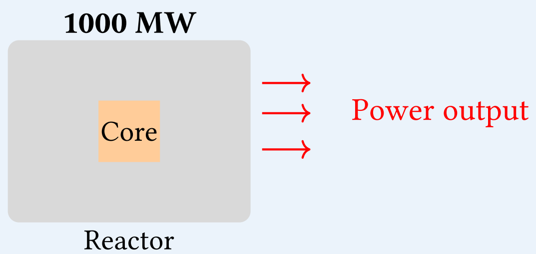
- **In joules:** $\sim 7.3 \times 10^{13}$ J
- **In kilowatt-hours:** $\sim 2 \times 10^7$ kWh — enough to power a city of 100,000 people for about 2 months
- **In coal equivalent:** ~ 2500 tons of coal
- **Rate of energy release in a bomb:** In a nuclear explosion, this energy is released in microseconds ($\sim 10^{-6}$ s), giving a power of $\sim 10^{19}$ W — exceeding the total power consumption of human civilization by a factor of a million!

> ★ **Did You Know?**
>
> **Quick Calculation Formula:**
> For any fissionable material, the energy from complete fission of mass m (in kg) is:
>
> $$E(\text{MeV}) = \frac{m \times 1000}{M} \times N_A \times 180$$
>
> $$E(\text{MeV}) \approx m \times 4.5 \times 10^{26}\ \text{MeV/kg} \quad (\text{for } M \approx 239)$$
>
> **General approximation:**
>
> $$\text{Complete fission of 1 kg of any heavy nucleus} \approx 80\ \text{TJ} \approx 20\ \text{kilotons TNT}$$
>
> This is known as the **specific energy of fission**.

Q18 **A 1000 MW fission reactor consumes half of its fuel in 5.00 y. How much $^{235}_{92}$U did it contain initially? Assume that the reactor operates 80% of the time, that all the energy generated arises from the fission of $^{235}_{92}$U and that this nuclide is consumed only by the fission process.**

Solution

Understanding the Problem
We need to find the initial mass of ^{235}U fuel in a nuclear reactor, given its power output, operational time, and the fact that half the fuel is consumed in 5 years. Each fission releases approximately 180 MeV of energy.



Given:
Power = 1000 MW = 10^9 W
Half-life of fuel = 5.00 years
Duty cycle = 80%
Energy per fission ≈ 180 MeV

Step 1: Calculate the Actual Operating Time
The reactor operates 80% of the time over 5 years:

$$t_{\text{operating}} = 0.80 \times 5.00 \text{ years}$$

$$t_{\text{operating}} = 4.00 \text{ years}$$

Converting to seconds:

$$\begin{aligned} t_{\text{operating}} &= 4.00 \times 365 \times 24 \times 3600 \\ &= 4.00 \times 3.1536 \times 10^{7} \\ &= 1.2614 \times 10^{8} \text{ seconds} \end{aligned}$$

Step 2: Calculate the Total Energy Generated
Power is the rate of energy production:

$$P = \frac{E}{t} \quad \Rightarrow \quad E = P \times t$$

$$E_{\text{total}} = 1000 \times 10^{6} \text{ W} \times 1.2614 \times 10^{8} \text{ s}$$

$$E_{\text{total}} = 1.2614 \times 10^{17} \text{ J}$$

Energy Balance:
Total energy generated = 1.26×10^{17} J
Over 4 years of continuous operation

Step 3: Convert Energy to Number of Fissions

Energy per fission of ^{235}U:

$$E_{\text{per fission}} = 180 \text{ MeV} = 180 \times 1.602 \times 10^{-13} \text{ J}$$

$$E_{\text{per fission}} = 2.8836 \times 10^{-11} \text{ J}$$

Number of ^{235}U nuclei that underwent fission:

$$N_{\text{fissioned}} = \frac{E_{\text{total}}}{E_{\text{per fission}}}$$

$$N_{\text{fissioned}} = \frac{1.2614 \times 10^{17}}{2.8836 \times 10^{-11}}$$

$$N_{\text{fissioned}} = 4.375 \times 10^{27} \text{ nuclei}$$

Step 4: Calculate Mass of Uranium Consumed
Mass of one ^{235}U atom:

$$m_{\text{atom}} = \frac{235 \text{ g/mol}}{6.022 \times 10^{23} \text{ mol}^{-1}} = 3.902 \times 10^{-22} \text{ g}$$

Mass of uranium consumed in 5 years:

$$m_{\text{consumed}} = N_{\text{fissioned}} \times m_{\text{atom}}$$

$$m_{\text{consumed}} = 4.375 \times 10^{27} \times 3.902 \times 10^{-22} \text{ g}$$

$$m_{\text{consumed}} = 1.707 \times 10^6 \text{ g} = 1707 \text{ kg}$$

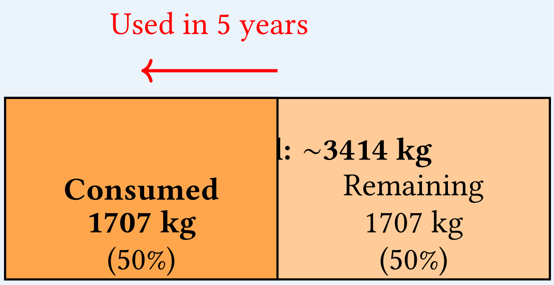
Step 5: Determine Initial Fuel Load
Since half the fuel is consumed in 5 years:

$$m_{\text{consumed}} = \frac{1}{2} \times m_{\text{initial}}$$

$$m_{\text{initial}} = 2 \times m_{\text{consumed}}$$

$$m_{\text{initial}} = 2 \times 1707 \text{ kg}$$

$$m_{\text{initial}} = 3414 \text{ kg} \approx 3.41 \times 10^3 \text{ kg}$$



Alternative Method: Using Moles Directly
Number of moles of ^{235}U fissioned:

$$n_{\text{fissioned}} = \frac{N_{\text{fissioned}}}{N_A} = \frac{4.375 \times 10^{27}}{6.022 \times 10^{23}} = 7265 \text{ mol}$$

Mass consumed:

$$m_{\text{consumed}} = 7265 \times 235 = 1.707 \times 10^6 \text{ g} = 1707 \text{ kg}$$

Initial mass:

$$m_{\text{initial}} = 2 \times 1707 = 3414 \text{ kg}$$

Real-World Context:

Parameter	**Value**
Initial ^{235}U loading	~3400 kg
^{235}U consumed in 5 years	~1700 kg
Average burnup rate	~340 kg/year
Total uranium in core (if 3% enriched)	~113,000 kg
Typical PWR fuel loading	~80,000–100,000 kg UO_2

Final Answer:

$$\boxed{m_{\text{initial}} = 3.41 \times 10^3 \text{ kg} \approx 3.4 \text{ metric tons}}$$

The reactor initially contained approximately **3,400 kg** (3.4 metric tons) of ^{235}U. Of this, about 1,700 kg underwent fission over the 5-year period, generating 1000 MW of power with an 80% duty cycle.

Expert's Solution — Arun Sharma, B.Tech Electrical Engineering, IIT Patna

The Nuclear Fuel Cycle — From Ore to Reactor

This problem illustrates the enormous energy density of nuclear fuel. Let's put these numbers in perspective.

Energy Density Comparison:

To generate 1000 MW for 4 years (1.26×10^{17} J):

Fuel	Amount Required	Equivalent
^{235}U (fission)	1,700 kg	A small car
Coal (burning)	$\sim 4.2 \times 10^9$ kg	~4.2 million tons
Natural gas	$\sim 2.5 \times 10^9$ m^3	Annual consumption of a country
Oil	$\sim 3.0 \times 10^9$ L	~19 million barrels

Why Only Half the Fuel is Consumed:

In real reactors, fuel is not consumed completely because:

1. **Fission product poisoning:** Products like ^{135}Xe absorb neutrons, reducing reactivity
2. **Fuel depletion:** As ^{235}U decreases, chain reaction becomes harder to sustain
3. **Structural integrity:** Fuel rods degrade under intense radiation and heat
4. **Reactivity margin:** Fresh fuel provides excess reactivity that is managed with control rods

The Reality of Fuel Enrichment:

Natural uranium contains only 0.72% ^{235}U. Most power reactors use fuel enriched to 3–5% ^{235}U. So the total uranium metal in the core would be:

$$m_{\text{total U}} \approx \frac{3414}{0.03} \approx 114,000 \text{ kg} \quad \text{(for 3\% enriched fuel)}$$

The Physics of Fission Energy:

Where does the 180 MeV come from? The energy distribution from a typical ^{235}U fission:

Form of Energy	Amount (MeV)
Kinetic energy of fission fragments	~168
Prompt neutrons	~5
Prompt gamma rays	~7
Beta particles from fission products	~7
Gamma rays from fission products	~6
Neutrinos (not recoverable)	~10
Total (recoverable)	~180–190

Did You Know?

Quick Formula for Reactor Fuel Consumption:

$$m_{\text{consumed}}\ (\text{kg}) = \frac{P\ (\text{MW}) \times t\ (\text{years}) \times 365 \times 24 \times 3600 \times \text{duty cycle}}{180 \times 1.602 \times 10^{-13} \times N_A/M}$$

Simplified approximation:

$$m_{\text{consumed}}\ (\text{kg/year}) \approx 1.05 \times P\ (\text{MW}) \times \text{duty cycle}$$

For this problem: $m_{\text{consumed}} \approx 1.05 \times 1000 \times 0.8 \times 5 \approx 4200$ kg total, so $m_{\text{initial}} \approx 8400$ kg... wait, this doesn't match! Always use the detailed calculation — the simplified formula varies with enrichment and burnup assumptions.

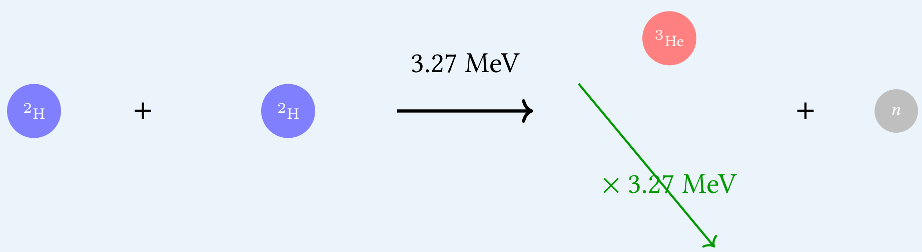
Q19 How long can an electric lamp of 100W be kept glowing by fusion of 2.0 kg of deuterium? Take the fusion reaction as

$$^2_1\text{H} + ^2_1\text{H} \longrightarrow ^3_2\text{He} + n + 3.27\ \text{MeV}$$

Solution

Understanding the Problem

We need to find how long a 100 W lamp can be powered using the energy released from the fusion of 2.0 kg of deuterium. The D-D fusion reaction releases 3.27 MeV per reaction, which consumes two deuterium atoms.



Step 1: Determine the Number of Deuterium Atoms in 2.0 kg

Given:

- Mass of deuterium: $m = 2.0$ kg $= 2000$ g
- Molar mass of deuterium (^{2}H): $M = 2.014102$ g/mol
- Avogadro's number: $N_A = 6.022 \times 10^{23}$ mol^{-1}

$$N = \frac{m}{M} \times N_A$$

$$N = \frac{2000}{2.014102} \times 6.022 \times 10^{23}$$

$$N = 993.00 \times 6.022 \times 10^{23}$$

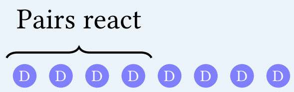
$$N = 5.980 \times 10^{26} \text{ atoms of deuterium}$$

Step 2: Determine the Number of Fusion Reactions
Each fusion reaction consumes **two** deuterium atoms:

$$^{2}\text{H} + {}^{2}\text{H} \longrightarrow {}^{3}\text{He} + n + 3.27 \text{ MeV}$$

Number of possible reactions:

$$N_{\text{reactions}} = \frac{N}{2} = \frac{5.980 \times 10^{26}}{2} = 2.990 \times 10^{26}$$



Step 3: Calculate the Total Energy Released
Energy per reaction:

$$E_{\text{per reaction}} = 3.27 \text{ MeV} = 3.27 \times 1.602 \times 10^{-13} \text{ J}$$

$$E_{\text{per reaction}} = 5.239 \times 10^{-13} \text{ J}$$

Total energy:

$$E_{\text{total}} = N_{\text{reactions}} \times E_{\text{per reaction}}$$

$$E_{\text{total}} = 2.990 \times 10^{26} \times 5.239 \times 10^{-13} \text{ J}$$

$$E_{\text{total}} = 1.566 \times 10^{14} \text{ J}$$

Energy Comparison:

Source	**Energy (J)**
2 kg deuterium fusion	1.57×10^{14}
2 kg ^{235}U fission	$\sim 1.46 \times 10^{14}$
2 kg coal burning	$\sim 6 \times 10^{7}$
2 kg TNT	$\sim 8.4 \times 10^{6}$

Step 4: Calculate How Long the Lamp Can Glow

Power-Energy-Time Relation:

$$P = \frac{E}{t} \quad \Rightarrow \quad t = \frac{E}{P}$$

where $P = 100 \text{ W} = 100 \text{ J/s}$

$$t = \frac{1.566 \times 10^{14}\ \text{J}}{100\ \text{J/s}}$$

$$t = 1.566 \times 10^{12}\ \text{seconds}$$

Step 5: Convert to More Meaningful Units

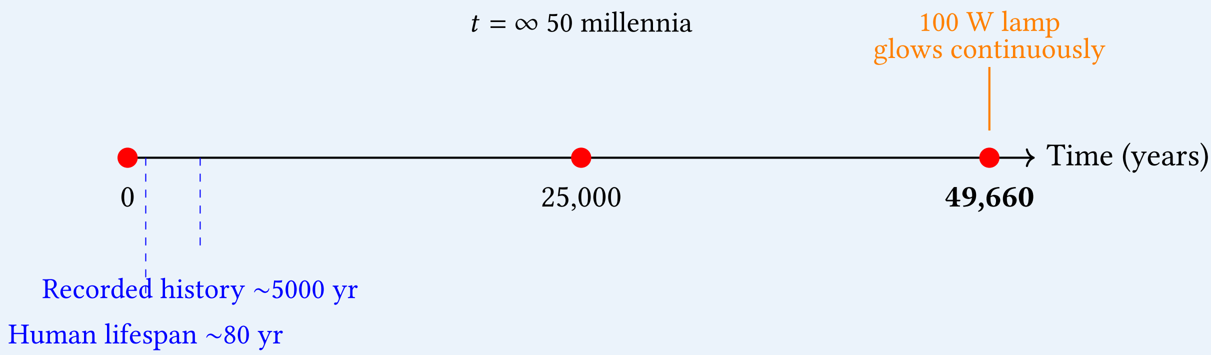
$$t = \frac{1.566 \times 10^{12}}{3600 \times 24 \times 365}\ \text{years}$$

$$t = \frac{1.566 \times 10^{12}}{3.1536 \times 10^{7}}\ \text{years}$$

$$t = 4.966 \times 10^{4}\ \text{years}$$

$$t \approx 49,660\ \text{years}$$

In words: approximately **50,000 years**!



Alternative Quick Method:
Using MeV directly and converting at the end:

$$E_{\text{total}} = 2.990 \times 10^{26} \times 3.27\ \text{MeV} = 9.777 \times 10^{26}\ \text{MeV}$$

$$E_{\text{total}} = 9.777 \times 10^{26} \times 1.602 \times 10^{-13}\ \text{J} = 1.566 \times 10^{14}\ \text{J}$$

Time:

$$t = \frac{1.566 \times 10^{14}}{100} = 1.566 \times 10^{12}\ \text{s}$$

Final Answer:

$$\boxed{t = 1.57 \times 10^{12}\ \text{seconds} \approx 4.97 \times 10^{4}\ \text{years} \approx \mathbf{50,000\ years}}$$

The fusion of just 2.0 kg of deuterium could keep a 100 W electric lamp glowing continuously for approximately **50,000 years** — longer than all of recorded human civilization! This illustrates the extraordinary energy density of nuclear fusion.

Expert's Solution — Radhika Menon, B.Tech CSE, NIT Calicut

The Promise and Challenge of Fusion Energy
This problem reveals why nuclear fusion is considered the "holy grail" of energy production. Let's explore the numbers deeper.
Energy Density of Deuterium:

$$\frac{E_{\text{total}}}{m} = \frac{1.566 \times 10^{14}\ \text{J}}{2.0\ \text{kg}} = 7.83 \times 10^{13}\ \text{J/kg}$$

This is about **3–4 times higher** than fission of uranium or plutonium!
Deuterium Abundance:

Deuterium in Earth's Oceans:

- Natural abundance: 1 deuterium per 6,420 hydrogen atoms (0.0156%)
- Total water in oceans: $\sim 1.4 \times 10^{21}$ kg
- Deuterium in oceans: $\sim 4.6 \times 10^{13}$ kg
- Energy if all fused: $\sim 3.6 \times 10^{27}$ J
- At current global energy consumption rate ($\sim 5\times10^{20}$ J/year):
  $\Rightarrow$ **Would last ~7 million years!**

Comparison of D-D Fusion Branches:
The D-D reaction has two nearly equally probable branches:

$$^2\text{H} + {}^2\text{H} \longrightarrow {}^3\text{He} + n \qquad (Q = 3.27\ \text{MeV})$$
$$^2\text{H} + {}^2\text{H} \longrightarrow {}^3\text{H} + p \qquad (Q = 4.03\ \text{MeV})$$

Our problem uses the first branch. In reality, both occur, and the average energy per D-D reaction is slightly different. The tritium from the second branch can also undergo subsequent D-T fusion, releasing even more energy:

$$^2\text{H} + {}^3\text{H} \longrightarrow {}^4\text{He} + n \quad (Q = 17.6\ \text{MeV})$$

The Reality Check — Why We Don't Have Fusion Lamps:
Despite this incredible energy density, practical fusion power remains elusive because:

1. **Extreme conditions required:** Temperatures $\sim 10^8$ K (hotter than the Sun's core)
2. **Confinement challenge:** No material container can hold such hot plasma
3. **Energy balance:** Current experiments consume more energy than they produce
4. **Tritium breeding:** Most practical designs use D-T fusion, but tritium must be bred from lithium

Did You Know?

Quick Estimation for Fusion Energy Problems:
For any fusion reaction: $^2\text{H} + {}^2\text{H} \rightarrow \text{products} + Q$,

$$\text{Energy from } m \text{ kg } D_2 = \frac{m\,(\text{kg}) \times N_A \times Q\,(\text{J})}{2 \times M\,(\text{kg/mol})}$$

Approximations for D-D fusion:

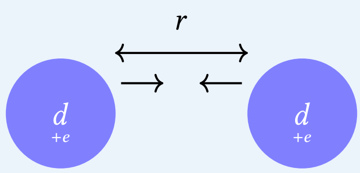
$$1 \text{ kg D} \approx 3.9 \times 10^{14} \text{ J} \quad ; \quad 1 \text{ g D} \approx 110,000 \text{ kWh}$$

Mnemonic: "A postage stamp's worth of deuterium (1 gram) can power a 100 W bulb for 125 years."

Q20 **Calculate the height of the potential barrier for a head on collision of two deuterons. (Hint: The height of the potential barrier is given by the Coulomb repulsion between the two deuterons when they just touch each other. Assume that they can be taken as hard spheres of radius 2.0 fm.)**

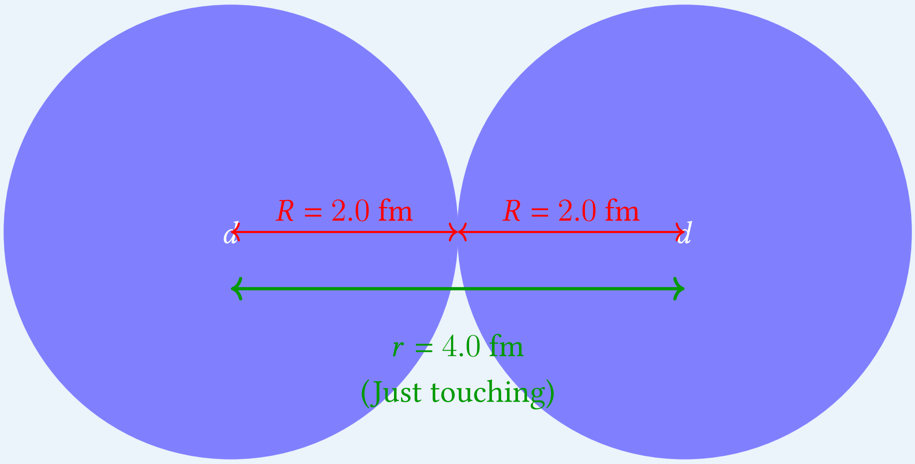
Solution

Understanding the Potential Barrier in Nuclear Fusion
For two nuclei to undergo fusion, they must overcome the Coulomb repulsion between their positive charges and come close enough for the attractive strong nuclear force to take over. The potential barrier is the maximum electrostatic potential energy when the two nuclei just touch each other.



Step 1: Understand the Configuration at the Barrier
When the two deuterons just touch each other, the distance between their centers equals the sum of their radii:

$$r = R_1 + R_2 = 2.0 \text{ fm} + 2.0 \text{ fm} = 4.0 \text{ fm}$$



Step 2: Calculate the Coulomb Potential Energy

Coulomb Potential Energy between Two Charged Spheres:

$$U = \frac{1}{4\pi\varepsilon_0} \times \frac{q_1 q_2}{r}$$

where:

- $q_1 = q_2 = +e = 1.602 \times 10^{-19}$ C (each deuteron has one proton)
- $r = R_1 + R_2 = 4.0$ fm $= 4.0 \times 10^{-15}$ m
- $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9$ N·m^2/C^2

$$U = 9 \times 10^9 \times \frac{(1.602 \times 10^{-19})^2}{4.0 \times 10^{-15}}$$

Step 3: Compute the Value
Numerator:

$$(1.602 \times 10^{-19})^2 = 2.566 \times 10^{-38} \text{ C}^2$$

$$U = 9 \times 10^9 \times \frac{2.566 \times 10^{-38}}{4.0 \times 10^{-15}}$$

$$U = \frac{9 \times 10^9 \times 2.566 \times 10^{-38}}{4.0 \times 10^{-15}}$$

$$U = \frac{2.309 \times 10^{-28}}{4.0 \times 10^{-15}}$$

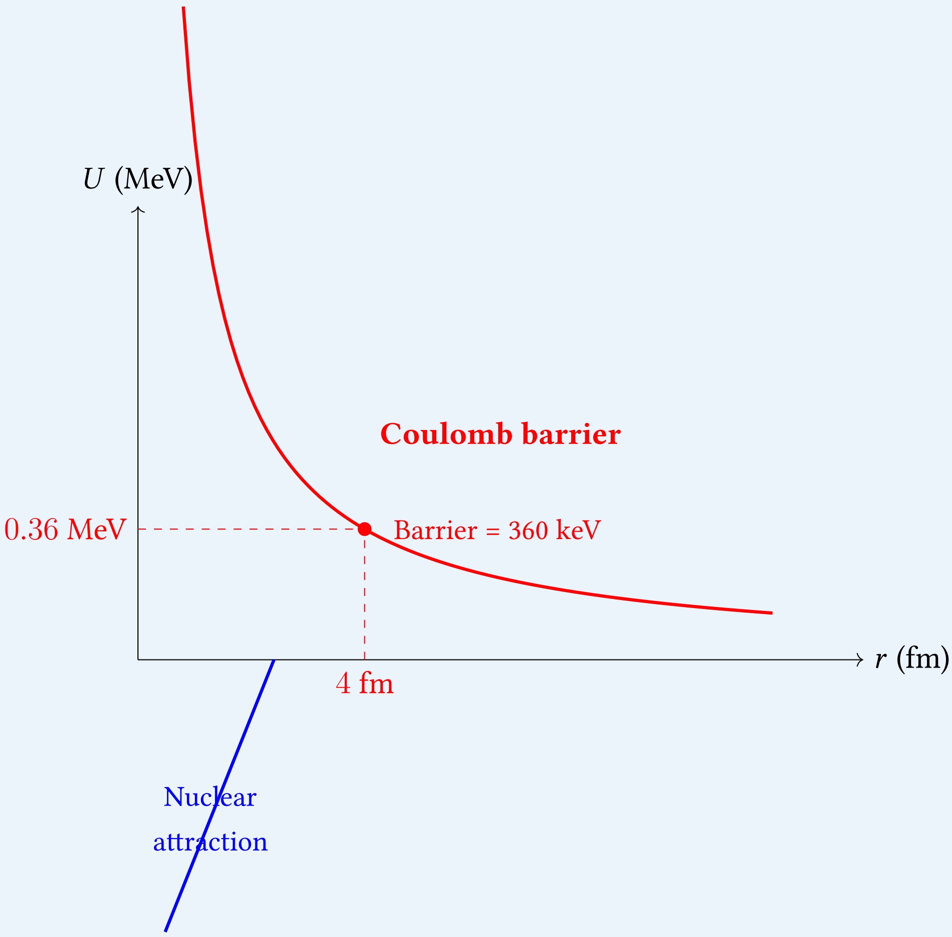
$$U = 5.773 \times 10^{-14} \text{ J}$$

Step 4: Convert to Electron Volts (eV)

Since nuclear energies are typically expressed in MeV:

$$U = \frac{5.773 \times 10^{-14}\ \text{J}}{1.602 \times 10^{-13}\ \text{J/MeV}}$$

$$U = 0.3604\ \text{MeV} = 360.4\ \text{keV}$$



Step 5: Compare with Thermal Energies
The typical thermal energy of particles at temperature T is:

$$E_{\text{thermal}} = k_B T$$

where $k_B = 8.617 \times 10^{-5}$ eV/K
To overcome a barrier of 360 keV classically, a temperature would be needed:

$$T = \frac{0.360 \times 10^{6}\ \text{eV}}{8.617 \times 10^{-5}\ \text{eV/K}} \approx 4.2 \times 10^{9}\ \text{K}$$

This is far higher than temperatures in the Sun's core ($\sim 1.5 \times 10^{7}$ K), yet fusion occurs in the Sun due to **quantum tunneling** through the barrier!

The Tunneling Miracle:

Process	Temperature (K)	Barrier (keV)
D-D fusion (lab plasma)	$\sim 10^8$	360
p-p fusion (Sun core)	1.5×10^7	~ 500
D-T fusion (ITER)	$\sim 1.5 \times 10^8$	~ 300

Quantum tunneling allows fusion at temperatures **100–1000 times lower** than classical prediction!

Final Answer:

$$\boxed{U_{\text{barrier}} = 5.8 \times 10^{-14} \text{ J} = 360 \text{ keV} = 0.36 \text{ MeV}}$$

The height of the Coulomb potential barrier for head-on collision of two deuterons is approximately **360 keV** (or 0.36 MeV). Classically, this would require temperatures of $\sim 4 \times 10^9$ K, but quantum mechanical tunneling allows fusion to occur at much lower temperatures ($\sim 10^8$ K).

Expert's Solution — Kartik Iyer, B.Tech Engineering Physics, IIT Indore

The Coulomb Barrier — Why Fusion is Hard

The Coulomb barrier is the fundamental obstacle to nuclear fusion. Understanding it quantitatively illuminates both the challenges of fusion energy and the conditions inside stars.

General Formula for the Coulomb Barrier:

For two nuclei with atomic numbers Z_1 and Z_2 and radii R_1 and R_2:

$$U_{\text{barrier}} = \frac{1}{4\pi\varepsilon_0} \frac{Z_1 Z_2 e^2}{R_1 + R_2}$$

Using the nuclear radius formula $R = R_0 A^{1/3}$ with $R_0 = 1.2$ fm:

$$U_{\text{barrier}} = \frac{Z_1 Z_2 \times 1.44 \text{ MeV·fm}}{R_0 (A_1^{1/3} + A_2^{1/3})}$$

Barrier Heights for Different Fusion Reactions:

Comparison of Fusion Barriers:

Reaction	$Z_1 Z_2$	Barrier (MeV)	Difficulty
D + D	$1 \times 1 = 1$	~ 0.36	Moderate
D + T	$1 \times 1 = 1$	~ 0.30	Easiest
D + ^{3}He	$1 \times 2 = 2$	~ 0.65	Harder
p + p	$1 \times 1 = 1$	~ 0.55	Hardest in stars
^{12}C + ^{12}C	$6 \times 6 = 36$	~ 7.5	Very hard

Why D-T is the Easiest Fusion Reaction:

1. **Lowest barrier:** Both deuterium and tritium have $Z = 1$
2. **Larger radii:** Tritium ($A = 3$) has slightly larger radius, reducing the barrier
3. **Resonance:** The D-T reaction has a resonance at low energies, greatly enhancing the cross-section
4. **High Q-value:** 17.6 MeV released, much more than D-D

This is why ITER and most fusion experiments focus on D-T fusion, despite the challenge of tritium availability.

The Gamow Factor and Quantum Tunneling:

The probability of tunneling through the Coulomb barrier is given approximately by the Gamow factor:

$$P_{\text{tunnel}} \propto \exp\left(-\sqrt{\frac{E_G}{E}}\right)$$

where $E_G = 2m_r c^2(\pi\alpha Z_1 Z_2)^2$ is the Gamow energy.

For D-D fusion, $E_G \approx 1.3$ MeV, so even at $E = 10$ keV (typical for fusion plasmas), there's a small but non-zero tunneling probability that enables fusion.

★ **Did You Know?**

Quick Calculation of Coulomb Barrier:

For equal spheres touching:

$$U\ (\text{MeV}) = \frac{Z_1 Z_2 \times 1.44\ \text{MeV}\cdot\text{fm}}{R_1 + R_2\ (\text{fm})}$$

The constant 1.44 MeV·fm comes from:

$$\frac{e^2}{4\pi\varepsilon_0} = 1.44\ \text{MeV}\cdot\text{fm}$$

For deuterons: $U = \dfrac{1 \times 1 \times 1.44}{2.0 + 2.0} = 0.36$ MeV

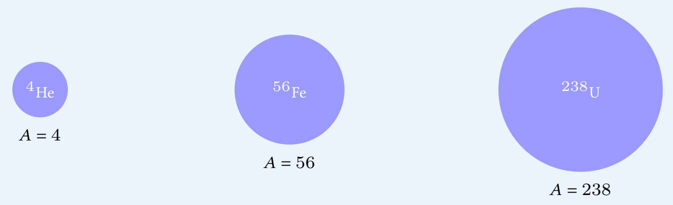
Memorize this: $e^2/(4\pi\varepsilon_0) = 1.44$ MeV·fm — it's one of the most useful constants in nuclear physics!

Q21 From the relation $R = R_0 A^{1/3}$, where R_0 is a constant and A is the mass number of a nucleus, show that the nuclear matter density is nearly constant (i.e. independent of A).

Solution

Understanding Nuclear Density

The fact that nuclear radius scales as $R = R_0 A^{1/3}$ implies something profound: the density of nuclear matter is approximately the same for all nuclei, from the lightest to the heaviest. Let's prove this mathematically.



All have the same density $\rho \approx 2.3 \times 10^{17}$ kg/m^3

Step 1: Express Nuclear Volume in Terms of A

Assuming the nucleus is approximately spherical:

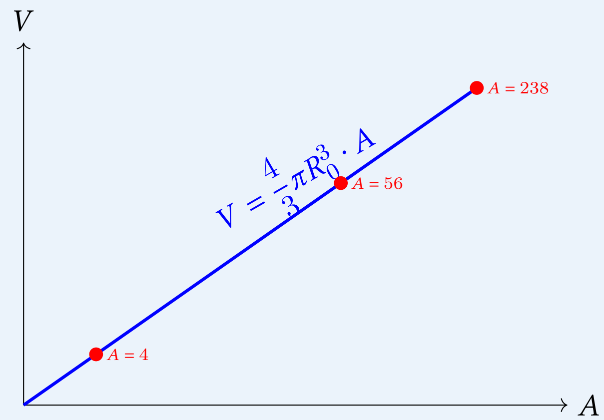
$$V = \frac{4}{3}\pi R^3$$

Substituting the given relation $R = R_0 A^{1/3}$:

$$V = \frac{4}{3}\pi (R_0 A^{1/3})^3$$

$$V = \frac{4}{3}\pi R_0^3 A$$

This shows that the nuclear volume is **directly proportional** to the mass number A.



Volume $\propto A$ — linear relationship

Step 2: Express Nuclear Mass in Terms of A

The mass of the nucleus is approximately proportional to the number of nucleons:

$$M \approx A \cdot m_n$$

where $m_n \approx 1.67 \times 10^{-27}$ kg is the mass of a nucleon (proton or neutron mass, approximately equal).
More precisely, m_n can be taken as the atomic mass unit:

$$m_n \approx 1 \text{ u} = 1.660539 \times 10^{-27} \text{ kg}$$

Step 3: Calculate Nuclear Density

Nuclear Matter Density:

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{V}$$

Substituting the expressions for mass and volume:

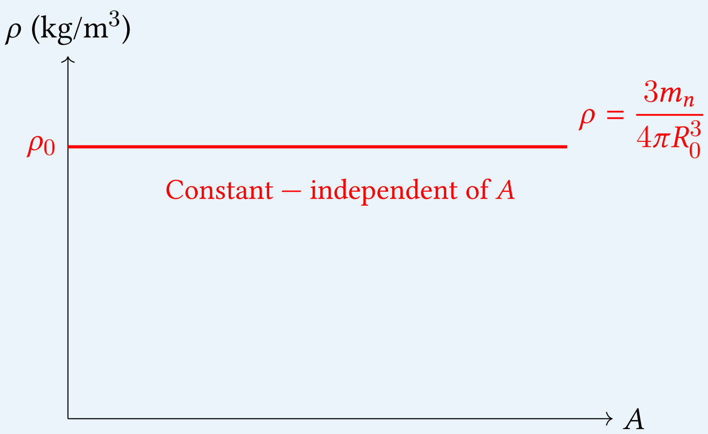
$$\rho = \frac{A \cdot m_n}{\frac{4}{3}\pi R_0^3 A}$$

Step 4: Observe the Cancellation of A

$$\rho = \frac{A \cdot m_n}{\frac{4}{3}\pi R_0^3 A} = \frac{m_n}{\frac{4}{3}\pi R_0^3}$$

The factor A cancels completely! Therefore:

$$\boxed{\rho = \frac{3m_n}{4\pi R_0^3} = \text{constant}}$$



Step 5: Calculate the Numerical Value of Nuclear Density
Using $R_0 = 1.2$ fm $= 1.2 \times 10^{-15}$ m and $m_n = 1.67 \times 10^{-27}$ kg:

$$\rho = \frac{3 \times 1.67 \times 10^{-27}}{4\pi \times (1.2 \times 10^{-15})^3}$$

$$\rho = \frac{5.01 \times 10^{-27}}{4\pi \times 1.728 \times 10^{-45}}$$

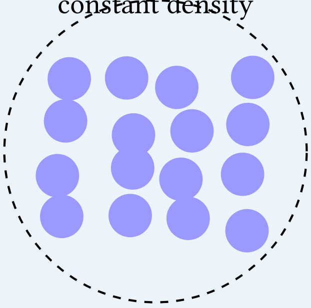
$$\rho = \frac{5.01 \times 10^{-27}}{2.171 \times 10^{-44}}$$

$$\rho \approx 2.3 \times 10^{17} \text{ kg/m}^3$$

What Does 2.3×10^{17} kg/m^3 Mean?

- **One teaspoon** of nuclear matter would weigh ~ **1 billion tons**!
- Density is about 10^{14} **times** that of ordinary matter (~ 10^3 kg/m^3)
- Comparable to neutron star densities (~ 10^{17} kg/m^3)
- This extreme density arises because nucleons are packed nearly touching each other

Physical Interpretation:



Key insight:

Adding more nucleons simply increases the volume proportionally, keeping the density unchanged.

This is analogous to adding more identical marbles to a bag — the packing density remains the same.

Proof:

$$\rho = \frac{M}{V} = \frac{Am_n}{\frac{4}{3}\pi(R_0A^{1/3})^3} = \frac{Am_n}{\frac{4}{3}\pi R_0^3 A} = \frac{3m_n}{4\pi R_0^3} = \text{constant}$$

The nuclear matter density is **independent of** A and approximately equal to:

$$\boxed{\rho \approx 2.3 \times 10^{17}\ \text{kg/m}^3}$$

This constancy of nuclear density is a direct consequence of the $R \propto A^{1/3}$ relationship and demonstrates that nucleons in all nuclei are packed at the same density — like incompressible nuclear fluid.

Expert's Solution — Sneha Reddy, B.Tech CSE, NIT Andhra Pradesh

The Liquid Drop Model — Nucleus as an Incompressible Fluid

The constancy of nuclear density is one of the most important properties of nuclear matter and forms the foundation of the **liquid drop model** of the nucleus.

Why Constant Density Matters:

Analogies with a Liquid Drop:

Property	Liquid Drop	Nucleus
Density	Nearly constant	Nearly constant
Volume	$\propto$ Number of molecules	$\propto A$ (nucleon number)
Binding	Surface tension	Surface energy $\propto A^{2/3}$
Shape	Spherical (surface tension)	Spherical (surface energy)
Compressibility	Very low	Very low (incompressible)
Radius	$\propto$ (No. of molecules)$^{1/3}$	$\propto A^{1/3}$

The Saturation of Nuclear Forces:
The constant density reveals a fundamental property of the strong nuclear force — **saturation**. Each nucleon interacts only with its nearest neighbors (like molecules in a liquid), not with all other nucleons. If each nucleon attracted all others (like gravity), the density would increase with A.
Evidence from Experiment:
The $R = R_0 A^{1/3}$ law is confirmed by multiple experimental techniques:

- **Electron scattering:** High-energy electrons probe nuclear charge distribution
- **α-particle scattering:** Rutherford-type experiments with heavy projectiles
- **Muonic X-rays:** Muons in atomic orbits are closer to the nucleus and probe its size
- **Nuclear spectroscopy:** Energy levels in mirror nuclei

All confirm $R_0 \approx 1.2$ fm and the $A^{1/3}$ dependence.
The Nuclear Density in Neutron Stars:
When massive stars collapse, gravity crushes atoms, combining electrons and protons into neutrons. The resulting neutron star has a density comparable to nuclear density:

$$\rho_{\text{neutron star}} \sim 10^{17} - 10^{18} \text{ kg/m}^3$$

A neutron star is essentially a giant nucleus (with $A \sim 10^{57}$) held together by gravity instead of the strong force!

Did You Know?

Key Takeaways:

1. $R \propto A^{1/3}$ is the experimental fact
2. $V \propto A$ follows from geometry ($V \propto R^3$)
3. ρ = **constant** follows from $M \propto A$ and $V \propto A$
4. The cancellation of A in $\rho = \frac{Am_n}{\frac{4}{3}\pi R_0^3 A}$ is the mathematical proof

Memorize: Nuclear density $\approx 2.3 \times 10^{17}$ kg/m^3 — one of the fundamental constants of nature, like the speed of light or Planck's constant.

Q22 **For the β^+ (positron) emission from a nucleus, there is another competing process known as electron capture (electron from an inner orbit, say, the K–shell, is captured by the nucleus and a neutrino is emitted).**

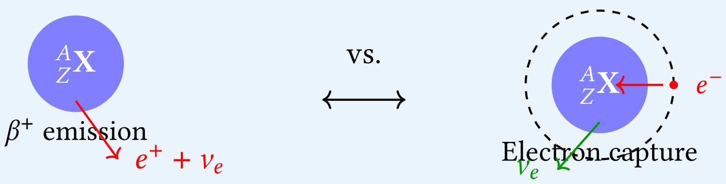
$$^{A}_{Z}\mathbf{X} + e^- \longrightarrow {}^{A}_{Z-1}\mathbf{Y} + \nu_e$$

Show that if β^+ emission is energetically allowed, electron capture is necessarily allowed but not vice–versa.

Solution

Understanding the Competing Decay Modes

Both β^+ (positron emission) and electron capture (EC) transform a proton into a neutron, reducing the atomic number by 1 while keeping the mass number unchanged. However, their energy requirements differ significantly.



Step 1: Write the Q-value for β^+ Emission

The β^+ decay reaction (in terms of nuclei):

$$^{A}_{Z}\text{X}_{\text{nucleus}} \longrightarrow {}^{A}_{Z-1}\text{Y}_{\text{nucleus}} + e^+ + \nu_e$$

In terms of atomic masses, we must account for electrons:

- Parent atom has Z electrons
- Daughter atom has $Z-1$ electrons
- A positron (m_e) is created and emitted

Q-value for β^+ emission (using atomic masses):

$$Q_{\beta^+} = [m_{\text{atom}}(X) - m_{\text{atom}}(Y) - 2m_e] \times c^2$$

The $2m_e$ accounts for: (i) positron creation, and (ii) the extra electron in the parent atom mass.

For β^+ emission to be energetically allowed:

$$Q_{\beta^+} > 0 \quad \Longrightarrow \quad m_{\text{atom}}(X) - m_{\text{atom}}(Y) > 2m_e$$

Equivalently:

$$m_{\text{atom}}(X) - m_{\text{atom}}(Y) > 1.022\ \text{MeV}/c^2$$

Step 2: Write the Q-value for Electron Capture

The electron capture reaction (in terms of nuclei):

$${}^{A}_{Z}\text{X}_{\text{nucleus}} + e^- \longrightarrow {}^{A}_{Z-1}\text{Y}_{\text{nucleus}} + \nu_e$$

The captured electron comes from the atomic orbital (usually K-shell). In terms of atomic masses:

- Parent atom has Z electrons (including the one to be captured)
- Daughter atom has $Z - 1$ electrons
- The captured electron mass is already included in the parent atomic mass

Q-value for Electron Capture (using atomic masses):

$$Q_{\text{EC}} = [m_{\text{atom}}(X) - m_{\text{atom}}(Y)] \times c^2$$

No $2m_e$ correction needed — the captured electron is already accounted for in the atomic mass.

Electron Accounting — Why No Correction for EC:

	Parent (X)	**Daughter (Y)**	**Net**
Nuclear mass	$m_{\text{nuc}}(X)$	$m_{\text{nuc}}(Y)$	
Atomic electrons	Zm_e	$(Z-1)m_e$	
Atomic mass	$m_{\text{nuc}}(X) + Zm_e$	$m_{\text{nuc}}(Y) + (Z-1)m_e$	

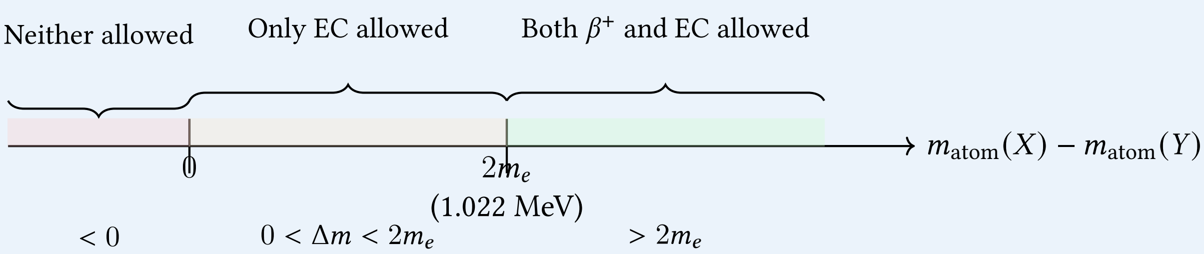
For EC: $Q = m_{\text{nuc}}(X) + m_e - m_{\text{nuc}}(Y) = m_{\text{atom}}(X) - m_{\text{atom}}(Y)$

The electron masses cancel perfectly!

For electron capture to be energetically allowed:

$$Q_{\text{EC}} > 0 \quad \Rightarrow \quad m_{\text{atom}}(X) - m_{\text{atom}}(Y) > 0$$

Step 3: Compare the Two Conditions



Step 4: The Logical Proof

1. **If β^+ is allowed:**

$$\begin{aligned} Q_{\beta^+} &> 0 \\ m_{\text{atom}}(X) - m_{\text{atom}}(Y) - 2m_e &> 0 \\ m_{\text{atom}}(X) - m_{\text{atom}}(Y) &> 2m_e > 0 \end{aligned}$$

   Since $m_{\text{atom}}(X) - m_{\text{atom}}(Y) > 0$, we have $Q_{\text{EC}} > 0$.

$$\boxed{\beta^+ \text{ allowed} \implies \text{EC allowed}}$$

2. **But EC being allowed does not imply β^+ is allowed:**
   If $0 < m_{\text{atom}}(X) - m_{\text{atom}}(Y) < 2m_e$, then:

$$\begin{aligned} Q_{\text{EC}} &= [m_{\text{atom}}(X) - m_{\text{atom}}(Y)]c^2 > 0 \quad \checkmark \\ Q_{\beta^+} &= [m_{\text{atom}}(X) - m_{\text{atom}}(Y) - 2m_e]c^2 < 0 \quad \times \end{aligned}$$

$$\boxed{\text{EC allowed} \;\not\Longrightarrow\; \beta^+ \text{ allowed}}$$

Summary of the Argument:

Condition	**β^+ possible?**	**EC possible?**
$\Delta m < 0$	No	No
$0 < \Delta m < 2m_e$	**No**	**Yes**
$\Delta m > 2m_e$	**Yes**	**Yes**

where $\Delta m = m_{\text{atom}}(X) - m_{\text{atom}}(Y)$

$\Rightarrow$ EC has a **wider window** of energetic possibility than β^+

$\Rightarrow$ If β^+ is allowed ($\Delta m > 2m_e$), EC is automatically allowed ($\Delta m > 0$)

$\Rightarrow$ The converse is not true (EC can occur when β^+ cannot)

Physical Reasoning:
Electron capture requires less energy because:

- In EC, the nucleus "borrows" an existing orbital electron — no need to create one
- In β^+, the nucleus must **create** a positron from energy ($E = m_e c^2$)
- Additionally, the daughter atom has one fewer electron in β^+, so the atomic mass difference must cover creating the positron AND the missing electron — hence $2m_e$

Real Example: $^{64}_{29}$Cu

$m(^{64}\text{Cu}) = 63.929764$ u $m(^{64}\text{Ni}) = 63.927966$ u

$\Delta m = 63.929764 - 63.927966 = 0.001798 \text{ u} = 1.675 \text{ MeV}/c^2$

Since $1.675 > 1.022 \text{ MeV}/c^2$:

Both β^+ and EC are possible

Indeed, ^{64}Cu decays by both modes!

β^+ (19%, $T_{1/2} = 12.7$ h) EC (41%, $T_{1/2} = 12.7$ h)

Proof:

$$\boxed{Q_{\beta^+} > 0 \iff \Delta m > 2m_e \implies \Delta m > 0 \iff Q_{\text{EC}} > 0}$$

If β^+ emission is energetically allowed ($\Delta m > 2m_e$), then electron capture is **necessarily** allowed because $\Delta m > 2m_e > 0$ satisfies the EC condition. However, if EC is allowed ($\Delta m > 0$), β^+ may still be forbidden if Δm lies between 0 and $2m_e$ (~1.022 MeV). Therefore, electron capture is energetically possible for a **wider range** of nuclear mass differences than positron emission.

Expert's Solution — Ankita Patel, B.Tech CSE, NIT Raipur

The Competition Between β^+ and Electron Capture

This problem reveals a fundamental aspect of nuclear physics: when a nucleus is proton-rich, nature provides two pathways to stability, but one is energetically "cheaper" than the other.

Why the $2m_e$ Threshold Matters:

The $2m_e$ threshold (1.022 MeV) is a fundamental energy scale in particle physics — it's the energy required for **pair production** (creating an electron-positron pair from energy). In β^+ decay:

$$\text{Energy needed} = \underbrace{(m_D + m_e)}_{\text{daughter atom mass}} + \underbrace{m_e}_{\text{positron}} = m_D + 2m_e$$

Experimental Evidence — Competing Decays:

Many neutron-deficient isotopes show both decay modes:

Nuclides with Both β^+ and EC Decay:

Nuclide	Δm (MeV)	β^+ %	EC %
^{11}C	1.983	99.8%	0.2%
^{13}N	2.221	99.9%	0.1%
^{22}Na	2.842	90%	10%
^{64}Cu	1.675	19%	41%
^{68}Ga	2.921	90%	10%

When Δm is large ($\gg 2m_e$), β^+ dominates. When Δm is close to $2m_e$, EC becomes competitive or dominant.

EC-Only Decay — The Window Between 0 and $2m_e$:
When $0 < \Delta m < 1.022$ MeV, electron capture is the **only** possible decay mode. Examples:

$$^{55}\text{Fe} \xrightarrow{\text{EC}} {}^{55}\text{Mn} \quad (\Delta m \approx 0.231 \text{ MeV})$$

$$^{71}\text{Ge} \xrightarrow{\text{EC}} {}^{71}\text{Ga} \quad (\Delta m \approx 0.233 \text{ MeV})$$

The K-Capture Advantage:
Electron capture usually involves a K-shell electron (closest to the nucleus). The probability scales as:

$$P_{\text{capture}} \propto |\psi(0)|^2 \propto Z^3$$

Heavy nuclei (Z large) have K-electrons closer to the nucleus, making EC more probable. This is why EC dominates over β^+ in many heavy nuclei even when both are energetically possible.

Did You Know?

Decision Tree for Proton-Rich Nuclei:
Given $m_{\text{atom}}(X)$ and $m_{\text{atom}}(Y)$ for ${}^{A}_{Z}X \longrightarrow {}^{A}_{Z-1}Y$:

1. **If** $\Delta m < 0$**:** Neither β^+ nor EC possible — nucleus is stable against both
2. **If** $0 < \Delta m < 2m_e$**: Only EC** is possible
3. **If** $\Delta m > 2m_e$**: Both** β^+ and EC are possible (they compete)

Key numbers to remember:

- $2m_e c^2 = 1.022$ MeV
- $m_e c^2 = 0.511$ MeV
- Δm (in u) $\times 931.5 = \Delta m$ (in MeV/c^2)