Chapter 7

Alternating Current

Voltage or current that changes its sign periodically with time is called alternating.
Most common form : sinusoidal AC.

Why AC ?

1. Easily stepped up / down by transformers.
2. Long - distance transmission with low losses.
3. AC generators are simpler than DC ones.

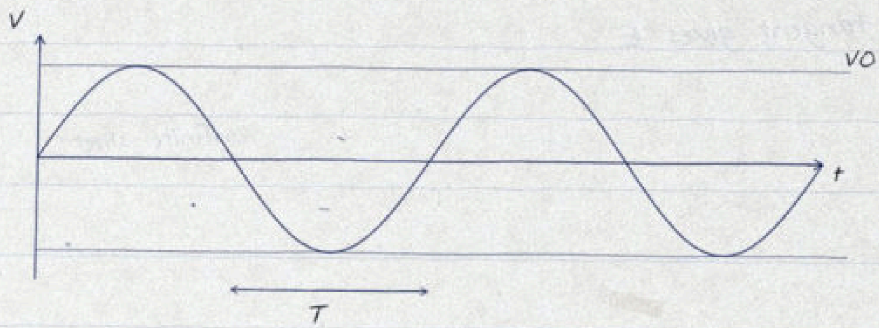


Fig. sinusoidal AC voltage waveform

Topics ahead

RMS ; R , L , C circuits ; LCR ; transformer.

AC Source - Instantaneous V & I

Sinusoidal form

An AC source delivers an EMF that varies as :

$$\boxed{V(t) = V_0 \sin \omega t}$$

<- V0 = peak
<- voltage

$$\boxed{I(t) = I_0 \sin (\omega t + \phi)}$$

<- phi = phase
<- difference

Useful quantities

Angular frequency : $\omega = 2 \pi f$

Time period : $T = 1 / f = 2 \pi / \omega$

In India : f = 50 Hz , omega = 100 pi rad/s.

Average over full cycle

ω V dt over 0 to T = 0 ; same for I.

Mean of pure sine over one cycle is zero ;

so we use half - cycle mean or RMS instead.

Phase relations (coming up)

Pure R : V , I in phase (phi = 0)

Pure L : V leads I by pi/2

Pure C : V lags I by pi/2

LCR : phase depends on X_L vs X_C.

Mean (Average) Value over Half Cycle

For positive half of cycle (0 to T/2) :

$I_{avg} = (1/(T/2))$ sum $I_0 \sin wt \, dt$

Limits : $t = 0$ to $t = T/2$.

$= (2 I_0 / T) \; [- \cos wt / w]$ from 0 to T/2

$= (2 I_0 / T) \, . \, (2 / w)$

$= (2 I_0 / T) \, . \, (T / \pi)$

$$\boxed{I_{avg} = 2 I_0 / \pi \qquad 0.637 \, I_0}$$

<- half - cycle
<- mean

Similarly : V_avg = 2 VO / pi   0.637 VO.

Why this* matters

DC ammeters cannot measure AC because mean over full cycle is ~~not zero~~ zero. Hot - wire meters / RMS meters are used instead.

Hence the RMS value - next page.

RMS Value (Root - Mean - Square)

Definition -

RMS current = steady DC current that produces the same heat in R over the same time.

Heat in R over one cycle :

$$H = \text{sum } I^2 R \, dt \quad \text{from } 0 \text{ to } T$$

$$= R \, I_0^2 \ \text{sum} \ \sin^2 wt \, dt$$

use $\sin^2 x = (1 - \cos 2x) / 2$:

$$= R \, I_0^2 \, . \, T / 2$$

Equate to I_rms 2 R T :

$$\boxed{I_{rms}^2 = I_0^2 / 2}$$

$$\boxed{I_{rms} = I_0 / \text{sqrt}(2) \qquad 0.707 \, I_0}$$

<- key result
<- for sinusoid

Similarly V_rms = VO /. sqrt(2).

Indian mains : 220 V is V_rms , so VO 311 V.

AC Through Pure Resistor

. Source $V = VO \sin omega\ t$ across resistance R.

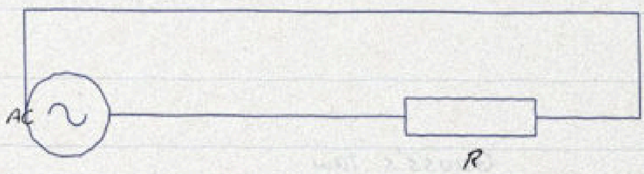


Fig. AC source + pure resistor

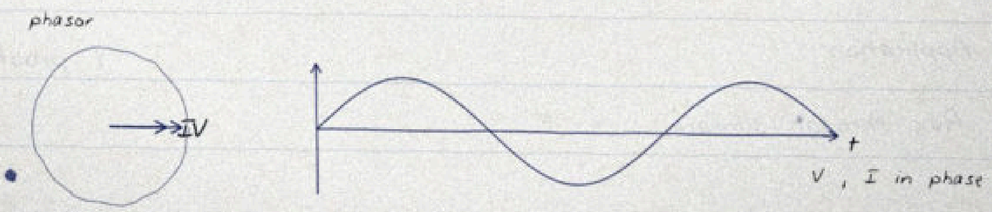
Derivation

KVL .: $V = I R \rightarrow I = V/R$

$= (V_0/R) \sin w t$

$$\boxed{I = I_0 \sin w t \ ; \ I_0 = V_0 / R}$$

<- phi = 0
<- (in phase)



AC Through Pure Inductor

Source V = VO sin omega t across an ideal inductor of inductance L (no resistance).

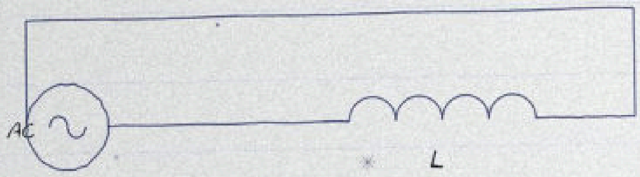


Fig. AC across pure inductor

Derivation

Self - induced EMF : $E_L = - L\, dI / dt$

KVL : $V - L\, dI/dt = 0$ -> $L\, dI/dt = V$

$dI / dt = (V_0/L) \sin \omega t$

Integrate : $I = - (V_0/\omega L) \cos \omega t$

$= (V_0/\omega L) \sin (\omega t - \pi/2)$

$\boxed{I = I_0 \sin (\omega t - \pi/2)}$ <- I lags V
<- by pi/2

$\boxed{X_L = \omega L \ ; \ I_0 = V_0 / X_L}$

Inductive Reactance & Phasor (L)

Inductive reactance X_L

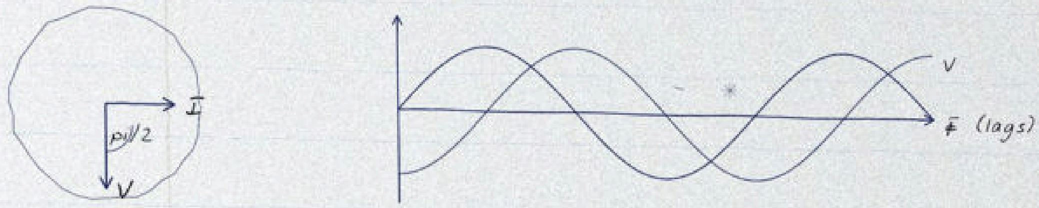
$$\boxed{X_L = wL = 2\pi f L}$$ <- unit : ohm

X_L grows with frequency.

DC (f = 0) : X_L = 0 -> L acts as plain wire.

High f : X_L large -> L ~~passes~~ blocks AC.

phasor diagram



Power in pure L (average)

$P = V I = V_0 I_0 \sin wt \cos wt$

$= (V_0 I_0/2) \sin 2wt$

Average over full cycle = 0 -> no real power.

L is a wattless component .

AC Through Pure Capacitor

Source V = VO sin omega t ; across capacitor C.

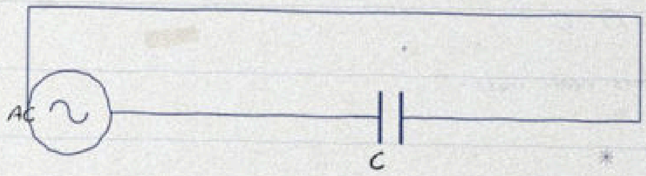


Fig. AC across pure capacitor

Derivation

Charge on C : $q = C V = C V_0 \sin w t$

$I = dq / dt = C V_0 w \cos w t$

$= C V_0 w \sin (w t + \Pi/2)$

$$\boxed{I = I_0 \sin (w t + \Pi/2)}$$ <- I leads V <- by pi/2

$$\boxed{X_c = 1 / (w C) \;\; ; \;\; I_0 = V_0 / X_c}$$ <- capacitive <- reactance

DC (f = 0) : X_C -> infinity (C blocks DC).

High f : X_C small -> C passes AC easily.

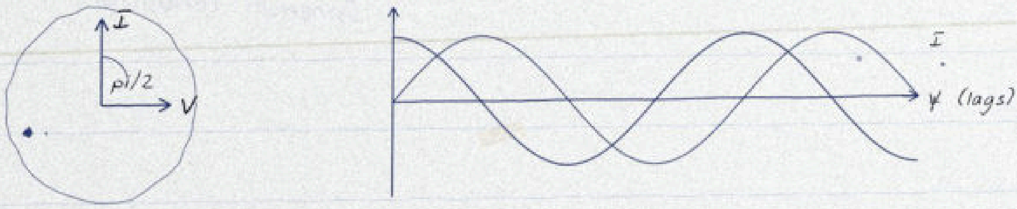
Average power in pure C is also zero (wattless).

Capacitive Reactance & Phasor (C)

$$\boxed{X_C = 1 / (\omega C) = 1 / (2 \pi f C)}$$ <- unit : ohm

X_C is INVERSE to omega ; doubling f halves X_C.

phasor (C)



Comparison R - L - C

R : phi = 0 (in phase)

L : phi = + pi/2 (V leads I)

C : phi = - pi/2 (V lags I)

Mnemonic : ELI the ICE man - !

Series LCR Circuit - Setup

R , L and C connected in series across an AC source $V = V0 \sin \omega t$.

Fig. series LCR circuit

Same current , different V across each

Let $I(t) = I0 \sin(\omega t - \phi)$.

$V_R = I R$ (in phase with I)

$V_L = I X_L$ (leads I by $\pi/2$)

$V_C = I X_C$ (lags I by $\pi/2$) .

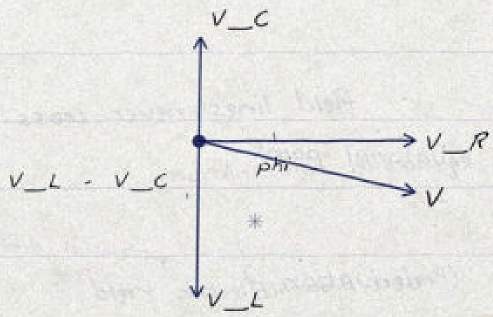
Source V must equal vector sum of all three.

$V = V_R + V_L + V_C$ (phasor sum)

Next page : phasor derivation of impedance Z.

Phasor Derivation of Impedance Z

Phasor diagram



Magnitudes

$$V^2 = V_R^2 + (V_L - V_C)^2$$
$$= I^2 [R^2 + (X_L - X_C)^2]$$

Define impedance Z by $V = I Z$:

$$\boxed{Z = sqrt(R^2 + (X_L - X_C)^2)}$$ <- unit : ohm

$$\boxed{\tan \phi = (X_L - X_C) / R}$$

Cases & Sign of phi

Three regimes

(1) X_L > X_C : inductive circuit

phi > 0 -> V leads I (like pure L).

(2) X_L < X_C : capacitive circuit

phi < 0 -> V lags I (like pure C).

(3) X_L = X_C : RESONANCE

phi = 0 ; Z = R (purely resistive).

Peak current

$$\boxed{I_0 = V_0 / Z}$$

<- and similarly
<- I_rms = V_rms / Z

Numeric example

R = 30 ohm , X_L = 60 ohm , X_C = 20 ohm :

$X_L - X_C = 40$ ohm

$Z = \sqrt{30^2 + 40^2} = 50$ ohm

$\tan \phi = 40 / 30 = 4 / 3$

ϕ 53 deg (V leads I)

Cross - check : cos phi = R/Z = 30/50 = 0.6.

Resonance in Series LCR

Condition

$X_L = X_C$ -> $wL = 1 / (wC)$

$$\boxed{w_0 = 1 / \sqrt{LC}}$$

<- resonant
<- angular freq

$$\boxed{f_0 = 1 / (2 \pi \sqrt{LC})}$$

At resonance

$Z_{min} = R$ (smallest possible)

$I_{max} = V_0 / R$

phi = 0 -> V , I in phase (pure resistive).

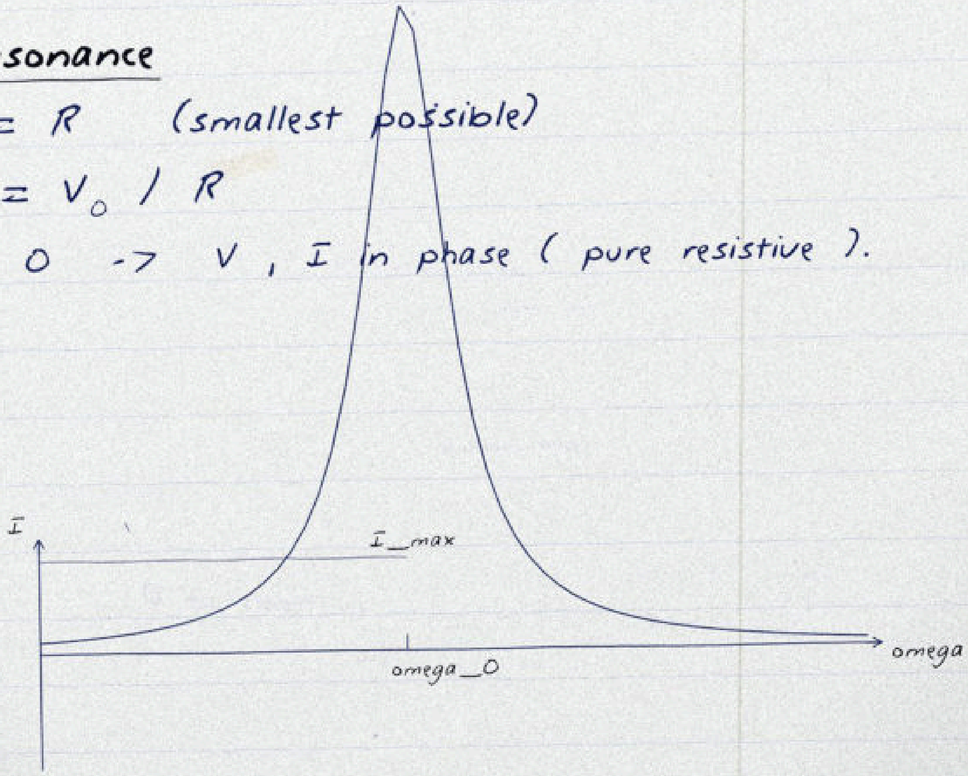


Fig. I vs omega - resonance curve

Sharpness of peak -> Q - factor (next page).

Sharpness & Q - Factor

Bandwidth

Bandwidth = 2 d omega = range of omega over which I drops to $I_{max} / \sqrt{2}$ (half - power points).

At these points : $\omega L - 1/(\omega C) = \pm R$

$d\omega = R / (2L)$ (half - width)

Quality factor Q

$$\boxed{Q = \omega_0 / (2\, d\omega) = \omega_0 L / R}$$

$$\boxed{Q = 1 / (\omega_0 R C) = (1 / R) \sqrt{L / C}}$$ <- key

<- res

Meaning

High Q : sharp peak -> good selectivity (radio tuning).

Low Q : broad peak -> poor selectivity.

High Q needs small R and ~~small~~ large L.

Q is dimensionless ; typical radio Q 100 .

Power in AC Circuits

Instantaneous power

$p(t) = V(t) \cdot I(t)$

$= V_0 I_0 \sin wt \sin(wt - \phi)$

Use $\sin A \sin B = (1/2)[\cos(A-B) - \cos(A+B)]$:

$p(t) = (V_0 I_0 / 2)[\cos\phi - \cos(2wt - \phi)]$

Average power

Average of cos(2 omega t - phi) over one cycle = 0.

$$\boxed{P_{avg} = (V_0 I_0 / 2) \cos\phi}$$

$$\boxed{P_{avg} = V_{rms} I_{rms} \cos\phi}$$

<- this is the
<- real power

Apparent vs real power

Apparent : V_rms I_rms (unit : V . A)

Real : V_rms I_rms cos phi (unit : W) .

Difference is stored / returned by L and C.

Ratio cos phi : power factor.

Power Factor & Wattless Current

Power factor cos phi

$$\boxed{\cos \phi = R / Z}$$

<- from phasor
<- triangle

Pure R : cos phi = 1 -> full power.
Pure L or C : cos phi = 0 -> P_avg = 0.
LCR : 0 < cos phi < 1 .

Wattless current

Component of I that is perpendicular to V in the phasor diagram does NO net work.

$I_{real} = I \cos \phi$ (in phase with V)
$I_{wattless} = I \sin \phi$ (perpendicular to V)

Why power factor matters

1. Low cos phi -> large I for same P : more I 2 R loss in transmission lines.

2. Industry uses capacitor banks to ~~decrease~~ improve $\cos \phi$ to 0.95 .

(Cancels inductive lag of motors.)

LC Oscillations - Setup

Charged capacitor C is connected to a pure inductor L (no resistance).

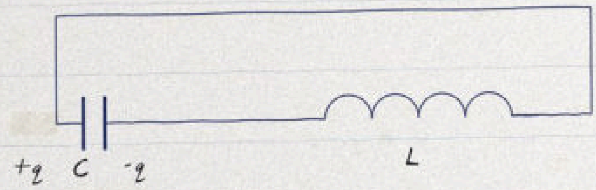


Fig. LC tank circuit (ideal)

Equation of motion

$V_C = V_L$ -> $q/C = -L\, dI/dt$

$I = dq/dt$ -> $L\, d^2q/dt^2 + q/C = 0$

$$\boxed{d^2q/dt^2 + (1/LC)\, q = 0}$$ <- SHM in q

Compare with $x'' + w^2 x = 0$:

$$\boxed{w = 1 / \mathrm{sqrt}(L\ C)}$$ <- natural <- frequency

LC Oscillations - Energy

Solutions

$q(t) = q_0 \cos wt$

$I(t) = dq/dt = -q_0 w \sin wt$

$I_0 = q_0 w = q_0 / \mathrm{sqrt}(LC)$

Energy at any t

$U_E = q^2 / (2C)$ (in C)

$U_B = (1/2) L I^2$ (in L)

Total $U = q_0^2 / (2C) = \text{const}$.

Energy sloshes between C and L like KE and PE of a SHM. At any instant, total is conserved.

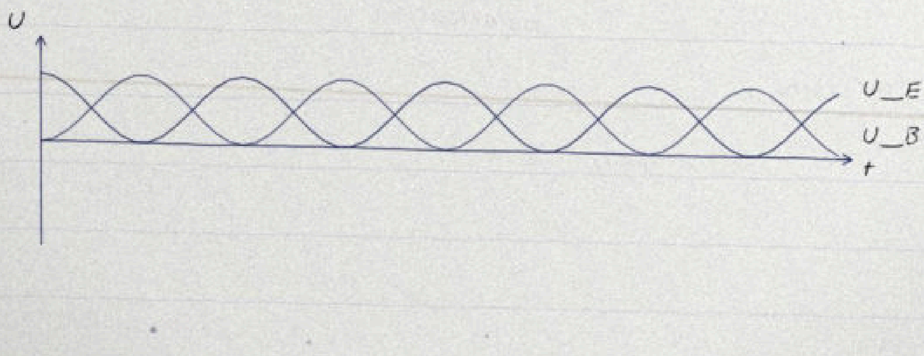


Fig. U_E and U_B exchange ; total constant

Real circuits : R causes damping (heat loss).

Transformer - Construction

Static device for changing AC voltage level by mutual induction between two coils on a common soft - iron core.

Parts

① Primary coil - connected to AC source.

② Secondary coil - delivers output.

③ Soft iron core - laminated ; carries common flux through both coils.

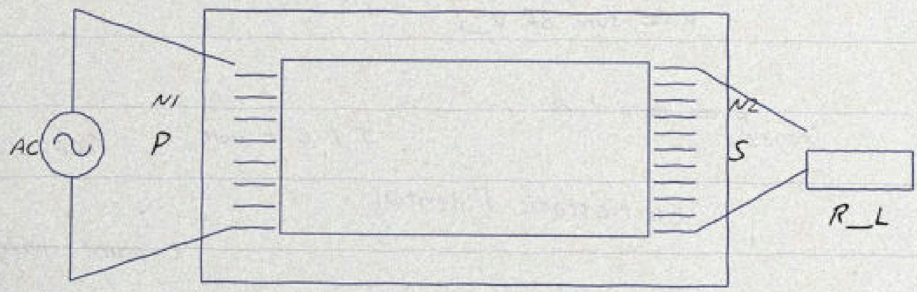


Fig. transformer : P + S coils on soft-iron core

Laminations -> reduce eddy currents in core.

Transformer - Working

Flux is common

Same flux phi (t) links every turn of P and S (no leakage , ideal case).

EMF in P : $E_1 = - N_1 \, d\phi / dt$

EMF in S : $E_2 = - N_2 \, d\phi / dt$

Divide :

$\boxed{V_2 / V_1 = N_2 / N_1}$ <- turns ratio

Ideal transformer (no losses)

Power in = Power out : $V_1 I_1 = V_2 I_2$

$\boxed{I_2 / I_1 = N_1 / N_2}$ <- I reciprocal <- of V

Step - up vs step - down

Step - up : $N2 > N1 \rightarrow V2 > V1$, $I2 < I1$.

Step - down : $N2 < N1 \rightarrow V2 < V1$, $I2 > I1$.

High V is used to transmit ; low V to ~~waste~~ use .

Power line : step - up at plant , step - down at home.

Energy Losses in Transformers

Four main losses

(1) Copper loss - $I^2 R$ in coil windings
Remedy : use thick low - R copper wire.

(2) Iron loss - in the core. Two parts :
(a) Hysteresis : energy lost per B - H cycle.
Remedy : use soft iron (narrow loop).
(b) Eddy currents : swirling I in core.
Remedy : laminate core into thin sheets.

(3) Flux leakage - not all flux links S.
Remedy : wind both coils on the same limb.

(4) Humming - magnetostriction of core
(small mechanical loss as audible sound).

Efficiency

$$\boxed{eta = P_{out} / P_{in} \times 100\ \%}$$

Typical real transformer : eta - 95 - 99 %.
Highest efficiency device in electrical eng. !
(No moving parts -> small frictional loss.)

Long - Distance Transmission

Power generated at V_gen (11 kV).
Stepped up to 220 kV (or more) before being sent through transmission lines.

Why high V ?

Power transmitted : $P = V I$

Line loss : $P_{loss} = I^2 R = (P / V)^2 R$

Higher V -> smaller I -> much smaller I 2 R loss.
Loss scales as 1 / V 2 .

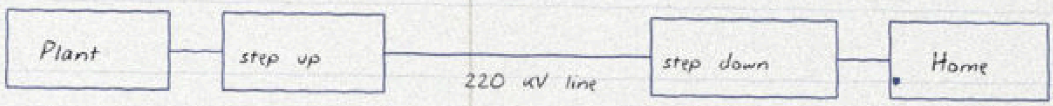


Fig. step-up before line ; step-down at user

Trade - off

Higher V needs better insulation , taller pylons.
Balance line loss vs cost of high - V hardware.

Worked Example I : Resonance

Problem : $L = 2.0$ H , $C = 32$ uF , $R = 10$ ohm , V_rms = 230 V . Find :
(a) f0 , (b) I_max at resonance , (c) Q factor.

(a) Resonant frequency

$f_0 = 1 / (2 \pi \text{ sqrt}(L\ C))$

$\text{sqrt}(L\ C) = \text{sqrt}(2.0 \cdot 32 \times 10^{-6})$

$= \text{sqrt}(6.4 \times 10^{-5}) = 8.0 \times 10^{-3}$ s

$f_0 = 1 / (2 \pi \cdot 8.0 \times 10^{-3})$

19.9 Hz

(b) I_max

At resonance $Z = R = 10$ ohm

$I_{rms} = V_{rms} / R = 230 / 10 = 23$ A

$I_0 = \text{sqrt}(2) \cdot I_{rms}$ 32.5 A

(c) Q factor

$Q = (1/R) \text{ sqrt}(L/C)$

$L/C = 2.0 / (32 \times 10^{-6}) = 6.25 \times 10^{4}$

$\text{sqrt}(L/C) = 250$ ohm

$Q = 250 / 10 = 25$

Bandwidth f0 / Q 19.9 / 25 0.8 Hz.

Very sharp peak -> selective circuit.

Worked Example II : Power

Problem : 220 V AC source feeds R = 100 ohm in series with L = 0.5 H at f = 50 Hz . Find :
(a) X_L , (b) Z , (c) I_rms , (d) P_avg.

(a) X_L

$X_L = 2 \pi f L = 2 \cdot 3.14 \cdot 50 \cdot 0.5$

157 ohm

(b) Z

$Z = \text{sqrt}(R^2 + X_L^2)$

$= \text{sqrt}(100^2 + 157^2)$

$= \text{sqrt}(10000 + 24649)$

186 ohm

(c) I_rms

$I_{rms} = 220 / 186$ 1.18 A

(d) P_avg

$\cos \phi = R / Z = 100 / 186$ 0.54

$P_{avg} = V_{rms} I_{rms} \cos \phi$

$= 220 \cdot 1.18 \cdot 0.54$

140 W

Note : apparent power = 220 . 1.18 260 V . A.

Worked Example III : Transformer

Problem : Step - down transformer N1 = 2000, N2 = 100 . Primary connected to 220 V , 50 Hz. Secondary feeds a 5 ohm load. Assume ideal.
Find (a) V2 , (b) I2 , (c) I1 , (d) P.

(a) V2

$V_2 / V_1 = N_2 / N_1 = 100 / 2000 = 1/20$

$V_2 = 220 / 20 = 11\ V$

(b) I2

$I_2 = V_2 / R = 11 / 5 = 2.2\ A$

(c) I1

$I_1 / I_2 = N_2 / N_1 = 1/20$

$I_1 = 2.2 / 20 = 0.11\ A$

(d) Power

$P_{in} = V_1 I_1 = 220 \cdot 0.11 = 24.2\ W$

$P_{out} = V_2 I_2 = 11 \cdot 2.2 = 24.2\ W$

P_in = P_out (ideal , no losses).

Real transformer : P_out ~~=~~ < P_in ; eta < 1 .

Typical efficiency 96 % for power-grid units.

Important Sub - Results

RMS for non - sinusoidal AC

Square wave : $I_{rms} = I0$. (peak = rms)

Triangular : $I_{rms} = I0 / \sqrt{3}$.

Half sine : $I_{rms} = I0 / 2$.

Choke coil

An inductor used in AC circuits to limit current without dissipating energy (replaces resistor).

Energy efficient ; used in tube - light starters.

AC ammeter & voltmeter

Hot - wire type or moving - iron type :

deflection prop to I^2 -> reads I_{rms} .

Phasor algebra cheat

V perpendicular I in pure L or C -> $P_{avg} = 0$

V parallel I in pure R -> $P_{avg} = V_{rms} I_{rms}$

General : $P_{avg} = V_{rms} I_{rms} \cos \phi$

Common pitfalls

1. Always use rms (not peak) in $V = IZ$.
2. Z is not algebraic sum of $R + X_L + X_C$.
3. At resonance Z is MIN , I is MAX.
4. Power in pure L or C is zero (wattless).

AC vs DC - Quick Comparison

Generation

AC : by rotating a coil in B (Faraday's law).

DC : by chemical cells , solar cells , rectifiers.

Transmission

AC : step up by transformer -> low I -> low lo

DC : no transformer ; HVDC uses costly converters.

Devices

AC : induction motor , transformer , fan , bulb.

DC : battery , dc motor , electronics (after SMPS).

Effects

AC : reactance (X_L , X_C) , phase , impedance.

DC : no reactance ; only ohmic resistance.

Safety

AC at same V is more dangerous (muscle grip).

DC tends to throw the victim off (one twitch).

Form factor

$$F = I_{rms} / I_{avg} = (I0 / \text{sqrt } 2) / (2\ I0 / \pi)$$

$$= \pi / (2 \text{ sqrt } 2) \quad 1.11$$

Peak factor = I0 / I_rms = sqrt 2 1.414 .

Summary - All Key Formulas

Sinusoidal AC

$V = V_0 \sin wt$; $I = I_0 \sin(wt - \phi)$

$w = 2\pi f$; $T = 1/f$

Mean / RMS

$I_{avg} = 2 I_0 / \pi$ (half cycle)

$I_{rms} = I_0 / \text{sqrt } 2$; $V_{rms} = V_0 / \text{sqrt } 2$

Reactance & impedance

$X_L = wL$; $X_C = 1 / wC$

$Z = \text{sqrt}(R^2 + (X_L - X_C)^2)$

$\tan\phi = (X_L - X_C) / R$

Resonance & Q

$w_0 = 1 / \text{sqrt}(LC)$; $f_0 = 1 / (2\pi \text{ sqrt}(LC))$

$Q = w_0 L / R = (1/R) \text{ sqrt}(L/C)$

Power & power factor

$P_{avg} = V_{rms} I_{rms} \cos\phi$

$\cos\phi = R / Z$ (power factor)

Transformer

$V_2 / V_1 = N_2 / N_1$; $I_2 / I_1 = N_1 / N_2$

$P_{in} = V_1 I_1 = V_2 I_2 = P_{out}$ (ideal)