

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) **All** questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If for any 2×2 square matrix A , $A(\text{adj } A) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, then write the value of $|A|$. [1 mark]

Solution:

Concept:

- For a square matrix, $A \text{adj } A = |A|I$, where I is the identity matrix of the same order.
- Equal matrices have equal corresponding entries; the scalar multiplying the identity is the determinant itself.

Step 1: Use the adjugate identity For the given matrix of order two, the identity gives

$$A \text{adj } A = \begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix}.$$

This relation holds for every square matrix, whether singular or nonsingular.

Step 2: Compare a diagonal entry The printed product is $8I$. Comparing its top-left entry with the top-left entry of $|A|I$ gives $|A| = 8$. Taking the determinant of the product instead would unnecessarily introduce a square and obscure the sign.

Step 3: Conclusion $|A| = 8$.

Quick Tip to Solve: Compare one diagonal entry of $A \text{adj } A$ directly with $|A|I$.

2. Determine the value of k for which the following function is continuous at $x = 3$: $f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$. [1 mark]

Solution:

Concept:

- Continuity at $x = a$ requires $\lim_{x \rightarrow a} f(x) = f(a)$.
- Factor a difference of squares before cancelling; cancellation here is valid only for $x \neq 3$.

Step 1: Simplify near the point For $x \neq 3$,

$$(x+3)^2 - 36 = (x+3-6)(x+3+6) = (x-3)(x+9).$$

Therefore the first branch of the function equals $x+9$ at every nearby point other than 3.

Step 2: Apply continuity The simplified expression has the same limit from both sides:

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (x+9) = 12.$$

Quick Tip to Solve: Cancel the factor $x-3$ before taking the limit; do not substitute into the original fraction.

The separately defined value is $f(3) = k$, so continuity requires this value to equal the limit.

Step 3: Conclusion $k = 12$.

Quick Tip to Solve: Cancel the factor $x - 3$ before taking the limit; do not substitute into the original fraction.

3. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$.

[1 mark]

Solution:

Concept:

- Split the numerator term by term to express the integrand using $\tan x$ and $\cot x$.
- The standard integrals are $\int \tan x dx = -\ln |\cos x| + C$ and $\int \cot x dx = \ln |\sin x| + C$.

Step 1: Rewrite the integrand On any interval where $\sin x \cos x \neq 0$,

$$\frac{\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \tan x - \cot x.$$

Step 2: Integrate the two terms Hence an antiderivative is

$$-\ln |\cos x| - \ln |\sin x| + C.$$

The logarithms are defined throughout each interval on which the original integrand is defined, and their sum combines using the product rule for logarithms.

Step 3: Conclusion $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = -\ln |\sin x \cos x| + C$.

Quick Tip to Solve: Divide each numerator term separately by $\sin x \cos x$ to obtain two standard integrals.

4. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

[1 mark]

Solution:

Concept:

- Parallel planes must first be written with identical coefficients of x, y, z .
- For $ax + by + cz = d_1$ and $ax + by + cz = d_2$, their distance is $|d_2 - d_1|/\sqrt{a^2 + b^2 + c^2}$.

Step 1: Match the normal vectors Divide the entire second equation by 2.5, including its right-hand side. It becomes

$$2x - y + 2z = 8.$$

Both planes now have the same normal vector $(2, -1, 2)$, so they are parallel.

Step 2: Compute the perpendicular separation Use the common normal coefficients, whose length is

Quick Tip to Solve: Scale the whole plane equation, including its constant, before subtracting the constants.

$\sqrt{2^2 + (-1)^2 + 2^2} = 3$. Thus the perpendicular distance is

$$d = \frac{|8 - 5|}{3} = 1.$$

The absolute value ensures a nonnegative distance regardless of plane order.

Step 3: Conclusion The distance between the planes is 1 unit.

Quick Tip to Solve: Scale the whole plane equation, including its constant, before subtracting the constants.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. If A is a skew-symmetric matrix of order 3, then prove that $\det A = 0$.

[2 marks]

Solution:

Concept:

- A skew-symmetric matrix satisfies $A^T = -A$, where A^T is its transpose.
- Transposition preserves a determinant: $\det(A^T) = \det A$.
- Multiplying every row of a matrix of order n by -1 multiplies its determinant by $(-1)^n$.

Step 1: Take determinants of the defining identity Since A is skew-symmetric, $A^T = -A$. Taking determinants of both sides gives

$$\det(A^T) = \det(-A).$$

The left side is $\det A$, since a determinant has the same value after interchanging its rows and columns.

Step 2: Use the odd order The matrix has three rows. In $-A$, each of these rows is multiplied by -1 , so multilinearity of the determinant yields

$$\det(-A) = (-1)^3 \det A = -\det A.$$

Consequently $\det A = -\det A$, which implies $2 \det A = 0$. This argument does not assume that A has an inverse; it establishes that every such matrix is singular.

Step 3: Conclusion Therefore $\det A = 0$, as required.

Quick Tip to Solve: Combine $A^T = -A$ with determinant invariance under transpose; odd order forces a sign change.

6. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$.

[2 marks]

Solution:

Concept:

- Rolle's theorem applies when f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$.

Quick Tip to Solve: After solving $f'(c) = 0$, retain only the root strictly inside the given interval.

- It guarantees a point $c \in (a, b)$ for which $f'(c) = 0$; only roots inside the open interval are admissible.

Step 1: Check the hypotheses The function $f(x) = x^3 - 3x$ is a polynomial, so it is continuous on $[-\sqrt{3}, 0]$ and differentiable on $(-\sqrt{3}, 0)$. At the endpoints,

$$f(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0, \quad f(0) = 0.$$

Thus the endpoint values agree and all hypotheses hold.

Step 2: Find the stationary point in the interval Differentiating gives $f'(x) = 3x^2 - 3$. Hence

$$f'(c) = 0 \implies 3c^2 - 3 = 0 \implies c = \pm 1.$$

Of these two roots, -1 lies strictly between $-\sqrt{3}$ and 0 ; 1 lies outside this interval and cannot be selected.

Step 3: Conclusion The required value is $\boxed{c = -1}$.

Quick Tip to Solve: After solving $f'(c) = 0$, retain only the root strictly inside the given interval.

7. The volume of a cube is increasing at the rate of $9 \text{ cm}^3/\text{s}$. How fast is its surface area increasing when the length of an edge is 10 cm ? [2 marks]

Solution:

Concept:

- For a cube with edge a , its volume is $V = a^3$ and total surface area is $S = 6a^2$.
- Differentiate with respect to time t using the chain rule: $dV/dt = 3a^2 da/dt$ and $dS/dt = 12a da/dt$.

Step 1: Find the rate of increase of the edge Let a , measured in centimetres, vary with time t , measured in seconds. Since $V = a^3$,

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}.$$

At the specified instant, $a = 10$ and $dV/dt = 9$, so

$$\frac{da}{dt} = \frac{9}{3(10)^2} = \frac{3}{100} \text{ cm/s}.$$

Step 2: Differentiate the total surface area The cube has six square faces, so $S = 6a^2$. Applying the chain rule and substituting the instantaneous values gives

$$\frac{dS}{dt} = 12a \frac{da}{dt} = 12(10) \frac{3}{100} = \frac{18}{5} \text{ cm}^2/\text{s}.$$

The positive result is consistent with an expanding cube. The edge value is substituted only after differentiation, because it varies with time.

Step 3: Conclusion The surface area is increasing at $\boxed{\frac{18}{5} \text{ cm}^2/\text{s} = 3.6 \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Eliminate the edge rate to get $dS/dt = (4/a)(dV/dt)$, then substitute the edge length.

8. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $\mathbb{R}$.

[2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive at every point of that interval.
- Completing the square expresses a quadratic as a nonnegative square plus a constant, making its sign easy to determine.

Step 1: Differentiate the polynomial The function $f(x) = x^3 - 3x^2 + 6x - 100$ is a polynomial, so it is differentiable for every real x . Differentiating each term gives

$$f'(x) = 3x^2 - 6x + 6.$$

The constant term contributes zero to the derivative.

Step 2: Establish positivity on the whole real line Complete the square in the derivative:

$$f'(x) = 3(x^2 - 2x + 2) = 3((x - 1)^2 + 1).$$

Since $(x - 1)^2 \geq 0$ for every real x , we have $f'(x) \geq 3 > 0$. Thus the derivative never vanishes or changes sign. By the derivative test for monotonicity, the value of the function increases as its argument increases throughout its domain.

Step 3: Conclusion Hence $f(x) = x^3 - 3x^2 + 6x - 100$ is strictly increasing on $\mathbb{R}$.

Quick Tip to Solve: Complete the square in the derivative; the positive constant remaining proves positivity for every real input.

9. The x -coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z -coordinate.

[2 marks]

Solution:

Concept:

- A point on the line through P and Q can be written as $P + t(Q - P)$, where t is a real parameter.
- The same parameter must be used in all three coordinates of a point on the line.

Step 1: Write a parametric point on the line The direction vector from $P(2, 2, 1)$ to $Q(5, 1, -2)$ is

$$Q - P = (5 - 2, 1 - 2, -2 - 1) = (3, -1, -3).$$

Therefore a general point R on the line has coordinates

$$R = (2 + 3t, 2 - t, 1 - 3t), \quad t \in \mathbb{R}.$$

This description gives P when $t = 0$ and Q when $t = 1$, confirming the line used.

Quick Tip to Solve: Use the x -coordinate to find the line parameter first, then substitute the same parameter into the z -coordinate.

Step 2: Use the given x-coordinate The specified point has $x = 4$, so

$$2 + 3t = 4 \implies t = \frac{2}{3}.$$

Substitute this value into the z-coordinate expression:

$$z = 1 - 3t = 1 - 3\left(\frac{2}{3}\right) = 1 - 2 = -1.$$

Since the x-component of the direction vector is nonzero, the given x-coordinate fixes a unique point on this line.

Step 3: Conclusion The required z-coordinate is $\boxed{-1}$.

Quick Tip to Solve: Use the x-coordinate to find the line parameter first, then substitute the same parameter into the z-coordinate.

10. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event “number obtained is even” and B be the event “number obtained is red”. Find if A and B are independent events. [2 marks]

Solution:

Concept:

- Two events A and B are independent exactly when $P(A \cap B) = P(A)P(B)$.
- For an ordinary fair die, the six outcomes are equally likely, and an event probability is its number of outcomes divided by six.

Step 1: List the events and their intersection The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The even-number event is $A = \{2, 4, 6\}$, while the red-number event is $B = \{1, 2, 3\}$. A number must be both even and red to belong to their intersection, giving

$$A \cap B = \{2\}.$$

In particular, the green even numbers 4 and 6 do not belong to this intersection.

Step 2: Compare the intersection probability with the product Counting the equally likely outcomes gives

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}.$$

But

$$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq \frac{1}{6} = P(A \cap B).$$

Thus the equality required for independent events does not hold. Equivalently, knowing that the outcome is red leaves only one even outcome among three red outcomes.

Step 3: Conclusion The events A and B are $\boxed{\text{not independent}}$.

Quick Tip to Solve: List the intersection explicitly: only 2 is both red and even, so compare $1/6$ with the product $1/4$.

11. Two tailors, A and B , earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP. [2 marks]

Solution:

Concept:

- The decision variables specify the quantities to be chosen; here they are the numbers of working days for the two tailors.
- An "at least" production requirement gives a greater-than-or-equal-to constraint. The objective is the total daily labour cost summed over the working days.

Step 1: Define the variables and labour cost Let x be the number of days tailor A works, and y the number of days tailor B works. Since working time cannot be negative, $x \geq 0$ and $y \geq 0$. At the given daily rates, the total labour cost, in rupees, is

$$Z = 300x + 400y.$$

The desired objective is to minimise this cost.

Step 2: Translate the production requirements Tailor A produces 6 shirts per day and tailor B produces 10 shirts per day. Their combined shirt production is therefore $6x + 10y$, and at least 60 shirts requires

$$6x + 10y \geq 60.$$

Each tailor produces 4 pairs of trousers per day, so their combined production is $4x + 4y$. At least 32 pairs of trousers requires

$$4x + 4y \geq 32, \quad \text{or equivalently} \quad x + y \geq 8.$$

These inequalities allow production above the stated minimums.

Step 3: Conclusion The LPP is: minimise $Z = 300x + 400y$, subject to

$$6x + 10y \geq 60, \quad 4x + 4y \geq 32, \quad x \geq 0, \quad y \geq 0.$$

Quick Tip to Solve: Write one production inequality for each garment and use greater-than-or-equal signs because the demand is "at least".

12. Find: $\int \frac{dx}{5 - 8x - x^2}.$

[2 marks]

Solution:

Concept:

- Completing the square gives $5 - 8x - x^2 = 21 - (x + 4)^2$.

Quick Tip to Solve: Complete the square before choosing an integration formula; the denominator is 21 minus a square, not a sum of squares.

- For $a > 0$, $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$, on intervals excluding the denominator zeros.

Step 1: Complete the square and substitute Let $I = \int \frac{dx}{5 - 8x - x^2}$. Rewrite the denominator carefully:

$$5 - 8x - x^2 = -(x^2 + 8x - 5) = 21 - (x + 4)^2.$$

Put $u = x + 4$, so $du = dx$, and write $a = \sqrt{21}$. Then

$$I = \int \frac{du}{a^2 - u^2}.$$

The original integrand is defined where $x \neq -4 \pm \sqrt{21}$.

Step 2: Integrate using partial fractions Use the decomposition

$$\frac{1}{a^2 - u^2} = \frac{1}{2a} \left(\frac{1}{a+u} + \frac{1}{a-u} \right).$$

The second term integrates with a minus sign because the derivative of $a - u$ is -1 . Therefore

$$\begin{aligned} I &= \frac{1}{2a} (\ln |a+u| - \ln |a-u|) + C \\ &= \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C. \end{aligned}$$

Substituting $u = x + 4$ and $a = \sqrt{21}$ gives the required expression, valid on any interval avoiding the two poles.

Step 3: Conclusion

$$\int \frac{dx}{5 - 8x - x^2} = \frac{1}{2\sqrt{21}} \ln \left| \frac{\sqrt{21} + x + 4}{\sqrt{21} - x - 4} \right| + C.$$

Quick Tip to Solve: Complete the square before choosing an integration formula; the denominator is 21 minus a square, not a sum of squares.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. If $\tan^{-1} \left(\frac{x-3}{x-4} \right) + \tan^{-1} \left(\frac{x+3}{x+4} \right) = \frac{\pi}{4}$, then find the value of x .

[4 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} t$ lies in $(-\pi/2, \pi/2)$. Thus a tangent equation must be checked against the original angle equation.

Quick Tip to Solve: After taking tangents of an inverse-tangent sum, check the signs of both arguments to locate the sum in the correct principal-angle interval.

- For angles A, B , $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, provided the denominator is nonzero.

Step 1: Introduce the two principal angles The expressions are defined only for $x \neq 4, -4$. Put

$$A = \tan^{-1}\left(\frac{x-3}{x-4}\right), \quad B = \tan^{-1}\left(\frac{x+3}{x+4}\right).$$

The given equation is $A + B = \pi/4$, so its tangent is 1. We first obtain the possible values of x , then verify the principal-value condition; taking tangents alone can lose information about multiples of π .

Step 2: Apply the addition formula and solve With $u = (x-3)/(x-4)$ and $v = (x+3)/(x+4)$, we have

$$u + v = \frac{(x-3)(x+4) + (x+3)(x-4)}{x^2 - 16} = \frac{2x^2 - 24}{x^2 - 16},$$

$$uv = \frac{x^2 - 9}{x^2 - 16}, \quad 1 - uv = \frac{-7}{x^2 - 16}.$$

This last expression is nonzero throughout the allowed domain. Hence

$$\frac{u + v}{1 - uv} = \frac{24 - 2x^2}{-7} = 1, \quad 2x^2 = 17.$$

The two candidates are $x = \sqrt{17/2}$ and $x = -\sqrt{17/2}$, neither of which is an excluded value.

Step 3: Verify that both candidates give the required angle Since $17/2 < 9$, both candidates satisfy $-3 < x < 3$. Therefore the numerator and denominator of u are both negative, while those of v are both positive. Thus $u, v > 0$, giving $0 < A, B < \pi/2$ and $0 < A + B < \pi$. For either candidate the calculation above gives $\tan(A + B) = 1$. Within $(0, \pi)$, the only angle with tangent 1 is $\pi/4$. Consequently both candidates satisfy the original inverse-tangent equation, with no extraneous root.

Step 4: Conclusion The required values are $x = \pm\sqrt{17/2} = \pm\frac{\sqrt{34}}{2}$.

Quick Tip to Solve: After taking tangents of an inverse-tangent sum, check the signs of both arguments to locate the sum in the correct principal-angle interval.

14(a). Using properties of determinants, prove that $\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$. [4 marks]

Solution:

Concept:

- Subtracting a multiple of one row from another leaves the determinant unchanged.
- A common factor in a row can be taken outside the determinant. Expansion along a column containing two zeros reduces a third-order determinant to a second-order determinant.

Step 1: Create zeros in the third column Let the given determinant be D . Since all three entries of the third column are 1, apply $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$, keeping R_3 unchanged. Neither

Quick Tip to Solve: When a determinant has a column of ones, subtract one row from the others first; the resulting zeros reveal common factors immediately.

operation changes the value of the determinant. Thus

$$D = \begin{vmatrix} a^2 + 2a - 3 & 2a - 2 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

In particular, the second entry of the second row becomes $(a + 2) - 3 = a - 1$. The third column now permits a single-term cofactor expansion.

Step 2: Extract the common factors from the first two rows Factor the entries using

$$a^2 + 2a - 3 = (a - 1)(a + 3), \quad 2a - 2 = 2(a - 1).$$

Both the first and second rows have the common factor $a - 1$. Taking one factor from each row gives

$$D = (a - 1)^2 \begin{vmatrix} a + 3 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

This uses the multiplication property of determinants, without dividing by $a - 1$; consequently the argument is valid even when $a = 1$.

Step 3: Expand along the third column Only the entry in row 3 contributes, and its cofactor sign is $(-1)^{3+3} = +1$. Therefore

$$\begin{aligned} D &= (a - 1)^2 \begin{vmatrix} a + 3 & 2 \\ 2 & 1 \end{vmatrix} \\ &= (a - 1)^2 ((a + 3) \cdot 1 - 2 \cdot 2) \\ &= (a - 1)^2 (a - 1). \end{aligned}$$

The remaining factor arises from the two-by-two determinant itself. All row operations used above preserve the original determinant, so the identity holds for every value of a .

Step 4: Conclusion Hence $\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$, as required.

Quick Tip to Solve: When a determinant has a column of ones, subtract one row from the others first; the resulting zeros reveal common factors immediately.

14(b). Find matrix A such that $\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$.

[4 marks]

Solution:**Concept:**

- A matrix of order 3×2 multiplied by A gives a matrix of order 3×2 , so A has order 2×2 .
- Each product entry is the scalar product of a row and a column. Equal matrices have equal corresponding entries.

Step 1: Introduce the unknown matrix Write $A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, because the inner dimensions must agree and the product must have two columns. Multiplying gives

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} 2p - r & 2q - s \\ p & q \\ -3p + 4r & -3q + 4s \end{pmatrix}.$$

Compare this product with the given right-hand matrix. This provides six scalar equations for the four entries. The left factor is rectangular, so the direct comparison method is appropriate.

Step 2: Determine the entries from the first two rows The second row gives $p = 1$ and $q = -2$ immediately, since multiplication by the row $(1, 0)$ selects the first row of A . The first row now gives

$$2p - r = -1, \quad 2q - s = -8.$$

On substitution,

$$2 - r = -1 \Rightarrow r = 3, \quad -4 - s = -8 \Rightarrow s = 4.$$

Thus these two rows determine all four entries uniquely. The remaining row must still be checked because the equation requires agreement in every position.

Step 3: Verify the remaining row and the complete product Substituting into the third row gives

$$-3p + 4r = -3 + 12 = 9, \quad -3q + 4s = 6 + 16 = 22.$$

Both agree with the right-hand side. Indeed,

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}.$$

Therefore the values satisfy all six entry conditions, and no other matrix can satisfy the first two rows.

Step 4: Conclusion The required matrix is $A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$.

Quick Tip to Solve: Use the row $(1, 0)$ first: it gives the first row of A directly. Then determine the second row and check the unused equation.

15(a). If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- Logarithmic differentiation handles a variable base raised to a variable exponent: $u^v = e^{v \ln u}$ for $u > 0$.
- Treat y as a differentiable function of x , and use the chain and product rules. The quantities a and b are constants, so a^b has derivative zero.

Step 1: Differentiate the first exponential term Work on the usual real domain $x > 0$, $y > 0$, where logarithms of both bases exist. Put $u = x^y$. Taking logarithms gives $\ln u = y \ln x$. Differentiating both sides with respect to x , remembering that y varies with x , gives

$$\frac{u'}{u} = \frac{dy}{dx} \ln x + \frac{y}{x}.$$

Consequently,

$$\frac{d}{dx}(x^y) = x^y \left(\ln x \frac{dy}{dx} + \frac{y}{x} \right).$$

The second term arises from differentiating $\ln x$, whereas the first arises from differentiating the exponent.

Step 2: Differentiate the second term and the equation Similarly, if $v = y^x$, then $\ln v = x \ln y$. The product rule and chain rule give

$$\frac{v'}{v} = \ln y + \frac{x}{y} \frac{dy}{dx}.$$

Hence differentiating the original equation $x^y + y^x = a^b$ gives

$$x^y \left(\ln x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\ln y + \frac{x}{y} \frac{dy}{dx} \right) = 0.$$

We differentiate the two terms separately; taking the logarithm of their sum would not split it into a sum of logarithms.

Step 3: Collect the derivative terms Move the terms that do not contain dy/dx to the right and factor the derivative on the left:

$$(x^y \ln x + xy^{x-1}) \frac{dy}{dx} = -(yx^{y-1} + y^x \ln y).$$

Dividing by the coefficient gives the derivative wherever $x^y \ln x + xy^{x-1} \neq 0$. This condition is necessary for the displayed quotient to define a finite derivative on the implicit curve.

Step 4: Conclusion

$$\frac{dy}{dx} = -\frac{yx^{y-1} + y^x \ln y}{x^y \ln x + xy^{x-1}}$$

This formula holds on the positive real domain wherever its denominator is nonzero.

Quick Tip to Solve: For each variable power, differentiate its logarithm first; in y^x , both the exponent x and the base $y(x)$ contribute a term.

15(b). If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

[4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x ; hence $\frac{d}{dx}e^y = e^y \frac{dy}{dx}$.
- The product rule is $(uv)' = u'v + uv'$. Apply it to $e^y(x+1)$, and then differentiate the resulting first-derivative relation once more.
- Since $e^y > 0$ and $e^y(x+1) = 1$, the real domain has $x+1 > 0$. Thus cancellation of e^y and division by $x+1$ are justified.

Step 1: Differentiate the given relation Write $y' = \frac{dy}{dx}$. Differentiating both sides of $e^y(x+1) = 1$ with respect to x , using the chain rule on e^y and the product rule on the left, gives

$$e^y y'(x+1) + e^y = 0.$$

Cancelling the nonzero factor e^y , we obtain

$$(x+1)y' + 1 = 0, \quad y' = -\frac{1}{x+1}.$$

The derivative of the constant on the right is zero; retaining this fact is essential when differentiating the implicit equation.

Step 2: Differentiate the first-derivative equation Let $y'' = \frac{d^2y}{dx^2}$. Differentiate $(x+1)y' + 1 = 0$ again. Both $x+1$ and y' depend on x , so the product rule yields

$$(x+1)y'' + y' = 0.$$

Consequently,

$$y'' = -\frac{y'}{x+1} = -\frac{-1/(x+1)}{x+1} = \frac{1}{(x+1)^2}.$$

On the other hand, squaring the first derivative gives

$$(y')^2 = \left(-\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2}.$$

The two expressions coincide throughout the real domain of the given relation.

Step 3: Conclusion Therefore $\boxed{\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2}$, as required, for $x > -1$.

Quick Tip to Solve: Cancel the positive factor e^y after the first differentiation; the simpler equation $(x+1)y' + 1 = 0$ makes the second differentiation immediate.

16. Find: $\int \frac{\cos \theta d\theta}{(4 + \sin^2 \theta)(5 - 4 \cos^2 \theta)}.$

[4 marks]

Solution:

Concept:

- Use $\cos^2 \theta = 1 - \sin^2 \theta$ to express both denominator factors in terms of $\sin \theta$.
- The substitution $t = \sin \theta$ gives $dt = \cos \theta d\theta$, reducing the integral to a rational function of t .
- Partial fractions separate the quadratic factors. The standard formula $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$, with $a > 0$, then applies.

Step 1: Rewrite the denominator and substitute Denote the required integral by I . The second factor becomes

$$5 - 4 \cos^2 \theta = 5 - 4(1 - \sin^2 \theta) = 1 + 4 \sin^2 \theta.$$

With $t = \sin \theta$, so that $dt = \cos \theta d\theta$, we have

$$I = \int \frac{dt}{(t^2 + 4)(4t^2 + 1)}.$$

Both quadratic factors are positive for real t , so this substitution introduces no denominator singularity.

Step 2: Resolve into partial fractions Set

$$\frac{1}{(t^2 + 4)(4t^2 + 1)} = \frac{A}{t^2 + 4} + \frac{B}{4t^2 + 1}.$$

Multiplying by the common denominator yields

$$1 = A(4t^2 + 1) + B(t^2 + 4) = (4A + B)t^2 + (A + 4B).$$

Comparing coefficients, $4A + B = 0$ and $A + 4B = 1$. Thus $B = -4A$, so $-15A = 1$, giving $A = -1/15$ and $B = 4/15$. Therefore

$$I = -\frac{1}{15} \int \frac{dt}{t^2 + 4} + \frac{4}{15} \int \frac{dt}{1 + 4t^2}.$$

Step 3: Integrate the two standard terms For the first term, the standard inverse-tangent formula with $a = 2$ gives

$$\int \frac{dt}{t^2 + 4} = \frac{1}{2} \tan^{-1} \left(\frac{t}{2} \right).$$

For the second term, put $u = 2t$, so $dt = du/2$. Hence

$$\int \frac{dt}{1 + 4t^2} = \frac{1}{2} \int \frac{du}{1 + u^2} = \frac{1}{2} \tan^{-1}(2t).$$

Combining the coefficients and reverting to $t = \sin \theta$, we get

$$I = \frac{2}{15} \tan^{-1}(2 \sin \theta) - \frac{1}{30} \tan^{-1} \left(\frac{\sin \theta}{2} \right) + C.$$

Quick Tip to Solve: Convert $5 - 4 \cos^2 \theta$ to $1 + 4 \sin^2 \theta$ first; the numerator then matches the differential of $\sin \theta$ exactly.

Step 4: Conclusion The required integral is $\boxed{\frac{2}{15} \tan^{-1}(2 \sin \theta) - \frac{1}{30} \tan^{-1}\left(\frac{\sin \theta}{2}\right) + C}$, where C is an arbitrary constant.

Quick Tip to Solve: Convert $5 - 4 \cos^2 \theta$ to $1 + 4 \sin^2 \theta$ first; the numerator then matches the differential of $\sin \theta$ exactly.

17(a). Evaluate: $\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx.$

[4 marks]

Solution:

Concept:

- The reflection property is $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. For a function symmetric about $a/2$, adding the two forms of $\int_0^a x f(x) dx$ replaces the factor x by $a/2$.
- Multiplying numerator and denominator by $\cos x$ changes $\tan x / (\sec x + \tan x)$ to $\sin x / (1 + \sin x)$. This form has a continuous extension at $x = \pi/2$, so that isolated removable point does not affect the integral.

Step 1: Simplify the integrand and use reflection Denote the required integral by I . Wherever the original trigonometric expression is defined,

$$\frac{\tan x}{\sec x + \tan x} = \frac{\sin x}{1 + \sin x}.$$

On $[0, \pi]$, the denominator of the simplified expression is positive. We may therefore evaluate the integral using its continuous extension. Since $\sin(\pi - x) = \sin x$, reflection gives

$$I = \int_0^\pi \frac{x \sin x}{1 + \sin x} dx = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \sin x} dx.$$

Adding these equal integrals and dividing by two yields

$$I = \frac{\pi}{2} \int_0^\pi \frac{\sin x}{1 + \sin x} dx = \frac{\pi}{2} (\pi - J), \quad J = \int_0^\pi \frac{dx}{1 + \sin x}.$$

Step 2: Evaluate the remaining elementary integral A primitive that remains well defined throughout this interval is $-\cos x / (1 + \sin x)$. Indeed, the quotient rule gives

$$\frac{d}{dx} \left(-\frac{\cos x}{1 + \sin x} \right) = \frac{\sin x(1 + \sin x) + \cos^2 x}{(1 + \sin x)^2} = \frac{1}{1 + \sin x}.$$

Consequently,

$$J = \left[-\frac{\cos x}{1 + \sin x} \right]_0^\pi = 1 - (-1) = 2.$$

Substituting this value into the expression obtained by reflection gives $I = \frac{\pi}{2}(\pi - 2)$.

Quick Tip to Solve: First replace the tangent and secant quotient by sine; reflection then removes the factor x and leaves only an elementary integral.

Step 3: Conclusion The required value is $\boxed{\frac{\pi(\pi - 2)}{2}}$.

Quick Tip to Solve: First replace the tangent and secant quotient by sine; reflection then removes the factor x and leaves only an elementary integral.

OR

17(b). Evaluate: $\int_1^4 \{|x - 1| + |x - 2| + |x - 4|\} dx$.

[4 marks]

Solution:

Concept:

- For a real number u , $|u| = u$ when $u \geq 0$, while $|u| = -u$ when $u < 0$. Each modulus must be opened according to its sign on the interval being integrated.
- Definite integrals are additive over adjacent intervals: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$. Split at each interior point where a modulus changes its sign.

Step 1: Identify the signs and split the interval Write the required integral as I . The expressions inside the moduli vanish at $x = 1, 2, 4$. Of these, only $x = 2$ is strictly inside the integration interval $[1, 4]$, so it is the only point at which we must split. Throughout this interval, $x - 1 \geq 0$ and $x - 4 \leq 0$. Thus

$$|x - 1| = x - 1, \quad |x - 4| = 4 - x.$$

Their sum is always 3. The remaining expression $x - 2$ is nonpositive on $[1, 2]$ and nonnegative on $[2, 4]$. Therefore

$$|x - 1| + |x - 2| + |x - 4| = \begin{cases} 5 - x, & 1 \leq x \leq 2, \\ x + 1, & 2 \leq x \leq 4. \end{cases}$$

Both formulas give 3 at $x = 2$, so the endpoint shared by the two intervals causes no inconsistency.

Step 2: Integrate each expression on its interval Using interval additivity and the preceding sign analysis,

$$\begin{aligned} I &= \int_1^2 (5 - x) dx + \int_2^4 (x + 1) dx \\ &= \left[5x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} + x \right]_2^4 \\ &= \left(10 - 2 - 5 + \frac{1}{2} \right) + (8 + 4 - 2 - 2) \\ &= \frac{7}{2} + 8. \end{aligned}$$

As a check, we can integrate the constant contribution 3 separately. It contributes $3(4 - 1) = 9$, while $\int_1^2 (2 - x) dx = 1/2$ and $\int_2^4 (x - 2) dx = 2$. The sum is again $9 + 1/2 + 2$.

Quick Tip to Solve: On the entire interval, the first and third moduli add to 3; split only the remaining modulus at $x = 2$.

Step 3: Conclusion Hence, $\int_1^4 \{|x-1| + |x-2| + |x-4|\} dx = \frac{23}{2}$.

Quick Tip to Solve: On the entire interval, the first and third moduli add to 3; split only the remaining modulus at $x = 2$.

18. Solve the differential equation $(\tan^{-1} x - y) dx = (1 + x^2) dy$.

[4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $IF = e^{\int P(x) dx}$.
- Multiplication by the integrating factor makes the left side the derivative of $y IF$, so $y IF = \int Q(x) IF dx + C$, where C is an arbitrary constant.
- Here the substitution $t = \tan^{-1} x$ gives $dt = \frac{dx}{1+x^2}$, and integration by parts evaluates $\int te^t dt$.

Step 1: Express the equation in linear form Since $1 + x^2 > 0$ for every real x , division by this expression is valid. The given equation becomes

$$\frac{dy}{dx} = \frac{\tan^{-1} x - y}{1 + x^2}, \quad \frac{dy}{dx} + \frac{y}{1 + x^2} = \frac{\tan^{-1} x}{1 + x^2}.$$

Thus $P(x) = \frac{1}{1+x^2}$ and $Q(x) = \frac{\tan^{-1} x}{1+x^2}$. The sign of the coefficient of y is positive after bringing its term to the left.

Step 2: Find and apply the integrating factor The integrating factor is

$$IF = e^{\int \frac{dx}{1+x^2}} = e^{\tan^{-1} x}.$$

Multiplying the linear equation by this factor and using the product rule gives

$$\frac{d}{dx} (ye^{\tan^{-1} x}) = \frac{\tan^{-1} x}{1 + x^2} e^{\tan^{-1} x}.$$

Therefore $ye^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1 + x^2} e^{\tan^{-1} x} dx + C$.

Step 3: Evaluate the integral and verify Put $t = \tan^{-1} x$. Then the integral is $\int te^t dt$. Taking t as the first factor in integration by parts,

$$\int te^t dt = te^t - \int e^t dt = (t - 1)e^t.$$

Hence $ye^t = (t - 1)e^t + C$. Dividing by the nonzero factor e^t yields $y = t - 1 + Ce^{-t}$. To check,

Quick Tip to Solve: Use $t = \tan^{-1} x$ immediately: the equation becomes $dy/dt + y = t$, whose integrating factor is simply e^t .

differentiate this expression:

$$\frac{dy}{dx} = \frac{1 - Ce^{-t}}{1 + x^2}, \quad t - y = 1 - Ce^{-t}.$$

These identities give $(1 + x^2)\frac{dy}{dx} = \tan^{-1} x - y$, exactly the original equation. The arbitrary constant remains unrestricted.

Step 4: Conclusion The general solution is $y = \tan^{-1} x - 1 + Ce^{-\tan^{-1} x}$, where $C \in \mathbb{R}$.

Quick Tip to Solve: Use $t = \tan^{-1} x$ immediately: the equation becomes $dy/dt + y = t$, whose integrating factor is simply e^t .

19. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle. [4 marks]

Solution:

Concept:

- If the position vectors of points P, Q are $\vec{p}, \vec{q}$, then $\overrightarrow{PQ} = \vec{q} - \vec{p}$. Subtract position vectors in the order endpoint minus starting point.
- Two nonzero vectors $\vec{u}, \vec{v}$ are perpendicular when $\vec{u} \cdot \vec{v} = 0$. To locate a right angle, choose both side vectors from the same vertex.
- For perpendicular sides of lengths a, b , a triangle has area $\frac{1}{2}ab$. A vector's length is the square root of the sum of the squares of its components.

Step 1: Obtain the side vectors The given position vectors correspond to $A = (2, -1, 1)$, $B = (1, -3, -5)$ and $C = (3, -4, -4)$. Subtracting the coordinates gives

$$\begin{aligned}\overrightarrow{AB} &= (-1, -2, -6), \\ \overrightarrow{AC} &= (1, -3, -5), \\ \overrightarrow{BC} &= (2, -1, 1).\end{aligned}$$

All these vectors are nonzero. Their squared lengths are respectively 41, 35 and 6, suggesting that AB is the hypotenuse and the right angle is at C .

Step 2: Prove that the angle at C is a right angle Use the two vectors starting at C :

$$\overrightarrow{CA} = (-1, 3, 5), \quad \overrightarrow{CB} = (-2, 1, -1).$$

Their scalar product is

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-1)(-2) + (3)(1) + (5)(-1) = 2 + 3 - 5 = 0.$$

Since both vectors are nonzero, their scalar product can be zero only when their included angle is 90° . Thus $\angle ACB = 90^\circ$, and the three points form a nondegenerate right-angled triangle. As an

Quick Tip to Solve: Compare squared side lengths first: $35 + 6 = 41$ identifies the right-angle vertex, then confirm with a scalar product at that vertex.

independent check, $AC^2 + BC^2 = 35 + 6 = 41 = AB^2$, in agreement with Pythagoras' theorem.

Step 3: Calculate the area using the perpendicular sides The base and perpendicular height can be taken as BC and CA , because these sides meet at the established right angle. Their lengths are

$$CA = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}, \quad CB = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}.$$

Consequently,

$$\text{Area} = \frac{1}{2}(CA)(CB) = \frac{1}{2}\sqrt{35}\sqrt{6} = \frac{\sqrt{210}}{2}.$$

No length unit is specified in the question, so the corresponding area is expressed in square units. The hypotenuse AB is not used as a perpendicular side in this calculation.

Step 4: Conclusion The triangle is right-angled at C , and its area is $\boxed{\frac{\sqrt{210}}{2} \text{ square units}}$.

Quick Tip to Solve: Compare squared side lengths first: $35 + 6 = 41$ identifies the right-angle vertex, then confirm with a scalar product at that vertex.

20. Find the value of λ , if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ are coplanar. [4 marks]

Solution:

Concept:

- Four points A, B, C, D are coplanar when the scalar triple product $[\vec{AB}, \vec{AC}, \vec{AD}]$ is zero. It represents the signed volume of the parallelepiped formed by the three displacement vectors.
- A displacement vector is obtained by subtracting the position vector of its starting point from that of its endpoint. All three vectors in the coplanarity test must start at the same point.

Step 1: Form three vectors from one point Name the given points $A(3, 6, 9)$, $B(1, 2, 3)$, $C(2, 3, 1)$ and $D(4, 6, \lambda)$. Subtract the coordinates of A in each case:

$$\begin{aligned}\vec{AB} &= -2\hat{i} - 4\hat{j} - 6\hat{k}, \\ \vec{AC} &= -\hat{i} - 3\hat{j} - 8\hat{k}, \\ \vec{AD} &= \hat{i} + (\lambda - 9)\hat{k}.\end{aligned}$$

The zero coefficient of $\hat{j}$ in $\vec{AD}$ follows because both A and D have second coordinate 6.

Step 2: Apply the scalar triple product condition First calculate a normal to the plane through

Quick Tip to Solve: Subtract one common position vector from the other three before taking the triple product; using three position vectors alone tests a plane through the origin instead.

A, B, C :

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & -4 & -6 \\ -1 & -3 & -8 \end{vmatrix} \\ &= (32 - 18)\hat{i} - (16 - 6)\hat{j} + (6 - 4)\hat{k} \\ &= 14\hat{i} - 10\hat{j} + 2\hat{k}.\end{aligned}$$

This normal is nonzero, so the first three points determine a plane. Coplanarity requires $\vec{AD}$ to be perpendicular to this normal:

$$\begin{aligned}(\vec{AB} \times \vec{AC}) \cdot \vec{AD} &= 0, \\ 14(1) - 10(0) + 2(\lambda - 9) &= 0, \\ 2\lambda - 4 &= 0.\end{aligned}$$

Step 3: Check the resulting plane Divide the normal by 2 to use $(7, -5, 1)$. The plane through A is

$$7(x - 3) - 5(y - 6) + (z - 9) = 0, \quad 7x - 5y + z = 0.$$

The first three points satisfy this equation. For D , its left side is $28 - 30 + \lambda = \lambda - 2$, which vanishes for the value obtained from the triple product. This also verifies the coplanarity directly.

Step 4: Conclusion The required value is $\boxed{\lambda = 2}$.

Quick Tip to Solve: Subtract one common position vector from the other three before taking the triple product; using three position vectors alone tests a plane through the origin instead.

21. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- For a discrete random variable X , the mean is $E(X) = \sum_x xP(X = x)$, where the sum runs over its distinct possible values.
- The variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$, with $E(X^2) = \sum_x x^2P(X = x)$.
- Drawing two cards without replacement produces distinct cards. Since each unordered pair is equally likely, its probability is $1/\binom{4}{2} = 1/6$.

Step 1: List the equally likely pairs and their sums The sample space of unordered pairs is

$$\{\{1, 3\}, \{1, 5\}, \{1, 7\}, \{3, 5\}, \{3, 7\}, \{5, 7\}\}.$$

There are six equally likely pairs. Their respective sums are 4, 6, 8, 8, 10, 12. The sum 8 occurs twice, from $\{1, 7\}$ and $\{3, 5\}$, while each other sum occurs once. Thus the five distinct sums themselves are

Quick Tip to Solve: Count the pairs that produce each sum before assigning probabilities: here sum 8 has twice the probability of any other sum.

not equally likely. The probability distribution is

x	4	6	8	10	12
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probabilities add to $(1 + 1 + 2 + 1 + 1)/6 = 1$, as required.

Step 2: Calculate the mean Multiply each possible value by its probability and add:

$$\begin{aligned} E(X) &= 4 \left(\frac{1}{6} \right) + 6 \left(\frac{1}{6} \right) + 8 \left(\frac{2}{6} \right) + 10 \left(\frac{1}{6} \right) + 12 \left(\frac{1}{6} \right) \\ &= \frac{4 + 6 + 16 + 10 + 12}{6} = \frac{48}{6} = 8. \end{aligned}$$

Step 3: Calculate the second moment and variance Square each possible sum before applying its probability:

$$\begin{aligned} E(X^2) &= \frac{4^2 + 6^2 + 2(8^2) + 10^2 + 12^2}{6} \\ &= \frac{16 + 36 + 128 + 100 + 144}{6} = \frac{212}{3}. \end{aligned}$$

Therefore,

$$\text{Var}(X) = \frac{212}{3} - 8^2 = \frac{212 - 192}{3} = \frac{20}{3}.$$

Step 4: Conclusion The mean is $E(X) = 8$ and the variance is $\text{Var}(X) = \frac{20}{3}$.

Quick Tip to Solve: Count the pairs that produce each sum before assigning probabilities: here sum 8 has twice the probability of any other sum.

22. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer. [4 marks]

Solution:

Concept:

- Bayes' theorem finds the probability of a cause when its outcome is known: $P(R | A) = P(R)P(A | R)/P(A)$.
- If R and I partition the student population into regular and irregular students, the total probability is $P(A) = P(R)P(A | R) + P(I)P(A | I)$.

Step 1: Define the events and identify the data Let R denote the event that the selected student has 100% attendance, I the event that the student is irregular, and A the event that the student attains an A grade. Every student belongs to exactly one of R and I , so these events are mutually exclusive and

Quick Tip to Solve: Form the two A-grade joint probabilities first, then divide the regular A-grade share by their sum.

exhaustive. The given data are

$$P(R) = 0.30, \quad P(I) = 0.70, \quad P(A | R) = 0.70, \quad P(A | I) = 0.10.$$

The student is already known to have an A grade. Thus the required probability is $P(R | A)$, rather than the given probability $P(A | R)$.

Step 2: Find the total probability of an A grade An A grade can come from either attendance group. The probabilities of these two disjoint possibilities are

$$\begin{aligned} P(R \cap A) &= P(R)P(A | R) = 0.30(0.70) = 0.21, \\ P(I \cap A) &= P(I)P(A | I) = 0.70(0.10) = 0.07. \end{aligned}$$

Hence, by the law of total probability, $P(A) = 0.21 + 0.07 = 0.28$. This is positive, so conditioning on A is well-defined.

Step 3: Apply Bayes' theorem and address regularity Among students attaining an A grade, the proportion with full attendance is

$$P(R | A) = \frac{P(R \cap A)}{P(A)} = \frac{0.21}{0.28} = \frac{21}{28} = \frac{3}{4}.$$

Regularity is also valuable outside school: consistent study, exercise and fulfilment of responsibilities build discipline, strengthen good habits and support steady progress. Thus regularity is required in everyday life as well as at school; attending regularly is one application of this broader principle.

Step 4: Conclusion The required probability is $\frac{3}{4}$, or 75%. Regularity is required beyond school too, because consistent effort and discipline help us fulfil responsibilities and achieve goals.

Quick Tip to Solve: Form the two A-grade joint probabilities first, then divide the regular A-grade share by their sum.

23. Maximise $Z = x + 2y$ subject to the constraints $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x, y \geq 0$. Solve the above LPP graphically. [4 marks]

Solution:

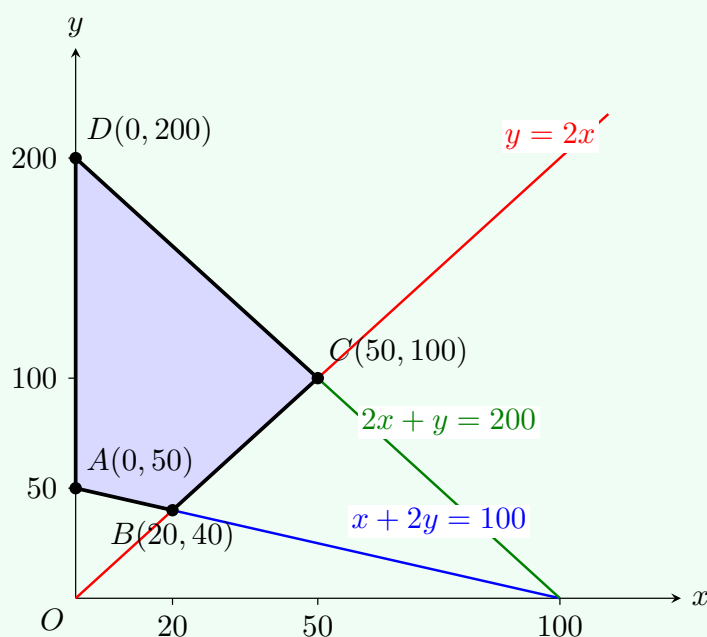
Concept:

- The feasible region is the intersection of the half-planes given by all constraints, restricted here to the first quadrant.
- A linear objective on a nonempty bounded polygon attains its maximum at a corner point. Evaluate $Z = x + 2y$ at each vertex.

Step 1: Draw the boundary lines and select the half-planes The line $x + 2y = 100$ has intercepts $(100, 0)$ and $(0, 50)$. Since the origin fails $x + 2y \geq 100$, select its side away from the origin. The inequality $2x - y \leq 0$ gives $y \geq 2x$, so select the region above $y = 2x$. The line $2x + y = 200$ has intercepts $(100, 0)$ and $(0, 200)$; its allowed side contains the origin. With $x, y \geq 0$, the intersection

Quick Tip to Solve: Rewrite the second constraint as y at least $2x$ before shading; it excludes the tempting intercept $(100, 0)$.

of these half-planes is the shaded quadrilateral below. All boundary lines are included because the inequalities are non-strict.



Shaded feasible region ABCD with its four corner points

Step 2: Determine the vertices of the feasible region On the y -axis, the lower and upper bounds give $A = (0, 50)$ and $D = (0, 200)$. To find the intersection of the lower boundaries, substitute $y = 2x$ into $x + 2y = 100$:

$$x + 4x = 100 \implies B = (20, 40).$$

Intersecting $y = 2x$ with the upper boundary gives

$$2x + 2x = 200 \implies C = (50, 100).$$

Each of A, B, C, D satisfies all the inequalities. The intersection $(100, 0)$ of the two intercept lines fails $y \geq 2x$, and the origin fails $x + 2y \geq 100$; neither is feasible. Thus the shaded bounded region has precisely these four vertices.

Step 3: Evaluate the objective at the corner points Apply the corner-point method to this nonempty bounded feasible region:

Vertex (x, y)	$Z = x + 2y$
$(0, 50)$	100
$(20, 40)$	100
$(50, 100)$	250
$(0, 200)$	400

As a check, $2x + y \leq 200$ implies $y \leq 200 - 2x$, so every feasible point satisfies $Z \leq 400 - 3x \leq 400$. Equality requires $x = 0, y = 200$, which is feasible and agrees with the corner-point comparison.

Step 4: Conclusion The maximum value is $Z_{\max} = 400$, attained at $(x, y) = (0, 200)$.

Quick Tip to Solve: Rewrite the second constraint as y at least $2x$ before shading; it excludes the tempting intercept $(100, 0)$.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24. Determine the product $\begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix}$ and use it to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$. [6 marks]

Solution:

Concept:

- For matrices B and A , the entry in row i , column j of BA is the scalar product of row i of B and column j of A . The order of the factors must be retained.
- A system of linear equations can be written as $AX = C$, where A is the coefficient matrix, X is the column of unknowns and C is the column of constants. If $BA = kI$, multiplying by B gives $kX = BC$.

Step 1: Calculate the ordered matrix product Write the given first and second matrices as B and A , respectively. Taking the scalar products of the rows of B with the columns of A , we obtain

$$BA = \begin{pmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} = 8I.$$

Thus every off-diagonal entry is zero and every diagonal entry is eight. In particular, A is nonsingular and $A^{-1} = \frac{1}{8}B$.

Step 2: Express and solve the equations in matrix form The coefficients of the printed equations occur in the rows of A . Hence the system is

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \\ 1 \end{pmatrix}.$$

Premultiply both sides by B , using the product already calculated:

$$8 \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 9 \\ 1 \end{pmatrix} = \begin{pmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{pmatrix} = \begin{pmatrix} 24 \\ -16 \\ -8 \end{pmatrix}.$$

Divide each component by eight to determine the unknowns. Since A has an inverse, this procedure gives the unique solution of the system.

Step 3: Verify the values in the original equations Substituting $x = 3$, $y = -2$, $z = -1$ into the three

Quick Tip to Solve: Keep the coefficient matrix as the second factor: the given product immediately supplies eight times the unknown column.

left sides gives

$$\begin{aligned}x - y + z &= 3 - (-2) + (-1) = 4, \\x - 2y - 2z &= 3 - 2(-2) - 2(-1) = 9, \\2x + y + 3z &= 2(3) + (-2) + 3(-1) = 1.\end{aligned}$$

All three constants agree with the printed system, so the matrix multiplication and signs in the solution are consistent.

Step 4: Conclusion The given matrix product is $8I$, and the unique solution is $x = 3, y = -2, z = -1$.

Quick Tip to Solve: Keep the coefficient matrix as the second factor: the given product immediately supplies eight times the unknown column.

25(a). Consider $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R} - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$. [6 marks]

Solution:

Concept:

- A function is bijective when it is both one-one and onto. One-one means $f(a) = f(b)$ forces $a = b$; onto means every element of the specified codomain has a preimage in the domain.
- To find an inverse, write $y = f(x)$, solve for x , and then interchange the variable names. Check all excluded values so the inverse has exactly the required domain and range.

Step 1: Check the function is well-defined The domain excludes $-\frac{4}{3}$, so $3x + 4 \neq 0$ and the formula is defined for every allowed x . Moreover,

$$f(x) - \frac{4}{3} = \frac{3(4x+3) - 4(3x+4)}{3(3x+4)} = \frac{-7}{3(3x+4)} \neq 0.$$

Thus no domain element maps to the excluded codomain value $\frac{4}{3}$, and the stated domain and codomain are appropriate.

Step 2: Prove that the function is one-one Let a, b belong to the domain and suppose $f(a) = f(b)$. Both denominators are nonzero, so cross-multiplication is valid:

$$\begin{aligned}(4a+3)(3b+4) &= (4b+3)(3a+4), \\12ab + 16a + 9b + 12 &= 12ab + 16b + 9a + 12, \\7a &= 7b.\end{aligned}$$

Therefore $a = b$, which proves that f is one-one.

Step 3: Prove that the function is onto and obtain its inverse Take any $y \in \mathbb{R} \setminus \{\frac{4}{3}\}$. Solving $y = \frac{4x+3}{3x+4}$ gives

$$3xy + 4y = 4x + 3, \quad (3y - 4)x = 3 - 4y, \quad x = \frac{3 - 4y}{3y - 4}.$$

Quick Tip to Solve: For the equation involving the inverse, apply f to both sides: the required input is simply $f(2)$.

This candidate exists because $3y - 4 \neq 0$. It also belongs to the domain, since

$$x + \frac{4}{3} = \frac{3(3 - 4y) + 4(3y - 4)}{3(3y - 4)} = \frac{-7}{3(3y - 4)} \neq 0.$$

For this value of x , direct substitution gives

$$4x + 3 = \frac{-7y}{3y - 4}, \quad 3x + 4 = \frac{-7}{3y - 4},$$

so their quotient is y . Every codomain element therefore has a preimage. Combined with the one-one result, this establishes bijectivity. The unique preimage formula gives

$$f^{-1} : \mathbb{R} \setminus \left\{\frac{4}{3}\right\} \longrightarrow \mathbb{R} \setminus \left\{-\frac{4}{3}\right\}, \quad f^{-1}(t) = \frac{3 - 4t}{3t - 4}.$$

Step 4: Evaluate the requested inverse values At zero the inverse formula gives

$$f^{-1}(0) = \frac{3}{-4} = -\frac{3}{4}.$$

To find the input for which the inverse has value two, solve

$$\frac{3 - 4x}{3x - 4} = 2 \implies 3 - 4x = 6x - 8 \implies 10x = 11.$$

The resulting input $\frac{11}{10}$ is different from $\frac{4}{3}$, so it is allowed. Equivalently it is $f(2) = \frac{11}{10}$, confirming the inverse relation.

Step 5: Conclusion f is bijective, with $f^{-1}(x) = \frac{3 - 4x}{3x - 4}$. Furthermore, $f^{-1}(0) = -\frac{3}{4}$, and $f^{-1}(x) = 2$ for $x = \frac{11}{10}$.

Quick Tip to Solve: For the equation involving the inverse, apply f to both sides: the required input is simply $f(2)$.

25(b). Let $A = \mathbb{Q} \times \mathbb{Q}$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to $*$ on A :
 (i) Find the identity element in A . (ii) Find the invertible elements of A . [6 marks]

Solution:

Concept:

- A binary operation is commutative if $x * y = y * x$ for all elements, and associative if $(x * y) * z = x * (y * z)$ for all elements.
- An identity e must satisfy $x * e = e * x = x$. An element is invertible when one element gives the identity on both sides.

Step 1: Check closure and commutativity Since products and sums of rational numbers are rational, $(ac, b + ad)$ belongs to A . Reversing the order gives

$$(c, d) * (a, b) = (ca, d + cb).$$

Although the first coordinates agree, the second coordinates need not agree. For example, both $(2, 0)$ and $(1, 1)$ lie in A , but

$$(2, 0) * (1, 1) = (2, 2), \quad (1, 1) * (2, 0) = (2, 1).$$

These unequal products supply a counterexample to commutativity.

Step 2: Verify associativity for arbitrary elements Let $(a, b), (c, d), (u, v)$ be any three elements of A . Applying the definition twice on each side gives

$$\begin{aligned} ((a, b) * (c, d)) * (u, v) &= (ac, b + ad) * (u, v) \\ &= (acu, b + ad + acv), \\ (a, b) * ((c, d) * (u, v)) &= (a, b) * (cu, d + cv) \\ &= (acu, b + a(d + cv)) \\ &= (acu, b + ad + acv). \end{aligned}$$

The two ordered pairs agree for every choice of the six rational coordinates, establishing associativity.

Step 3: Find and verify the identity element Write the identity as (r, s) . The right-identity requirement is

$$(a, b) * (r, s) = (ar, b + as) = (a, b).$$

Taking $a = 1$ gives $r = 1$ and $s = 0$. This candidate also works on the left, since

$$(1, 0) * (a, b) = (a, b), \quad (a, b) * (1, 0) = (a, b).$$

Quick Tip to Solve: For this operation, solve the first-coordinate equation $ac = 1$ first; it immediately excludes $a = 0$, after which the second coordinate determines the inverse.

Thus both identity requirements are satisfied.

Step 4: Determine all invertible elements and their inverses For (c, d) to be an inverse of (a, b) , the equation

$$(a, b) * (c, d) = (ac, b + ad) = (1, 0)$$

requires $ac = 1$. Hence $a \neq 0$, $c = 1/a$, and $d = -b/a$. These coordinates are rational whenever a, b are rational and $a \neq 0$. Check the other order:

$$\left(\frac{1}{a}, -\frac{b}{a}\right) * (a, b) = \left(1, -\frac{b}{a} + \frac{b}{a}\right) = (1, 0).$$

Conversely, if $a = 0$, the first coordinate of every product $(0, b) * (c, d)$ is zero, so it cannot equal the first coordinate of the identity. Such an element has no inverse.

Step 5: Conclusion The operation is not commutative, but is associative. Its identity is $(1, 0)$. The invertible elements are exactly $\{(a, b) \in \mathbb{Q} \times \mathbb{Q} : a \neq 0\}$, with $(a, b)^{-1} = (1/a, -b/a)$.

Quick Tip to Solve: For this operation, solve the first-coordinate equation $ac = 1$ first; it immediately excludes $a = 0$, after which the second coordinate determines the inverse.

26. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. [6 marks]

Solution:

Concept:

- For a cuboid with square base of side $x > 0$ and height $h > 0$, volume is $V = x^2h$ and total surface area is $S = 2x^2 + 4xh$.
- For a fixed positive volume, eliminate one dimension and minimise the resulting function of one positive variable. A positive second derivative at a stationary point confirms a local minimum; a sign change of the first derivative establishes the global minimum here.

Step 1: Express the surface area using the fixed volume Let the given volume be $V > 0$, the side of the square base be x , and the height be h . The volume condition gives

$$x^2h = V, \quad h = \frac{V}{x^2}, \quad x > 0.$$

Since the cuboid is closed, both square faces and all four rectangular side faces contribute to its surface area. Therefore

$$S = 2x^2 + 4xh = 2x^2 + \frac{4V}{x}.$$

The volume V is constant, while x may vary through positive values; the height is then determined by the volume equation.

Step 2: Find the stationary value of the base side Differentiating the surface-area function with respect to x ,

$$\frac{dS}{dx} = 4x - \frac{4V}{x^2} = \frac{4(x^3 - V)}{x^2}.$$

Quick Tip to Solve: Include both square faces because the cuboid is closed, then use $h = V/x^2$ before differentiating so that the fixed-volume constraint is respected.

At an interior minimum the derivative must vanish. Thus

$$4x - \frac{4V}{x^2} = 0 \implies x^3 = V \implies x = V^{1/3}.$$

As x is positive, this is the only stationary point in the allowed domain.

Step 3: Establish that this gives the absolute minimum The second derivative is

$$\frac{d^2S}{dx^2} = 4 + \frac{8V}{x^3}.$$

At $x = V^{1/3}$, it equals $4 + 8 = 12 > 0$, confirming a strict local minimum. To verify the required minimum over all possible cuboids, observe that $x^2 > 0$, so the sign of $S'(x)$ is the sign of $x^3 - V$. Consequently $S'(x) < 0$ for $0 < x < V^{1/3}$ and $S'(x) > 0$ for $x > V^{1/3}$. The area decreases up to the stationary point and increases afterwards, making it the absolute minimum on $(0, \infty)$.

Step 4: Identify the dimensions at the minimum Substituting the minimising base side in the volume constraint yields

$$h = \frac{V}{(V^{1/3})^2} = V^{1/3} = x.$$

The length and breadth are already equal to x , because the base is square. Thus all three dimensions are equal. The corresponding least surface area, as a check, is

$$S_{\min} = 2V^{2/3} + 4V^{2/3} = 6V^{2/3}.$$

Step 5: Conclusion For the given volume, the surface area of the closed cuboid is minimum when its height equals the side of its square base; hence the cuboid is a cube.

Quick Tip to Solve: Include both square faces because the cuboid is closed, then use $h = V/x^2$ before differentiating so that the fixed-volume constraint is respected.

27(a). Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$. [6 marks]

Solution:

Concept:

- The equation of a line through two points is obtained from $y - y_1 = m(x - x_1)$, where $m = (y_2 - y_1)/(x_2 - x_1)$ is its slope.
- The area between two boundaries is $\int_a^b [y_{\text{upper}} - y_{\text{lower}}] dx$. If the upper boundary changes at a vertex, divide the interval there before integrating.
- A vertical strip inside this triangle has lower boundary AC . Its upper boundary is AB up to $x = 6$, and BC after $x = 6$.

Step 1: Obtain the equations of the three sides For AB , the slope is $(6 - 1)/(6 - 4) = 5/2$. For BC , it

Quick Tip to Solve: Split the vertical-strip integral at the middle abscissa, here $x = 6$, where the upper side changes from AB to BC .

is $(4 - 6)/(8 - 6) = -1$, while for AC it is $(4 - 1)/(8 - 4) = 3/4$. Thus the required side equations are

$$AB : y - 1 = \frac{5}{2}(x - 4), \quad y = \frac{5}{2}x - 9,$$

$$BC : y - 6 = -(x - 6), \quad y = 12 - x,$$

$$AC : y - 1 = \frac{3}{4}(x - 4), \quad y = \frac{3}{4}x - 2.$$

Each equation passes through both endpoints of its corresponding side, so these are the boundaries of the given triangle.

Step 2: Identify the strip heights and split the integral The triangle extends from $x = 4$ to $x = 8$. The vertex B has abscissa 6, at which the upper side changes. On $[4, 6]$, the vertical height is

$$\left(\frac{5}{2}x - 9\right) - \left(\frac{3}{4}x - 2\right) = \frac{7}{4}x - 7 = \frac{7}{4}(x - 4).$$

On $[6, 8]$, the height is

$$(12 - x) - \left(\frac{3}{4}x - 2\right) = 14 - \frac{7}{4}x = \frac{7}{4}(8 - x).$$

Both heights are non-negative over their respective intervals, as required for an area integral. Hence, writing the area as S ,

$$S = \int_4^6 \left(\frac{7}{4}x - 7\right) dx + \int_6^8 \left(14 - \frac{7}{4}x\right) dx.$$

Step 3: Evaluate the two contributions The first contribution is

$$\left[\frac{7}{8}x^2 - 7x\right]_4^6 = \left(\frac{63}{2} - 42\right) - (14 - 28) = \frac{7}{2}.$$

The second contribution is

$$\left[14x - \frac{7}{8}x^2\right]_6^8 = (112 - 56) - \left(84 - \frac{63}{2}\right) = \frac{7}{2}.$$

Adding these two non-overlapping portions completes the integration over the entire triangle.

Step 4: Conclusion Therefore, the area of triangle ABC is $S = \frac{7}{2} + \frac{7}{2} = 7$ square units.

Quick Tip to Solve: Split the vertical-strip integral at the middle abscissa, here $x = 6$, where the upper side changes from AB to BC .

27(b). Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

[6 marks]

Solution:

Concept:

- The intersection points of two curves are found by solving their equations simultaneously; their abscissae provide the limits of integration for the enclosed region.
- For vertical strips, the enclosed area is $S = \int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$, with a and b the intersection abscissae.
- Before integrating, check which curve lies above the other throughout the interval. Integrating the difference in the opposite order would give a negative signed integral rather than the required area.

Step 1: Express both boundaries as functions of x The parabola and the straight line can be written as

$$y_{\text{parabola}} = \frac{3}{4}x^2, \quad y_{\text{line}} = \frac{3}{2}x + 6.$$

The parabola opens upwards, whereas the second equation describes a straight line. To determine the bounded portion between them, we first find where these two ordinates are equal. The enclosed region ends at those common points.

Step 2: Find the intersection points Equating the two expressions for y gives

$$\begin{aligned} \frac{3}{4}x^2 &= \frac{3}{2}x + 6, \\ 3x^2 &= 6x + 24, \\ x^2 - 2x - 8 &= 0, \\ (x - 4)(x + 2) &= 0. \end{aligned}$$

Hence the limits are $x = -2$ and $x = 4$. Substituting in $y = 3x^2/4$, the common points are $(-2, 3)$ and $(4, 12)$. These values also satisfy the straight-line equation, providing a direct check of the limits.

Step 3: Establish the upper curve and form the area integral The line ordinate minus the parabola ordinate is

$$\frac{3}{2}x + 6 - \frac{3}{4}x^2 = \frac{3}{4}(4 - x)(x + 2).$$

For every x in $[-2, 4]$, both factors $4 - x$ and $x + 2$ are non-negative. Therefore the line is the upper boundary over the entire region, and the parabola is the lower boundary. No further splitting is needed. Consequently,

$$S = \int_{-2}^4 \left(\frac{3}{2}x + 6 - \frac{3}{4}x^2 \right) dx.$$

The integrand vanishes at both endpoints and is positive between them, consistent with the bounded region.

Quick Tip to Solve: Factor the line-minus-parabola expression as $\frac{3}{4}(4 - x)(x + 2)$; this confirms both the intersection limits and the correct order of subtraction.

Step 4: Evaluate the definite integral Integrating each term separately gives

$$S = \left[\frac{3}{4}x^2 + 6x - \frac{1}{4}x^3 \right]_{-2}^4.$$

At the upper limit the antiderivative is

$$\frac{3}{4}(16) + 24 - \frac{1}{4}(64) = 12 + 24 - 16 = 20.$$

At the lower limit it is

$$\frac{3}{4}(4) - 12 - \frac{1}{4}(-8) = 3 - 12 + 2 = -7.$$

Subtract the lower-limit value from the upper-limit value; the negative lower-limit value must retain its sign in this subtraction.

Step 5: Conclusion Thus the enclosed area is

$$S = 20 - (-7) = 27 \text{ square units}.$$

Quick Tip to Solve: Factor the line-minus-parabola expression as $\frac{3}{4}(4-x)(x+2)$; this confirms both the intersection limits and the correct order of subtraction.

28. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx} = (x+2y)$, given that $y = 0$ when $x = 1$. [6 marks]

Solution:

Concept:

- A differential equation is homogeneous when its right-hand side is a function of the ratio y/x . The substitution $v = y/x$, or $y = vx$, gives $dy/dx = v + x dv/dx$.
- After separating variables, integrate both sides and determine the integration constant from the given initial condition. Complete the square to integrate a quadratic denominator.

Step 1: Recognise homogeneity and substitute In a neighbourhood of the initial point $(1, 0)$, we have $x \neq 0$ and $x - y \neq 0$. Hence the given equation becomes

$$\frac{dy}{dx} = \frac{x+2y}{x-y} = \frac{1+2(y/x)}{1-y/x}.$$

Put $y = vx$. Differentiating with respect to x , using the product rule, gives $dy/dx = v + x dv/dx$. Therefore

$$(1-v) \left(v + x \frac{dv}{dx} \right) = 1 + 2v.$$

Quick Tip to Solve: Split $1-v$ into $-\frac{1}{2}(2v+1) + \frac{3}{2}$ to obtain a logarithm and an inverse tangent directly.

On moving the term $v - v^2$ to the right, we obtain

$$x(1 - v) \frac{dv}{dx} = 1 + v + v^2.$$

Step 2: Separate variables and integrate The polynomial $v^2 + v + 1 = (v + 1/2)^2 + 3/4$ is positive for all real v , so division by it does not discard a real constant-ratio solution. Separating variables gives

$$\frac{1 - v}{v^2 + v + 1} dv = \frac{dx}{x}.$$

Write the numerator as $1 - v = -\frac{1}{2}(2v + 1) + \frac{3}{2}$. Thus

$$\begin{aligned} \int \frac{1 - v}{v^2 + v + 1} dv &= -\frac{1}{2} \ln(v^2 + v + 1) + \frac{3}{2} \int \frac{dv}{(v + 1/2)^2 + 3/4} \\ &= -\frac{1}{2} \ln(v^2 + v + 1) + \sqrt{3} \tan^{-1} \left(\frac{2v + 1}{\sqrt{3}} \right). \end{aligned}$$

Consequently this expression equals $\ln |x| + C$. Here $\tan^{-1}$ denotes the principal inverse tangent, with the branch continued from the initial point.

Step 3: Apply the initial condition and return to the original variables When $x = 1$ and $y = 0$, the substituted variable is $v = 0$. Since $\tan^{-1}(1/\sqrt{3}) = \pi/6$, the integrated equation yields $C = \pi/(2\sqrt{3})$. Multiplying the equation by -2 and rearranging gives

$$\ln(v^2 + v + 1) + 2 \ln |x| = 2\sqrt{3} \tan^{-1} \left(\frac{2v + 1}{\sqrt{3}} \right) - \frac{\pi}{\sqrt{3}}.$$

Now $x^2(v^2 + v + 1) = x^2 + xy + y^2$. Substituting $v = y/x$ therefore expresses the solution entirely in x, y . This relation is understood on the connected solution interval through $(1, 0)$, where the differential equation is defined. At the initial point the left side is $\ln 1 = 0$, and the right side is also zero, checking the constant.

Step 4: Verify the differential equation Set $D = x^2 + xy + y^2$ and $u = (x + 2y)/(\sqrt{3}x)$. Implicit differentiation of the resulting relation gives

$$\frac{2x + y + (x + 2y)y'}{D} = \frac{3(xy' - y)}{D}.$$

Indeed, $u' = 2(xy' - y)/(\sqrt{3}x^2)$ and $1 + u^2 = 4D/(3x^2)$. Cancelling D , then collecting terms, gives

$$2x + 4y = 2(x - y)y',$$

which is precisely the printed differential equation.

Step 5: Conclusion The required particular solution is

$$\ln(x^2 + xy + y^2) = 2\sqrt{3} \tan^{-1} \left(\frac{x + 2y}{\sqrt{3}x} \right) - \frac{\pi}{\sqrt{3}}.$$

Quick Tip to Solve: Split $1 - v$ into $-\frac{1}{2}(2v + 1) + \frac{3}{2}$ to obtain a logarithm and an inverse tangent directly.

29(a). Find the coordinates of the point where the line through the points $(3, -4, -5)$ and $(2, -3, 1)$, crosses the plane determined by the points $(1, 2, 3)$, $(4, 2, -3)$ and $(0, 4, 3)$. [6 marks]

Solution:

Concept:

- A plane through three non-collinear points has a normal vector given by the cross product of two vectors joining those points.
- A line through points P, Q has vector equation $\mathbf{r} = \mathbf{P} + t(\mathbf{Q} - \mathbf{P})$. Substitution in the plane equation determines the parameter at their intersection.

Step 1: Find two vectors in the plane Let the three given points of the plane be $A = (1, 2, 3)$, $B = (4, 2, -3)$, and $C = (0, 4, 3)$. The vectors joining A to the other two points are

$$\overrightarrow{AB} = (3, 0, -6), \quad \overrightarrow{AC} = (-1, 2, 0).$$

Their cross product is perpendicular to both vectors and therefore perpendicular to the plane:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= (0 \cdot 0 - (-6) \cdot 2, (-6)(-1) - 3 \cdot 0, 3 \cdot 2 - 0(-1)) \\ &= (12, 6, 6) = 6(2, 1, 1).\end{aligned}$$

This nonzero cross product also verifies that the points are not collinear, so they determine a unique plane. We may use the simpler normal vector $(2, 1, 1)$.

Step 2: Write and check the plane equation Using the normal vector and the point A , the plane equation is

$$2(x - 1) + (y - 2) + (z - 3) = 0, \quad \text{or} \quad 2x + y + z = 7.$$

For completeness, the other two defining points also satisfy it: at B , $2(4) + 2 - 3 = 7$, and at C , $2(0) + 4 + 3 = 7$. Thus the plane equation uses all three printed points consistently.

Step 3: Parametrise the given line Let $P = (3, -4, -5)$ and $Q = (2, -3, 1)$. A direction vector of the line is

$$\overrightarrow{PQ} = Q - P = (-1, 1, 6).$$

Hence every point on the entire line has coordinates

$$(x, y, z) = (3, -4, -5) + t(-1, 1, 6),$$

or, equivalently,

$$x = 3 - t, \quad y = -4 + t, \quad z = -5 + 6t.$$

The parameter t can be any real number; the requested line is not restricted to the segment joining the two given points. Its direction has dot product $2(-1) + 1 + 6 = 5 \neq 0$ with the plane normal, so there is one intersection.

Step 4: Find the parameter and verify the intersection Substitute the parametric coordinates into

Quick Tip to Solve: Use the simplified plane normal $(2, 1, 1)$; substitute the parametric line directly to avoid solving three simultaneous equations.

$$2x + y + z = 7:$$

$$\begin{aligned} 2(3 - t) + (-4 + t) + (-5 + 6t) &= 7, \\ -3 + 5t &= 7, \\ t &= 2. \end{aligned}$$

Using this value in all three coordinate equations gives $x = 3 - 2 = 1$, $y = -4 + 2 = -2$, and $z = -5 + 12 = 7$. The candidate lies on the plane because $2(1) + (-2) + 7 = 7$. It also lies on the line, since its displacement from P is $(-2, 2, 12) = 2(-1, 1, 6)$, exactly twice the direction vector.

Step 5: Conclusion The line crosses the plane at $(1, -2, 7)$.

Quick Tip to Solve: Use the simplified plane normal $(2, 1, 1)$; substitute the parametric line directly to avoid solving three simultaneous equations.

OR

29(b). A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C . Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$. [6 marks]

Solution:

Concept:

- A plane with finite nonzero intercepts a, b, c on the coordinate axes has intercept form $X/a + Y/b + Z/c = 1$. Capital letters denote coordinates of a general point on the plane.
- The centroid of a triangle is the coordinatewise average of its three vertices. Thus the centroid of $(a, 0, 0), (0, b, 0), (0, 0, c)$ is $(a/3, b/3, c/3)$.
- The perpendicular distance from the origin to $lX + mY + nZ + d = 0$ is $|d|/\sqrt{l^2 + m^2 + n^2}$. Here the fixed distance is $3p$, with $p > 0$.

Step 1: Express the plane through its intercepts Let the variable plane meet the axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$. Since these are the three intercepts forming the stated triangle, take a, b, c to be finite and nonzero. Their signs need not be positive: the intercepts can lie on either side of the origin. The plane is

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} - 1 = 0.$$

Substitution of each of A, B, C verifies that it lies on this plane.

Step 2: Apply the fixed perpendicular distance In the distance formula the coefficients are $l = 1/a$, $m = 1/b$, $n = 1/c$ and $d = -1$. Hence the given distance condition gives

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 3p.$$

Quick Tip to Solve: Write each axis intercept as three times its centroid coordinate before eliminating parameters; the factor of nine then cancels immediately.

Squaring and rearranging, using that the distance and p are positive, we obtain

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2}.$$

This relation describes the allowed intercepts of the moving plane, independently of their signs.

Step 3: Replace the intercepts by centroid coordinates Let $G(x, y, z)$ be the centroid of ABC . The centroid formula yields

$$x = \frac{a}{3}, \quad y = \frac{b}{3}, \quad z = \frac{c}{3}.$$

Therefore $a = 3x$, $b = 3y$, $c = 3z$. Substitution into the intercept relation gives

$$\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = \frac{1}{9p^2}.$$

Multiplying both sides by 9 eliminates the intercept parameters.

Step 4: Verify that the relation is sufficient Conversely, take any point (x, y, z) satisfying the resulting relation, with $xyz \neq 0$. Construct the plane with intercepts $a = 3x$, $b = 3y$, $c = 3z$. Its triangle of intercepts has centroid (x, y, z) . Its perpendicular distance from the origin is

$$\frac{1}{\sqrt{\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2}}} = \frac{1}{\sqrt{1/(9p^2)}} = 3p.$$

Thus every point of the stated surface is attained by an admissible plane, establishing both directions of the locus claim.

Step 5: Conclusion Hence the locus of the centroid is

$$\boxed{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}}, \quad xyz \neq 0.$$

Quick Tip to Solve: Write each axis intercept as three times its centroid coordinate before eliminating parameters; the factor of nine then cancels immediately.