

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) This question paper contains **29** questions.
- (iii) Questions **1-4** in Section A are very short-answer type questions carrying **1** mark each.
- (iv) Questions **5-12** in Section B are short-answer type questions carrying **2** marks each.
- (v) Questions **13-23** in Section C are long-answer I type questions carrying **4** marks each.
- (vi) Questions **24-29** in Section D are long-answer II type questions carrying **6** marks each.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If a line makes angles 90° and 60° respectively with the positive directions of x and y axes, find the angle which it makes with the positive direction of z -axis. [1 mark]

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$, and $n = \cos \gamma$, where the angles are measured from the positive coordinate axes.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$.

Step 1: Substitute the known angles Here $\alpha = 90^\circ$ and $\beta = 60^\circ$, so $l = 0$ and $m = \frac{1}{2}$. Thus

$$0^2 + \left(\frac{1}{2}\right)^2 + \cos^2 \gamma = 1.$$

Step 2: Find both possible direction angles It follows that $\cos^2 \gamma = \frac{3}{4}$, hence $\cos \gamma = \pm \frac{\sqrt{3}}{2}$. A direction angle lies between 0° and 180° ; both signs are allowed because the given information does not specify the sign of the z -component.

Step 3: Conclusion The possible angles with the positive z -axis are $\boxed{30^\circ \text{ or } 150^\circ}$.

Quick Tip to Solve: After using the sum of squared direction cosines, retain both signs of the remaining cosine.

2. Evaluate: $\int_2^3 3^x dx$. [1 mark]

Solution:

Concept:

- For a positive constant $a \neq 1$, $\frac{d}{dx} a^x = a^x \ln a$.
- The definite integral is the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Find an antiderivative Since the base is 3, division by $\ln 3$ compensates for the factor introduced by differentiation:

$$\int 3^x dx = \frac{3^x}{\ln 3} + C.$$

Here $\ln$ denotes the natural logarithm.

Step 2: Apply the limits Using the fundamental theorem of calculus gives

$$\int_2^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_2^3 = \frac{3^3 - 3^2}{\ln 3} = \frac{27 - 9}{\ln 3}.$$

The order of subtraction is upper limit first; the result is positive, as expected for a positive integrand.

Quick Tip to Solve: For a constant-base exponential, divide by the natural logarithm of its base before applying limits.

Step 3: Conclusion $\int_2^3 3^x dx = \frac{18}{\ln 3}.$

Quick Tip to Solve: For a constant-base exponential, divide by the natural logarithm of its base before applying limits.

3. If A is a 3×3 invertible matrix, then what will be the value of k if $\det(A^{-1}) = (\det A)^k$. [1 mark]

Solution:

Concept:

- An invertible matrix has $\det A \neq 0$, and $AA^{-1} = I$.
- Determinants multiply: $\det(AB) = \det A \det B$; also $\det I = 1$.

Step 1: Take determinants of the inverse identity For the given invertible 3×3 matrix, take determinants on both sides of $AA^{-1} = I_3$:

$$\det A \det(A^{-1}) = \det I_3 = 1.$$

Invertibility permits division by $\det A$.

Step 2: Write the reciprocal as a power Consequently

$$\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}.$$

Comparing with the stated general identity $\det(A^{-1}) = (\det A)^k$ identifies the exponent. The size of the matrix does not change this inverse-determinant rule.

Step 3: Conclusion The exponent in the general inverse-determinant identity is $k = -1$.

Quick Tip to Solve: Take determinants of the product with the inverse; the reciprocal corresponds to exponent minus one.

4. Determine the value of the constant k so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$. [1 mark]

Solution:

Concept:

- Continuity at $x = 0$ requires the left-hand limit, right-hand limit and $f(0)$ to agree.
- For negative x , $|x| = -x$, so $x/|x| = -1$.

Quick Tip to Solve: Replace the absolute value by minus x on the negative branch before taking the left-hand limit.

Step 1: Calculate the left-hand limit For every $x < 0$,

$$f(x) = \frac{kx}{|x|} = \frac{kx}{-x} = -k.$$

Thus $\lim_{x \rightarrow 0^-} f(x) = -k$. The cancellation is valid because negative x is nonzero.

Step 2: Impose continuity at zero The branch for $x \geq 0$ gives both $f(0) = 3$ and $\lim_{x \rightarrow 0^+} f(x) = 3$. Therefore continuity requires $-k = 3$. With this value, both branches are constantly 3 near zero, so the equality of limits and the function value is verified.

Step 3: Conclusion The required constant is $k = -3$.

Quick Tip to Solve: Replace the absolute value by minus x on the negative branch before taking the left-hand limit.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. Prove that if E and F are independent events, then the events E and F' are also independent.

[2 marks]

Solution:

Concept:

- Independence of E and F means $P(E \cap F) = P(E)P(F)$.
- The complement F' satisfies $P(F') = 1 - P(F)$. Disjoint events have additive probabilities.

Step 1: Partition the event E Every outcome of E either belongs to F or to its complement, and cannot belong to both. Hence

$$E = (E \cap F) \cup (E \cap F'),$$

where the two events in the union are disjoint. By probability addition,

$$P(E) = P(E \cap F) + P(E \cap F').$$

This partition remains valid even if one of the events has probability zero.

Step 2: Use the given independence Rearrange the preceding equality and substitute the independence assumption:

$$\begin{aligned} P(E \cap F') &= P(E) - P(E \cap F) \\ &= P(E) - P(E)P(F) \\ &= P(E)[1 - P(F)] \\ &= P(E)P(F'). \end{aligned}$$

The last expression is exactly the product of the probabilities of the two events whose independence must be established. No division by a probability is needed, so the proof covers all possible probability values.

Quick Tip to Solve: Split E into its intersections with F and its complement, then subtract the known intersection probability.

Step 3: Conclusion Since $P(E \cap F') = P(E)P(F')$, the events E and F' are independent.

Quick Tip to Solve: Split E into its intersections with F and its complement, then subtract the known intersection probability.

6. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P. for finding how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced. [2 marks]

Solution:

Concept:

- Let x be the number of necklaces and y the number of bracelets produced per day.
- The objective is total daily profit, and each stated capacity gives an inequality constraint.

Step 1: Translate the production capacities The firm handles at most 24 articles in total, giving $x + y \leq 24$. A necklace requires half an hour and a bracelet requires one hour, so the daily time requirement is $\frac{1}{2}x + y$ hours. Since at most 16 hours are available,

$$\frac{1}{2}x + y \leq 16, \quad \text{equivalently } x + 2y \leq 32.$$

At least one of each article must be made; thus $x \geq 1$ and $y \geq 1$, which are stronger than nonnegativity.

Step 2: Form the profit objective The profit from x necklaces is $100x$ rupees and that from y bracelets is $300y$ rupees. The total daily profit is therefore $Z = 100x + 300y$. The requested formulation seeks to maximize Z over the simultaneous capacity and minimum-production constraints. Counts of actual articles are whole numbers, while the usual L.P.P. formulation uses these linear inequalities to describe the feasible region; no graphical optimization is requested.

Step 3: Conclusion The L.P.P. is

$$\begin{aligned} \text{Maximize } Z &= 100x + 300y, \\ x + y &\leq 24, \quad x + 2y \leq 32, \\ x &\geq 1, \quad y \geq 1. \end{aligned}$$

Here x and y denote daily necklaces and bracelets respectively; actual production counts are integers.

Quick Tip to Solve: Keep the half-hour coefficient with necklaces, and use lower bounds of one because both products are compulsory.

7. Find: $\int \frac{dx}{x^2 + 4x + 8}$.

[2 marks]

Solution:

Concept:

- Completing the square changes $x^2 + 4x + 8$ into $(x + 2)^2 + 4$.
- The standard formula is $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1}(u/a) + C$, for $a > 0$.

Step 1: Complete the square and substitute Write

$$x^2 + 4x + 8 = x^2 + 4x + 4 + 4 = (x + 2)^2 + 2^2.$$

Set $u = x + 2$. Then $du = dx$, since the derivative of $x+2$ is 1. The integral becomes

$$\int \frac{du}{u^2 + 2^2}.$$

The denominator is always positive for real x , so the antiderivative applies on the entire real line.

Step 2: Apply the inverse-tangent formula Using the standard form with $a = 2$, the antiderivative is $\frac{1}{2} \tan^{-1}(u/2) + C$. Substitute $u = x + 2$. To check the coefficient by differentiation,

$$\frac{d}{dx} \left[\frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) \right] = \frac{1}{4} \frac{1}{1 + (x+2)^2/4} = \frac{1}{(x+2)^2 + 4}.$$

This equals the original integrand, confirming both the factor and the shift.

Step 3: Conclusion $\int \frac{dx}{x^2 + 4x + 8} = \frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) + C$, where C is an arbitrary constant.

Quick Tip to Solve: Complete the square first; the square root of the remaining constant sets both inverse-tangent factors.

8. Show that all the diagonal elements of a skew symmetric matrix are zero.

[2 marks]

Solution:

Concept:

- A real square matrix A is skew symmetric when $A^T = -A$, where A^T denotes its transpose.
- Transposition interchanges row and column indices: $(A^T)_{ij} = a_{ji}$. A diagonal entry has equal indices, so it stays in the same position.

Step 1: Compare corresponding entries Let $A = [a_{ij}]$ be a skew symmetric matrix of order n . By definition, $A^T = -A$. Equality of matrices means equality of entries at every corresponding position. Thus, for every pair i, j , we have $a_{ji} = -a_{ij}$. This relation applies in particular to the diagonal positions.

Step 2: Apply the relation to each diagonal position Put $j = i$ in the entry relation. Then

$$a_{ii} = -a_{ii} \implies 2a_{ii} = 0 \implies a_{ii} = 0.$$

Quick Tip to Solve: Compare the diagonal entries of $A^T = -A$; transposition does not move those entries.

Since i was any index from 1 to n , the argument applies separately to every diagonal entry, regardless of the entries away from the diagonal.

Step 3: Conclusion Hence every diagonal element of a skew symmetric matrix is zero: $a_{11} = a_{22} = \dots = a_{nn} = 0$.

Quick Tip to Solve: Compare the diagonal entries of $A^T = -A$; transposition does not move those entries.

9. Find $\frac{dy}{dx}$ at $x = 1, y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.

[2 marks]

Solution:

Concept:

- In implicit differentiation, treat y as a function of x , and differentiate the constant K to zero.
- The chain and product rules give $\frac{d}{dx} \cos(xy) = -\sin(xy)(y + x dy/dx)$.

Step 1: Differentiate the given relation Write $y' = dy/dx$. Differentiating $\sin^2 y + \cos(xy) = K$ with respect to x gives

$$2 \sin y \cos y y' - \sin(xy)(y + xy') = 0.$$

The factor $y + xy'$ comes from differentiating the entire product xy , while $2 \sin y \cos y = \sin 2y$.

Step 2: Collect the derivative and substitute the point Rearranging the equation,

$$(\sin 2y - x \sin(xy))y' = y \sin(xy).$$

At $x = 1, y = \pi/4$, we have $\sin 2y = 1$ and $\sin(xy) = \sqrt{2}/2$. Therefore

$$y' = \frac{\pi\sqrt{2}/8}{1 - \sqrt{2}/2} = \frac{\pi\sqrt{2}}{4(2 - \sqrt{2})}.$$

The denominator is nonzero, so division is valid. Multiplying numerator and denominator by $2 + \sqrt{2}$ simplifies this value.

Step 3: Conclusion

$$\left. \frac{dy}{dx} \right|_{(1, \pi/4)} = \frac{\pi(1 + \sqrt{2})}{4}.$$

Quick Tip to Solve: Keep xy together inside the cosine and use $(xy)' = y + xy'$ before substituting.

10. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on $\mathbb{R}$.

[2 marks]

Solution:

Concept:

- A differentiable function is nondecreasing on an interval if its derivative is nonnegative throughout that interval.
- For strict increase, one may rewrite the function as a shifted cubic; the real cubic function is strictly increasing.

Step 1: Differentiate and factor The given polynomial is differentiable at every real number. Its derivative is

$$f'(x) = 12x^2 - 36x + 27 = 3(2x - 3)^2.$$

A square is never negative, so $f'(x) \geq 0$ for all $x \in \mathbb{R}$. It is positive except at the single point $x = 3/2$, where it is zero.

Step 2: Verify strict increase even across the stationary point The derivative never changes to a negative value, so there is no interval of decrease. To make strict increase explicit, complete the cube:

$$f(x) = 4 \left(x - \frac{3}{2} \right)^3 + \frac{13}{2}.$$

If $x_1 < x_2$, then $x_1 - 3/2 < x_2 - 3/2$. Cubing preserves strict order on the real line; multiplying by the positive number 4 and adding $13/2$ also preserve it. Thus $f(x_1) < f(x_2)$.

Step 3: Conclusion Hence $f(x) = 4x^3 - 18x^2 + 27x - 7$ is strictly increasing on $\mathbb{R}$, including across $x = 3/2$.

Quick Tip to Solve: A derivative zero at one isolated point does not prevent increase; check its sign on both sides.

11. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line

$$5x - 25 = 14 - 7y = 35z.$$

[2 marks]

Solution:

Concept:

- A line through position vector $\vec{a}$ with nonzero direction vector $\vec{b}$ has equation $\vec{r} = \vec{a} + \lambda \vec{b}$, $\lambda \in \mathbb{R}$.
- Parallel lines have proportional direction vectors, so a common nonzero scalar multiple may be chosen for convenient direction ratios.

Step 1: Extract a direction vector from the given line Let the common value in $5x - 25 = 14 - 7y = 35z$ be t . Solving each equality gives

$$x = 5 + \frac{t}{5}, \quad y = 2 - \frac{t}{7}, \quad z = \frac{t}{35}.$$

Thus a direction vector is $(1/5, -1/7, 1/35)$. Multiplying all three components by 35 gives the simpler parallel direction vector $(7, -5, 1)$. The negative sign in its second component follows from $14 - 7y = t$.

Quick Tip to Solve: Set all three expressions equal to a parameter; solve for coordinates before reading direction ratios.

Step 2: Use the required point and direction The point $A(1, 2, -1)$ has position vector $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$. Using $\vec{b} = 7\hat{i} - 5\hat{j} + \hat{k}$, the required line is obtained from $\vec{r} = \vec{a} + \lambda\vec{b}$. At $\lambda = 0$, this gives A ; for varying λ , its direction remains parallel to the given line.

Step 3: Conclusion

$$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.$$

Quick Tip to Solve: Set all three expressions equal to a parameter; solve for coordinates before reading direction ratios.

12. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then find the rate of change of the slope of the curve when $x = 3$. [2 marks]

Solution:

Concept:

- The slope of a curve at a point is $m = dy/dx$; its rate of change in time is dm/dt .
- If x varies with time, the chain rule gives $\frac{dm}{dt} = \frac{dm}{dx} \frac{dx}{dt}$.

Step 1: Express the slope as a function of the coordinate For $y = 5x - 2x^3$, differentiation with respect to x gives the slope

$$m = \frac{dy}{dx} = 5 - 6x^2.$$

Differentiate the slope once more with respect to x :

$$\frac{dm}{dx} = -12x.$$

This derivative measures the change in slope per unit increase in x , whereas the question asks for the change per second.

Step 2: Convert the change with coordinate into the change with time Let t denote time in seconds. We are given $dx/dt = 2$ units per second. Apply the chain rule to the function $m(x(t))$:

$$\frac{dm}{dt} = \frac{dm}{dx} \frac{dx}{dt} = (-12x)(2) = -24x.$$

At the instant when $x = 3$, this is $-24 \times 3 = -72$. The negative sign means that the slope is becoming smaller as time increases; it must be retained in the signed rate.

Step 3: Conclusion

$$\frac{dm}{dt} = -72 \text{ s}^{-1}.$$

Thus the slope decreases at the rate of 72 per second when $x = 3$.

Quick Tip to Solve: For a rate of change of slope, differentiate dy/dx and then multiply by the supplied dx/dt .

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13(a). Evaluate: $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx.$

[4 marks]

Solution:

Concept:

- Reflection property: $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, obtained by substituting $u = a - x$ and reversing the limits.
- If $g(a-x) = g(x)$, adding reflected integrals gives $\int_0^a xg(x) dx = (a/2) \int_0^a g(x) dx$.
- For $t = \cos x$, $dt = -\sin x dx$, while $\int dt/(1+t^2) = \tan^{-1} t + C$.

Step 1: Reflect the integral about the midpoint Denote the required integral by I . Its denominator is at least 1, so its integrand is continuous on the whole interval. Applying the reflection property with $a = \pi$, and using $\sin(\pi - x) = \sin x$, $\cos^2(\pi - x) = \cos^2 x$, we obtain

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$

The trigonometric quotient stays unchanged, whereas the linear factor becomes $\pi - x$. This is why reflection simplifies this particular integral.

Step 2: Add the original and reflected expressions Adding the two equal expressions for I combines their numerator factors as $x + (\pi - x) = \pi$:

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx.$$

Write the remaining integral as J . Then $I = (\pi/2)J$. The factor of two must be retained because the left side is the sum of two copies of the required integral.

Step 3: Evaluate the trigonometric integral Put $t = \cos x$, so $dt = -\sin x dx$. At the lower endpoint $x = 0$, $t = 1$; at the upper endpoint $x = \pi$, $t = -1$. Hence

$$\begin{aligned} J &= - \int_1^{-1} \frac{dt}{1+t^2} = \int_{-1}^1 \frac{dt}{1+t^2} \\ &= [\tan^{-1} t]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

Reversing the limits cancels the negative sign from the differential. The positive value also agrees with the nonnegative original integrand on $[0, \pi]$.

Step 4: Conclusion Thus $I = (\pi/2)(\pi/2)$, and the required value is $\boxed{\pi^2/4}$.

Quick Tip to Solve: Reflect x to $\pi - x$ and add whenever a factor x multiplies a quotient symmetric about $\pi/2$.

13(b). Evaluate: $\int_0^{3/2} |x \sin \pi x| dx$.

[4 marks]

Solution:

Concept:

- The modulus satisfies $|f(x)| = f(x)$ when $f(x) \geq 0$, and $|f(x)| = -f(x)$ when $f(x) < 0$. Split the integration interval at every sign change.
- Integration by parts gives $\int u dv = uv - \int v du$. Here choose $u = x$ and $dv = \sin(\pi x) dx$.
- By the chain rule, $\int \sin(\pi x) dx = -\cos(\pi x)/\pi$; the factor π must be included.

Step 1: Determine the sign and remove the modulus Let the required integral be I . Since $x \geq 0$ throughout $[0, 3/2]$, the sign of $x \sin(\pi x)$ is determined by $\sin(\pi x)$. On $[0, 1]$, its argument runs from 0 to π , where sine is nonnegative. On $[1, 3/2]$, its argument runs from π to $3\pi/2$, where sine is nonpositive. The sign change is therefore at $x = 1$, and

$$I = \int_0^1 x \sin(\pi x) dx - \int_1^{3/2} x \sin(\pi x) dx.$$

The second contribution has a minus sign because the modulus makes the originally negative part positive.

Step 2: Find a common antiderivative by parts Taking $u = x$, $du = dx$, $dv = \sin(\pi x) dx$, and $v = -\cos(\pi x)/\pi$, we get

$$\begin{aligned} \int x \sin(\pi x) dx &= -\frac{x \cos(\pi x)}{\pi} + \frac{1}{\pi} \int \cos(\pi x) dx \\ &= -\frac{x \cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2} + C. \end{aligned}$$

Denote the displayed expression without C by $F(x)$. Differentiating it gives $-\cos(\pi x)/\pi + x \sin(\pi x) + \cos(\pi x)/\pi = x \sin(\pi x)$, verifying the antiderivative.

Step 3: Evaluate the two interval contributions At the three required endpoints,

$$F(0) = 0, \quad F(1) = \frac{1}{\pi}, \quad F(3/2) = -\frac{1}{\pi^2},$$

since $\cos \pi = -1$, $\sin \pi = 0$, $\cos(3\pi/2) = 0$, and $\sin(3\pi/2) = -1$. Therefore

$$\begin{aligned} \int_0^1 x \sin(\pi x) dx &= \frac{1}{\pi}, \\ - \int_1^{3/2} x \sin(\pi x) dx &= - \left(-\frac{1}{\pi^2} - \frac{1}{\pi} \right) = \frac{1}{\pi^2} + \frac{1}{\pi}. \end{aligned}$$

Both contributions are positive, as required for an integral of an absolute value.

Step 4: Conclusion Adding the contributions, $I = \frac{2}{\pi} + \frac{1}{\pi^2} = \frac{2\pi + 1}{\pi^2}$.

Quick Tip to Solve: Split at $x = 1$, where $\sin(\pi x)$ changes sign; use the same antiderivative on both intervals and negate the second contribution.

14. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2) dx = (y^3 - 3x^2y) dy$, where c is a parameter. [4 marks]

Solution:

Concept:

- A homogeneous differential equation is reduced by $y = vx$, with $dy = v dx + x dv$.
- The general solution of a first-order equation contains one arbitrary constant. Differentiation and elimination of that constant verifies the resulting family.

Step 1: Separate the variables On an interval where $x \neq 0$, set $y = vx$. The equation becomes

$$x^3(1 - 3v^2) dx = x^3(v^3 - 3v)(v dx + x dv).$$

Cancel x^3 and collect the terms containing dx :

$$(1 - v^4) dx = x(v^3 - 3v) dv.$$

For $v^2 \neq 1$, therefore,

$$\frac{dx}{x} = \frac{v^3 - 3v}{1 - v^4} dv = \left(-\frac{v}{1 - v^2} - \frac{2v}{1 + v^2} \right) dv.$$

The last decomposition follows by taking the common denominator $(1 - v^2)(1 + v^2)$.

Step 2: Integrate and substitute back Integration yields

$$\ln |x| = \frac{1}{2} \ln |1 - v^2| - \ln(1 + v^2) + K.$$

Combining the logarithms, exponentiating, and absorbing the constant and the fixed sign of $1 - v^2$ into a real constant c , we get

$$\frac{1 - v^2}{(1 + v^2)^2} = cx^2.$$

Since $v = y/x$, this becomes

$$\frac{x^2(x^2 - y^2)}{(x^2 + y^2)^2} = cx^2.$$

Cancel x^2 and multiply by the denominator. The excluded constant ratios $v = 1$ and $v = -1$ give $y = x$ and $y = -x$; substitution verifies both in the original equation, and both belong to the resulting family with $c = 0$.

Step 3: Verify by differentiating the family Differentiate $x^2 - y^2 = c(x^2 + y^2)^2$, keeping c constant, and divide by 2:

$$x dx - y dy = 2c(x^2 + y^2)(x dx + y dy).$$

Away from the origin, replace c by $(x^2 - y^2)/(x^2 + y^2)^2$ and multiply by $x^2 + y^2$:

$$(x^2 + y^2)(x dx - y dy) = 2(x^2 - y^2)(x dx + y dy).$$

Quick Tip to Solve: Factor $1 - v^4$ after setting $y = vx$; the two resulting logarithms combine into precisely the required squared denominator.

Expansion and collection of coefficients gives

$$(-x^3 + 3xy^2) dx + (y^3 - 3x^2y) dy = 0,$$

which rearranges to the given equation. This verification also covers points with $x = 0$, $y \neq 0$, where the substitution used above is unavailable. At the origin both coefficients of the original equation vanish.

Step 4: Conclusion Hence $x^2 - y^2 = c(x^2 + y^2)^2$, containing one arbitrary real constant c , is the general solution of the given differential equation.

Quick Tip to Solve: Factor $1 - v^4$ after setting $y = vx$; the two resulting logarithms combine into precisely the required squared denominator.

15. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

(a) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

[4 marks]

Solution:

Concept:

- Three vectors are coplanar if and only if their scalar triple product vanishes: $[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = 0$.
- The cross product of two nonparallel vectors is normal to their plane. A third vector lies in that plane precisely when its component along this normal is zero.

Step 1: Obtain the coplanarity condition The given vectors have components $\vec{a} = (1, 1, 1)$, $\vec{b} = (1, 0, 0)$, and $\vec{c} = (c_1, c_2, c_3)$. The first two are nonparallel and therefore span a plane. Their cross product is

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \hat{j} - \hat{k}.$$

Taking its dot product with $\vec{c}$ gives

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (\hat{j} - \hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) = c_2 - c_3.$$

Thus the necessary and sufficient condition for the three vectors to be coplanar is $c_2 = c_3$. Notice that c_1 does not occur because $\hat{i}$ already lies in the plane spanned by the first two vectors.

Step 2: Solve part (a) Substitute $c_1 = 1$ and $c_2 = 2$ into the coplanarity condition:

$$2 - c_3 = 0.$$

For this value, the third vector is $\vec{c} = \hat{i} + 2\hat{j} + 2\hat{k}$. As an independent check,

$$2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + \hat{k}) - \hat{i} = \hat{i} + 2\hat{j} + 2\hat{k} = \vec{c}.$$

Quick Tip to Solve: When one vector is $\hat{i}$, compute the cross product first: here the coplanarity test immediately reduces to equality of the other two components.

It is therefore a linear combination of $\vec{a}$ and $\vec{b}$, as required for coplanarity.

Step 3: Prove part (b) With $c_2 = -1$ and $c_3 = 1$, the scalar triple product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -1 - 1 = -2 \neq 0.$$

This value is independent of c_1 , so varying the coefficient of $\hat{i}$ cannot make it zero. Equivalently, every vector in the span has the form

$$\lambda \vec{a} + \mu \vec{b} = (\lambda + \mu)\hat{i} + \lambda \hat{j} + \lambda \hat{k},$$

whose $\hat{j}$ and $\hat{k}$ coefficients are equal. The coefficients -1 and 1 cannot satisfy this equality for any choice of the $\hat{i}$ coefficient.

Step 4: Conclusion (a) $c_3 = 2$. (b) No value of c_1 makes the three vectors coplanar, since their scalar triple product is always -2 .

Quick Tip to Solve: When one vector is $\hat{i}$, compute the cross product first: here the coplanarity test immediately reduces to equality of the other two components.

16. Often it is taken that a truthful person commands more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society? [4 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability: if S denotes an actual six and R denotes a report of six, then $P(S | R) = P(S)P(R | S)/P(R)$.
- The events S and $\bar{S}$ partition the die outcomes, and total probability gives $P(R) = P(S)P(R | S) + P(\bar{S})P(R | \bar{S})$.

Step 1: Identify the events and their probabilities A fair die has six equally likely faces, so $P(S) = 1/6$ and $P(\bar{S}) = 5/6$. The probability that the man speaks the truth is $4/5$, and the probability that he speaks falsely is $1/5$. In the usual truthful-or-false report model of this question, reporting six is truthful when six occurs and false when six does not occur. Hence

$$P(R | S) = \frac{4}{5}, \quad P(R | \bar{S}) = \frac{1}{5}.$$

The requested probability is conditional on hearing his report; it is not merely the original chance of six.

Step 2: Compute the posterior probability A report of six can arise from an actual six with a truthful report or from a non-six with a false report. These two possibilities are disjoint, and therefore

$$P(R) = \frac{1}{6} \cdot \frac{4}{5} + \frac{5}{6} \cdot \frac{1}{5} = \frac{4}{30} + \frac{5}{30} = \frac{9}{30}.$$

Quick Tip to Solve: For Bayes' theorem, put genuine-six reports in the numerator and all ways of reporting six in the denominator.

By Bayes' theorem,

$$P(S | R) = \frac{\frac{1}{6} \cdot \frac{4}{5}}{\frac{9}{30}} = \frac{4/30}{9/30} = \frac{4}{9}.$$

The numerator counts reports arising from a genuine six; the denominator includes every way of reporting six. Thus the man's truth probability $4/5$ is not itself the answer.

Step 3: Discuss the value of truthfulness Yes, I agree. Truthfulness makes a person dependable, since others can trust that person's statements and promises. Consistent honesty builds confidence, strengthens relationships and earns respect in society. It also encourages responsibility: a truthful person accepts facts and mistakes openly instead of misleading others.

Step 4: Conclusion The required probability is $\boxed{\frac{4}{9}}$. Yes, truthfulness promotes trust and reliability and thereby earns greater respect in society.

Quick Tip to Solve: For Bayes' theorem, put genuine-six reports in the numerator and all ways of reporting six in the denominator.

17. Prove that $\tan\left[\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right] + \tan\left[\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right] = \frac{2b}{a}$. [4 marks]

Solution:

Concept:

- The principal value $\theta = \cos^{-1}(a/b)$ lies in $[0, \pi]$, with $\cos \theta = a/b$. For the stated identity to be defined over the reals, require $b \neq 0$, $|a/b| \leq 1$, and $a \neq 0$.
- Use $\tan u + \tan v = \sin(u+v)/(\cos u \cos v)$ when both tangents exist, together with $2 \cos u \cos v = \cos(u+v) + \cos(u-v)$.

Step 1: Introduce angles and check the domain Let $\theta = \cos^{-1}(a/b)$, and put

$$u = \frac{\pi}{4} + \frac{\theta}{2}, \quad v = \frac{\pi}{4} - \frac{\theta}{2}.$$

Then $u+v = \pi/2$, $u-v = \theta$, and $\cos \theta = a/b$. The inverse cosine is real when $|a/b| \leq 1$ and $b \neq 0$. If $a = 0$, then $\theta = \pi/2$, so the first tangent is undefined and the right-hand side is also undefined. Thus the identity is understood on its domain with $a \neq 0$. For $0 \leq \theta \leq \pi$, the only possible undefined first tangent occurs at this excluded value; the second angle lies between $-\pi/4$ and $\pi/4$, where tangent is defined.

Step 2: Combine the tangent terms Express the tangents as sine divided by cosine and combine them over a common denominator:

$$\begin{aligned} \tan u + \tan v &= \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v} \\ &= \frac{\sin(u+v)}{\cos u \cos v} = \frac{1}{\cos u \cos v}. \end{aligned}$$

Here the numerator is one because $u+v = \pi/2$. The common denominator is nonzero on the domain

Quick Tip to Solve: Name the two tangent angles: their sum is $\pi/2$ and their difference is $\cos^{-1}(a/b)$, making product-to-sum immediate.

established above, so the combination is valid.

Step 3: Evaluate the cosine product By the product-to-sum identity,

$$\begin{aligned} 2 \cos u \cos v &= \cos(u + v) + \cos(u - v) \\ &= \cos \frac{\pi}{2} + \cos \theta = 0 + \frac{a}{b}. \end{aligned}$$

Consequently $\cos u \cos v = a/(2b)$. Substituting this into the combined expression gives

$$\tan u + \tan v = \frac{1}{a/(2b)} = \frac{2b}{a}.$$

This calculation uses the principal inverse-cosine value directly and does not require a sign choice for a square root or an inverse-tangent conversion.

Step 4: Conclusion Therefore $\tan[\pi/4 + \frac{1}{2} \cos^{-1}(a/b)] + \tan[\pi/4 - \frac{1}{2} \cos^{-1}(a/b)] = 2b/a$, on the real domain where both sides are defined. Hence proved.

Quick Tip to Solve: Name the two tangent angles: their sum is $\pi/2$ and their difference is $\cos^{-1}(a/b)$, making product-to-sum immediate.

18(a). Using properties of determinants, prove that $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y).$ [4 marks]

Solution:

Concept:

- Adding another column to a column leaves the determinant unchanged. A common factor in a whole column can be taken outside the determinant.
- Subtracting one row from another preserves the determinant. Expansion along a column with two zeros then reduces the calculation to a second-order minor, evaluated by $ad - bc$.

Step 1: Use the equal row sums Let the given determinant be Δ . Each row has sum $3x + 3y$. Apply $C_1 \rightarrow C_1 + C_2 + C_3$, leaving the other two columns unchanged:

$$\Delta = \begin{vmatrix} 3(x+y) & x+y & x+2y \\ 3(x+y) & x & x+y \\ 3(x+y) & x+2y & x \end{vmatrix} = 3(x+y) \begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}.$$

The column addition can be performed as two successive additions, each preserving the value. Extracting the common factor is an identity; it requires no division by $x + y$, so it is valid also when $x + y = 0$.

Step 2: Create zeros by row subtraction Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to the last determinant. The first row is unchanged. In the second row, $x - (x + y) = -y$ and $(x + y) - (x + 2y) = -y$. In the

Quick Tip to Solve: Equal row sums suggest adding all columns into one column; row subtraction then creates two zeros for a short cofactor expansion.

third row, $(x + 2y) - (x + y) = y$ and $x - (x + 2y) = -2y$. Thus

$$\Delta = 3(x + y) \begin{vmatrix} 1 & x + y & x + 2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}.$$

All occurrences of x have cancelled from the last two rows, and the first column now has only one nonzero entry.

Step 3: Evaluate the remaining minor Expand along the first column. The top entry has positive cofactor sign, while the other entries contribute zero:

$$\begin{aligned} \Delta &= 3(x + y) \begin{vmatrix} -y & -y \\ y & -2y \end{vmatrix} \\ &= 3(x + y) [(-y)(-2y) - (-y)y] \\ &= 3(x + y)(2y^2 + y^2). \end{aligned}$$

The off-diagonal product is $-y^2$; subtracting it adds y^2 . The minor is therefore $3y^2$. No division by y has been used, so this calculation includes $y = 0$ as well.

Step 4: Conclusion Hence $\Delta = 9y^2(x + y)$, which proves the required identity for all x, y .

Quick Tip to Solve: Equal row sums suggest adding all columns into one column; row subtraction then creates two zeros for a short cofactor expansion.

OR

18(b). Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$; find a matrix D such that $CD - AB = O$.

[4 marks]

Solution:

Concept:

- Matrix multiplication uses the row-column rule: entry (i, j) of AB is the scalar product of row i of A and column j of B .
- If C is invertible, $CD = AB$ gives $D = C^{-1}AB$ by multiplication on the left. Matrix products must retain their order.
- For $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, provided $ad - bc \neq 0$.

Step 1: Compute the product on the right The given equation $CD - AB = O$ is equivalent to

Quick Tip to Solve: Isolate the product CD first, then multiply by the inverse of C on the left; verify the result by checking CD against AB .

$CD = AB$. Using the row-column rule,

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2(5) - 7 & 2(2) - 4 \\ 3(5) + 4(7) & 3(2) + 4(4) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix}. \end{aligned}$$

The negative entry in row one of A contributes to both entries of row one in AB .

Step 2: Check that the coefficient matrix is invertible For the printed matrix C ,

$$|C| = \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2(8) - 5(3) = 1 \neq 0.$$

Thus the inverse exists and is

$$C^{-1} = \frac{1}{1} \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}.$$

Multiply $CD = AB$ on the left by this inverse. By associativity, $(C^{-1}C)D = C^{-1}(AB)$, so $D = C^{-1}(AB)$. Multiplying on the right would not cancel the factor C on the left of D .

Step 3: Obtain the entries of the unknown matrix Therefore

$$\begin{aligned} D &= \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} \\ &= \begin{pmatrix} 24 - 215 & 0 - 110 \\ -9 + 86 & 0 + 44 \end{pmatrix} = \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}. \end{aligned}$$

There is a unique solution, since the inverse operation applied to any solution gives this same matrix.

Step 4: Verify the original equation Direct multiplication gives

$$CD = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix} \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix} = \begin{pmatrix} -382 + 385 & -220 + 220 \\ -573 + 616 & -330 + 352 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix}.$$

This is exactly the product AB found above, so every entry of $CD - AB$ vanishes.

Step 5: Conclusion The required matrix is $D = \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}$.

Quick Tip to Solve: Isolate the product CD first, then multiply by the inverse of C on the left; verify the result by checking CD against AB .

19(a). Differentiate the function $(\sin x)^x + \sin^{-1} \sqrt{x}$ with respect to x .

[4 marks]

Solution:

Concept:

- A variable base raised to a variable exponent is differentiated logarithmically: taking a logarithm converts the exponent into a multiplier, after which the product rule applies.
- For the inverse sine function, the chain rule is $\frac{d}{dx} \sin^{-1} v = \frac{v'}{\sqrt{1-v^2}}$. Here $v = \sqrt{x}$; $\sin^{-1}$ denotes inverse sine.

Step 1: Differentiate the variable power Put $u = (\sin x)^x$. On an interval with $\sin x > 0$, take natural logarithms to obtain $\log u = x \log(\sin x)$. Differentiating both sides using the product and chain rules,

$$\begin{aligned}\frac{u'}{u} &= \log(\sin x) + x \frac{\cos x}{\sin x} \\ &= \log(\sin x) + x \cot x.\end{aligned}$$

Multiplication by u therefore yields

$$u' = (\sin x)^x [\log(\sin x) + x \cot x].$$

The logarithm term arises from differentiating the exponent x , and the cotangent term arises from differentiating the base $\sin x$. Treating the exponent as constant would incorrectly omit the first term.

Step 2: Differentiate the inverse sine term and add Since $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$, the chain rule gives

$$\begin{aligned}\frac{d}{dx} \sin^{-1} \sqrt{x} &= \frac{1}{\sqrt{1-(\sqrt{x})^2}} \frac{1}{2\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}\sqrt{1-x}}.\end{aligned}$$

Differentiation distributes over a sum, so add this expression to u' . For the given real function, the common differentiable domain is $0 < x < 1$: the inverse sine argument is in its domain, neither square-root denominator vanishes, and $\sin x > 0$ allows the logarithmic calculation. The endpoint values do not belong to this differentiable interval.

Step 3: Conclusion Thus, for $0 < x < 1$,

$$\frac{d}{dx} [(\sin x)^x + \sin^{-1} \sqrt{x}] = (\sin x)^x [\log(\sin x) + x \cot x] + \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Quick Tip to Solve: Use logarithmic differentiation for $(\sin x)^x$; for $\sin^{-1} \sqrt{x}$, remember to multiply the inverse sine derivative by $1/(2\sqrt{x})$.

19(b). If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2 y}{dx^2} = 0$.

[4 marks]

Solution:

Concept:

- Logarithmic differentiation transforms products and powers into sums. Implicit differentiation treats y as a function of x , with $y' = dy/dx$.
- The quotient rule gives $\frac{d}{dx}(y/x) = \frac{xy' - y}{x^2}$. If $y' = y/x$, this derivative vanishes and a further differentiation gives $y'' = 0$.

Step 1: Logarithmically differentiate the given relation On a differentiable branch where logarithms are defined, take logarithms of $x^m y^n = (x + y)^{m+n}$, with the exponents fixed:

$$m \log x + n \log y = (m + n) \log(x + y).$$

Differentiating with respect to x gives

$$\frac{m}{x} + \frac{n}{y} y' = \frac{m + n}{x + y} (1 + y').$$

Multiplying by $xy(x + y)$ and collecting the terms containing y' ,

$$\begin{aligned} my(x + y) + nx(x + y)y' &= (m + n)xy(1 + y'), \\ my^2 - nxy + (nx^2 - mxy)y' &= 0, \\ (my - nx)(y - xy') &= 0. \end{aligned}$$

This factorisation also identifies the exceptional case that must be checked before cancelling a factor.

Step 2: Obtain the first derivative and differentiate again Wherever $my - nx \neq 0$, the preceding identity implies $y - xy' = 0$, so $y' = y/x$. Differentiating this quotient,

$$y'' = \frac{xy' - y}{x^2} = \frac{x(y/x) - y}{x^2} = 0.$$

If instead $my - nx = 0$ throughout a branch and $m \neq 0$, then $y = (n/m)x$, which also has zero second derivative. At an isolated point where this factor vanishes, the same result follows by continuity of the second derivative on a smooth branch. If $m = 0$ and $n \neq 0$, the factor is $-nx$, so the ordinary cancellation applies for $x \neq 0$. Thus the exceptional factor does not invalidate the result for nondegenerate exponents. As usual, the relation is considered on a smooth branch where its powers and the differentiation are defined; m, n are not both zero, since that choice would make the given equation an identity independent of y .

Step 3: Conclusion Therefore, on every smooth nondegenerate branch of the given relation,

$$\boxed{\frac{d^2 y}{dx^2} = 0}.$$

Quick Tip to Solve: After logarithmic differentiation, factor the equation as $(my - nx)(y - xy') = 0$; check the linear exceptional branch before cancelling $my - nx$.

20. The random variable X can take only the values 0, 1, 2, 3. Given that $P(2) = P(3) = p$ and $P(0) = 2P(1)$. If $\sum p_i x_i^2 = 2 \sum p_i x_i$, find the value of p . [4 marks]

Solution:

Concept:

- A probability distribution satisfies $\sum_i p_i = 1$, with all probabilities nonnegative.
- The moments are weighted sums: $E(X) = \sum_i p_i x_i$ and $E(X^2) = \sum_i p_i x_i^2$. Here the given condition is $E(X^2) = 2E(X)$.

Step 1: Express and normalize the distribution Put $P(X = 1) = a$. Then $P(X = 0) = 2a$, while $P(X = 2) = P(X = 3) = p$. Since these are the only possible values, the sum of their probabilities is one:

$$2a + a + p + p = 1, \quad 3a + 2p = 1.$$

The notation $P(2)$ and $P(3)$ denotes the probabilities of those values, rather than the values themselves. The probability at zero is essential in this normalization even though it contributes nothing to the moments.

Step 2: Calculate and compare the moments Weight each value and its square by the corresponding probability:

$$\begin{aligned} E(X) &= 0(2a) + 1(a) + 2p + 3p = a + 5p, \\ E(X^2) &= 0^2(2a) + 1^2(a) + 2^2p + 3^2p = a + 13p. \end{aligned}$$

The given equality therefore becomes

$$a + 13p = 2(a + 5p) = 2a + 10p,$$

which gives $a = 3p$. Both sides have been calculated from the same distribution; in particular the square is applied to each value of X , not to its probability.

Step 3: Find the parameter and verify the distribution Substituting $a = 3p$ into $3a + 2p = 1$ gives

$$3(3p) + 2p = 1, \quad 11p = 1.$$

The four probabilities, in the order $X = 0, 1, 2, 3$, are consequently $6/11, 3/11, 1/11, 1/11$. They are all nonnegative, sum to 1, and satisfy $P(0) = 2P(1)$. Moreover $E(X) = 8/11$ and $E(X^2) = 16/11$, verifying the printed moment condition exactly.

Step 4: Conclusion Therefore, $p = \frac{1}{11}$.

Quick Tip to Solve: Write the moment condition as $\sum p_i x_i (x_i - 2) = 0$: only the values 1 and 3 contribute, giving $P(1) = 3p$ immediately.

21. Using vectors find the area of triangle ABC with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

[4 marks]

Solution:

Concept:

- A side vector is found by subtracting the coordinates of its initial point from those of its terminal point.
- For two side vectors $\mathbf{u}$ and $\mathbf{v}$ starting at the same vertex, $|\mathbf{u} \times \mathbf{v}|$ is the area of their parallelogram. The triangle has half this area.

Step 1: Form two vectors from the same vertex Taking A as the common initial point, subtract its coordinates from those of B and C :

$$\begin{aligned}\overrightarrow{AB} &= (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k} \\ &= \hat{i} - 3\hat{j} + \hat{k}, \\ \overrightarrow{AC} &= (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} \\ &= 3\hat{i} + 3\hat{j} - 4\hat{k}.\end{aligned}$$

Using two sides with the same initial point ensures that their cross product gives the parallelogram associated with this triangle.

Step 2: Evaluate the cross product Expand the determinant along its first row, remembering the negative sign of the middle cofactor:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} \\ &= \hat{i}\{(-3)(-4) - (1)(3)\} \\ &\quad - \hat{j}\{(1)(-4) - (1)(3)\} \\ &\quad + \hat{k}\{(1)(3) - (-3)(3)\} \\ &= 9\hat{i} + 7\hat{j} + 12\hat{k}.\end{aligned}$$

In particular, the middle minor equals -7 , so the negative cofactor makes the $\hat{j}$ component positive.

Step 3: Take the magnitude and halve it The magnitude of this vector gives the parallelogram area:

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{9^2 + 7^2 + 12^2} = \sqrt{274}.$$

Its diagonal divides the parallelogram into two congruent triangles. Hence the required triangle area is $\frac{1}{2}\sqrt{274}$. As a check, the cross-product vector is perpendicular to both sides: $9 - 21 + 12 = 0$ and $27 + 21 - 48 = 0$.

Step 4: Conclusion The area of triangle ABC is $\boxed{\frac{\sqrt{274}}{2} \text{ square units}}.$

Quick Tip to Solve: Form both side vectors from A , then check the middle cofactor sign before taking half the cross-product magnitude.

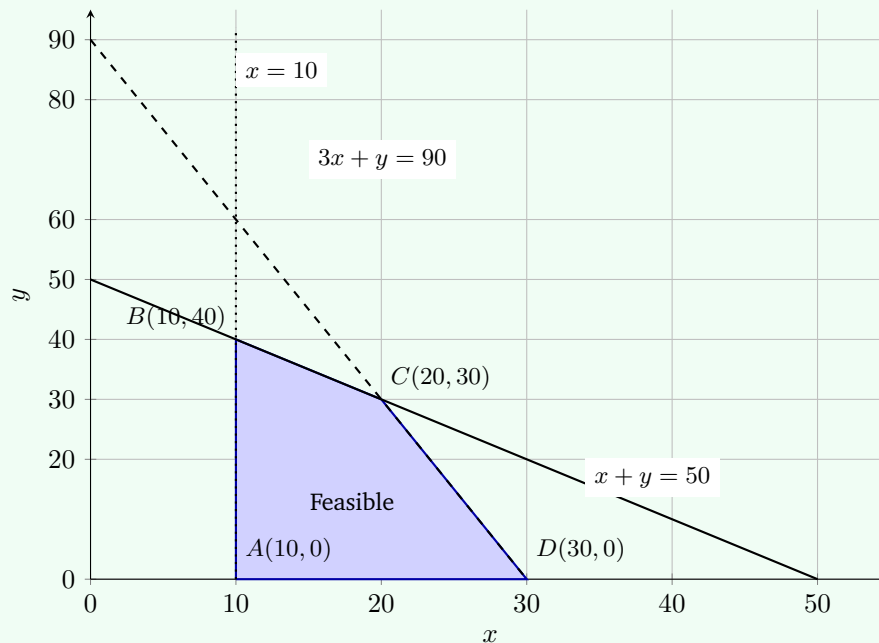
22. Solve the following L.P.P. graphically: Maximise $Z = 4x + y$ subject to constraints $x + y \leq 50$, $3x + y \leq 90$, $x \geq 10$, $x, y \geq 0$. [4 marks]

Solution:

Concept:

- Each linear inequality selects a half-plane bounded by its corresponding straight line. The feasible region is the intersection of all these half-planes, including the restrictions $x \geq 10$ and $y \geq 0$.
- For a nonempty bounded polygonal feasible region, a linear objective function attains its maximum at a corner point. Evaluate $Z = 4x + y$ at every vertex to identify the maximum.

Step 1: Plot the boundary lines and select the feasible side The line $x + y = 50$ passes through $(0, 50)$ and $(50, 0)$. The line $3x + y = 90$ passes through $(0, 90)$ and $(30, 0)$. Since the origin satisfies both upper-bound inequalities, their permitted half-planes lie below these lines in the first quadrant. Draw the vertical line $x = 10$ and retain its right-hand side. The condition $y \geq 0$ retains the region on or above the horizontal axis; $x \geq 0$ already follows from $x \geq 10$. The shaded common region in the graph is the feasible region $ABCD$.



The shaded polygon $ABCD$ is the feasible region.

Step 2: Find the vertices of the feasible polygon On $x = 10$, the bounds on y are $y \leq 40$ and $y \leq 60$; hence the upper vertex there is $B = (10, 40)$, and the lower vertex is $A = (10, 0)$. At the intersection of the two sloping boundaries,

$$\begin{aligned} x + y &= 50, \\ 3x + y &= 90, \\ 2x &= 40 \Rightarrow x = 20, \quad y = 30. \end{aligned}$$

Thus $C = (20, 30)$. On $y = 0$, the restrictions give $x \leq 50$ and $x \leq 30$, so the rightmost feasible vertex is $D = (30, 0)$. All four vertices satisfy every constraint. The common region is closed and bounded,

Quick Tip to Solve: On the x-axis, use the tighter bound $x \leq 30$; the intercept $(50, 0)$ of the other line is outside the feasible region.

so the corner-point method applies.

Step 3: Compare the objective values at all corner points Substituting the four vertices gives

Vertex (x, y)	$Z = 4x + y$
$A(10, 0)$	40
$B(10, 40)$	80
$C(20, 30)$	110
$D(30, 0)$	120

The greatest entry occurs at D . At this point $3x + y = 90$, $x + y = 30 \leq 50$, and $x \geq 10$, confirming its feasibility. The values at the neighbouring vertices differ, so the objective is not constant along either edge meeting D .

Step 4: Conclusion The maximum value is $Z_{\max} = 120$, attained uniquely at $x = 30, y = 0$.

Quick Tip to Solve: On the x -axis, use the tighter bound $x \leq 30$; the intercept $(50, 0)$ of the other line is outside the feasible region.

23. Find: $\int \frac{2x}{(x^2 + 1)(x^4 + 4)} dx.$

[4 marks]

Solution:

Concept:

- When the numerator contains $2x dx$ and the denominator is a polynomial in x^2 , the substitution $t = x^2$, $dt = 2x dx$, reduces the integral to a rational function of t .
- Resolve a rational function with a linear factor and an irreducible quadratic into partial fractions. Use $\int dt/(t + a) = \ln |t + a|$ and $\int dt/(t^2 + a^2) = a^{-1} \tan^{-1}(t/a)$, with $a > 0$.

Step 1: Substitute the square of x Let the required integral be I . Put $t = x^2$, so that $dt = 2x dx$, $x^2 + 1 = t + 1$ and $x^4 + 4 = t^2 + 4$. Consequently,

$$I = \int \frac{dt}{(t + 1)(t^2 + 4)}.$$

This substitution accounts for the entire numerator; no extra factor of two remains. The quadratic $t^2 + 4$ has no real linear factors, so its partial-fraction numerator must be linear in t .

Step 2: Determine the partial-fraction coefficients Write

$$\frac{1}{(t + 1)(t^2 + 4)} = \frac{A}{t + 1} + \frac{Bt + D}{t^2 + 4}.$$

Multiplication by the denominator yields the polynomial identity

$$\begin{aligned} 1 &= A(t^2 + 4) + (Bt + D)(t + 1) \\ &= (A + B)t^2 + (B + D)t + (4A + D). \end{aligned}$$

Quick Tip to Solve: Set $t = x$ squared before using partial fractions; this reduces the degree and prevents an unnecessarily long decomposition in x .

Comparing coefficients gives $A + B = 0$, $B + D = 0$ and $4A + D = 1$. Hence $B = -A$, $D = A$, and $5A = 1$. Thus $A = 1/5$, $B = -1/5$, $D = 1/5$, giving

$$I = \frac{1}{5} \int \frac{dt}{t+1} - \frac{1}{5} \int \frac{t dt}{t^2+4} + \frac{1}{5} \int \frac{dt}{t^2+4}.$$

Step 3: Integrate and return to x In the second integral, the derivative of the denominator is $2t$, so $\int t dt/(t^2 + 4) = \frac{1}{2} \ln(t^2 + 4)$. In the last integral, use the inverse-tangent formula with $a = 2$. Therefore

$$I = \frac{1}{5} \ln|t+1| - \frac{1}{10} \ln(t^2+4) + \frac{1}{10} \tan^{-1}\left(\frac{t}{2}\right) + C.$$

Substituting $t = x^2$ makes $t+1 = x^2+1 > 0$, so the absolute-value signs may be omitted. The other logarithm also has a strictly positive argument for every real x . As a check, differentiation of the expression in t gives

$$\frac{1}{5(t+1)} + \frac{1-t}{5(t^2+4)} = \frac{(t^2+4) + (1-t)(t+1)}{5(t+1)(t^2+4)} = \frac{1}{(t+1)(t^2+4)}.$$

Multiplying by $dt/dx = 2x$ recovers the original integrand.

Step 4: Conclusion

$$I = \frac{1}{5} \ln(x^2+1) - \frac{1}{10} \ln(x^4+4) + \frac{1}{10} \tan^{-1}\left(\frac{x^2}{2}\right) + C.$$

Quick Tip to Solve: Set $t = x$ squared before using partial fractions; this reduces the degree and prevents an unnecessarily long decomposition in x .

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Using integration, find the area of region bounded by the triangle whose vertices are $(-2, 1)$, $(0, 4)$ and $(2, 3)$. [6 marks]

Solution:

Concept:

- The area between two curves is the integral of the upper ordinate minus the lower ordinate over their common horizontal interval.
- A straight line through two points has slope $m = (y_2 - y_1)/(x_2 - x_1)$. Split an area integral wherever its upper boundary changes.

Step 1: Find the equations of the three sides Denote the vertices by $A(-2, 1)$, $B(0, 4)$ and $C(2, 3)$. The slope of AB is $(4 - 1)/(0 + 2) = 3/2$, so its equation is $y = 3x/2 + 4$. The slope of BC is $(3 - 4)/(2 - 0) = -1/2$, giving $y = 4 - x/2$. Finally, the slope of AC is $(3 - 1)/(2 + 2) = 1/2$, giving

Quick Tip to Solve: Split at the middle vertex's abscissa: here the upper side changes at $x = 0$, while the lower side stays the same.

$y = x/2 + 2$. Each equation satisfies both endpoints of its side.

Step 2: Identify the upper and lower boundaries Throughout $-2 \leq x \leq 2$, the lower side is AC . On $[-2, 0]$, the upper side is AB , whereas on $[0, 2]$, it is BC . The change occurs at vertex B , whose abscissa is zero. The two heights are therefore $x + 2$ and $2 - x$, respectively. Both heights are nonnegative on their stated intervals and vanish at the outer endpoints, as required for the triangular region.

Step 3: Integrate the two vertical heights Adding the two strips of area gives

$$\begin{aligned} A &= \int_{-2}^0 \left[\left(\frac{3x}{2} + 4 \right) - \left(\frac{x}{2} + 2 \right) \right] dx \\ &\quad + \int_0^2 \left[\left(4 - \frac{x}{2} \right) - \left(\frac{x}{2} + 2 \right) \right] dx \\ &= \left[\frac{x^2}{2} + 2x \right]_{-2}^0 + \left[2x - \frac{x^2}{2} \right]_0^2 \\ &= (0 - (-2)) + (2 - 0) = 4. \end{aligned}$$

The integration covers every vertical slice exactly once and uses the boundary lines rather than a geometric area formula.

Step 4: Conclusion The required area of the triangle is 4 square units.

Quick Tip to Solve: Split at the middle vertex's abscissa: here the upper side changes at $x = 0$, while the lower side stays the same.

OR

24(b). Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration. [6 marks]

Solution:

Concept:

- In the first quadrant the circle has upper ordinate $y = \sqrt{16 - x^2}$, and the line has ordinate $y = x/\sqrt{3}$. Integrate the difference of ordinates to obtain the region above the line.
- The standard integral is $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$, where the inverse sine is in radians.

Step 1: Locate the first-quadrant intersection Substitute $y = x/\sqrt{3}$ into the circle:

$$x^2 + \frac{x^2}{3} = 16 \implies x^2 = 12.$$

The first-quadrant intersection is therefore $P(2\sqrt{3}, 2)$. The circle meets the positive y -axis at $B(0, 4)$, and the line passes through the origin O . The region between the upper arc BP and the line segment OP has the positive y -axis as its remaining boundary.

Quick Tip to Solve: Solve the line-circle intersection first; distinguish the side of the line before choosing the integration limits.

Step 2: Set up the area integral For $0 \leq x \leq 2\sqrt{3}$, the circular ordinate is at least the line ordinate. Thus the region above the line in the first quadrant has area

$$A = \int_0^{2\sqrt{3}} \left(\sqrt{16 - x^2} - \frac{x}{\sqrt{3}} \right) dx.$$

The upper and lower ordinates agree at the intersection, so there is no need to split this integral. The subtraction removes the triangular portion below the line from the area under the circular arc over this interval.

Step 3: Evaluate the definite integral Applying the antiderivative with $a = 4$,

$$A = \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \left(\frac{x}{4} \right) - \frac{x^2}{2\sqrt{3}} \right]_0^{2\sqrt{3}}.$$

At the upper limit, $\sqrt{16 - 12} = 2$ and $\sin^{-1}(\sqrt{3}/2) = \pi/3$. The lower-limit terms all vanish. Hence

$$A = 2\sqrt{3} + \frac{8\pi}{3} - \frac{12}{2\sqrt{3}} = \frac{8\pi}{3}.$$

Step 4: Identify the complementary first-quadrant portion The line divides the first-quadrant disk into two portions. If the intended boundary uses the positive x -axis instead, its area is the complementary portion below the line. Integration gives the whole quadrant as

$$\int_0^4 \sqrt{16 - x^2} dx = 4\pi.$$

Consequently that lower portion has area $4\pi - 8\pi/3 = 4\pi/3$. Stating the boundary axis explicitly distinguishes the two regions, since the printed question supplies no shaded figure.

Step 5: Conclusion The first-quadrant region above the line has area $\boxed{\frac{8\pi}{3} \text{ square units}}$. The complementary region below the line, bounded also by the positive x -axis, has area $\boxed{\frac{4\pi}{3} \text{ square units}}$.

Quick Tip to Solve: Solve the line-circle intersection first; distinguish the side of the line before choosing the integration limits.

25. Solve the differential equation $x \frac{dy}{dx} + y = x \cos x + \sin x$, **given that** $y = 1$ **when** $x = \frac{\pi}{2}$. [6 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $e^{\int P(x) dx}$.

Quick Tip to Solve: Recognise both sides as product derivatives: the equation is $(xy)' = (x \sin x)'$, so integrate once before applying the given value.

- The product rule gives $\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$. A given value of y fixes the constant in the general solution.

Step 1: Put the equation in linear form Since the initial value is specified at $x = \pi/2 > 0$, work first on an interval containing this point with $x > 0$. Dividing the given equation by x gives

$$\frac{dy}{dx} + \frac{1}{x}y = \cos x + \frac{\sin x}{x}.$$

Thus $P(x) = 1/x$ and $Q(x) = \cos x + (\sin x)/x$. The integrating factor on this interval is

$$\text{I.F.} = e^{\int (1/x) dx} = e^{\log x} = x.$$

Multiplication by this factor makes the left side an exact derivative of a product.

Step 2: Integrate the product derivative Multiplying the linear equation by x , we obtain

$$\frac{d}{dx}(xy) = x \cos x + \sin x.$$

To integrate the right side, use integration by parts on its first term, taking $u = x$ and $dv = \cos x dx$. Then $du = dx$, $v = \sin x$, and

$$\int x \cos x dx = x \sin x - \int \sin x dx = x \sin x + \cos x.$$

Hence

$$\begin{aligned} xy &= \int (x \cos x + \sin x) dx \\ &= x \sin x + \cos x - \cos x + C \\ &= x \sin x + C. \end{aligned}$$

The cosine terms cancel; only one arbitrary constant C is needed for the first-order equation.

Step 3: Apply the initial condition and verify Substituting $x = \pi/2$ and $y = 1$ into the integrated relation gives

$$\frac{\pi}{2} = \frac{\pi}{2} \sin \frac{\pi}{2} + C = \frac{\pi}{2} + C, \quad C = 0.$$

Thus the relation reduces to $xy = x \sin x$. For $x > 0$, division by x gives $y = \sin x$. Its derivative is $dy/dx = \cos x$, so substitution into the original equation gives $x \cos x + \sin x$, exactly its right side. Also $\sin(\pi/2) = 1$, confirming the specified initial value. The same expression extends smoothly through $x = 0$ and satisfies the original undivided equation there as well.

Step 4: Conclusion The required particular solution is $y = \sin x$.

Quick Tip to Solve: Recognise both sides as product derivatives: the equation is $(xy)' = (x \sin x)'$, so integrate once before applying the given value.

26(a). Find the equation of the plane through the line of intersection of $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$. Hence find whether the plane thus obtained contains the line $x - 1 = 2y - 4 = 3z - 12$. [6 marks]

Solution:

Concept:

- If two intersecting planes are $P_1 = 0$ and $P_2 = 0$, then $P_1 + \lambda P_2 = 0$ represents planes through their common line, with the second original plane considered separately if necessary.
- Two planes are perpendicular when their normal vectors have zero scalar product. A line lies in a plane when one point on it lies in the plane and its direction vector is perpendicular to the plane's normal.

Step 1: Form the family through the intersection Writing $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the two given planes become

$$P_1 = 2x - 3y + 4z - 1 = 0, \quad P_2 = x - y + 4 = 0.$$

Every common point satisfies both equations, and therefore satisfies their linear combination

$$(2x - 3y + 4z - 1) + \lambda(x - y + 4) = 0.$$

The normal to this family member is

$$\vec{n} = (2 + \lambda)\hat{i} + (-3 - \lambda)\hat{j} + 4\hat{k}.$$

The second original plane has normal $(1, -1, 0)$. Its scalar product with $(2, -1, 1)$ is 3, so that plane is not the required perpendicular plane; the family above includes the required member.

Step 2: Impose perpendicularity to the third plane The plane $2x - y + z + 8 = 0$ has normal $\vec{m} = 2\hat{i} - \hat{j} + \hat{k}$. Perpendicularity requires

$$\begin{aligned} \vec{n} \cdot \vec{m} &= 0, \\ 2(2 + \lambda) + (-1)(-3 - \lambda) + 4 &= 0, \quad \lambda = -\frac{11}{3}. \\ 11 + 3\lambda &= 0, \end{aligned}$$

Substitute this value into the family equation and multiply throughout by 3:

$$\begin{aligned} 3(2x - 3y + 4z - 1) - 11(x - y + 4) &= 0, \\ -5x + 2y + 12z - 47 &= 0. \end{aligned}$$

Its normal $(-5, 2, 12)$ has scalar product $-10 - 2 + 12 = 0$ with the third plane's normal, verifying the perpendicularity. Its construction also ensures that it contains the intersection of the first two planes.

Step 3: Test the entire given line Let the common value in $x - 1 = 2y - 4 = 3z - 12$ be t . Then every point of the line has coordinates

$$x = 1 + t, \quad y = 2 + \frac{t}{2}, \quad z = 4 + \frac{t}{3}.$$

Quick Tip to Solve: Set the common expression equal to t : the line direction is $(1, 1/2, 1/3)$, not $(1, 2, 3)$. Check both constant and parameter terms in the plane.

Substitution into the obtained plane gives

$$\begin{aligned}-5x + 2y + 12z - 47 &= -5(1 + t) + 2\left(2 + \frac{t}{2}\right) \\ &\quad + 12\left(4 + \frac{t}{3}\right) - 47 \\ &= (-5 + 4 + 48 - 47) + (-5 + 1 + 4)t \\ &= 0.\end{aligned}$$

Both the constant term and the coefficient of t vanish. Therefore the plane equation holds for every point on the line, rather than only at a single intersection point.

Step 4: Conclusion The required plane is

$$-5x + 2y + 12z - 47 = 0.$$

Equivalently, its vector equation is

$$\vec{r} \cdot (-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47.$$

It contains the entire line $x - 1 = 2y - 4 = 3z - 12$.

Quick Tip to Solve: Set the common expression equal to t : the line direction is $(1, 1/2, 1/3)$, not $(1, 2, 3)$. Check both constant and parameter terms in the plane.

OR

26(b). Find the vector and Cartesian equations of a line passing through $(1, 2, -4)$ and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. [6 marks]

Solution:

Concept:

- In a symmetric line equation, the denominators are direction ratios. Thus the two given lines have direction vectors $\vec{b}_1 = (3, -16, 7)$ and $\vec{b}_2 = (3, 8, -5)$.
- A direction vector $\vec{d}$ perpendicular to both given directions satisfies $\vec{d} \cdot \vec{b}_1 = \vec{d} \cdot \vec{b}_2 = 0$. For non-parallel directions, their cross product supplies such a vector.
- A line through position vector $\vec{a}$ with nonzero direction vector $\vec{d}$ is $\vec{r} = \vec{a} + t\vec{d}$, where $t \in \mathbb{R}$. Multiplying all direction ratios by the same nonzero number leaves the line unchanged.

Step 1: Extract the directions of the given lines Comparing each printed equation with $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$, we obtain

$$\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}, \quad \vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}.$$

Quick Tip to Solve: Take the cross product of the denominator triples, simplify its common factor, and check both dot products before inserting the given point.

The numerators specify points on the given lines; the perpendicularity condition depends on the denominators, so those points do not enter the direction calculation.

Step 2: Calculate a common perpendicular direction Taking the cross product and expanding with the alternating signs gives

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= [80 - 56]\hat{i} - [-15 - 21]\hat{j} + [24 + 48]\hat{k} \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}.\end{aligned}$$

Dividing by 12, we may use $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. This is nonzero, so it defines a valid line direction.

Step 3: Form the line through the specified point The position vector of $(1, 2, -4)$ is $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$. Substitution in the point-direction form gives

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}.$$

Componentwise, $x = 1 + 2t$, $y = 2 + 3t$, and $z = -4 + 6t$. Eliminating t gives the required symmetric Cartesian form.

Step 4: Verify the point and perpendicularity At $t = 0$, the parametric coordinates are exactly $(1, 2, -4)$. Moreover,

$$\begin{aligned}(2, 3, 6) \cdot (3, -16, 7) &= 6 - 48 + 42 = 0, \\ (2, 3, 6) \cdot (3, 8, -5) &= 6 + 24 - 30 = 0.\end{aligned}$$

Thus the constructed direction is perpendicular to each given line direction, verifying both independent conditions.

Step 5: Conclusion The vector equation is $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + t(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}$. The Cartesian equation is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$.

Quick Tip to Solve: Take the cross product of the denominator triples, simplify its common factor, and check both dot products before inserting the given point.

27(a). Consider $f : \mathbb{R}_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$. Hence find (i) $f^{-1}(10)$ (ii) y if $f^{-1}(y) = \frac{4}{3}$, where $\mathbb{R}_+$ is the set of all non-negative real numbers.

[6 marks]

Solution:

Concept:

- A function has an inverse on its stated codomain precisely when it is bijective: distinct inputs give distinct outputs (injectivity), and every element of the codomain is attained (surjectivity).
- Completing the square gives $9x^2 + 6x - 5 = (3x + 1)^2 - 6$. On the specified domain $x \geq 0$, the quantity $3x + 1$ is positive, determining the correct square-root branch.

Step 1: Prove that the function is one-one Let $a, b \in \mathbb{R}_+$ and suppose $f(a) = f(b)$. Then

$$\begin{aligned} 9a^2 + 6a - 5 &= 9b^2 + 6b - 5, \\ (a - b)[9(a + b) + 6] &= 0. \end{aligned}$$

Since $a, b \geq 0$, the factor $9(a + b) + 6 \geq 6$ cannot vanish. Therefore $a = b$, proving injectivity. The restriction to non-negative inputs is essential here because it ensures the second factor is strictly positive.

Step 2: Prove that every codomain value is attained First, for every $x \geq 0$, $9x^2 + 6x \geq 0$, so $f(x) \geq -5$, as required by the codomain. Now choose any $y \in [-5, \infty)$ and solve $f(x) = y$:

$$(3x + 1)^2 = y + 6.$$

Because $3x + 1 \geq 1$, we take the positive square root, obtaining

$$x = \frac{\sqrt{y+6}-1}{3}.$$

Here $y + 6 \geq 1$, so the square root exists and this $x \geq 0$ belongs to the domain. Substitution gives $f(x) = (\sqrt{y+6})^2 - 6 = y$. Thus every y in the codomain has a preimage and f is onto. Together with injectivity, this proves that f is invertible.

Step 3: Establish the inverse and its domain The unique preimage found above yields

$$f^{-1} : [-5, \infty) \longrightarrow \mathbb{R}_+, \quad f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}.$$

The negative square-root branch would give $x = (-\sqrt{y+6}-1)/3 < 0$, which is excluded by the original domain. As a further check, for $x \geq 0$,

$$f^{-1}(f(x)) = \frac{\sqrt{(3x+1)^2-1}-1}{3} = \frac{3x+1-1}{3} = x.$$

Positivity of $3x + 1$ justifies removing the square root without a negative sign.

Quick Tip to Solve: Rewrite the quadratic as $(3x + 1)^2 - 6$; the non-negative domain selects the positive root. To find an inverse input from its output, apply the original function.

Step 4: Evaluate the two requested quantities For part (i), substitute $y = 10$ in the inverse formula:

$$f^{-1}(10) = \frac{\sqrt{16} - 1}{3} = \frac{4 - 1}{3} = 1.$$

This agrees with $f(1) = 9 + 6 - 5 = 10$. For part (ii),

$$\begin{aligned}\frac{\sqrt{y+6} - 1}{3} &= \frac{4}{3}, \\ \sqrt{y+6} &= 5, \\ y+6 &= 25, \quad y = 19.\end{aligned}$$

The value 19 lies in the inverse domain, and $f(4/3) = 16 + 8 - 5 = 19$, which independently checks this evaluation.

Step 5: Conclusion The function is bijective and hence invertible, with $f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$ for $y \geq -5$. The requested values are (i) $f^{-1}(10) = 1$ and (ii) $y = 19$.

Quick Tip to Solve: Rewrite the quadratic as $(3x + 1)^2 - 6$; the non-negative domain selects the positive root. To find an inverse input from its output, apply the original function.

OR

27(b). Discuss the commutativity and associativity of binary operation $*$ defined on $A = \mathbb{Q} - \{1\}$ by the rule $a * b = a - b + ab$ for all $a, b \in A$. Also find the identity element of $*$ in A and hence find the invertible elements of A . [6 marks]

Solution:

Concept:

- A binary operation on A must be closed: its value must belong to A for every pair of elements of A .
- Commutativity requires $a * b = b * a$, while associativity requires $(a * b) * c = a * (b * c)$ for every admissible triple.
- An identity e must satisfy both $a * e = a$ and $e * a = a$ for every $a \in A$. An inverse is defined relative to such a two-sided identity.

Step 1: Check the printed rule and closure The printed rule is $a * b = a - b + ab$, on $A = \mathbb{Q} \setminus \{1\}$. In particular, the middle term has a minus sign. Take $a = 0$ and $b = -1$, both of which belong to A . Then

$$0 * (-1) = 0 - (-1) + 0(-1) = 1 \notin A.$$

Thus the rule as printed is not closed on A , so strictly it does not define a binary operation on this set. Nevertheless, its requested algebraic properties can be tested directly; the following counterexamples have every intermediate result in A .

Quick Tip to Solve: For this rule, test the pairs $(0, -1)$ and $(0, 2)$ first; they reveal the closure failure and distinguish a right identity from a two-sided identity.

Step 2: Test commutativity For arbitrary rational inputs,

$$(a * b) - (b * a) = (a - b + ab) - (b - a + ab) = 2(a - b).$$

Equality is therefore not true for all pairs. For example, $0 * 2 = -2$, whereas $2 * 0 = 2$. Both inputs and both outputs lie in A , and the two outputs differ.

Step 3: Test associativity Use the admissible triple $a = 0, b = 0, c = 2$. Direct substitution gives

$$\begin{aligned}(0 * 0) * 2 &= 0 * 2 = -2, \\ 0 * (0 * 2) &= 0 * (-2) = 2.\end{aligned}$$

The intermediate values 0 and -2 belong to A . Since the two bracketings give different answers, the associativity condition fails even where both expressions are defined on the stated set.

Step 4: Investigate the identity and inverses If a two-sided identity $e \in A$ existed, the right-identity condition applied to $a = 0$ would give

$$0 * e = -e = 0, \quad e = 0.$$

This candidate does satisfy $a * 0 = a$ for every a ; however, $0 * a = -a$, which is not a in general. In particular, $0 * 2 = -2 \neq 2$. Thus 0 is only a right identity, and no two-sided identity exists. Since invertibility requires a two-sided identity, invertible elements cannot be defined for the printed rule as an operation on A . Solving an equation with the right identity alone would not supply an inverse in the required sense.

Step 5: Conclusion As printed, the rule is not closed on A . It is neither commutative nor associative, has no two-sided identity, and consequently has no invertible elements in the usual sense. The element 0 is only a right identity.

Quick Tip to Solve: For this rule, test the pairs $(0, -1)$ and $(0, 2)$ first; they reveal the closure failure and distinguish a right identity from a two-sided identity.

28. A metal box with a square base and vertical sides is to contain 1024 cm^3 . The material for the top and bottom costs ₹ 5 per cm^2 and the material for the sides costs ₹ 2.50 per cm^2 . Find the least cost of the box. [6 marks]

Solution:

Concept:

- If a closed box has square base side x and height h , its volume is x^2h , its combined top and bottom area is $2x^2$, and its four sides have total area $4xh$.
- Multiply each surface area by its own material rate to obtain the cost. Use the fixed volume to express that cost in one positive variable.
- A stationary value satisfies $C'(x) = 0$. The sign of $C'(x)$, or a positive second derivative throughout the domain, establishes a minimum.

Quick Tip to Solve: Include both square faces at the higher rate before substituting $h = 1024/x^2$; the resulting cost is $10x^2 + 10240/x$.

Step 1: Express the height in terms of the base side Let the side of the square base be x cm and the vertical height be h cm, where $x > 0$ and $h > 0$. The fixed capacity gives

$$x^2h = 1024, \quad h = \frac{1024}{x^2}.$$

The box has both a top and a bottom, so both of these square surfaces must be included in the cost. Its four vertical surfaces are rectangles with area xh each.

Step 2: Form the total cost function The top and bottom together cost $5(2x^2) = 10x^2$ rupees. The material for all four sides costs $2.50(4xh) = 10xh$ rupees. Consequently, the total cost in rupees is

$$\begin{aligned} C(x) &= 10x^2 + 10xh \\ &= 10x^2 + 10x \left(\frac{1024}{x^2} \right) \\ &= 10x^2 + \frac{10240}{x}, \quad x > 0. \end{aligned}$$

This expression allows every possible positive base side, with the corresponding height determined by the capacity.

Step 3: Find the stationary base side Differentiate with respect to x :

$$C'(x) = 20x - \frac{10240}{x^2} = \frac{20(x^3 - 512)}{x^2}.$$

Since $x^2 > 0$, the stationary condition is $x^3 = 512$, giving $x = 8$. The associated height is

$$h = \frac{1024}{8^2} = \frac{1024}{64} = 16 \text{ cm}.$$

Step 4: Verify the minimum and calculate the cost For $0 < x < 8$, $x^3 - 512 < 0$, so $C'(x) < 0$; for $x > 8$, $C'(x) > 0$. Thus the cost decreases up to $x = 8$ and increases afterwards, proving a global minimum over the positive domain. Equivalently, $C''(x) = 20 + 20480/x^3 > 0$ throughout that domain. At $x = 8$ and $h = 16$, the top and bottom area is 128 cm^2 , and the total side area is 512 cm^2 . Their respective costs are $128 \times 5 = 640$ rupees and $512 \times 2.50 = 1280$ rupees. Therefore the total minimum cost is $640 + 1280 = 1920$ rupees. The dimensions also verify the required volume, since $8 \times 8 \times 16 = 1024 \text{ cm}^3$.

Step 5: Conclusion The least cost is ₹ 1920, attained by a box with square base $8 \text{ cm} \times 8 \text{ cm}$ and height 16 cm .

Quick Tip to Solve: Include both square faces at the higher rate before substituting $h = 1024/x^2$; the resulting cost is $10x^2 + 10240/x$.

29. If $A = \begin{pmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{pmatrix}$, find A^{-1} . Using A^{-1} solve the system of equations $\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2$, $\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5$, $\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = -4$. [6 marks]

Solution:

Concept:

- A square matrix has an inverse exactly when its determinant is nonzero. For such a matrix, $A^{-1} = \frac{1}{|A|} \text{adj } A$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ uses the minor M_{ij} obtained by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.
- Put $u = 1/x$, $v = 1/y$, and $w = 1/z$, where $x, y, z \neq 0$. The given equations then form the linear system $AU = B$, so $U = A^{-1}B$.

Step 1: Calculate the determinant Expand along the first row, keeping the alternating signs:

$$\begin{aligned} |A| &= 2\{(-6)(-20) - 5(9)\} \\ &\quad - 3\{4(-20) - 5(6)\} + 10\{4(9) - (-6)(6)\} \\ &= 2(75) - 3(-110) + 10(72) \\ &= 150 + 330 + 720 = 1200. \end{aligned}$$

Since $|A| \neq 0$, the matrix is invertible and the inverse method gives a unique solution for the reciprocal variables.

Step 2: Find all cofactors and the adjugate Applying the cofactor signs to the nine minors gives

$$\begin{aligned} C_{11} &= 120 - 45 = 75, & C_{12} &= -(-80 - 30) = 110, \\ C_{13} &= 36 + 36 = 72, & C_{21} &= -(-60 - 90) = 150, \\ C_{22} &= -40 - 60 = -100, & C_{23} &= -(18 - 18) = 0, \\ C_{31} &= 15 + 60 = 75, & C_{32} &= -(10 - 40) = 30, \\ C_{33} &= -12 - 12 = -24. \end{aligned}$$

Thus the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{pmatrix}.$$

Therefore

$$A^{-1} = \frac{1}{1200} \begin{pmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{pmatrix}.$$

Step 3: Transform and solve the system using the inverse With $u = 1/x$, $v = 1/y$, and $w = 1/z$, the

Quick Tip to Solve: Introduce reciprocal variables before applying the matrix inverse, and remember to take reciprocals again at the end; the vector from the matrix product is not the vector of original unknowns.

system becomes

$$A \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix}.$$

Premultiplying both sides by the inverse just obtained,

$$\begin{aligned} \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \frac{1}{1200} \begin{pmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -4 \end{pmatrix} \\ &= \frac{1}{1200} \begin{pmatrix} 150 + 750 - 300 \\ 220 - 500 - 120 \\ 144 + 96 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 \\ -1/3 \\ 1/5 \end{pmatrix}. \end{aligned}$$

Each reciprocal is nonzero, so take reciprocals to recover the original unknowns: $x = 2$, $y = -3$, and $z = 5$.

Step 4: Check the original equations Substitution checks the signs and the reciprocal transformation:

$$\begin{aligned} 2/2 + 3/(-3) + 10/5 &= 1 - 1 + 2 = 2, \\ 4/2 - 6/(-3) + 5/5 &= 2 + 2 + 1 = 5, \\ 6/2 + 9/(-3) - 20/5 &= 3 - 3 - 4 = -4. \end{aligned}$$

All three equations are satisfied, and the inverse guarantees uniqueness for the transformed system.

Step 5: Conclusion

$$A^{-1} = \frac{1}{1200} \begin{pmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{pmatrix}, \quad \boxed{x = 2, y = -3, z = 5}.$$

Quick Tip to Solve: Introduce reciprocal variables before applying the matrix inverse, and remember to take reciprocals again at the end; the vector from the matrix product is not the vector of original unknowns.