

# CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-1-1

## Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



**Time Allowed: 3 hours**

**Maximum Marks: 100**

### General Instructions

- (i) All questions are compulsory.
- (ii) This question paper contains 29 questions.
- (iii) Questions 1-4 in Section A are very short-answer type questions carrying 1 mark each.
- (iv) Questions 5-12 in Section B are short-answer type questions carrying 2 marks each.
- (v) Questions 13-23 in Section C are long-answer I type questions carrying 4 marks each.
- (vi) Questions 24-29 in Section D are long-answer II type questions carrying 6 marks each.

## Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If  $A$  is a  $3 \times 3$  invertible matrix, then what will be the value of  $k$  if  $\det(A^{-1}) = (\det A)^k$ . [1 mark]

**Solution:**

**Concept:**

- For an invertible square matrix  $A$ ,  $AA^{-1} = I$ , where  $I$  is the identity matrix.
- The determinant of a product equals the product of determinants:  $\det(BC) = \det B \det C$ .

**Step 1: Take determinants of the inverse identity** Applying determinants to  $AA^{-1} = I_3$ , we obtain

$$\det A \det(A^{-1}) = \det I_3 = 1.$$

Invertibility ensures that  $\det A \neq 0$ , so division by this determinant is valid.

**Step 2: Compare the powers** Thus

$$\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}.$$

Comparing this standard determinant identity with the given expression  $(\det A)^k$  identifies the required exponent. The order of the matrix does not introduce a factor of three into this exponent.

**Step 3: Conclusion**  $k = -1$ .

**Quick Tip to Solve:** Take determinants of  $AA^{-1} = I$ ; the reciprocal determinant gives the exponent immediately.

2. Determine the value of the constant  $k$  so that the function  $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$  is continuous at  $x = 0$ . [1 mark]

**Solution:**

**Concept:**

- Continuity at  $x = 0$  requires the left limit, the right limit and  $f(0)$  to agree.
- For  $x < 0$ ,  $|x| = -x$ , so  $x/|x| = -1$ .

**Step 1: Evaluate the left-hand limit** For every negative  $x$ ,

$$f(x) = \frac{kx}{-x} = -k.$$

Hence  $\lim_{x \rightarrow 0^-} f(x) = -k$ . The cancellation is performed for  $x < 0$ , not at zero, so it does not involve division by zero.

**Step 2: Match the value and the right-hand limit** The second branch applies at zero and to its right.

**Quick Tip to Solve:** On the negative side replace  $|x|$  by  $-x$  before taking the limit.

Therefore

$$f(0) = 3, \quad \lim_{x \rightarrow 0^+} f(x) = 3.$$

The continuity condition is consequently  $-k = 3$ . With this choice the left branch also has value 3 throughout its domain.

**Step 3: Conclusion**  $k = -3$ .

**Quick Tip to Solve:** On the negative side replace  $|x|$  by  $-x$  before taking the limit.

**3. Evaluate:**  $\int_2^3 3^x dx$ .

[1 mark]

**Solution:**

**Concept:**

- For  $a > 0, a \neq 1, \int a^x dx = a^x / \ln a + C$ .
- A definite integral is evaluated by subtracting the antiderivative at its lower endpoint from that at its upper endpoint.

**Step 1: Find an antiderivative** Since  $\frac{d}{dx} 3^x = 3^x \ln 3$ , an antiderivative of the given integrand is

$$F(x) = \frac{3^x}{\ln 3}.$$

The denominator is needed because the base is 3 rather than the natural exponential base.

**Step 2: Apply the limits** Using the limits in their printed order,

$$\int_2^3 3^x dx = \left[ \frac{3^x}{\ln 3} \right]_2^3 = \frac{3^3 - 3^2}{\ln 3} = \frac{27 - 9}{\ln 3}.$$

The result is positive, consistent with a positive integrand and an increasing interval.

**Step 3: Conclusion**  $\frac{18}{\ln 3}$ .

**Quick Tip to Solve:** For an exponential with constant base, divide by the natural logarithm of that base.

**4. If a line makes angles  $90^\circ$  and  $60^\circ$  respectively with the positive directions of  $x$  and  $y$  axes, find the angle which it makes with the positive direction of  $z$ -axis.**

[1 mark]

**Solution:**

**Concept:**

- If  $\alpha, \beta, \gamma$  are direction angles, their direction cosines satisfy  $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ .
- Direction angles lie between  $0^\circ$  and  $180^\circ$ ; a squared cosine may allow two angles.

**Quick Tip to Solve:** After using the sum of squared direction cosines, retain both signs of the remaining cosine.

**Step 1: Substitute the given angles** Let  $\gamma$  be the angle with the positive  $z$ -axis. Then

$$\cos^2 90^\circ + \cos^2 60^\circ + \cos^2 \gamma = 1,$$

$$\text{so } 0 + \frac{1}{4} + \cos^2 \gamma = 1.$$

**Step 2: Retain both possible signs** Therefore  $\cos^2 \gamma = \frac{3}{4}$ , giving  $\cos \gamma = \pm \frac{\sqrt{3}}{2}$ . Within the direction-angle range, the positive sign corresponds to  $30^\circ$  and the negative sign to  $150^\circ$ . The given angles do not fix the sign of the component along the  $z$ -axis.

**Step 3: Conclusion**  $\gamma = 30^\circ$  or  $150^\circ$ .

**Quick Tip to Solve:** After using the sum of squared direction cosines, retain both signs of the remaining cosine.

## Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

**5. Show that all the diagonal elements of a skew symmetric matrix are zero.**

[2 marks]

**Solution:**

**Concept:**

- A real square matrix  $A$  is skew symmetric when  $A^T = -A$ .
- Transposition interchanges row and column indices:  $(A^T)_{ij} = a_{ji}$ ; a diagonal entry has equal indices.

**Step 1: Compare corresponding matrix entries** Let  $A = (a_{ij})$  be any skew symmetric matrix of order  $n$ . The equality  $A^T = -A$  is an equality entry by entry. Therefore, for every pair of indices  $i, j$ ,

$$a_{ji} = -a_{ij}.$$

This relates each off-diagonal entry to its reflected entry and also applies when the indices are equal.

**Step 2: Apply the relation on the diagonal** Choose  $j = i$ . The diagonal entry is unchanged by transposition, so

$$a_{ii} = -a_{ii} \implies 2a_{ii} = 0 \implies a_{ii} = 0.$$

This holds independently for each  $i = 1, 2, \dots, n$ . Thus every diagonal position, not just the first one, satisfies the required condition. No restriction on the off-diagonal entries is needed beyond skew symmetry.

**Step 3: Conclusion** Hence all diagonal elements of a skew symmetric matrix are zero:

$$a_{11} = a_{22} = \dots = a_{nn} = 0.$$

**Quick Tip to Solve:** Compare a diagonal entry in  $A^T = -A$ ; transposition leaves that entry in the same position.

6. Find  $\frac{dy}{dx}$  at  $x = 1, y = \frac{\pi}{4}$  if  $\sin^2 y + \cos xy = K$ .

[2 marks]

**Solution:**

**Concept:**

- In implicit differentiation,  $y$  is a function of  $x$ ; write  $y' = dy/dx$ .
- Use the chain rule for trigonometric functions and the product rule  $d(xy)/dx = y + xy'$ . The constant  $K$  has derivative zero.

**Step 1: Differentiate the equation** Differentiating  $\sin^2 y + \cos(xy) = K$  with respect to  $x$ ,

$$2 \sin y \cos y y' - \sin(xy)(y + xy') = 0.$$

The minus sign comes from differentiating cosine; the factor  $y + xy'$  accounts for both changing factors inside  $xy$ .

**Step 2: Collect and substitute** Collecting the derivative terms gives

$$[2 \sin y \cos y - x \sin(xy)]y' = y \sin(xy).$$

At  $x = 1, y = \pi/4$ , we have  $2 \sin y \cos y = 1$  and  $\sin(xy) = 1/\sqrt{2}$ . Therefore

$$(1 - 1/\sqrt{2})y' = \frac{\pi}{4\sqrt{2}}.$$

The coefficient is nonzero, and hence

$$y' = \frac{\pi}{4(\sqrt{2} - 1)} = \frac{\pi(\sqrt{2} + 1)}{4}.$$

**Step 3: Conclusion**

$$\left. \frac{dy}{dx} \right|_{(1, \pi/4)} = \frac{\pi(\sqrt{2} + 1)}{4}.$$

**Quick Tip to Solve:** Differentiate  $xy$  as  $y + xy'$ , then collect every derivative term before substitution.

7. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm.

[2 marks]

**Solution:**

**Concept:**

- For a sphere of radius  $r$ , volume  $V = \frac{4}{3}\pi r^3$  and surface area  $S = 4\pi r^2$ .
- Both quantities depend on time through  $r$ ; the chain rule connects their rates to  $dr/dt$ .

**Step 1: Find the instantaneous change in radius** Differentiate the volume formula with respect to time  $t$ :

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

**Quick Tip to Solve:** Eliminate  $dr/dt$  to obtain  $dS/dt = (2/r)(dV/dt)$ , then insert the specified radius.

The given volume rate is  $3 \text{ cm}^3/\text{s}$ . At  $r = 2 \text{ cm}$ ,

$$3 = 16\pi \frac{dr}{dt}, \quad \frac{dr}{dt} = \frac{3}{16\pi} \text{ cm/s}.$$

This is the radius rate at the specified instant.

**Step 2: Calculate the surface-area rate** Differentiating  $S = 4\pi r^2$  yields

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

Substituting the same instantaneous radius and radius rate,

$$\frac{dS}{dt} = 8\pi(2) \frac{3}{16\pi} = 3 \text{ cm}^2/\text{s}.$$

The rate is positive because the radius is increasing. The area rate has square-centimetre units, unlike the given volume rate, which has cubic-centimetre units.

**Step 3: Conclusion** The surface area increases at  $3 \text{ cm}^2/\text{s}$ .

**Quick Tip to Solve:** Eliminate  $dr/dt$  to obtain  $dS/dt = (2/r)(dV/dt)$ , then insert the specified radius.

**8. Show that the function  $f(x) = 4x^3 - 18x^2 + 27x - 7$  is always increasing on  $\mathbb{R}$ .**

[2 marks]

**Solution:**

**Concept:**

- A differentiable function is non-decreasing on an interval when its derivative is non-negative throughout that interval.
- To prove strict increase despite an isolated zero of the derivative, compare the function values at any two distinct ordered inputs.

**Step 1: Differentiate and factor** For the given polynomial, differentiation gives

$$f'(x) = 12x^2 - 36x + 27 = 3(2x - 3)^2.$$

A square is non-negative for every real input. Thus  $f'(x) \geq 0$  for all  $x \in \mathbb{R}$ , with equality only at  $x = \frac{3}{2}$ . The derivative is positive on both sides of this isolated point, so there is no interval on which the function decreases or remains constant.

**Step 2: Confirm strict increase** The same polynomial can be written as

$$f(x) = 4 \left( x - \frac{3}{2} \right)^3 + \frac{13}{2}.$$

If  $a < b$ , then  $a - \frac{3}{2} < b - \frac{3}{2}$ . The cube function is strictly increasing, so cubing preserves this strict inequality. Multiplying by the positive number 4 and adding  $\frac{13}{2}$  gives  $f(a) < f(b)$ . This also explains why the single stationary point does not prevent strict increase.

**Step 3: Conclusion** Hence  $f(x) = 4x^3 - 18x^2 + 27x - 7$  is always increasing on  $\mathbb{R}$ .

**Quick Tip to Solve:** Factor the derivative as a square; an isolated zero does not imply a decreasing interval.

9. Find the vector equation of the line passing through the point  $A(1, 2, -1)$  and parallel to the line  $5x - 25 = 14 - 7y = 35z$ . [2 marks]

**Solution:**

**Concept:**

- A line through position vector  $\vec{a}$ , with non-zero direction vector  $\vec{b}$ , has equation  $\vec{r} = \vec{a} + \lambda\vec{b}$ , where  $\lambda \in \mathbb{R}$ .
- Parallel lines have proportional direction vectors; extract these from a common parameter in the given line.

**Step 1: Extract a direction vector** Put each expression in the given equation equal to a common real parameter  $t$ :

$$5x - 25 = 14 - 7y = 35z = t.$$

Solving separately gives

$$x = 5 + \frac{t}{5}, \quad y = 2 - \frac{t}{7}, \quad z = \frac{t}{35}.$$

Thus a direction vector is  $\frac{1}{5}\hat{i} - \frac{1}{7}\hat{j} + \frac{1}{35}\hat{k}$ . Multiplying by 35 yields the simpler proportional direction vector  $7\hat{i} - 5\hat{j} + \hat{k}$ . The negative sign in the second component comes from the term  $-7y$ .

**Step 2: Use the specified point** The position vector of  $A(1, 2, -1)$  is  $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ . The required line is parallel to the given one, so it can use the direction vector found above. Substituting these into the standard vector form gives

$$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.$$

At  $\lambda = 0$  this passes through  $A$ , and its direction is proportional to that of the given line, verifying both requirements.

**Step 3: Conclusion** The required equation is  $\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda(7\hat{i} - 5\hat{j} + \hat{k}), \lambda \in \mathbb{R}$ .

**Quick Tip to Solve:** Set the three expressions equal to one parameter before reading the direction ratios.

10. Prove that if  $E$  and  $F$  are independent events, then the events  $E$  and  $F'$  are also independent. [2 marks]

**Solution:**

**Concept:**

- Independence of events  $E$  and  $F$  means  $P(E \cap F) = P(E)P(F)$ .
- The complement  $F'$  has probability  $P(F') = 1 - P(F)$ ; probabilities of disjoint events add.

**Step 1: Partition the event E** Each outcome in  $E$  either belongs to  $F$  or does not belong to  $F$ . Therefore

$$E = (E \cap F) \cup (E \cap F').$$

These two events are disjoint, because an outcome cannot lie in both  $F$  and its complement. By the

**Quick Tip to Solve:** Partition E into its intersections with F and its complement, then subtract probabilities.

addition rule,

$$P(E) = P(E \cap F) + P(E \cap F').$$

Rearranging isolates the probability required for the independence test:  $P(E \cap F') = P(E) - P(E \cap F)$ .

**Step 2: Apply the given independence** As  $E$  and  $F$  are independent, replace  $P(E \cap F)$  by  $P(E)P(F)$ . Hence

$$\begin{aligned} P(E \cap F') &= P(E) - P(E)P(F) \\ &= P(E)[1 - P(F)] \\ &= P(E)P(F'). \end{aligned}$$

This identity is exactly the criterion for independence of  $E$  and  $F'$ . No division by any probability was used, so the proof also covers events with probability zero or one.

**Step 3: Conclusion** Hence the events  $E$  and  $F'$  are independent.

**Quick Tip to Solve:** Partition  $E$  into its intersections with  $F$  and its complement, then subtract probabilities.

**11. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P. for finding how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced. [2 marks]**

**Solution:**

**Concept:**

- Define decision variables as the daily quantities to be produced, and express total profit as a linear objective.
- An upper limit on a resource gives a  $\leq$  constraint; a requirement of at least one gives a lower bound of 1.

**Step 1: Define the variables and profit** Let  $x$  be the number of necklaces and  $y$  the number of bracelets produced per day. A necklace earns a profit of 100 rupees, while a bracelet earns 300 rupees. Therefore the total daily profit, in rupees, is

$$Z = 100x + 300y.$$

The objective is to maximize  $Z$ . Using the variables in this order is important because necklaces take less time and have the smaller profit per unit.

**Step 2: Translate each production condition** The firm can handle at most 24 articles in total, giving  $x + y \leq 24$ . Each necklace takes half an hour and each bracelet takes one hour. Thus the available 16 hours impose

$$\frac{x}{2} + y \leq 16, \quad \text{equivalently } x + 2y \leq 32.$$

At least one of each article must be produced, so  $x \geq 1$  and  $y \geq 1$ . These lower bounds already imply non-negativity. The quantities represent whole articles in practice; the requested linear programming formulation uses the stated linear objective and inequalities.

**Quick Tip to Solve:** Keep the necklace coefficient at one-half in the time constraint before clearing fractions.

**Step 3: Conclusion** The L.P.P. is: maximize  $Z = 100x + 300y$ , subject to

$$x + y \leq 24, \quad x + 2y \leq 32, \quad x \geq 1, \quad y \geq 1.$$

**Quick Tip to Solve:** Keep the necklace coefficient at one-half in the time constraint before clearing fractions.

12. Find:  $\int \frac{dx}{x^2 + 4x + 8}.$

[2 marks]

**Solution:**

**Concept:**

- Complete the square in a quadratic denominator to obtain a sum of two squares.
- The standard formula is  $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$ , where  $a > 0$ .

**Step 1: Complete the square** The denominator is

$$x^2 + 4x + 8 = (x + 2)^2 + 4 = (x + 2)^2 + 2^2.$$

Consequently the integral becomes

$$I = \int \frac{dx}{(x + 2)^2 + 2^2}.$$

The denominator is strictly positive for every real  $x$ ; completing the square identifies a sum of squares, so the inverse tangent formula is appropriate rather than the logarithmic formula for a difference of squares.

**Step 2: Substitute and use the standard formula** Let  $u = x + 2$ , so  $du = dx$ . There is no extra factor in the differential because the coefficient of  $x$  in this substitution is 1. Taking  $a = 2$  in the standard formula gives

$$I = \int \frac{du}{u^2 + 2^2} = \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) + C.$$

Substituting back  $u = x + 2$ , the antiderivative is  $\frac{1}{2} \tan^{-1}((x + 2)/2) + C$ . As a check, its derivative is  $\frac{1}{4} / [1 + (x + 2)^2/4] = 1 / [(x + 2)^2 + 4]$ , exactly the original integrand.

**Step 3: Conclusion**  $\int \frac{dx}{x^2 + 4x + 8} = \frac{1}{2} \tan^{-1}\left(\frac{x + 2}{2}\right) + C.$

**Quick Tip to Solve:** Complete the square first, and retain the outside factor  $1/a$  in the inverse tangent formula.

## Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

**13. Prove that**  $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right] = \frac{2b}{a}$ . [4 marks]

**Solution:**

**Concept:**

- The principal value  $\theta = \cos^{-1}(a/b)$  lies in  $[0, \pi]$ . For real, defined expressions in this identity,  $b \neq 0$ ,  $|a/b| \leq 1$ , and  $a \neq 0$ .
- The sum formula  $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$  follows by writing each tangent as sine divided by cosine.
- The product-to-sum identity is  $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$ . It allows the two denominators to be combined without introducing a half-angle square root.

**Step 1: Introduce the principal angle and check the denominators** Let  $\theta = \cos^{-1}(a/b)$ , so that  $\cos \theta = a/b$ . Write

$$A = \frac{\pi}{4} + \frac{\theta}{2}, \quad B = \frac{\pi}{4} - \frac{\theta}{2}.$$

Then  $A + B = \pi/2$  and  $A - B = \theta$ . Since  $0 \leq \theta \leq \pi$ , we have  $\pi/4 \leq A \leq 3\pi/4$  and  $-\pi/4 \leq B \leq \pi/4$ . Thus  $\cos B$  is nonzero;  $\cos A$  vanishes only at  $\theta = \pi/2$ , which corresponds to  $a = 0$  and is excluded because the right-hand side would also be undefined.

**Step 2: Combine the tangent terms into one fraction** Applying the sine addition formula to the numerator gives

$$\begin{aligned} \tan A + \tan B &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B} \\ &= \frac{\sin(A+B)}{\cos A \cos B} = \frac{1}{\cos A \cos B}. \end{aligned}$$

Here  $\sin(A+B) = \sin(\pi/2) = 1$ ; no choice of sign for a half-angle value is required.

**Step 3: Evaluate the product in the denominator** Using the product-to-sum formula and the values of the sum and difference,

$$\begin{aligned} 2 \cos A \cos B &= \cos(A+B) + \cos(A-B) \\ &= \cos \frac{\pi}{2} + \cos \theta = \frac{a}{b}. \end{aligned}$$

Therefore  $\cos A \cos B = a/(2b)$ . Substitution into the preceding fraction gives  $\tan A + \tan B = 1/[a/(2b)]$ .

**Step 4: Conclusion** Hence, throughout the real domain where both sides are defined,

$$\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right] = \frac{2b}{a}.$$

**Quick Tip to Solve:** Observe that the two angles add to  $\pi/2$ ; combine their tangents first, then use the product-to-sum formula for their cosine product.

**14(a).** Using properties of determinants, prove that  $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$  [4 marks]

**Solution:**

**Concept:**

- Adding multiples of other columns to one column does not change the determinant. Thus  $C_1 \rightarrow C_1 + C_2 + C_3$  can be performed by two successive column additions.
- A common factor in a column may be taken outside the determinant. Adding or subtracting one row from another also leaves its value unchanged.
- Expansion along a column with two zero entries reduces a third-order determinant to a second-order determinant, whose value is  $ad - bc$ .

**Step 1:** Replace the first column by the sum of all three columns Denote the given determinant by  $\Delta$ . In each row, the sum of the entries is  $x + (x+y) + (x+2y) = 3(x+y)$ . Applying  $C_1 \rightarrow C_1 + C_2 + C_3$ , we obtain

$$\Delta = \begin{vmatrix} 3(x+y) & x+y & x+2y \\ 3(x+y) & x & x+y \\ 3(x+y) & x+2y & x \end{vmatrix}.$$

This operation preserves the determinant because each of the two added columns is different from the first column.

**Step 2:** Take out the common first-column factor All three entries of the first column have the factor  $3(x+y)$ . By the scaling property of a determinant,

$$\Delta = 3(x+y) \begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}.$$

This is a factorisation identity and does not require division by  $x+y$ ; it therefore also holds when  $x+y=0$ .

**Step 3:** Create zero entries below the first entry Now perform  $R_2 \rightarrow R_2 - R_1$  and  $R_3 \rightarrow R_3 - R_1$ , keeping  $R_1$  fixed. The second row becomes  $(0, -y, -y)$ , while the third row becomes  $(0, y, -2y)$ . Hence

$$\Delta = 3(x+y) \begin{vmatrix} 1 & x+y & x+2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}.$$

Each row replacement preserves the determinant and makes expansion along the first column immediate.

**Step 4:** Expand along the first column Only the entry in position  $(1, 1)$  contributes, with positive

**Quick Tip to Solve:** Equal row sums suggest adding all columns into one column; then subtract the first row from the others to obtain a single nonzero entry in that column.

cofactor sign. Thus

$$\begin{aligned}\Delta &= 3(x+y) \begin{vmatrix} -y & -y \\ y & -2y \end{vmatrix} \\ &= 3(x+y)[(-y)(-2y) - (-y)(y)] \\ &= 3(x+y)(2y^2 + y^2).\end{aligned}$$

The subtraction of the negative product is essential: the remaining second-order determinant is  $3y^2$ .

**Step 5: Conclusion** Therefore,

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$$

The identity holds for all real  $x$  and  $y$ , including the cases in which the determinant is zero.

**Quick Tip to Solve:** Equal row sums suggest adding all columns into one column; then subtract the first row from the others to obtain a single nonzero entry in that column.

OR

**14(b).** Let  $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ ,  $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$ ,  $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$ ; find a matrix  $D$  such that  $CD - AB = O$ .

[4 marks]

**Solution:**

**Concept:**

- Matrix multiplication is order-sensitive. The equation  $CD - AB = O$ , where  $O$  is the zero matrix, gives  $CD = AB$ .
- If  $C$  is invertible, left multiplication by  $C^{-1}$  gives  $D = C^{-1}AB$ . For  $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , its inverse is

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

**Step 1: Calculate the product AB** Multiplying rows of  $A$  by columns of  $B$ ,

$$AB = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 2(5) - 7 & 2(2) - 4 \\ 3(5) + 4(7) & 3(2) + 4(4) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix}.$$

Hence  $CD$  must equal this product. The multiplication order remains  $AB$ , since exchanging the order generally changes a matrix product.

**Step 2: Find the inverse of C and isolate D** The determinant of  $C$  is  $2(8) - 5(3) = 1$ , which is nonzero. Thus its inverse exists and

$$C^{-1} = \frac{1}{1} \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}.$$

**Quick Tip to Solve:** Cancel  $C$  in  $CD$  by multiplying on the left by its inverse, preserving the order of all matrix factors.

Left multiplication of  $CD = AB$  by  $C^{-1}$  gives  $C^{-1}CD = C^{-1}AB$ . Since  $C^{-1}C = I$ , we obtain

$$D = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} = \begin{pmatrix} 24 - 215 & -110 \\ -9 + 86 & 44 \end{pmatrix}.$$

Invertibility also ensures uniqueness: any matrix satisfying the original equation must equal this same product  $C^{-1}AB$ .

**Step 3: Check the result in the original equation** Substituting the entries obtained above,

$$CD = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix} = \begin{pmatrix} -382 + 385 & -220 + 220 \\ -573 + 616 & -330 + 352 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} = AB.$$

Every entry of  $CD$  equals its corresponding entry of  $AB$ ; consequently  $CD - AB = O$ , as required.

**Step 4: Conclusion** The unique required matrix is  $D = \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}$ .

**Quick Tip to Solve:** Cancel  $C$  in  $CD$  by multiplying on the left by its inverse, preserving the order of all matrix factors.

**15(a). Differentiate the function  $(\sin x)^x + \sin^{-1} \sqrt{x}$  with respect to  $x$ .**

[4 marks]

**Solution:**

**Concept:**

- When both the base and the exponent depend on  $x$ , logarithmic differentiation is useful: for  $u = f(x)^{g(x)} > 0$ , take  $\ln u = g(x) \ln f(x)$  before differentiating.
- The chain rule gives  $\frac{d}{dx} \sin^{-1} v = \frac{v'}{\sqrt{1-v^2}}$ , for  $-1 < v < 1$ . Here  $\sin^{-1}$  denotes the inverse sine function.

**Step 1: Differentiate the variable power by taking logarithms** Put  $u = (\sin x)^x$ . On an interval where  $\sin x > 0$ , take natural logarithms to write

$$\ln u = x \ln(\sin x).$$

Apply the chain rule to the left side and the product rule to the right side:

$$\frac{u'}{u} = \ln(\sin x) + x \frac{\cos x}{\sin x} = \ln(\sin x) + x \cot x.$$

Multiplying by  $u$ , then substituting its original expression, gives

$$u' = (\sin x)^x [\ln(\sin x) + x \cot x].$$

The usual constant-exponent power rule cannot be applied directly here, because the exponent  $x$  also varies. The logarithm makes both sources of variation explicit.

**Quick Tip to Solve:** For a variable base raised to a variable exponent, take logarithms first; retain the inner square-root derivative in the inverse sine term.

**Step 2: Differentiate the inverse sine term using the chain rule** Put  $v = \sqrt{x}$ , so that  $v' = 1/(2\sqrt{x})$ . For  $0 < x < 1$ , both the square root and the inverse sine derivative are defined, and

$$\frac{d}{dx} (\sin^{-1} \sqrt{x}) = \frac{1}{\sqrt{1 - (\sqrt{x})^2}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

The factor  $1/(2\sqrt{x})$  is essential: it accounts for differentiating the inner function  $\sqrt{x}$ , rather than treating the inverse sine argument as  $x$ .

**Step 3: Add the two derivatives and state the interval** Differentiation is linear, so the derivative of the given sum is the sum of the two derivatives just obtained. The common open interval  $0 < x < 1$  has  $\sin x > 0$ , making the logarithmic differentiation valid, and also keeps both radicals in the second derivative positive. Hence no denominator vanishes on this interval. The endpoint values do not fall within the interval on which this derivative formula is asserted.

**Step 4: Conclusion** For  $0 < x < 1$ , the required derivative is

$$(\sin x)^x [\ln(\sin x) + x \cot x] + \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

**Quick Tip to Solve:** For a variable base raised to a variable exponent, take logarithms first; retain the inner square-root derivative in the inverse sine term.

OR

**15(b). If  $x^m y^n = (x + y)^{m+n}$ , prove that  $\frac{d^2 y}{dx^2} = 0$ .**

[4 marks]

**Solution:**

**Concept:**

- Logarithmic differentiation converts products of powers into sums: on an interval where the given powers and logarithms are defined,  $\log(x^m y^n) = m \log |x| + n \log |y|$ .
- In implicit differentiation,  $y$  is a function of  $x$ . Thus  $\frac{d}{dx} \log |y| = \frac{y'}{y}$ , where  $y' = \frac{dy}{dx}$ .
- A relation homogeneous in  $x, y$  can determine a constant ratio  $y/x$ . A straight-line branch  $y = kx$ , with constant  $k$ , has zero second derivative.

**Step 1: Take logarithms and differentiate** Starting with the printed relation  $x^m y^n = (x + y)^{m+n}$ , take logarithms on a regular interval of the curve:

$$m \log |x| + n \log |y| = (m + n) \log |x + y|.$$

Differentiating with respect to  $x$ , treating  $m, n$  as fixed exponents, gives

$$\frac{m}{x} + \frac{n}{y} y' = \frac{m + n}{x + y} (1 + y').$$

**Quick Tip to Solve:** After logarithmic differentiation, factor the equation as  $(my - nx)(y - xy') = 0$ ; differentiate  $xy' = y$ , and check the straight-line factor separately.

Here the factor  $1 + y'$  comes from differentiating  $x + y$ ; omitting it would invalidate the implicit differentiation.

**Step 2: Collect the derivative terms and factor** Multiply through by  $xy(x + y)$  and expand:

$$\begin{aligned}my(x + y) + nx(x + y)y' &= (m + n)xy(1 + y'), \\my^2 - nxy + (nx^2 - mxy)y' &= 0, \\(my - nx)(y - xy') &= 0.\end{aligned}$$

Consequently, on a branch where  $my - nx \neq 0$ , we obtain  $xy' = y$ . Differentiating this identity directly, without an unnecessary quotient expansion, yields

$$y' + xy'' = y', \quad xy'' = 0.$$

Hence the second derivative vanishes at every regular point of this branch with  $x \neq 0$ .

**Step 3: Check the straight-line case** The factor  $my - nx$  must be considered before division. If it vanishes identically on an interval, then  $my = nx$ , which describes a straight line through the origin (for  $m \neq 0$ ,  $y = (n/m)x$ ). Its first derivative is constant and its second derivative is zero as well. The same straight-line nature is also visible by putting  $v = y/x$  in the original equation: the common power of  $x$  cancels, leaving  $v^n = (1 + v)^{m+n}$ , an equation involving only the constant ratio. The calculation is understood on a differentiable branch where the original relation defines  $y$ ; at an included boundary point the result extends whenever the second derivative exists continuously.

**Step 4: Conclusion** Thus every regular differentiable branch of the given implicit relation has

$$\boxed{\frac{d^2y}{dx^2} = 0}.$$

**Quick Tip to Solve:** After logarithmic differentiation, factor the equation as  $(my - nx)(y - xy') = 0$ ; differentiate  $xy' = y$ , and check the straight-line factor separately.

**16. Find:**  $\int \frac{2x}{(x^2 + 1)(x^2 + 2)^2} dx.$

[4 marks]

**Solution:**

**Concept:**

- If the numerator contains the derivative of an expression in the denominator, substitution reduces the integral to a rational function: take  $t = x^2 + 2$ , so that  $dt = 2x dx$ .
- A repeated linear factor requires a separate partial-fraction term for each power. For the denominator  $(t - 1)t^2$ , use  $\frac{A}{t - 1} + \frac{B}{t} + \frac{D}{t^2}$ .
- The standard antiderivatives are  $\int \frac{dt}{t - a} = \log |t - a| + C$  and  $\int t^{-2} dt = -t^{-1} + C$ .

**Step 1: Substitute the quadratic expression** Denote the required integral by  $I$ . The factor  $2x dx$

**Quick Tip to Solve:** Set  $t = x^2 + 2$  first; include both  $1/t$  and  $1/t^2$  in the partial fractions, and remember that integrating  $-t^{-2}$  gives  $+1/t$ .

suggests putting  $t = x^2 + 2$ . Then  $dt = 2x dx$  and  $x^2 + 1 = t - 1$ , so

$$I = \int \frac{dt}{(t-1)t^2}.$$

This substitution removes the quadratic expressions simultaneously. We now have a rational function with one simple factor and one repeated factor, suitable for partial fractions.

**Step 2: Determine the partial fractions** Write

$$\frac{1}{(t-1)t^2} = \frac{A}{t-1} + \frac{B}{t} + \frac{D}{t^2}.$$

Multiplying by  $(t-1)t^2$  gives the polynomial identity

$$1 = At^2 + Bt(t-1) + D(t-1).$$

Setting  $t = 1$  gives  $A = 1$ , and setting  $t = 0$  gives  $D = -1$ . These substitutions determine coefficients of a polynomial identity, so they are valid even though these values are not in the substituted integral's domain. Comparing coefficients of  $t^2$  gives  $A + B = 0$ , hence  $B = -1$ . Thus

$$\frac{1}{(t-1)t^2} = \frac{1}{t-1} - \frac{1}{t} - \frac{1}{t^2}.$$

**Step 3: Integrate and return to the original variable** Integrating each term, with one arbitrary constant  $C$ , we obtain

$$\begin{aligned} I &= \log |t-1| - \log |t| + \frac{1}{t} + C \\ &= \log \left( \frac{x^2+1}{x^2+2} \right) + \frac{1}{x^2+2} + C. \end{aligned}$$

Both quadratic expressions are positive for every real  $x$ , so absolute-value signs are unnecessary in the final logarithm. As a check, differentiation gives

$$\begin{aligned} I' &= \frac{2x}{x^2+1} - \frac{2x}{x^2+2} - \frac{2x}{(x^2+2)^2} \\ &= \frac{2x[(x^2+2)^2 - (x^2+1)(x^2+2) - (x^2+1)]}{(x^2+1)(x^2+2)^2} \\ &= \frac{2x}{(x^2+1)(x^2+2)^2}, \end{aligned}$$

which is exactly the printed integrand.

**Step 4: Conclusion**

$$\int \frac{2x dx}{(x^2+1)(x^2+2)^2} = \log \left( \frac{x^2+1}{x^2+2} \right) + \frac{1}{x^2+2} + C.$$

**Quick Tip to Solve:** Set  $t = x^2 + 2$  first; include both  $1/t$  and  $1/t^2$  in the partial fractions, and remember that integrating  $-t^{-2}$  gives  $+1/t$ .

**17(a). Evaluate:**  $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$ .

[4 marks]

**Solution:**

**Concept:**

- The reflection property of a definite integral is  $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ . It follows by putting  $t = a - x$ , reversing the limits, and cancelling the negative differential.
- Here  $\sin(\pi - x) = \sin x$  and  $\cos^2(\pi - x) = \cos^2 x$ . Thus the factor  $\sin x/(1 + \cos^2 x)$  is unchanged by reflection about  $\pi/2$ .
- The standard antiderivative  $\int du/(1 + u^2) = \tan^{-1} u + C$  applies after the substitution  $u = \cos x$ , for which  $du = -\sin x dx$ .

**Step 1: Reflect the integral in its interval** Let  $I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$ . The denominator is at least one, so the integrand is continuous throughout the interval and the definite integral exists. Apply the reflection property with  $a = \pi$ :

$$I = \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$

The square on the cosine is essential: although cosine changes sign under this reflection, its square does not.

**Step 2: Add the original and reflected forms** Both expressions represent the same number  $I$ . Adding them combines the two linear factors  $x$  and  $\pi - x$ , leaving the constant  $\pi$ :

$$\begin{aligned} 2I &= \int_0^\pi \frac{[x + (\pi - x)] \sin x}{1 + \cos^2 x} dx \\ &= \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx. \end{aligned}$$

This removes the troublesome factor  $x$  without introducing integration by parts. Write the remaining integral as  $J$ , so that  $I = \pi J/2$ .

**Step 3: Evaluate the remaining integral by substitution** Put  $u = \cos x$ . The limits become  $u = 1$  at  $x = 0$ , and  $u = -1$  at  $x = \pi$ . Since  $du = -\sin x dx$ , reversing these limits gives

$$\begin{aligned} J &= - \int_1^{-1} \frac{du}{1 + u^2} = \int_{-1}^1 \frac{du}{1 + u^2} \\ &= [\tan^{-1} u]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

These inverse-tangent values lie in the principal interval  $(-\pi/2, \pi/2)$ . Substituting this value of  $J$  into the relation from the preceding step determines the required integral. Its positive sign also agrees with the nonnegative integrand on  $[0, \pi]$ .

**Step 4: Conclusion** Therefore,  $I = \frac{\pi}{2} \cdot \frac{\pi}{2} = \boxed{\frac{\pi^2}{4}}$ .

**Quick Tip to Solve:** Reflect  $x$  to  $\pi - x$  first: the other factor is symmetric, so adding the two integrals replaces  $x$  by the constant  $\pi/2$ .

**17(b). Evaluate:**  $\int_0^{3/2} |x \sin \pi x| dx$ .

[4 marks]

**Solution:**

**Concept:**

- For a real number  $v$ ,  $|v| = v$  when  $v \geq 0$ , and  $|v| = -v$  when  $v < 0$ . A modulus integral must therefore be split at the points where its inner expression changes sign.
- Since  $x \geq 0$  throughout this interval, the sign of  $x \sin \pi x$  is determined by  $\sin \pi x$ . The sine is nonnegative on  $[0, 1]$  and nonpositive on  $[1, 3/2]$ .
- Integration by parts uses  $\int u dv = uv - \int v du$ . Taking  $u = x$  and  $dv = \sin \pi x dx$  makes the remaining integral a simple cosine integral.

**Step 1: Remove the modulus on each interval** Denote the required integral by  $I$ . The zeros of  $\sin \pi x$  in the given interval occur at  $x = 0$  and  $x = 1$ . On  $(0, 1)$ , its argument lies between 0 and  $\pi$ , where sine is positive. On  $(1, 3/2]$ , its argument lies between  $\pi$  and  $3\pi/2$ , where sine is negative. Hence

$$I = \int_0^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx.$$

The minus sign on the second integral converts its negative signed contribution into a positive area.

**Step 2: Find a common antiderivative** Use integration by parts with  $u = x$ ,  $du = dx$ ,  $dv = \sin \pi x dx$ , and  $v = -\cos \pi x / \pi$ . Thus

$$\begin{aligned} \int x \sin \pi x dx &= -\frac{x \cos \pi x}{\pi} + \frac{1}{\pi} \int \cos \pi x dx \\ &= -\frac{x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2} + C. \end{aligned}$$

Let  $F(x) = -x \cos \pi x / \pi + \sin \pi x / \pi^2$ . Differentiating checks the result: the cosine terms  $-\cos \pi x / \pi$  and  $+\cos \pi x / \pi$  cancel, leaving  $F'(x) = x \sin \pi x$ .

**Step 3: Substitute the three endpoints** At the relevant endpoints, use  $\cos 0 = 1$ ,  $\cos \pi = -1$ ,  $\cos(3\pi/2) = 0$ , and  $\sin(3\pi/2) = -1$ . This gives

$$F(0) = 0, \quad F(1) = \frac{1}{\pi}, \quad F(3/2) = -\frac{1}{\pi^2}.$$

Therefore the two signed integrals are

$$\int_0^1 x \sin \pi x dx = \frac{1}{\pi}, \quad \int_1^{3/2} x \sin \pi x dx = -\frac{1}{\pi^2} - \frac{1}{\pi}.$$

Subtract the second signed integral from the first; both contributions to the modulus integral are positive.

**Step 4: Conclusion** Hence,  $I = \frac{1}{\pi} - \left(-\frac{1}{\pi^2} - \frac{1}{\pi}\right) = \boxed{\frac{2}{\pi} + \frac{1}{\pi^2}}$ .

**Quick Tip to Solve:** Split at  $x = 1$  before integrating. Use one antiderivative for both pieces and subtract the signed integral on  $[1, 3/2]$ .

**18. Prove that  $x^2 - y^2 = c(x^2 + y^2)^2$  is the general solution of the differential equation  $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$ , where  $c$  is a parameter.** [4 marks]

**Solution:**

**Concept:**

- A general solution of a first-order differential equation contains one arbitrary constant. Along a solution curve, a function whose total differential vanishes remains constant.
- The product and quotient rules allow us to differentiate an implicit relation. The differential of  $x^2 + y^2$  is  $2x dx + 2y dy$ .

**Step 1: Differentiate the proposed family** Treat  $c$  as a constant in the given family

$$x^2 - y^2 = c(x^2 + y^2)^2.$$

Differentiating both sides gives

$$2x dx - 2y dy = 2c(x^2 + y^2)(2x dx + 2y dy).$$

Dividing by 2, we obtain

$$x dx - y dy = 2c(x^2 + y^2)(x dx + y dy).$$

This differentiation is implicit: both  $x$  and  $y$  vary along a member of the family, while its parameter  $c$  stays fixed.

**Step 2: Eliminate the arbitrary constant** Away from the origin,  $x^2 + y^2 \neq 0$ . The original relation gives

$$c = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Substituting into the differentiated relation and multiplying by  $x^2 + y^2$ ,

$$(x^2 + y^2)(x dx - y dy) = 2(x^2 - y^2)(x dx + y dy).$$

Collecting the coefficients of the two differentials gives

$$\begin{aligned} (-x^3 + 3xy^2) dx + (y^3 - 3x^2y) dy &= 0, \\ (x^3 - 3xy^2) dx &= (y^3 - 3x^2y) dy. \end{aligned}$$

Thus every member of the proposed family satisfies the printed differential equation.

**Step 3: Verify that the equation integrates to this family** To establish generality, set

$$F(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Direct use of the quotient rule yields

$$dF = \frac{-2(x^3 - 3xy^2) dx + 2(y^3 - 3x^2y) dy}{(x^2 + y^2)^3}.$$

**Quick Tip to Solve:** Differentiate the proposed family first, then substitute for its constant; this avoids a lengthy homogeneous substitution.

The numerator is zero by the given equation, hence  $dF = 0$ . Integration on a connected solution interval therefore gives  $F = c$ , with one arbitrary constant. Multiplying by  $(x^2 + y^2)^2$  recovers the proposed family. The quotient calculation is made away from the origin; the resulting implicit polynomial relation itself is also defined at the origin.

**Step 4: Conclusion** Hence  $\boxed{x^2 - y^2 = c(x^2 + y^2)^2}$ , containing one arbitrary constant  $c$ , is the general solution of the given differential equation.

**Quick Tip to Solve:** Differentiate the proposed family first, then substitute for its constant; this avoids a lengthy homogeneous substitution.

**19. Let  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{b} = \hat{i}$  and  $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ , then**

**(a) Let  $c_1 = 1$  and  $c_2 = 2$ , find  $c_3$  which makes  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  coplanar.**

**(b) If  $c_2 = -1$  and  $c_3 = 1$ , show that no value of  $c_1$  can make  $\vec{a}$ ,  $\vec{b}$  and  $\vec{c}$  coplanar.**

[4 marks]

**Solution:**

**Concept:**

- Three vectors are coplanar if and only if their scalar triple product is zero:  $[\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$ .
- The scalar triple product is the determinant with the three vectors' components as rows. A zero value means that the parallelepiped formed by them has zero volume.

**Step 1: Obtain the common condition for coplanarity** The vectors have components  $\vec{a} = (1, 1, 1)$ ,  $\vec{b} = (1, 0, 0)$  and  $\vec{c} = (c_1, c_2, c_3)$ . Consequently,

$$[\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Expanding along the second row, whose last two entries vanish, gives

$$[\vec{a} \vec{b} \vec{c}] = - \begin{vmatrix} 1 & 1 \\ c_2 & c_3 \end{vmatrix} = -(c_3 - c_2) = c_2 - c_3.$$

Therefore the necessary and sufficient condition for the three vectors to be coplanar is  $c_2 = c_3$ . In particular, this condition does not involve  $c_1$ .

**Step 2: Solve part (a)** Here  $c_1 = 1$  and  $c_2 = 2$ . Substituting in the condition above gives

$$2 - c_3 = 0, \quad c_3 = 2.$$

This value can also be checked by a linear combination. With  $c_3 = 2$ ,

$$\vec{c} = \hat{i} + 2\hat{j} + 2\hat{k} = 2(\hat{i} + \hat{j} + \hat{k}) - \hat{i} = 2\vec{a} - \vec{b}.$$

Hence  $\vec{c}$  lies in the plane spanned by  $\vec{a}$  and  $\vec{b}$ , confirming coplanarity.

**Quick Tip to Solve:** Expand the determinant along the row for the unit vector  $\hat{i}$ ; it immediately reduces coplanarity to equality of the other two components.

**Step 3: Prove part (b)** With  $c_2 = -1$  and  $c_3 = 1$ , the scalar triple product is

$$[\vec{a} \vec{b} \vec{c}] = -1 - 1 = -2 \neq 0.$$

It remains nonzero for every real value of  $c_1$ , so no such value can make the vectors coplanar. To see why varying  $c_1$  cannot help, observe that a vector in the span of  $\vec{a}$  and  $\vec{b}$  has the form

$$\lambda \vec{a} + \mu \vec{b} = (\lambda + \mu)\hat{i} + \lambda\hat{j} + \lambda\hat{k}.$$

Its  $\hat{j}$  and  $\hat{k}$  coefficients must be equal. In this part they are  $-1$  and  $1$ , respectively, so the required equality is impossible, regardless of the  $\hat{i}$  coefficient.

**Step 4: Conclusion** (a) The required value is  $c_3 = 2$ . (b) No real value of  $c_1$  makes the vectors coplanar, since their scalar triple product equals  $-2$ .

**Quick Tip to Solve:** Expand the determinant along the row for the unit vector  $\hat{i}$ ; it immediately reduces coplanarity to equality of the other two components.

**20. If  $\vec{a}, \vec{b}, \vec{c}$  are mutually perpendicular vectors of equal magnitudes, show that the vector  $\vec{a} + \vec{b} + \vec{c}$  is equally inclined to  $\vec{a}, \vec{b}$  and  $\vec{c}$ . Also, find the angle which  $\vec{a} + \vec{b} + \vec{c}$  makes with  $\vec{a}$  or  $\vec{b}$  or  $\vec{c}$ .** [4 marks]

**Solution:**

**Concept:**

- For two nonzero vectors  $\vec{u}, \vec{v}$ , their angle  $\theta$  satisfies  $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$ .
- Mutually perpendicular vectors have zero pairwise scalar products. The square of a vector's magnitude is its scalar product with itself.

**Step 1: Use the common magnitude and perpendicularity** Let  $|\vec{a}| = |\vec{b}| = |\vec{c}| = m > 0$ , and put  $\vec{s} = \vec{a} + \vec{b} + \vec{c}$ . The vectors must be nonzero for their angles to be defined. Since they are mutually perpendicular,

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0.$$

Thus every mixed term vanishes when we expand the scalar product of the sum with itself.

**Step 2: Find the magnitude of the resultant** By distributivity of the scalar product,

$$\begin{aligned} |\vec{s}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} \\ &= 3m^2. \end{aligned}$$

Hence  $|\vec{s}| = \sqrt{3}m$ , taking the positive square root because a magnitude is nonnegative.

**Step 3: Compare all three angles** Let  $\alpha, \beta, \gamma$  be the angles made by  $\vec{s}$  with  $\vec{a}, \vec{b}, \vec{c}$ , respectively. Taking scalar products gives

$$\vec{s} \cdot \vec{a} = m^2, \quad \vec{s} \cdot \vec{b} = m^2, \quad \vec{s} \cdot \vec{c} = m^2.$$

**Quick Tip to Solve:** First eliminate every mixed scalar product using perpendicularity; the common magnitude then cancels from each angle formula.

For example,  $\vec{s} \cdot \vec{a} = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} = m^2$ . Therefore,

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{m^2}{(\sqrt{3}m)m} = \frac{1}{\sqrt{3}}.$$

Each vector angle lies in  $[0, \pi]$ , where cosine is one-to-one. Equality of these cosines therefore proves equality of the angles, not merely equality of their scalar products.

**Step 4: Conclusion** The vector  $\vec{a} + \vec{b} + \vec{c}$  is equally inclined to all three vectors. Each angle is  $\cos^{-1}(1/\sqrt{3})$ , approximately  $54.74^\circ$ .

**Quick Tip to Solve:** First eliminate every mixed scalar product using perpendicularity; the common magnitude then cancels from each angle formula.

**21. The random variable  $X$  can take only the values 0, 1, 2, 3. Given that  $P(X = 0) = P(X = 1) = p$  and  $P(X = 2) = P(X = 3)$  such that  $\sum p_i x_i^2 = 2 \sum p_i x_i$ , find the value of  $p$ .** [4 marks]

**Solution:**

**Concept:**

- A discrete probability distribution has nonnegative probabilities whose sum is one. Here  $p_i$  denotes the probability corresponding to the value  $x_i$ .
- The sums  $\sum p_i x_i$  and  $\sum p_i x_i^2$  are  $E(X)$  and  $E(X^2)$ , respectively. The given condition is  $E(X^2) = 2E(X)$ , rather than  $E(X^2) = 2[E(X)]^2$ .

**Step 1: Express the four probabilities using two variables** Put  $P(X = 2) = P(X = 3) = q$ . The distribution has probabilities  $p, p, q, q$  at the values 0, 1, 2, 3, respectively. Since these are the only possible values of  $X$ , normalization gives

$$p + p + q + q = 1, \quad p + q = \frac{1}{2}.$$

Thus  $q = \frac{1}{2} - p$ ; no extra probability mass occurs at any other value.

**Step 2: Calculate the first and second moments** Multiply each value by its own probability to obtain the first moment, and multiply each squared value by that same probability for the second moment:

$$\begin{aligned} \sum p_i x_i &= 0p + 1p + 2q + 3q = p + 5q, \\ \sum p_i x_i^2 &= 0^2p + 1^2p + 2^2q + 3^2q = p + 13q. \end{aligned}$$

In particular, the probability at zero contributes to normalization even though it contributes zero to both moment sums.

**Quick Tip to Solve:** Name the equal probabilities at 2 and 3 as  $q$ ; normalization and the moment condition then give two simple linear equations.

**Step 3: Apply the moment condition and solve** Substitute the two sums into the stated condition:

$$p + 13q = 2(p + 5q),$$

$$p + 13q = 2p + 10q,$$

$$p = 3q.$$

Combining this relation with normalization gives  $3q + q = \frac{1}{2}$ , so  $q = \frac{1}{8}$ , and then  $p = \frac{3}{8}$ . These values are nonnegative and the four probabilities sum to  $\frac{3}{8} + \frac{3}{8} + \frac{1}{8} + \frac{1}{8} = 1$ .

**Step 4: Check the required equality directly** With these probabilities,

$$\sum p_i x_i = \frac{3}{8} + \frac{2}{8} + \frac{3}{8} = 1, \quad \sum p_i x_i^2 = \frac{3}{8} + \frac{4}{8} + \frac{9}{8} = 2.$$

Thus the second moment is exactly twice the first, as required. Normalization and the given moment condition determine the distribution uniquely.

**Step 5: Conclusion** Hence the required value is  $p = \frac{3}{8}$ .

**Quick Tip to Solve:** Name the equal probabilities at 2 and 3 as  $q$ ; normalization and the moment condition then give two simple linear equations.

**22.** Often it is taken that a truthful person commands more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society? [4 marks]

**Solution:**

**Concept:**

- Bayes' theorem reverses a conditional probability:  $P(E | R) = \frac{P(E)P(R | E)}{P(E)P(R | E) + P(E^c)P(R | E^c)}$ . Here  $E$  is the event that the die shows six and  $R$  is the event that the man reports six.
- A fair die has six equally likely outcomes, so  $P(E) = 1/6$  and  $P(E^c) = 5/6$ . The given truthfulness probability is  $4/5$ , and the probability of a false statement is  $1/5$ .

**Step 1: Define the events and conditional probabilities** Let  $E$  denote an actual six and  $R$  denote the report of a six. If a six actually appears, reporting six is a true statement. Thus  $P(R | E) = 4/5$ . If a six does not appear, the report of six is false, so in the standard truthful/false-report model of this question  $P(R | E^c) = 1/5$ . We distinguish the report from the actual die outcome: the report alone does not make the actual outcome certain.

**Step 2: Find the total probability of a report of six** The mutually exclusive events  $E$  and  $E^c$  exhaust

**Quick Tip to Solve:** For a reported outcome, form both joint contributions: actual six with truth and non-six with a false report.

all possible outcomes. By the law of total probability,

$$\begin{aligned}P(R) &= P(E)P(R | E) + P(E^c)P(R | E^c) \\&= \frac{1}{6} \cdot \frac{4}{5} + \frac{5}{6} \cdot \frac{1}{5} \\&= \frac{4}{30} + \frac{5}{30} = \frac{9}{30}.\end{aligned}$$

The first contribution corresponds to a truthful report of an actual six; the second corresponds to a false report when the die is not six.

**Step 3: Apply Bayes' theorem** The required probability is conditional on the report. Therefore

$$P(E | R) = \frac{P(E)P(R | E)}{P(R)} = \frac{(1/6)(4/5)}{9/30} = \frac{4/30}{9/30} = \frac{4}{9}.$$

The prior probability of six and the reliability of the report must both be included; using only  $4/5$  would ignore the five possible non-six outcomes.

**Step 4: Explain the value of truthfulness** Yes, I agree. A truthful person earns the trust of others because their words and actions are dependable. Such trust strengthens relationships and encourages people to respect the person. Truthfulness also reflects honesty and integrity, which are valued in society.

**Step 5: Conclusion** The probability that the die actually shows six is  $\boxed{\frac{4}{9}}$ . Yes, truthfulness promotes trust and respect in society.

**Quick Tip to Solve:** For a reported outcome, form both joint contributions: actual six with truth and non-six with a false report.

**23. Solve the following L.P.P. graphically: Minimise  $Z = 5x + 10y$  subject to constraints  $x + 2y \leq 120$ ,  $x + y \geq 60$ ,  $x - 2y \geq 0$  and  $x, y \geq 0$ .** [4 marks]

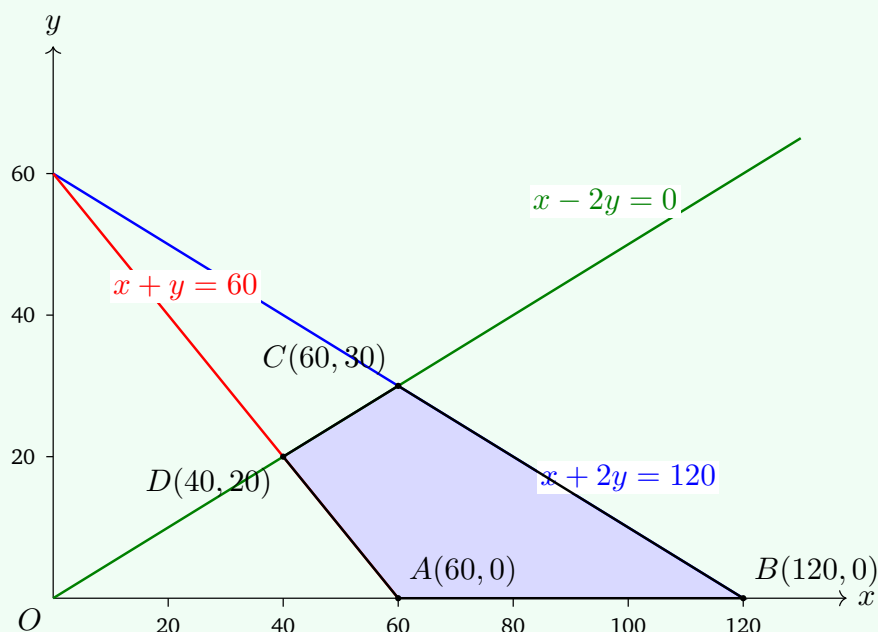
**Solution:**

**Concept:**

- The feasible region is the common part of the half-planes satisfying every constraint, including  $x \geq 0$  and  $y \geq 0$ .
- For a linear objective on a bounded feasible polygon, the minimum occurs at a vertex. Here  $Z = 5x + 10y = 5(x + 2y)$ .

**Step 1: Plot the boundary lines and choose the half-planes** The line  $x + 2y = 120$  passes through  $(120, 0)$  and  $(0, 60)$ ; take the side below it. The line  $x + y = 60$  passes through  $(60, 0)$  and  $(0, 60)$ ; take the side above it. The line  $x - 2y = 0$ , or  $y = x/2$ , passes through the origin and  $(60, 30)$ ; take the side  $y \leq x/2$ . Work only in the first quadrant. The shaded polygon in the graph is the intersection of these half-planes.

**Quick Tip to Solve:** Factor the objective as  $5(x + 2y)$ ; the bound  $x + y \geq 60$  and  $y \geq 0$  immediately checks the graphical minimum.



The shaded quadrilateral  $ABCD$  is the feasible region.

**Step 2: Find the vertices of the feasible polygon** On  $y = 0$ , the inequalities give  $60 \leq x \leq 120$ , giving  $A = (60, 0)$  and  $B = (120, 0)$ . At the intersection of  $x + 2y = 120$  and  $x = 2y$ , we get  $4y = 120$ , hence  $C = (60, 30)$ . At the intersection of  $x + y = 60$  and  $x = 2y$ , we get  $3y = 60$ , hence  $D = (40, 20)$ . These four points satisfy all the constraints. The intersection  $(0, 60)$  of the first two lines violates  $x - 2y \geq 0$ , so it is excluded.

**Step 3: Evaluate the objective at each vertex** The feasible region  $ABCD$  is bounded. Evaluating  $Z$  at its vertices gives

| Vertex      | $Z = 5x + 10y$ |
|-------------|----------------|
| $A(60, 0)$  | 300            |
| $B(120, 0)$ | 600            |
| $C(60, 30)$ | 600            |
| $D(40, 20)$ | 400            |

Thus the vertex comparison identifies  $A$  as the minimising vertex. For an independent check,  $x + 2y = (x + y) + y \geq 60$ , so  $Z \geq 300$ . Equality requires  $x + y = 60$  and  $y = 0$ , which uniquely gives  $(60, 0)$ .

**Step 4: Conclusion** The minimum value is  $Z_{\min} = 300$ , attained at  $x = 60, y = 0$ .

**Quick Tip to Solve:** Factor the objective as  $5(x + 2y)$ ; the bound  $x + y \geq 60$  and  $y \geq 0$  immediately checks the graphical minimum.

## Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24. Use product  $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix} \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$  to solve the system of equations  $x+3z=9$ ,  $-x+2y-2z=4$ ,  $2x-3y+4z=-3$ . [6 marks]

### Solution:

#### Concept:

- For square matrices  $A$  and  $B$ , the relation  $AB = I$  shows that  $B = A^{-1}$ , where  $I$  is the identity matrix.
- Transposition reverses the order of a product:  $(AB)^T = B^T A^T$ . Thus the inverse of  $A^T$  is  $B^T$ .
- Write a linear system as  $CX = D$ , with coefficient matrix  $C$ , unknown column  $X$  and constant column  $D$ . If  $C$  is invertible, then  $X = C^{-1}D$ .

**Step 1: Compute the supplied matrix product** Denote the first supplied matrix by  $A$  and the second by  $B$ . Multiplying each row of  $A$  by each column of  $B$  gives

$$AB = \begin{pmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Hence  $A^{-1} = B$ . In particular, this computation establishes the inverse directly, so there is no need to calculate cofactors separately.

**Step 2: Identify the coefficient matrix of the equations** Arrange the coefficients consistently in the order  $x, y, z$ , inserting a zero coefficient for  $y$  in the first equation. The given equations become

$$\begin{pmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ -3 \end{pmatrix}.$$

The coefficient matrix is  $A^T$ , rather than  $A$ . Transposing  $AB = I$  gives  $B^T A^T = I$ , so premultiplication of the system by  $B^T$  isolates the unknown column.

**Quick Tip to Solve:** Compare the equation coefficients with the supplied matrix before multiplying: here they form its transpose, so use the transpose of the supplied inverse.

**Step 3: Multiply by the appropriate inverse** We obtain

$$\begin{aligned}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= B^T \begin{pmatrix} 9 \\ 4 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{pmatrix} \begin{pmatrix} 9 \\ 4 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -18 + 36 - 18 \\ 0 + 8 - 3 \\ 9 - 12 + 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}.\end{aligned}$$

Since the coefficient matrix has an inverse, the resulting solution is unique.

**Step 4: Check all three original equations** Substitute the computed values into the left-hand sides:

$$\begin{aligned}x + 3z &= 0 + 3(3) = 9, \\ -x + 2y - 2z &= -0 + 2(5) - 2(3) = 4, \\ 2x - 3y + 4z &= 2(0) - 3(5) + 4(3) = -3.\end{aligned}$$

Each value agrees with its given right-hand side, confirming both the matrix orientation and the arithmetic.

**Step 5: Conclusion** The unique solution of the given system is  $x = 0, y = 5, z = 3$ .

**Quick Tip to Solve:** Compare the equation coefficients with the supplied matrix before multiplying: here they form its transpose, so use the transpose of the supplied inverse.

**25(a).** Consider  $f : \mathbb{R}_+ \rightarrow [-5, \infty)$  given by  $f(x) = 9x^2 + 6x - 5$ . Show that  $f$  is invertible with  $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$ . Hence find (i)  $f^{-1}(10)$  (ii)  $y$  if  $f^{-1}(y) = \frac{4}{3}$ , where  $\mathbb{R}_+$  is the set of all non-negative real numbers.

[6 marks]

**Solution:**

**Concept:**

- A function is invertible precisely when it is one-one and onto its stated codomain. Its inverse reverses each input-output pair.
- Here  $\mathbb{R}_+ = [0, \infty)$ . Completing the square gives  $f(x) = (3x + 1)^2 - 6$ , and  $3x + 1 \geq 1$  throughout the domain.
- To find an inverse, solve  $y = f(x)$  for the permitted value of  $x$ , keeping the domain restriction when taking square roots.

**Step 1: Show that the function is well-defined and one-one** For  $x \geq 0$ , the expression  $9x^2 + 6x$  is non-negative, so  $f(x) \geq -5$ . Thus every output belongs to the stated codomain. Now let  $a, b \in \mathbb{R}_+$

**Quick Tip to Solve:** Complete the square to obtain  $(3x + 1)^2 - 6$ ; the non-negative domain forces the positive square-root branch and simultaneously proves the inverse's range.

and suppose  $f(a) = f(b)$ . Then

$$\begin{aligned}9a^2 + 6a - 5 &= 9b^2 + 6b - 5, \\(a - b)(9(a + b) + 6) &= 0.\end{aligned}$$

Because  $a + b \geq 0$ , the second factor is at least 6 and cannot vanish. Consequently  $a = b$ . This proves that different inputs cannot have the same output; hence  $f$  is one-one.

**Step 2: Prove that every element of the codomain is attained** Let  $y$  be an arbitrary member of  $[-5, \infty)$ . Since  $y + 6 \geq 1$ , the number

$$x = \frac{\sqrt{y+6} - 1}{3}$$

is real and non-negative, so it lies in  $\mathbb{R}_+$ . For this input,  $3x + 1 = \sqrt{y+6}$ , and therefore

$$f(x) = (3x + 1)^2 - 6 = (\sqrt{y+6})^2 - 6 = y.$$

Thus every  $y$  in the codomain has a preimage in the domain, establishing that  $f$  is onto. Together with the preceding one-one proof, this makes  $f$  bijective and hence invertible.

**Step 3: Derive and verify the inverse formula** Starting with  $y = f(x)$ , completing the square yields

$$y + 6 = 9x^2 + 6x + 1 = (3x + 1)^2.$$

Since  $x \geq 0$ , the quantity  $3x + 1$  is positive. Taking the non-negative square root therefore gives  $3x + 1 = \sqrt{y+6}$ , and solving for  $x$  gives the stated formula. Its domain is  $[-5, \infty)$  and its codomain is  $\mathbb{R}_+$ . The other quadratic root  $(-1 - \sqrt{y+6})/3$  is negative and is outside the original domain. As an additional check, for  $x \geq 0$ ,

$$f^{-1}(f(x)) = \frac{\sqrt{(3x+1)^2} - 1}{3} = \frac{3x+1-1}{3} = x.$$

The previous step already verified the opposite composition for every permitted  $y$ .

**Step 4: Evaluate the two requested quantities** For part (i), substitute  $y = 10$  into the inverse expression:

$$f^{-1}(10) = \frac{\sqrt{10+6} - 1}{3} = \frac{4-1}{3} = 1.$$

For part (ii), the condition gives

$$\begin{aligned}\frac{\sqrt{y+6} - 1}{3} &= \frac{4}{3}, \\ \sqrt{y+6} &= 5, \\ y + 6 &= 25, \\ y &= 19.\end{aligned}$$

The square root equation has a positive right-hand side, so squaring introduces no extraneous solution here. Both 10 and 19 belong to the inverse's domain. Directly,  $f(1) = 10$  and  $f(4/3) = 16 + 8 - 5 = 19$ , confirming the evaluations.

**Quick Tip to Solve:** Complete the square to obtain  $(3x + 1)^2 - 6$ ; the non-negative domain forces the positive square-root branch and simultaneously proves the inverse's range.

**Step 5: Conclusion** The function is invertible, with  $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$  for  $y \geq -5$ . Hence (i)  $f^{-1}(10) = 1$ , and (ii)  $y = 19$ .

**Quick Tip to Solve:** Complete the square to obtain  $(3x+1)^2 - 6$ ; the non-negative domain forces the positive square-root branch and simultaneously proves the inverse's range.

OR

**25(b).** Discuss the commutativity and associativity of binary operation  $*$  defined on  $A = \mathbb{Q} - \{1\}$  by the rule  $a * b = a - b + ab$  for all  $a, b \in A$ . Also find the identity element of  $*$  in  $A$  and hence find the invertible elements of  $A$ . [6 marks]

**Solution:**

**Concept:**

- A binary operation on a set must be closed: its result must belong to that set for every pair of its elements.
- Commutativity requires  $a * b = b * a$ ; associativity requires  $(a * b) * c = a * (b * c)$  for every permissible triple.
- A two-sided identity  $e$  satisfies  $a * e = e * a = a$  for every  $a$ . An inverse is defined relative to such an identity.

**Step 1: Check the printed rule and closure** The printed rule is  $a * b = a - b + ab$ , on  $A = \mathbb{Q} \setminus \{1\}$ . Its values are rational, but they need not remain in  $A$ . In fact,  $0, -1 \in A$ , whereas

$$0 * (-1) = 0 - (-1) + 0(-1) = 1 \notin A.$$

Thus the rule, literally as printed, does not define a binary operation on  $A$ . We can nevertheless test the algebraic rule for the requested properties using examples whose intermediate results belong to  $A$ .

**Step 2: Test commutativity** Interchanging the inputs gives

$$a * b = a - b + ab, \quad b * a = b - a + ab,$$

so their difference is  $2(a - b)$ , which is not always zero. For example, take  $a = 2, b = 0$ . Both belong to  $A$ , and

$$2 * 0 = 2, \quad 0 * 2 = -2.$$

The rule is therefore not commutative.

**Step 3: Test associativity** Use  $a = 0, b = 0, c = 2$ ; every input and intermediate result in this example lies in  $A$ . First  $0 * 0 = 0$ , hence

$$(0 * 0) * 2 = 0 * 2 = -2.$$

**Quick Tip to Solve:** Check closure first, then test both identity equations separately: here  $a * 0 = a$ , but  $0 * a = -a$ , so a right identity is insufficient.

On the other hand,  $0 * 2 = -2$ , and so

$$0 * (0 * 2) = 0 * (-2) = 2.$$

These results differ. Consequently, even where the compositions are defined within  $A$ , the rule fails the associativity equality.

**Step 4: Examine the identity equations** A right identity must satisfy

$$a * e = a - e + ae = a \implies e(a - 1) = 0.$$

Since  $a \neq 1$ , this requires  $e = 0$ , and indeed  $a * 0 = a$ . But a left identity must satisfy  $e * a = a$  for every  $a$ . Setting  $a = 0$  forces  $e = 0$ ; with this value and  $a = 2$ , we get  $0 * 2 = -2 \neq 2$ . Thus 0 is only a right identity, and no two-sided identity exists.

**Step 5: Interpret the request for invertible elements** The usual definition of an invertible element requires a two-sided identity  $e$  and an element  $b \in A$  satisfying  $a * b = b * a = e$ . Because the printed rule has no such identity, invertibility in this standard sense is not defined. The right identity alone cannot be substituted for a two-sided identity. This obstruction is separate from, and additional to, the closure failure found above.

**Step 6: Conclusion** For the printed rule, closure fails, and the rule is neither commutative nor associative. It has right identity 0, but no two-sided identity in  $A$ ; hence there are no invertible elements in the usual identity-based sense. The printed premise calling this a binary operation on  $A$  is inconsistent with its formula.

**Quick Tip to Solve:** Check closure first, then test both identity equations separately: here  $a * 0 = a$ , but  $0 * a = -a$ , so a right identity is insufficient.

**26. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is  $\frac{\pi}{3}$ .** [6 marks]

**Solution:**

**Concept:**

- In a right angled triangle, if the hypotenuse is  $h$ , a side is  $x$ , and the other perpendicular side is  $y$ , then  $h^2 = x^2 + y^2$ .
- The area is  $\Delta = \frac{1}{2}xy$ . Maximising  $\Delta^2$  is equivalent to maximising  $\Delta$ , because both are non-negative.
- A derivative changing from positive to negative identifies a maximum. For the angle  $\theta$  between side  $x$  and the hypotenuse,  $\cos \theta = x/h$ .

**Step 1: Express the sides using the fixed sum** Let the given sum of the hypotenuse and the specified side be  $s > 0$ . Write the specified side as  $x$ , the hypotenuse as  $h$ , and the remaining perpendicular side as  $y$ . The condition gives  $h + x = s$ , so  $h = s - x$ . By the Pythagorean theorem,

$$y^2 = (s - x)^2 - x^2 = s^2 - 2sx.$$

**Quick Tip to Solve:** Use  $h = s - x$  and maximise the squared area to remove the square root; convert the resulting ratio  $x/h$  into the required angle.

A non-degenerate triangle requires  $x > 0$  and  $y > 0$ , giving the interval  $0 < x < s/2$ . Conversely, each  $x$  in this interval produces positive sides satisfying the required right triangle and fixed sum.

**Step 2: Write and differentiate the squared area** Since the two perpendicular sides are  $x$  and  $y$ , the area satisfies

$$\Delta = \frac{1}{2}x\sqrt{s^2 - 2sx}, \quad F(x) = \Delta^2 = \frac{1}{4}(s^2x^2 - 2sx^3).$$

The area is positive on the admissible interval, so its largest value occurs at exactly the same point as that of  $F$ . Differentiating the polynomial gives

$$F'(x) = \frac{1}{4}(2s^2x - 6sx^2) = \frac{sx}{2}(s - 3x).$$

Thus the only stationary point inside  $0 < x < s/2$  is  $x = s/3$ ; the root  $x = 0$  corresponds to a degenerate endpoint and is excluded.

**Step 3: Establish that this is the absolute maximum** Because  $s > 0$  and  $x > 0$ , the sign of  $F'(x)$  is the sign of  $s - 3x$ . Therefore

$$F'(x) > 0 \quad (0 < x < s/3), \quad F'(x) < 0 \quad (s/3 < x < s/2).$$

The squared area increases up to  $x = s/3$  and decreases thereafter. This proves the absolute maximum throughout the admissible interval, rather than merely locating a stationary point. Also, the area tends to zero at both degenerate endpoints, consistently with this maximum.

**Step 4: Determine the angle at the maximum** At  $x = s/3$ , the hypotenuse is  $h = s - x = 2s/3$ . Let  $\theta$  be the angle between this hypotenuse and the specified side  $x$ . The side  $x$  is adjacent to  $\theta$ , so

$$\cos \theta = \frac{x}{h} = \frac{s/3}{2s/3} = \frac{1}{2}.$$

This is an acute angle of a right triangle; hence  $\theta = \pi/3$ . As a check, the remaining side is  $y = s/\sqrt{3}$ , and the corresponding area is  $s^2/(6\sqrt{3})$ .

**Step 5: Conclusion** The maximum area occurs at  $x = s/3$ ,  $h = 2s/3$ . Therefore the angle between the given side and the hypotenuse is  $\boxed{\pi/3}$ , as required.

**Quick Tip to Solve:** Use  $h = s - x$  and maximise the squared area to remove the square root; convert the resulting ratio  $x/h$  into the required angle.

**27(a).** Using integration, find the area of region bounded by the triangle whose vertices are  $(-2, 1)$ ,  $(0, 4)$  and  $(2, 3)$ . [6 marks]

**Solution:**

**Concept:**

- The area between an upper curve  $y = f(x)$  and a lower curve  $y = g(x)$  is  $\int_a^b [f(x) - g(x)] dx$ , where  $a, b$  are the limits of the region.
- A straight line through two points has slope  $m = (y_2 - y_1)/(x_2 - x_1)$  and equation  $y - y_1 = m(x - x_1)$ .

**Quick Tip to Solve:** Split at the middle vertex's x-coordinate,  $x=0$ ; subtract the line AC from each of the two upper sides before integrating.

For a triangle, split the integral where its upper boundary changes from one side to another.

**Step 1: Find the equations of the three sides** Write the vertices as  $A(-2, 1)$ ,  $B(0, 4)$  and  $C(2, 3)$ . The slopes of  $AB$ ,  $BC$  and  $AC$ , respectively, are

$$\frac{4-1}{0-(-2)} = \frac{3}{2}, \quad \frac{3-4}{2-0} = -\frac{1}{2}, \quad \frac{3-1}{2-(-2)} = \frac{1}{2}.$$

Thus their equations are

$$AB : y = \frac{3}{2}x + 4, \quad BC : y = 4 - \frac{x}{2}, \quad AC : y = \frac{x}{2} + 2.$$

Each equation gives the stated coordinates at both endpoints of its side.

**Step 2: Identify the upper and lower boundaries** The side  $AC$  is the lower boundary throughout  $-2 \leq x \leq 2$ . The upper boundary is  $AB$  for  $-2 \leq x \leq 0$ , and  $BC$  for  $0 \leq x \leq 2$ . Consequently, the height of a vertical strip is  $x + 2$  on the first interval and  $2 - x$  on the second. Both heights are nonnegative on their respective intervals and vanish at the outer vertices. Hence, if  $S$  denotes the required area,

$$S = \int_{-2}^0 \left[ \left( \frac{3}{2}x + 4 \right) - \left( \frac{x}{2} + 2 \right) \right] dx + \int_0^2 \left[ \left( 4 - \frac{x}{2} \right) - \left( \frac{x}{2} + 2 \right) \right] dx.$$

**Step 3: Evaluate both definite integrals** The portion to the left of  $B$  has area

$$S_1 = \int_{-2}^0 (x + 2) dx = \left[ \frac{x^2}{2} + 2x \right]_{-2}^0 = 0 - (2 - 4) = 2.$$

The portion to the right of  $B$  has area

$$S_2 = \int_0^2 (2 - x) dx = \left[ 2x - \frac{x^2}{2} \right]_0^2 = (4 - 2) - 0 = 2.$$

These strips cover the triangle completely, meeting only at  $x = 0$ , so their areas are added without double counting.

**Step 4: Conclusion** The area of the triangle is  $S = S_1 + S_2 = 2 + 2 = \boxed{4 \text{ square units}}$ .

**Quick Tip to Solve:** Split at the middle vertex's x-coordinate,  $x=0$ ; subtract the line  $AC$  from each of the two upper sides before integrating.

**27(b).** Find the area bounded by the circle  $x^2 + y^2 = 16$  and the line  $\sqrt{3}y = x$  in the first quadrant, using integration. [6 marks]

**Solution:**

**Concept:**

- In the first quadrant the circle  $x^2 + y^2 = 16$  has upper boundary  $y = \sqrt{16 - x^2}$ , radius 4, and the line is  $y = x/\sqrt{3}$ .
- The area beneath a nonnegative boundary  $y = f(x)$  is  $\int_a^b f(x) dx$ . When a region's boundary changes at an intersection, evaluate its area in separate intervals.
- The standard antiderivative is  $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$ , valid for  $-a \leq x \leq a$ .

**Step 1: Find the intersection in the first quadrant** Put  $y = x/\sqrt{3}$  in  $x^2 + y^2 = 16$ :

$$x^2 + \frac{x^2}{3} = 16 \implies x = 2\sqrt{3}, \quad y = 2.$$

The first-quadrant intersection is  $P(2\sqrt{3}, 2)$ . Let  $O(0, 0)$  be the origin and  $A(4, 0)$  the circle's x-intercept. The sector adjoining the positive x-axis has boundary  $OP$ , arc  $PA$  and  $OA$ .

**Step 2: Set up the area integral** For  $0 \leq x \leq 2\sqrt{3}$ , the upper boundary is  $y = x/\sqrt{3}$ ; for  $2\sqrt{3} \leq x \leq 4$ , it is  $y = \sqrt{16 - x^2}$ . The lower boundary is  $y = 0$ . Hence

$$S = \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{2\sqrt{3}}^4 \sqrt{16 - x^2} dx.$$

Splitting at the intersection keeps every vertical strip within the sector.

**Step 3: Evaluate the two contributions** The first integral is

$$\left[ \frac{x^2}{2\sqrt{3}} \right]_0^{2\sqrt{3}} = 2\sqrt{3}.$$

For the second integral, use

$$F(x) = \frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1}(x/4).$$

Since  $\sin^{-1}(1) = \pi/2$  and  $\sin^{-1}(\sqrt{3}/2) = \pi/3$ ,

$$F(4) = 4\pi,$$

$$F(2\sqrt{3}) = 2\sqrt{3} + \frac{8\pi}{3},$$

$$F(4) - F(2\sqrt{3}) = \frac{4\pi}{3} - 2\sqrt{3}.$$

Thus  $S = 2\sqrt{3} + (4\pi/3 - 2\sqrt{3}) = 4\pi/3$ .

**Step 4: Check the complementary first-quadrant region** The full quadrant has area  $\pi(4)^2/4 = 4\pi$ . Therefore the complementary sector adjoining the positive y-axis has area  $4\pi - 4\pi/3 = 8\pi/3$ . This

**Quick Tip to Solve:** First locate the intersection  $(2\sqrt{3}, 2)$ ; the x-axis sector uses the line up to that x-value and the circle thereafter.

also equals the integral of circle minus line from  $x = 0$  to  $x = 2\sqrt{3}$ .

**Step 5: Conclusion** The sector between the line and the positive x-axis has area  $4\pi/3$  square units. The complementary sector between the line and the positive y-axis has area  $8\pi/3$  square units.

**Quick Tip to Solve:** First locate the intersection  $(2\sqrt{3}, 2)$ ; the x-axis sector uses the line up to that x-value and the circle thereafter.

**28. Solve the differential equation**  $x \frac{dy}{dx} + y = x \cos x + \sin x$ , **given that**  $y = 1$  **when**  $x = \frac{\pi}{2}$ . [6 marks]

**Solution:**

**Concept:**

- A linear differential equation has the form  $y' + P(x)y = Q(x)$ . Its integrating factor is  $e^{\int P(x) dx}$ , which turns the left-hand side into the derivative of a product.
- The product rule gives  $\frac{d}{dx}(xy) = xy' + y$ . Also,  $\frac{d}{dx}(x \sin x) = x \cos x + \sin x$ , so the two sides of this equation can be integrated directly.
- The constant of integration is determined by the specified value of  $y$ . Since the initial value is at  $x = \pi/2 > 0$ , the solution may first be obtained on the interval  $x > 0$ .

**Step 1: Recognise the linear equation and its integrating factor** The given equation is

$$x \frac{dy}{dx} + y = x \cos x + \sin x.$$

For  $x \neq 0$ , division by  $x$  gives

$$\frac{dy}{dx} + \frac{1}{x}y = \cos x + \frac{\sin x}{x}.$$

Thus  $P(x) = 1/x$ . On  $x > 0$ , an integrating factor is

$$\text{I.F.} = e^{\int dx/x} = e^{\log x} = x.$$

Multiplication by this factor returns the original equation and shows why its left-hand side is a product derivative. There is no need to separate the terms involving  $y$  or introduce an additional substitution.

**Step 2: Integrate the product derivative** Applying the product rule on both sides gives

$$\frac{d}{dx}(xy) = \frac{d}{dx}(x \sin x).$$

Integrating with respect to  $x$ ,

$$xy = x \sin x + C, \quad y = \sin x + \frac{C}{x} \quad (x \neq 0).$$

Equivalently, integration by parts yields  $\int x \cos x dx = x \sin x + \cos x$ , while  $\int \sin x dx = -\cos x$ . The cosine terms cancel, confirming the same general integral. Only one arbitrary constant is required because the differential equation is of first order.

**Quick Tip to Solve:** Before computing an integrating factor, check whether the left side is a product derivative: here both sides are derivatives of products with  $x$ .

**Step 3: Apply the given value and verify the particular solution** Substitute  $x = \pi/2$  and  $y = 1$  into the integrated equation:

$$\frac{\pi}{2} = \frac{\pi}{2} \sin \frac{\pi}{2} + C = \frac{\pi}{2} + C.$$

Hence  $C = 0$ . For this value the term  $C/x$  disappears. To check directly, the resulting function has derivative  $dy/dx = \cos x$ , so

$$x \frac{dy}{dx} + y = x \cos x + \sin x, \quad y\left(\frac{\pi}{2}\right) = 1.$$

Both the differential equation and the prescribed initial value are satisfied. Although division by  $x$  excluded zero during the linear-equation method, this particular function also satisfies the original undivided equation at zero and extends smoothly there.

**Step 4: Conclusion** The required particular solution is  $y = \sin x$ .

**Quick Tip to Solve:** Before computing an integrating factor, check whether the left side is a product derivative: here both sides are derivatives of products with  $x$ .

**29(a).** Find the equation of the plane through the line of intersection of  $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$  and  $\vec{r} \cdot (\hat{i} - \hat{j}) + 4 = 0$  and perpendicular to the plane  $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$ . Hence find whether the plane thus obtained contains the line  $x - 1 = 2y - 4 = 3z - 12$ . [6 marks]

**Solution:**

**Concept:**

- If two intersecting planes have equations  $F_1 = 0$  and  $F_2 = 0$ , a plane through their common line can be written as  $F_1 + \lambda F_2 = 0$ , where  $\lambda$  is a real parameter.
- Two planes are perpendicular when their normal vectors have zero dot product. For a plane  $ax + by + cz + d = 0$ , a normal vector is  $(a, b, c)$ .
- A line lies in a plane only if a point on the line belongs to the plane and its direction is parallel to the plane. Alternatively, substitution of its parametric coordinates must satisfy the plane equation for every parameter value.

**Step 1: Form the family through the intersection** Put  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ . The given planes become

$$F_1 = 2x - 3y + 4z - 1 = 0, \quad F_2 = x - y + 4 = 0.$$

Their normals  $(2, -3, 4)$  and  $(1, -1, 0)$  are not parallel, so they intersect in a line. Every common point satisfies  $F_1 + \lambda F_2 = 0$ . Thus the required plane has the form

$$(2 + \lambda)x - (3 + \lambda)y + 4z - 1 + 4\lambda = 0.$$

Its normal is  $\vec{n} = (2 + \lambda, -3 - \lambda, 4)$ . The exceptional member  $F_2 = 0$  need not be selected separately here: its normal has dot product 3, rather than zero, with the normal of the plane to which perpendicularity is required.

**Quick Tip to Solve:** Expand the dot product before solving for the family parameter; then substitute a common parameter in the line to check containment as an identity.

**Step 2: Impose perpendicularity and check the resulting plane** The plane  $2x - y + z + 8 = 0$  has normal  $\vec{m} = (2, -1, 1)$ . Therefore

$$\begin{aligned}\vec{n} \cdot \vec{m} &= 0, \\ 2(2 + \lambda) + (-3 - \lambda)(-1) + 4 &= 0, \\ 11 + 3\lambda &= 0, \quad \lambda = -\frac{11}{3}.\end{aligned}$$

Substituting this value in the family and multiplying by 3 gives

$$-5x + 2y + 12z - 47 = 0.$$

For a direct verification, its normal  $(-5, 2, 12)$  has dot product  $-10 - 2 + 12 = 0$  with  $(2, -1, 1)$ . Its equation is also precisely  $3F_1 - 11F_2 = 0$ , so every point on the original intersection belongs to this plane.

**Step 3: Test whether the given line is contained in the plane** Set the three equal expressions on the given line equal to  $t$ :

$$x - 1 = 2y - 4 = 3z - 12 = t.$$

Consequently,

$$x = 1 + t, \quad y = 2 + \frac{t}{2}, \quad z = 4 + \frac{t}{3}.$$

Substitute in the left side of the plane equation:

$$\begin{aligned}-5x + 2y + 12z - 47 &= -5(1 + t) + 2\left(2 + \frac{t}{2}\right) + 12\left(4 + \frac{t}{3}\right) - 47 \\ &= (-5 + 4 + 48 - 47) + (-5 + 1 + 4)t = 0.\end{aligned}$$

This identity holds for every real  $t$ . In particular, the point  $(1, 2, 4)$  lies in the plane, and the line direction  $(1, 1/2, 1/3)$  has zero dot product with its normal. Both conditions establish containment, not merely an intersection at a single point.

**Step 4: Conclusion** The required plane is  $\boxed{-5x + 2y + 12z - 47 = 0}$ , or  $\boxed{\vec{r} \cdot (-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47}$ . It contains the entire line  $\boxed{x - 1 = 2y - 4 = 3z - 12}$ .

**Quick Tip to Solve:** Expand the dot product before solving for the family parameter; then substitute a common parameter in the line to check containment as an identity.

**29(b).** Find the vector and Cartesian equations of a line passing through  $(1, 2, -4)$  and perpendicular to the two lines  $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$  and  $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$ . [6 marks]

**Solution:**

**Concept:**

- In the symmetric equation  $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ , the vector  $a\hat{i} + b\hat{j} + c\hat{k}$  gives the direction of the line;  $(x_0, y_0, z_0)$  is a point on it.
- A direction vector perpendicular to two nonparallel vectors is their cross product. Perpendicularity can be checked using  $\vec{d} \cdot \vec{u} = 0$  and  $\vec{d} \cdot \vec{v} = 0$ .
- The line through a point with position vector  $\vec{a}$  and nonzero direction vector  $\vec{d}$  has equation  $\vec{r} = \vec{a} + \lambda\vec{d}$ , where  $\lambda \in \mathbb{R}$ . Multiplying  $\vec{d}$  by a nonzero constant does not change the line.

**Step 1: Read the directions of the given lines** The denominators in the two symmetric equations give their respective direction vectors as

$$\vec{u} = 3\hat{i} - 16\hat{j} + 7\hat{k}, \quad \vec{v} = 3\hat{i} + 8\hat{j} - 5\hat{k}.$$

The constants in their numerators locate the given lines, but perpendicularity depends on these direction vectors. We seek a line through  $P(1, 2, -4)$  whose direction is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

**Step 2: Obtain and verify a perpendicular direction** Take the cross product:

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= (80 - 56)\hat{i} - (-15 - 21)\hat{j} + (24 + 48)\hat{k} \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}. \end{aligned}$$

This vector is nonzero, so the two given directions are nonparallel. Dividing by 12, we may use  $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ . The checks are

$$\begin{aligned} \vec{d} \cdot \vec{u} &= 2(3) + 3(-16) + 6(7) = 6 - 48 + 42 = 0, \\ \vec{d} \cdot \vec{v} &= 2(3) + 3(8) + 6(-5) = 6 + 24 - 30 = 0. \end{aligned}$$

Thus this direction satisfies both perpendicularity conditions.

**Step 3: Form the line through the prescribed point** The position vector of  $P$  is  $\hat{i} + 2\hat{j} - 4\hat{k}$ . Substituting this point and the direction just found into the point-direction form gives

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad \lambda \in \mathbb{R}.$$

Equating components gives  $x = 1 + 2\lambda$ ,  $y = 2 + 3\lambda$ , and  $z = -4 + 6\lambda$ . Eliminating the common parameter gives  $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$ . Taking  $\lambda = 0$  recovers  $(1, 2, -4)$ , verifying the required point.

**Quick Tip to Solve:** Use the denominators as direction ratios, take their cross product, and simplify all three components by the same factor before inserting the given point.

**Step 4: Conclusion** The required vector and Cartesian equations are, respectively,

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad \lambda \in \mathbb{R}$$

and

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

**Quick Tip to Solve:** Use the denominators as direction ratios, take their cross product, and simplify all three components by the same factor before inserting the given point.